

# **CHAOS, LOCALIZATION, AND TRANSPORT IN KICKED ROTOR AND ITS VARIANTS**

**A THESIS**

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**by**

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**2018**

*To*

*My family*



## ***CERTIFICATE***

I feel great pleasure in certifying that the thesis entitled, “**Chaos, localization, and transport in kicked rotor and its variants**” embodies a record of the results of investigations carried out by Sanku Paul under my guidance.

He has completed the following requirements as per Ph.D. regulations of the University.

- (a) Course work as per the university rules.
- (b) Residential requirements of the university.
- (c) Presented his work in the departmental committee.
- (d) Published/accepted minimum of two research papers in a referred research journal.

I am satisfied with the analysis of data, interpretation of results and conclusions drawn.

I recommend the submission of thesis.

Date : 14/5/2018.

Dr. M. S. Santhanam  
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## ***DECLARATION***

*I, Sanku Paul, hereby declare that the work incorporated in the present thesis entitled, “Chaos, localization, and transport in kicked rotor and its variants” is my own and original. This work (in part or in full) has not been submitted to any University for the award of a Degree or a Diploma.*

Date : 10/5/2018

*Sanku Paul.*  
(Sanku Paul)

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# ABSTRACT

Quantum chaos is the study of a quantum system whose classical counterpart is chaotic. One of the central questions of quantum chaos is the relation between the quantum and the classical regimes. In this work, periodically driven systems are used to study the transport and decoherence processes in quantum systems that display classically chaotic dynamics. The periodically kicked rotor, the motion of a particle on a ring experiencing kicks at a periodic time intervals, is a paradigmatic model of Hamiltonian chaos. Its classical dynamics can be reduced to a difference equation while its quantum dynamics can be determined using a Floquet operator. This system is experimentally realizable in cold atomic cloud interacting with a flashing optical lattice. Two different variants of kicked rotor system will be presented and will discuss the unusual properties that have been observed in these systems. (i) One of the variants considered is the kicked rotor in a periodic finite well potential. This system is special because it violates the assumptions of the Kolmogorov-Arnold-Moser (KAM) theorem and hence, abrupt transitions from integrability to chaos becomes possible. The quantum manifestations of the non-KAM chaotic system have been studied, and the effect of the non-KAM feature on the behavior of its spectra and also on its transport and localization properties are demonstrated. (ii) In another variant, we introduce decoherence in the quantum kicked rotor by suppressing kicks during certain time intervals drawn from Lévy distribution,  $\omega(\tau) \sim \tau^{-1-\alpha}$ . For,  $\alpha < 1$ , it is theoretically shown that this scenario leads to non-exponential coherence decay, thus effectively slowing down the decoherence to the classical domain. While, for  $\alpha > 1$ , an anomalous diffusion characteristic is observed. Both of these results have been experimentally realized and displays an excellent agreement with the theoretical results.

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## List of Publications

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1. *Barrier-induced chaos in a kicked rotor: Classical subdiffusion and quantum localization*  
Sanku Paul, Harinder Pal and M. S. Santhanam  
Phys. Rev. E **93**, 060203(R) (2016).
2. *Nonexponential Decoherence and Subdiffusion in Atom-Optics Kicked Rotor*  
Sumit Sarkar, Sanku Paul, Chetan Vishwakarma, Sunil Kumar, Gunjan Verma, M. Sainath, Umakant D. Rapol, and M. S. Santhanam  
Phys. Rev. Lett. **118**, 174101 (2017).
3. *Floquet states of a kicked particle in a singular potential: Exponential and power-law profiles*  
Sanku Paul and M. S. Santhanam  
Phys. Rev. E **97**, 032217 (2018).

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## Introduction

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### 1.1 Brief Overview

In nature, almost everything is dynamic. Historically, the concept of a dynamical system had existed for centuries, however, studying it mathematically became possible since the mid-1600s with the introduction of Newton's laws of motion [Newton 1987] and the universal gravitational force. By combining the laws of motion and the gravitational force, Newton solved a restricted form of Kepler's problem of planetary motion, namely, the motion of two bodies in the gravitational field. Given the exact position and velocity of a point particle, one can predict the past and the future states of motion using these laws of motion. Looking at the success of this theory, many physicists and mathematicians started extending this two-body problem to solve the three-body problem. But unfortunately, they faced difficulties in getting an exact analytical solution that could describe the motion.

In the late 19th century, Poincare's work contained novel ideas that gave first hints that it might be impossible to get an exact solution for the most general three-body problem. The approach to its approximate solution was laid down by considering two massive particles which were perturbed by a third less massive particle.

The question is whether this motion is stable. Poincare was the first person who had a glimpse about the existence of complicated dynamics or chaos as it is known today, “in which a deterministic system exhibits aperiodic behavior that depends sensitively on the initial conditions, thereby rendering long-term prediction impossible” [Strogatz]. The concept of chaos was relatively dormant till the development of modern-day computers using which it became possible to get the numerical solutions for the equations governing the dynamical system. It was only in 1963, Edward Lorenz, a meteorologist at MIT, with the help of Royal McBee LGP-30 computer discovered chaos on a strange attractor while he was trying to understand the unpredictable behavior of weather. He was studying an atmospheric convection model for weather forecasting, and every time he had given the same three significant digits as initial conditions, he always ended up with entirely different solutions which showed the sensitive dependence on initial conditions. As he was not specifying the higher significant digits, the computer itself was introducing some randomness in those digits that lead him to get such arbitrary solutions.

In the nineteenth century, another ground-breaking piece of work came up when W. R. Hamilton formulated the Hamiltonian mechanics [Hamilton 1834], where the system’s dynamics could be described in position-momentum phase space. The concept of Hamiltonian formalism is important because the phase space trajectories are volume-preserving and also if there is any symmetry present in the system, the phase space dimensions get reduced because of certain conserved quantities associated with the symmetries. This makes the problem easy to solve. In Hamiltonian mechanics, the motion is described by  $2n$  first order differential equations if a system has  $n$  degrees of freedom. These are Hamilton’s equations of motion and are represented as

$$\begin{aligned}\dot{\mathbf{x}} &= \frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} &= -\frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}\end{aligned}\tag{1.1}$$

where,  $H$  is the Hamiltonian of a system and  $\mathbf{x}$  and  $\mathbf{p}$  are the position and momen-

tum of a particle. Now, for a Hamiltonian system with  $n$  degrees of freedom having  $n$  constants of motion, and  $[F_i, H] = 0$ , where  $F_i$  are constants of motion, the equations of motion are completely integrable and such a system is called an integrable system. In such integrable systems, dynamics takes place on an  $n$ -dimensional torus. The motion of a free particle, 1-d harmonic oscillator are examples of integrable systems. But this class of systems are scarcely found in nature.

On the other hand, there is another class of system, which does not have  $n$  constants of motion and hence the equations are not completely integrable and are known as non-integrable systems. These systems are plenty in nature. The solutions of these systems can be obtained in a perturbative manner. The question that arises whether this perturbation series converge or not. Poincare showed that the solutions could diverge because of the small denominators in the perturbation series, which are a result of internal resonances of the system. This small denominator problem is called the small-divisor problem in classical mechanics. The solution of this problem remained a question until 1968 when the results of Kolmogorov [Kolmogorov 1954], Arnold [Arnol'd 1963] and Moser [Moser 1968], now called KAM theory, pointed out that there are convergent solutions associated with orbits whose frequencies are not close to the resonance frequencies of the system which gives rise to the small-divisor problem. The chaos is the result of the orbits associated with these internal resonance frequencies.

### 1.1.1 Kolmogorov-Arnold-Moser (KAM) Theory

The KAM theory holds for a class of non-integrable Hamiltonian systems which satisfy the KAM conditions. Consider a non-integrable Hamiltonian,  $H$ , such that it can be decomposed into an integrable part,  $H_0$  and a perturbation,  $H_1$ , and can be written as [Tabor 1989]

$$H = H_0 + \epsilon H_1, \quad (1.2)$$

where,  $\epsilon$  is the strength of the perturbation. KAM theory says that if the Hamiltonian  $H$  is analytic, however, recently it is proved that if the third derivative of the Hamiltonian exists [Herman 1983] and the unperturbed motion is non-degenerate (in the action-angle representation, the non-degeneracy condition is

$$\det \left| \frac{\partial \omega_i}{\partial I_j} \right| = \det \left| \frac{\partial^2 H_0}{\partial I_i \partial I_j} \right| \neq 0, \quad (1.3)$$

where,  $I_j$  and  $\omega$  are action and frequency), then, such a system (often called KAM system) obeys KAM theorem [Tabor 1989] which can be stated as follows:

*If  $H_1$  is small enough, then for almost all  $\omega^*$  (incommensurate frequencies or away from resonance) there exists an invariant torus  $T(\omega^*)$  of the perturbed system such that  $T(\omega^*)$  is close to  $T_0(\omega^*)$ .*

For the KAM systems, with an increase in the strength of the perturbation, invariant tori gradually break down giving rise to global chaos. The last invariant torus to break down is the one having the highest incommensurate frequency. The kicked rotor model and the three-body celestial problem are examples of KAM system. We will discuss the kicked rotor model in more detail in the later part of this chapter.

There also exists another class of systems, which fails to obey either of these two conditions, analyticity or non-degeneracy, or both, for applicability of the KAM theorem. Such systems are called non-KAM systems. For these systems, even with small perturbation, all the tori breaks down instantly to give rise to global chaos. This class of systems is not yet well studied but are of great importance because of their physical applications or realizations such as in semi-conductor superlattices [Fromhold 2000], tokamak fusion [Karney 1997], plasma physics [Chia 1996] and ion traps [Gardiner 1997]. There are also several models, like kicked harmonic oscillator [Gardiner 1997], Sinai billiards [Sinai 1963], the kicked rotor in an infinite well potential [Hu 1999, Sankaranarayanan 2001] and Fermi accelerator model [Jose 1986] that have aided our understanding of the dynamics of non-KAM systems. In the case of the kicked harmonic oscillator, the system breaks the non-

degeneracy condition while for the others, the analyticity condition of the KAM theory is violated. For non-integrable Hamiltonian systems, typically, there can exist solutions which give rise to chaotic dynamics. Notable exceptions range from the pendulum to more sophisticated ones such as the Fermi-Pasta-Ulam problem.

## 1.2 Quantum Chaos: A Brief History

In the 19th century, when most of the physicists were working on classical physics problems, a solution given by Max Planck related to black body radiation in 1900 created a revolution that led to the beginning of a new subject, the quantum mechanics. He proposed a hypothesis, “energy exchange takes place in discrete units”. Even though the wave nature of light was already known since the 18th century, when Robert Hooke, Christiaan Huygens proposed a wave theory of light based on experimental observations, but the particle nature of light came up as a result of Max Planck’s work. In 1905, Einstein’s theoretical solution using the concept of particle nature of light to the problem of the “Photoelectric effect experiment” performed by Heinrich Rudolf Hertz in 1887 [Hertz 1887] boosted the field into even higher esteem. This field got a definitive form after the seminal work of Niels Bohr, Louis de Broglie, Max Born, Werner Heisenberg, Wolfgang Pauli, Erwin Schrödinger, and many others. Quantum mechanics solved several problems related to atomic scales, such as photoelectric effects, quantum interference effects, and several phenomena in nuclear physics and quantum-optics.

With the progress in both the fields, physicists started asking questions whether the mechanics of these two fields are related ? In quantum mechanics, one talks about energy levels and wavefunctions while in classical systems, everything is related to classical trajectories which do not exist in the quantum system due to Heisenberg’s uncertainty principle. Then, how the dynamics of these systems are related ? How the wavefunctions in quantum system behaves when its classical counterpart is chaotic ? All these questions ultimately led to the beginnings of a new subject, the quantum chaos. Even though chaos in quantum systems does

not exist in the same sense of classical systems because the trajectories cannot be defined and the Schrödinger equation is linear, yet it is a study that endeavors to find the connection between classical chaos and quantum mechanics.

In 1970's, a semiclassical approach, Gutzwiller's periodic orbit quantization theory, which is a semiclassical approximation of Feynman's path integral formalism, was introduced [Gutzwiller 1967, Gutzwiller 1971, Gutzwiller 1990]. Even though for integrable systems, semiclassical methods such as WKB approximations or Einstein-Brillouin-Keller methods were known but, in 1917, Einstein realized that these methods fail for non-integrable classical systems [Einstein 1917]. Gutzwiller's periodic orbit theory, on the other hand, holds for both integrable and non-integrable systems (including chaotic system). It computes the spectra in terms of the superposition of the classical periodic orbits and its properties. Wintgen in 1988 applied this theory on the hydrogen atom in a uniform magnetic field and calculated the approximate eigenvalues of this quantum system [Wintgen 1988].

In 1973, in another pioneering work, Percival conjectured that two kinds of spectra exist in a bound system: i) regular spectrum associated with the completely integrable classical system and ii) irregular spectrum when the classical system is non-integrable [Percival 1973]. M. V. Berry and M. Tabor in 1977 showed that the level spacing distribution associated with regular spectrum follows the Poissonian distribution  $P(S) \sim \exp(-S)$ , where  $S$  represents spacings and shows level clustering [Berry 1977]. Since 1950s, a parallel and independent work in the field of statistical theory of level spectra led by Wigner was being established. Wigner began to apply random matrix theory to study the fluctuation properties of nuclear spectra [Wigner 1951]. This theory was further developed by Dyson [Dyson 1962] and Mehta [Mehta 1960] and later by several others. But it was in 1979 that a connection between classical chaotic dynamics and the random matrix theory was seen. McDonald and Kaufman [McDonald 1979] found that the level spacing distribution of quantum energy levels of classically chaotic quantum billiards also shows level repulsion similar to that of random matrices. Later, similar results were also obtained by Casati, Valz-Gris, and Guarneri [Casati 1980] in 1980. In connection with

random matrix theory, Bohigas, Giannoni, and Schmit conjectured in 1984 [Bohigas 1984] that “Spectra of time-reversal invariant systems whose classical analogs are chaotic systems show the same fluctuation properties as predicted by GOE” which is now known today as BGS conjecture.

In chaotic quantum systems like quantum billiards [Friedman 2001], microwave cavity [Stöckmann 1990], hydrogen atom in magnetic field [Holle 1986, Main 1986], complex nuclei spectra [Wigner 1951], typical wave-functions tend to be delocalized over the entire basis states. This is associated with level repulsion property which is usually seen in the chaotic systems. However, in some chaotic quantum systems, wave-function spreading is attenuated and the wave-function is localized. One such example is the dynamical localization effect, where the level spacing distribution shows the level clustering. This phenomenon is observed in standard kicked rotor system [Casati 1979, Casati 1980, Chirikov 1981] which is a periodically driven system. The periodically driven system has huge applications in the field of condensed matter physics, quantum optics, quantum chaos and many others. Some of the periodically driven systems on which many experimental and theoretical studies have been conducted are kicked Harper model [Harper 1955, Artuso 1992], hydrogen atom in microwave field [Koch 1995, Bayfield 1974], etc. In this thesis also, we will concentrate on a periodically driven system and in this regard, we will consider a variant of kicked rotor system to understand the localization and transport properties. This kicked rotor model has also been studied in a variety of contexts ranging from metal-insulator transitions [Chabe 2008, Wang 2014, Garcia 2005, Garcia 2007], coherent control [Gong 2001, Gong 2003], entanglement measures [Matsui 2016, Lakshminarayan 2001, Miller 1999, Santhanam 2008], quantum resonances [Fishman 2002, Lepers 2008, Behinaein 2006, Dana 2006], quantum ratchets [Schanz 2001, Jones 2004, Monteiro 2002, Dana 2008] to quantum transport [Jones 2004, Jones 2007, Sadgrove 2005] and decoherence effects [Sarkar 2017, Klappauf 1998, Schomerus 2007, Schomerus 2008]. Hence, in the remaining portion of the introduction, we will give a brief review on the kicked rotor system.

### 1.3 The Standard Kicked Rotor

The standard kicked rotor is considered to be the fundamental model to understand the quantum chaotic properties of a periodically driven system. This model was introduced in 1977 at the Como conference by Chirikov, *et. al.* [Casati 1979]. This was also the first numerical experimental model to study the relation between classical chaos and its quantum dynamics in the field of periodically driven systems. Kicked rotor model describes the motion of a particle on a ring receiving kicks at a periodic interval of time. This is a one-dimensional non-interacting model. The Hamiltonian of a kicked rotor model is

$$\begin{aligned} H &= \frac{p^2}{2} + K \cos(x) \sum_n \delta(t - n) \\ &= H_0 + H_1 \end{aligned} \tag{1.4}$$

where  $x$  and  $p$  are the angular position and momentum of a particle,  $K$  is the kick strength of an external field which is applied at a periodic interval of time. If  $K = 0$ ,  $H$  represents Hamiltonian of a free particle. Here,  $H_0$  is the unperturbed Hamiltonian and  $H_1$  is the perturbation. Since the Hamiltonian,  $H$ , in Eq. (1.4) is analytic and the motion due to  $H_0$  is non-degenerate, this serves as a representative model to understand the classical dynamics of a KAM system.

With the increase in the kick strength  $K$ , the system shows a transition from regular to mixed and finally gives rise to chaos for very large kick strength which is shown in Fig. 1.1. Figure 1.1(a) illustrates that for  $K = 0.25$ , the phase space is covered with invariant tori. However, at  $K = K_c = 0.961534$  shown in Fig. 1.1(b), the last invariant tori are present shown by red lines that separates the two regions of phase space and hinders the transport of particles from one region of phase space to the other. While for kick strength just above  $K_c$ , these invariant tori breaks down to give rise to global chaos and hence, particles are free to move anywhere in this chaotic region of the phase space. With further increase in the kick strength,  $K = 6.4$ , phase space is filled with mixed structures which is shown in Fig. 1.1(c). The regular structures present act as an accelerator mode (which accelerates the



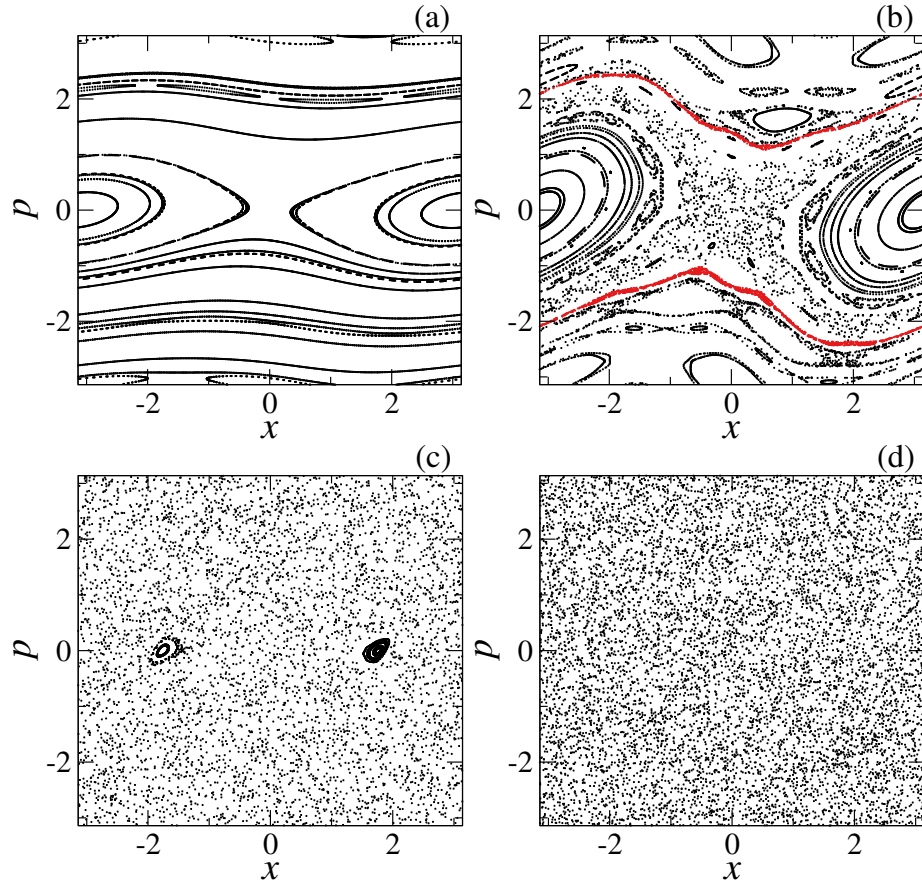


Figure 1.1: Stroboscopic sections of kicked rotor for (a)  $K = 0.25$ , (b)  $K = 0.961534$ , (c)  $K = 6.4$  and (d)  $K = 10.0$ .

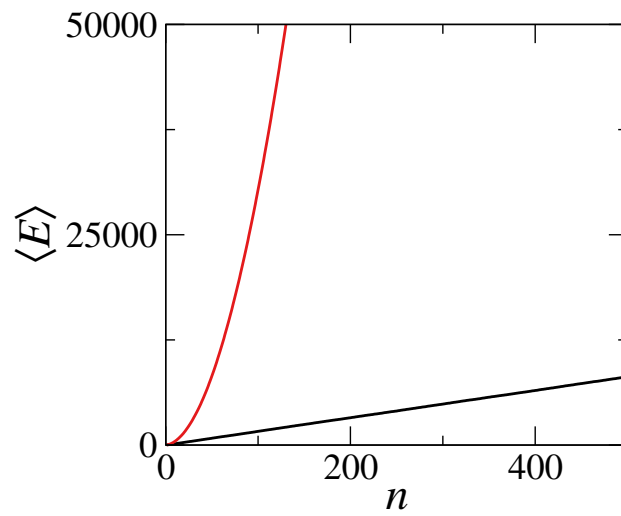


Figure 1.2: Classical energy growth of kicked rotor system for two different kick strengths,  $K = 6.4$  (red line) and  $K = 10.0$  (black line).

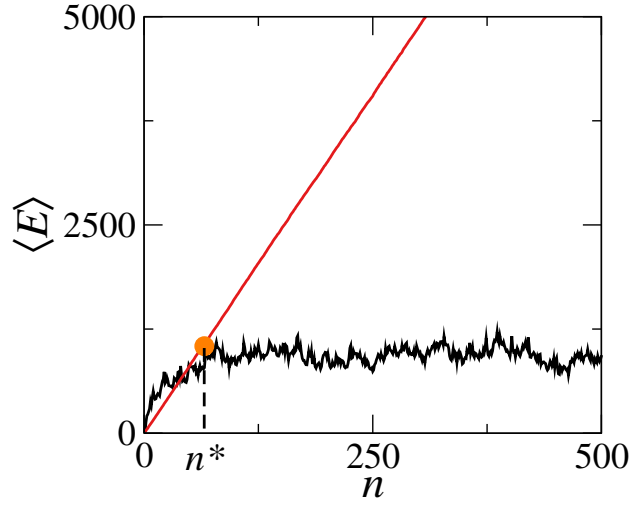


Figure 1.3: Classical and quantum mean energy growths of kicked rotor system for kick strengths  $K = 10.0$  represented by black and red solid lines respectively. Vertical dashed black line represents the break-time,  $n^*$ .

diffusion of a particle in momentum space) which speed up the diffusion process. While for  $K = 10$ , the phase space is fully chaotic as shown in Fig. 1.1(d) and a particle can occupy each and every part of the phase space. Thus, for the classically mixed and chaotic regime, the system shows diffusion in momentum space and is quantified by the average energy growth of the system as shown in Fig. 1.2. In general for the kicked rotor system with  $K > K_c$ , the system shows diffusive behaviour ( $\langle E \rangle = Dt$ ) with oscillating diffusion coefficient ( $D$ ) but for special values of  $K$  which are near multiples of  $2\pi$ , the system shows anomalous diffusion with respect to time,  $\langle E \rangle \sim t^\mu$ , where  $1 < \mu \leq 2$ . This anomalous diffusion is the result of the presence of accelerator modes and the exponent  $\mu$  depends on the kick strength [Benkadda 1997, White 1998, Sundaram 1999]. Red line in Fig. 1.2 illustrates that for the same kick strength as in Fig. 1.1(c), the system displays ballistic energy growth, while black line in Fig. 1.2 corresponds to the classically chaotic kicked rotor which demonstrates unbounded and diffusive energy growth.

On the other hand, the quantum counterpart of a classically chaotic kicked rotor shows the well known dynamical localization effect. The average energy growth saturates as shown in Fig. 1.3. Initially, the quantum energy grows till the time (called the break-time denoted by  $n^*$  in the Fig. 1.3) the system feels the spectrum

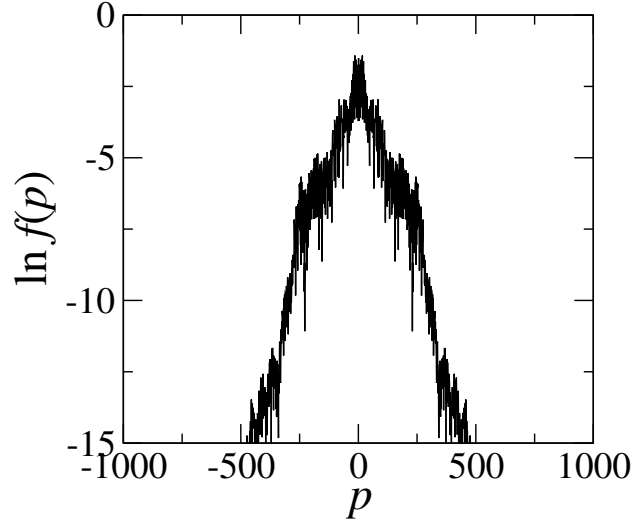


Figure 1.4: Semi-log plot of momentum distribution of the ground state of the kicked rotor after evolving it for 1000 kicks. For this plot  $K = 10.0$  and  $\hbar_s = 1.0$ .

to be continuous, and after the break-time, the system observes the discreteness of the spectrum and displays the localization effect. The localization observed is of exponential in nature. Due to the exponential decay profile of the wave-functions shown in Fig. 1.4, the level spacing distribution exhibits Poissonian profile and hence, shows level clustering.

To understand the origin of the exponential localization, S. Fishman, *et. al.*, in 1982 [Fishman 1982], mapped the kicked rotor system onto tight binding Anderson model. Hence, by drawing an analogy with the Anderson model, the exponential localization observed in the kicked rotor is explained [Fishman 1982, Grempel 1984, Fishman 1989]. The localization is a consequence of quantum interference effect. Several experiments have been performed on ultra-cold atoms [Moore 1994] and hydrogen atom in microwave field [Koch 1995] to support the results obtained on kicked rotor system.

On the other hand, kicked rotor with non-analytic potential such as the kicked rotor in an infinite well [Sankaranarayanan 2001, Hu 1999], and Fermi accelerator model [Jose 1986], represents a non-KAM system. In all of these systems, the quantum counterpart shows localization effects, however, the localization is of power-law in nature. Thus, we can see that kicked rotor satisfying KAM condition

shows exponential localization and kicked rotor with non-analytic potential exhibits power-law localization. Naturally, the question arises, can there be any system that will show a transition from power-law to exponential localization. Also, we have noticed that the classical kicked rotor shows only normal and super-diffusive energy growth. Is it possible to get a sub-diffusive energy growth ? Both of these questions motivated us to study some variants of the kicked rotor system.

## 1.4 Decoherence: A Brief History

Until now, we have discussed about the classical, and quantum dynamics of a one-dimensional non-interacting kicked rotor system without taking into account the effects of interaction with the environment. In general, it is very difficult to maintain experimentally a quantum system isolated from interacting with the external environment. It easily couples with the environment and loses its quantum coherence property. Hence, the system decoheres.

Although the term “decoherence” came to light with the work of Zurek [Zurek 1981, Zurek 1982, Zurek 2003], Zeh [Zeh 1970], Joos [Joos 1985, Joos 2003] and many others, but the concept was there even before. Decoherence is associated with the measurement problem in a quantum system. When measurement is performed on a quantum system, it makes the quantum system behave like classical. The measurement enforces a quantum system to interact with the environment, and the environment destroys the phase coherence between the quantum states and hence, inhibits coherent superposition. Thus, measurement results in collapse of wave function on a macroscopically localized state. Hence, the quantum system loses its quantum properties. Decoherence causes trouble in implementing devices for quantum information processing such as quantum computers. Recently, due to the development of several new experimental techniques, it has become possible to perform experiments on systems like in matter-wave interferometry [Hornberger 2012], superconducting systems [Leggett 2002], cavity QED [Raimond 2001], to observe decoherence.

Decoherence occurs not only during measurement but can also be introduced by the environment in terms of classical noise. This can be observed in the kicked rotor system. The effects of interaction with the environment can be studied by treating the kick strength [Klappauf 1998, Steck 2000] or the phase [White 2014] or the kicking time-period [Oskay 2003, Ott 1984] as a stochastic process or it could be due to spontaneous emissions [Ammann 1998, Lepers 2010] at every kick. All of these noisy cases induces decoherence in the system, and the system starts absorbing energy instead of showing localization effect. It is found that the rate of decay of coherence for all of these noisy cases is exponentially fast,  $D \sim \exp(t/\tau_c)$ , where  $D$  is the decoherence factor and  $t_c$  is the coherence time. Although, the rate can be slowed down by properly controlling the interaction between the environment and the system, but that only modifies the coherence time,  $\tau_c$ . Still, the loss of coherence remains exponentially fast. This rapid decay of coherence creates a problem in proper understanding of the quantum phenomenon. Hence, it is desirable to obtain some level of control to possibly slow down the decoherence process. This constitutes one of the motivations for the second part of this thesis.

## 1.5 Motivations

Thus, motivated by the existing results, the main questions that we addressed in this thesis work can be coined as follows:

- i) In kicked rotor system, normal and super-diffusive energy growth can be observed. Is there any mechanism to obtain sub-diffusive energy growth in the kicked rotor or its variants ?
- ii) What is the quantum manifestation of the classically chaotic non-KAM system ?
- iii) Standard kicked rotor which is a KAM system shows exponential localization while kicked rotor with non-analytic potential being a non-KAM system

shows power-law localization. Is there any system can be modelled to obtain a transition from power-law to exponential localization ? This will be interesting because the same system will then behave as a conductor in some parameter regime while insulator in some other.

- iv) We have seen that kicked rotor with usual noise scenario shows an exponential rate of decoherence and results in rapid loss of information about quantum properties. Thus, our question is, is there any possible way to modify the expression of a decoherence factor to make it non-exponential ?

## 1.6 Plan of the Thesis

With these motivations, we studied different variants of the kicked rotor system. We presented all the results of our study in this thesis.

Below is a brief summary of all the chapters:

- **Chapter 2 : Kicked Rotor in a Periodic Finite Well Potential : Classical Dynamics**

In this chapter, we considered a non-KAM system which comprises of kicked rotor placed in a periodic finite well potential. We studied the effect of the non-KAM feature on the classical dynamics. We demonstrated how the dynamic changes upon variation in the system parameters like well width, height of the well, etc. We have also addressed a question that we posed as (i) in the last section of this chapter.

- **Chapter 3 : Kicked Rotor in a Periodic Finite Well Potential : Quantum Dynamics**

In this chapter, we emphasized on understanding the quantum manifestation of the non-KAM chaotic system that is introduced in chapter 2. We studied the quantum dynamics using the Floquet operator and demonstrated, how the quantum chaotic properties get affected upon varying the system parameters.

In this chapter, we addressed the questions (ii) and (iii) that we posed in the motivation section of this chapter.

- **Chapter 4 : Kicked Rotor with Lévy Distributed Kicks**

Here, we discussed about the second part of our work and addressed the question (iv) that is related to the decoherence phenomenon observed in the kicked rotor system. We have studied a kicked rotor system which is subjected to interact with an external environment, and the effect of interaction is such that a particle instead of receiving periodic kicks now experience kicks at few selected times. The waiting time between the consecutive kicks is taken from Lévy waiting time distribution  $w(\tau) \sim \tau^{-1-\alpha}$ , where  $\alpha$  is the Lévy exponent.. The motivation behind this work is to slow down the rate of decoherence which is in general known to be exponentially faster [Klappauf 1998, White 2014]. We theoretically demonstrated that the rate of decoherence can be slowed down to non-exponential for a regime of Lévy exponents which is lower than a critical value [Sarkar 2017]. This work is important as it illustrates that slow decoherence is not difficult to achieve, but needs a tricky way to handle a system.

- **Chapter 5 : Future Plan**

In this chapter, we outline a plan that we will be carrying out in our near future. The first and foremost one would be to investigate whether kicked rotor in periodic finite well potential, is multifractal or not because it shows signatures that are usually seen in multifractal systems. Next, we will extend the work “Kicked rotor with Lévy distributed kicks” by taking some other distributions whose mean diverges in order to establish if non-stationarity implies non-exponential decoherence. Further, we can explore the possibility to work on many-body interacting systems.

- **Chapter 6 : Appendix**

Here, we have shown a detailed analytical calculation of the decoherence

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factor for a kicked rotor system with kicks taking place at few selected times. The waiting times between these consecutive kicks are drawn from a random uncorrelated distribution.



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### **Kicked Rotor in a Periodic Finite Well Potential : Classical Dynamics**

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Kicked rotor in recent years has attracted huge attention of the scientific community with its promising classical and quantum dynamical features. The kicked rotor comprises a particle rotating on a ring and receiving kicks at a periodic interval of time. It is a KAM system, and the imprint of its KAM nature is profound while increasing the kick strength of the kicks imparted to the particle. With the gradual increase in the kick strength, the regular structures which were predominantly occupied the phase space breaks down leading to global chaos above a certain critical kick strength. In the classically chaotic limit, the system shows an unbounded and diffusive energy growth,  $\langle E \rangle = \frac{K^2}{2}t$ , where  $K$  is the kick strength. While for some  $K$  near multiples of  $2\pi$ , the system exhibits super-diffusive energy growth,  $\langle E \rangle = t^\mu$ , where,  $1 < \mu \leq 2$  and the exponent  $\mu$  strongly depends on the kick strength  $K$  [Benkadda 1997, White 1998, Sundaram 1999]. This super-diffusion can be attributed to the presence of accelerator modes along with the sticky island in the phase space that enhances the growth. One important point to note is that sub-diffusion is not observed and in general, in the Hamiltonian system, the anomalous

transport is of super-diffusive in nature [Altmann 2008, Zaslavsky 2002].

The classical and quantum dynamics of kicked rotor are, by now, fairly well understood but it continues to be a model of choice to study physical phenomena in a variety of contexts ranging from classical and quantum ratchets [Schanz 2001], classical-quantum correspondence [Gadway 2008, Wang 2008], coherent quantum control [Gong 2001, Gong 2003], metal-insulator transition [Manai 2015, Garcia 2005, Qin 2014, Datta 2008, Wang 2009], nonlinearity effects [Ermann 2014, Ivanchenko 2014, Brambila 2013, Gligoric 2011], and decoherence [Sarkar 2017, Schomerus 2007, Schomerus 2008]. Many experiments have been performed on ultracold atoms [Moore 1994], hydrogen atom in microwave field [Koch 1995] to understand the dynamics of the kicked rotor system.

Going beyond the standard kicked rotor as paradigmatic KAM system, comparatively much less is known about a class of variants of the kicked rotor which violate the conditions of KAM theorem. Some of the examples include kicked rotor in an infinite well [Hu 1999, Sankaranarayanan 2001], the kicked harmonic oscillator [Gardiner 1997] and Fermi accelerator model [Jose 1986]. All of these systems display abrupt changes in the invariant tori as a function of kick strength  $K$ , and this results in a rather abrupt transition from integrability to chaos. This property of the non-KAM system has been experimentally realized in semiconductor super-lattices to control the electronic transport [Fromhold 2000] and also in coupled billiards in the form of two-dimensional electron gas in an external magnetic field [Brunner 2007, Brunner 2008]. Despite their physical applications in the field of plasma physics [Chia 1996], tokamak fusion [Karney 1997], ion trap [Gardiner 1997], these non-KAM variant of kicked rotor systems are not yet well explored and hence need careful attention. In particular, quantum manifestations of non-KAM classical dynamics have not been studied earlier.

Motivated by this, we have studied a non-KAM version of kicked rotor system. Beyond the question of non-KAM dynamical features, the additional interest is to somehow exploit these non-KAM features to observe sub-diffusive energy growth which is generally not observed in the standard kicked rotor.

## 2.1 The System

The system we consider is that of a particle placed in a periodic finite well potential and subjected to kicks at a periodic intervals of time. The periodic finite well potential introduced is non-analytic at the well boundaries and hence, it violates one of the KAM conditions which relies on the analyticity of the Hamiltonian. Thus, the Hamiltonian representing our system is

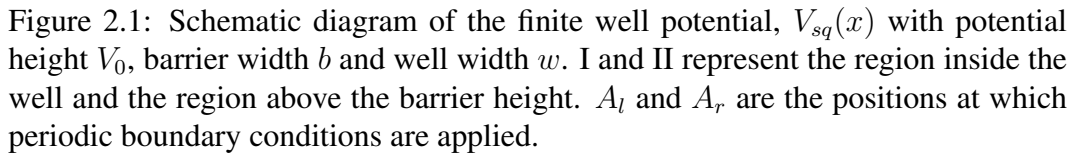
$$H = \frac{p^2}{2m} + V_{sq}(x) + K \cos\left(\frac{2\pi x}{\lambda} + \phi\right) \sum_n \delta(t - nT) \quad (2.1)$$

where,  $V_{sq}(x)$  is the stationary finite well potential shown in Fig. 2.1 and is mathematically expressed as

$$V_{sq} = V_0[\theta(x - w - nd) - \theta(x - w - b - nd)]. \quad (2.2)$$

In this,  $\theta$  represents the step function. In Eq. (2.2),  $w$  and  $b$  represent the well width and the barrier width, respectively, and  $d = w + b$  is the periodicity of the well which is taken to be  $2\pi$  throughout this work. In Fig. 2.1,  $A_l$  and  $A_r$  are the positions where periodic boundary condition is applied.

In the Hamiltonian in Eq. (2.1),  $p$  represents the momentum of a particle of mass  $m$ ,  $x$  is the position of the particle,  $K$  is the kick strength of the kicking field,  $\lambda$  and  $\phi$  are the wavelength and phase of the kicking field. The kicks are implemented through a series of delta functions in Eq. (2.1) with kicking period  $T$ . As seen from Eq. (2.1), several tunable parameters, namely,  $V_0$ ,  $K$ ,  $\phi$ ,  $b$  can be used to control the dynamical behavior of the system. In the limit  $V_0 \rightarrow 0$ , Eq. (2.1) coincides with the kicked rotor limit and the system behaves like a KAM system. At the other extreme, as barrier height increases, i.e.,  $V_0 \rightarrow \infty$ , this system will correspond to a kicked rotor in an infinite potential well. The latter system has been studied rigorously in Ref. [Sankaranarayanan 2001] and it is known that its dynamical properties are very different from that of the standard kicked rotor. In the present work, we maintain the potential height in between these limits,  $0 < V_0 < \infty$ ; the limiting values



Besides the potential height, barrier width,  $b$ , can also be used to control the dynamics of the system in Eq. (2.1). For instance, for barrier width  $b = 2\pi$  or  $w = 0$ , the potential discontinuities are effectively absent and all the particles are above the constant potential height,  $V_0$ . In this case, Eq. (2.1) represents a standard kicked rotor and displays the usual dynamical features associated with it. The only difference between the cases  $V_0 = 0$  and  $V_0 > 0$  (provided  $b = 2\pi$  or  $w = 0$ ) is that the energy of the system will increase by a constant amount  $V_0$  in the latter case. On the other hand, by decreasing the barrier width  $b$ , the system starts displaying dynamics both inside the well and above the well. This is a different scenario from that of the kicked rotor. In this case, the effect of non-KAM feature becomes apparent. Thus, considering these effects of  $b$ , we can speculate that by varying  $b \in [0, 2\pi]$  the system in Eq. (2.1) will display a wide variety of dynamics. There is another controllable parameter present in the system in Eq. (2.1), namely the phase  $\phi$  of

the external field. Varying  $\phi$  implies the introduction of an additional phase in the dynamics of the system through the external field. Thus, there are several controllable parameters present in the system and all of them have profound impact on the dynamics. We will discuss in detail the effect of these parameters in the subsequent section. The underlying physical mechanism responsible for the observed dynamics will also be discussed.

The dimensionless Hamiltonian of the system in Eq. (2.1) is

$$\tilde{H} = \frac{\tilde{p}^2}{2} + \tilde{V}_{sq}(\tilde{x}) + \tilde{K} \cos(\tilde{x}) \sum_n \delta(\tilde{t} - n), \quad (2.3)$$

where the scaled parameters are given by,

$$\tilde{x} = \frac{2\pi}{d}x, \quad \tilde{\lambda} = \frac{\lambda}{d}, \quad \bar{x} = \frac{\tilde{x}}{\tilde{\lambda}} + \phi, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{b} = \frac{2\pi}{d}b, \quad R = \frac{2w}{d}, \quad \tilde{\phi} = \tilde{\lambda}\phi,$$

$$\tilde{H} = \frac{4\pi^2 T^2}{md^2}H, \quad \tilde{p} = \frac{2\pi T}{md}p, \quad \tilde{V}_0 = \frac{4\pi^2 T^2}{md^2}V_0, \quad \tilde{K} = \frac{4\pi^2 T}{md^2}K,$$

and the scaled form of the potential is

$$\tilde{V}_{sq} = \tilde{V}_0[\theta(\tilde{\lambda}\bar{x} - R\pi - 2n\pi - \tilde{\phi}) - \theta(\tilde{\lambda}\bar{x} - R\pi - \tilde{b} - 2n\pi - \tilde{\phi})].$$

The Eq. (2.3) is the scaled Hamiltonian for the system composed of a particle of unit mass in a finite well potential receiving kicks at a periodic interval time of unit period. In further calculations, tilde and bar are removed from the scaled variables.

To understand the effect of the finite well potential on the particle's dynamics, consider a simple situation when the kick strength is zero and the particle has a finite momentum placed inside the well region, i.e., its energy  $E < V_0$  (note, different regions of the potential are specified in Fig. 2.2). As it evolves in time from the chosen initial conditions, it will collide with the barrier and will reflect back. This process of collision and reflection will happen repeatedly and thus will lead the particle to stay inside the well region. However, if  $K = 0$  and the particle has energy

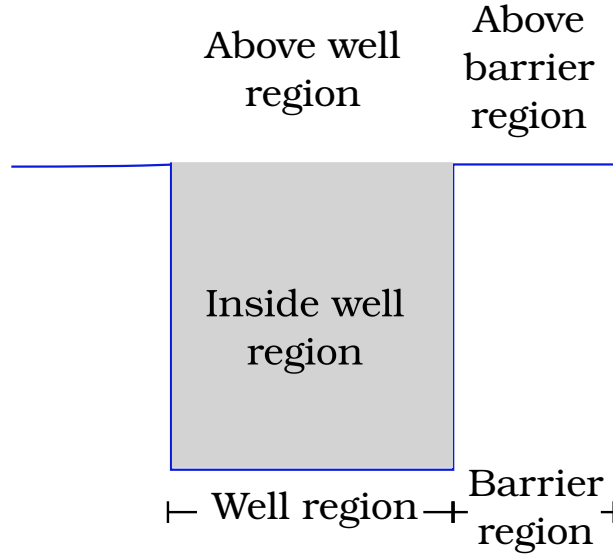


Figure 2.2: Different regions of the finite well potential  $V_{sq}(x)$ .

$E > V_0$ , i.e., above the well region, then it will suffer a change in the momentum at positions where  $V_{sq}(x)$  is singular. When the particle enters the above-barrier region (specified in Fig. 2.2) from the well region, its momentum will decrease to conserve the total energy,  $T = E - V$ . On the other hand, when it comes out of the above-barrier region, its momentum will increase. Thus, it is noticed that the particle will suffer a change in the momentum, irrespective of whether it encounters the barrier or goes in and out of the above-barrier region.

Now, if we introduce the kicking term by making the kick strength finite  $K > 0$ , the dynamics of the system becomes more intricate, but the effect of the finite well potential on the particle's dynamics remains the same as when the kick strength was zero. Thus, the effect of finite well potential  $V_{sq}$  can be accounted for by incorporating the appropriate boundary conditions on the kicked rotor dynamics governed by  $H_0 = \frac{p^2}{2} + K \cos(x) \sum_n \delta(t - n)$ . With this understanding, the Hamiltonian,  $H$  in Eq. (2.3) can be rewritten as

$$H = H_0 + V_{sq}(x). \quad (2.4)$$

## 2.2 The Classical Map

The map that dictates the dynamics of the system in Eq (2.3) is

$$p_n = p_{n-1} + K \sin(x_{n-1}), \quad (2.5a)$$

$$x_n = x_{n-1} + p_n, \quad (2.5b)$$

$$\begin{pmatrix} p_n^j \\ x_n^j \end{pmatrix} = \widehat{M} \begin{pmatrix} p_n \\ x_n \end{pmatrix}, \quad (2.6)$$

where, Eq. (2.5) represents the effect of  $H_0$  and the effect of  $V_{sq}$  are incorporated using Eq. (2.6). In Eq. (2.6),  $\widehat{M} = \widehat{M}_j \cdots \widehat{M}_1$  takes care of the boundary condition and  $j$  gives the number of times a particle encounters the singularity of the potential in between two consecutive kicks.

In the system in Eq. (2.3),  $V_{sq}$  represents a periodic potential, the dynamics of the particle in each potential well is the same. Thus for convenience, we will concentrate on the dynamics of the particle in the potential well described by the shaded region in Fig. 2.3. In Fig. 2.3,  $A_l$  and  $A_r$  represents the positions where periodic boundary condition is applied to bring the particle back inside the shaded region.

Consider a particle at time  $(n - 1)$  lying in the well region, i.e.,  $x_{n-1} \in [B_{-1}, B_1]$ , where  $B_{\pm 1}$  are the positions of the barriers shown in Fig. 2.3. After a unit time evolution, if the new position  $x_n$ , (obtained using Eq. (2.5b)), of the particle still lies in  $[B_{-1}, B_1]$ , then, the particle's motion will not get affected by the presence of  $V_{sq}$ . However, if  $x_n$  lies outside the well region, i.e.  $x_n \notin [B_{-1}, B_1]$ , then the effect of stationary potential comes into play through the boundary conditions that is taken care of by  $\widehat{M}$ . Depending on the energy of the particle  $E$  and the potential height  $V_0$ , two situations arise: i)  $E_n \leq V_0$  and ii)  $E_n > V_0$ .

Now, a particle with energy  $E < V_0$  whose position  $x_n$  obtained using Eq. (2.5) lies outside  $[B_{-1}, B_1]$ , it is necessary to bring the particle back inside  $[B_{-1}, B_1]$ ,

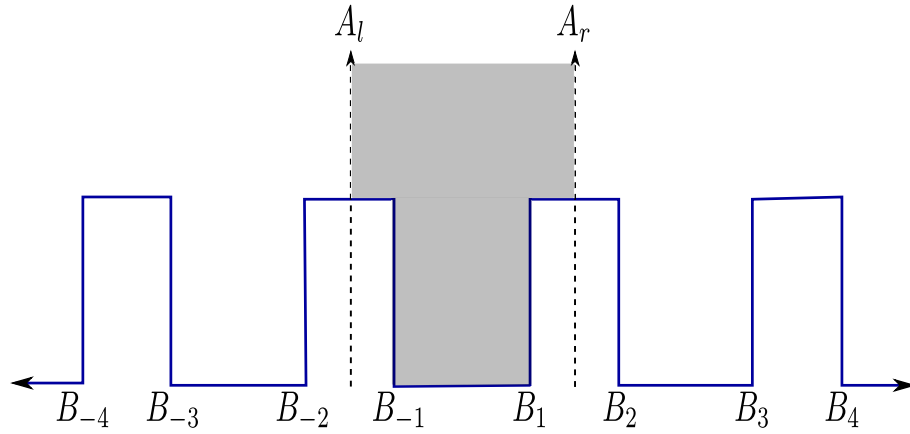


Figure 2.3: Schematic diagram representing the positions of barriers of a periodic finite well potential. Shaded region represents the potential well of our interest.  $A_l$  and  $A_r$  are the positions where the periodic boundary condition is applied.

since it is forbidden for a classical particle to tunnel through the barrier. This task can be performed by using reflective boundary condition and the name ‘reflective’ is because, a particle during its journey will hit the barrier and will reflect back, hence named reflective boundary condition. As there is no dissipative term in the Hamiltonian in Eq. (2.3), the collision with the wall is perfectly elastic and results in no energy loss. The only effect is that the momentum changes by  $2p_n$ . Thus, when  $E \leq V_0$ , taking care of the reflective boundary condition, the new equation of motion is

$$\begin{pmatrix} p_n^{i+1} \\ x_n^{i+1} \end{pmatrix} = \widehat{M}_1 \begin{pmatrix} p_n^i \\ x_n^i \end{pmatrix} = \begin{pmatrix} -p_n^i \\ 2(B_{\pm 1} \pm jR) - x_n^i \end{pmatrix}. \quad (2.7)$$

where  $B_{\pm 1} \pm jR$  represents the position closest to  $x_n^i$  with the additional condition  $|x_n^i| > |B_{\pm 1} \pm jR|$ . Here,  $j$  is an integer that makes  $B_{\pm 1} \pm jR$  closest to  $x_n^i$ . Equation (2.7) holds till the particle’s position is  $|B_{-1}| > |x_n^{i-1}| > |B_1|$ . The negative sign in the momentum equation,  $p \rightarrow -p$ , indicates reflective boundary condition.

Now, consider a particle with energy  $E > V_0$ , and if its position  $x_n$  obtained using Eq. (2.5) lies outside  $[B_{-1}, B_1]$ , a refractive boundary condition as well as



periodic boundary condition are applied to bring the particle back in the potential well of our interest. The name ‘refractive’ of the refractive boundary condition is due to its similarity to regular refraction of light where light’s velocity changes while entering a medium. Here also, a particle’s momentum changes when it enters the region above the barrier. This change in momentum is the result of conserving the total energy of the particle in between two consecutive kicks, i.e.,  $E = T + V$  or  $T = E - V$ , where,  $T$  is the kinetic energy and  $V$  is the potential energy of the particle. Thus, when a particle goes above the barrier, it gains a potential energy  $V_0$  and hence, to conserve the total energy  $E$ , its kinetic energy,  $T$ , gets reduced. As a result, its momentum changes. Thus, by taking into account the refractive boundary condition, we have

$$\begin{pmatrix} p_n^{i+1} \\ x_n^{i+1} \end{pmatrix} = \widehat{M}_1 \begin{pmatrix} p_n^i \\ x_n^i \end{pmatrix} = \begin{pmatrix} \text{sign}(p_n^i) \sqrt{(p_n^i)^2 - (\text{sign}(V_{diff})2V_0)} \\ 2B_i - x_n^i \end{pmatrix}, \quad (2.8)$$

where  $B_i$  represents the positions of the barrier and are shown in Fig. 2.3. It can be noticed from Eq. (2.8) that the magnitude of momentum changes depending on the direction of motion of the particle, whether it is entering the barrier region or going out of it. When the particle reaches the barrier region from the well region, it essentially goes from  $V_0 = 0$  region to  $V_0 > 0$  region. Thus,  $V_{diff} (= V_0 - 0)$  in Eq. (2.8), is positive. Hence, the magnitude of momentum decreases, which is apparent from the Eq. (2.8) and is consistent with the conservation of energy as the particle gains an extra potential energy  $V_0$ . On the other hand, when the particle goes out of the barrier region ( $V_0 > 0$ ) to the well region ( $V_0 = 0$ ),  $V_{diff} (= 0 - V_0)$  is negative and hence, the magnitude of momentum increases. The new position  $x_n^i$  attained by a particle is obtained by taking into account the change in momentum suffered by the particle going through the potential barrier. Equation (2.8) is used to calculate the new position of the particle. After computing the new position  $x_n^i$ , the periodic boundary condition is applied to bring the particle back in  $[-\frac{R}{2}\pi - \frac{b}{2} + \phi, \frac{R}{2}\pi + \frac{b}{2} + \phi]$ .

It can be noted that we have applied periodic boundary condition only on  $x$  as

we are interested in studying the dynamics on a cylinder. This boundary condition allows diffusion in the momentum direction which can be quantified by the average energy growth  $\langle E \rangle$  of the system. Depending on  $V_0$  in two different regimes, (i)  $V_0 = 0$  and (ii)  $V_0 \neq 0$ , the mean energy, at any instant of time, can be formally written down as,

$$\langle E \rangle = \frac{1}{N} \sum_{l=1}^N \frac{p_l^2}{2} \quad -\frac{R}{2}\pi + \phi < x < \frac{R}{2}\pi + \phi \quad (2.9a)$$

$$= \frac{1}{N} \left( \sum_{l=1}^N \frac{p_l^2}{2} + V_0 \right) \quad \begin{cases} -\frac{R}{2}\pi - \frac{b}{2} + \phi < x < -\frac{R}{2}\pi + \phi, \\ \frac{R}{2}\pi + \phi < x < \frac{R}{2}\pi + \frac{b}{2} + \phi. \end{cases} \quad (2.9b)$$

In this, average is taken over an ensemble of  $N$  particles and  $p_l$  is the momentum of  $l$ th particle.

We note that the model in Eq. (2.3) is different from the one presented in Ref. [Pal 2010]. In Ref. [Pal 2010], the authors considered a kicked rotor placed in a double barrier well while in the model presented in Eq. (2.3), the kicked rotor is placed in a periodic finite well potential. The additional periodicity in the potential in this model leads to a non-trivial dynamics and also make it different from the model considered in Ref. [Pal 2010].

Now, in the subsequent sections, we will study the classical dynamics of the system in Eq. (2.3). We will analyze the effects of the non-KAM feature on the dynamics of the system. To understand the effects, we will concentrate on the stroboscopic sections generated for different sets of parameter values and examine the rate of diffusion in momentum space by calculating the average energy growth of the system.

## 2.3 Surface of Section

In this section, we look at the surface of a section for various choices of parameters. Since the Hamiltonian in Eq. (2.3) is time-periodic with periodicity  $T$ , the values of  $x$  and  $p$  are recorded after intervals of time period  $T$ , typically just before or after a

kick had been imparted. In this thesis,  $T$  is considered to be 1.

To generate the surface of section numerically, the initial condition is taken to be an ensemble of particles uniformly distributed in  $x$  with momentum  $p \in [-0.5, 0.5]$ . The random numbers required to do this are generated using `rand()`/`RANDMAX()` functionality provided by the C compiler, `gcc 4.8.4`. The initial conditions thus generated are evolved using the map in Eq. (2.5) and by incorporating proper boundary conditions. The classical average energy of the system at each time is calculated using Eq. (2.9). The results obtained for different choices of system parameters are presented below.

### 2.3.1 Effect of Change in Barrier Width

In this section, we will discuss the effect of varying the barrier width  $b$  on the dynamics of the system in Eq. (2.3). For this, we plotted the stroboscopic sections in Fig. 2.4 for different choices of barrier widths while keeping all the other parameters,  $V_0$ ,  $K$ ,  $\phi$  constant. Figure 2.4 shows that with the increase in barrier width, the motion of particles becomes chaotic. The chaotic motion that emerges inside the well region with the increase in barrier width can be attributed to the increase in the number of collisions with the walls of the well. Note that in between two consecutive kicks the particle follows the invariant tori corresponding to the standard kicked rotor system ( $V_0 = 0$ ) except when interrupted by encounters with the potential barrier. For  $E < V_0$ , each of such encounter with the barrier leads to particle hopping from one invariant tori to another. This can be clearly understood from the Fig. 2.5 whose parameters are same as that of Fig. 2.4(c). Here we have plotted the dynamics inside the well region only. It can be noticed that a particle initially placed at  $a_0$  (denoted by green circle) evolves in time along  $a_1$ ,  $a_2$ ,  $a_3$  and  $a_4$ , all of these points  $(a_0, \dots, a_4)$  lie on a particular torus defined by winding number  $\omega_1$ . After encountering the barrier, it jumps onto another torus with winding number  $\omega_2$  and follows the path  $a_5$ ,  $a_6$ ,  $a_7$  and  $a_8$ . One can certainly notice that  $\omega_1 \neq \omega_2$ . Thus, torus hopping takes place after every encounter with the boundary. Hence, more such encounters with the barrier result in more tori hopping. This, effectively, re-

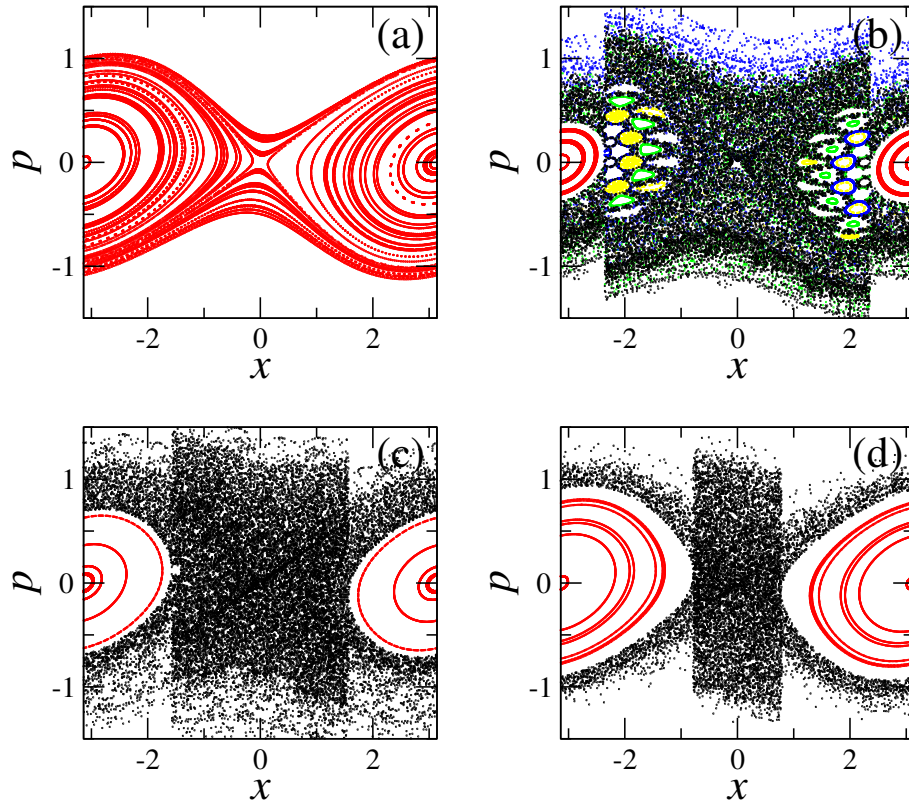


Figure 2.4: Stroboscopic sections of kicked rotor in a periodic finite well potential for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $\phi = 0$  and different barrier widths, a)  $b = 0.000002\pi$ , b)  $b = 0.5\pi$ , c)  $b = \pi$  and d)  $b = 1.5\pi$ .

sults in invariant tori being destroyed and thus, leads to chaotic motion. Hence, the increase in the barrier encounters results in rapid decay of autocorrelation function over time as shown in Fig. 2.6. Thus, the chaotic motion in this system arises due to the presence of potential barriers. Figure 2.6 shows the decay of auto-correlation as a function of time for different barrier widths. For small barrier widths such as  $b = 0.000002$ , auto-correlation does not decay. But with the increase in barrier width, the auto-correlation function decays faster as illustrated in Fig. 2.6.

Figure 2.4(a) shows that for barrier width  $b \rightarrow 0$ , the motion of a particle lie on regular invariant structures for kick strength  $K = 0.25$ . The phase space displayed in Fig. 2.4(a) is very much similar to that of a kicked rotor system at identical kick strength,  $K = 0.25$ . But, as we increase the barrier width to  $b = 0.5\pi$  as shown in Fig. 2.4(b), the regular invariant structures break down to give rise to the intricate

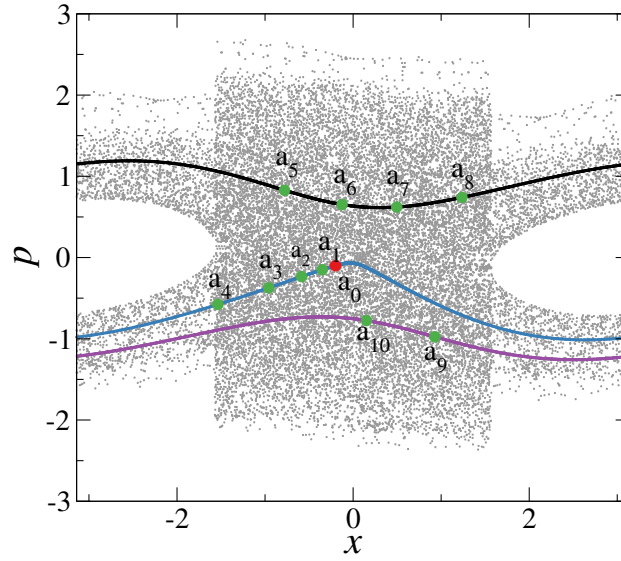


Figure 2.5: Plot represents the mechanism of tori hopping. The background stroboscopic section is for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $\phi = 0$ ,  $b = \pi$ . Black, blue and purple lines corresponds to the tori of a kicked rotor for  $K = 0.25$ . Green circles describes the trajectory of a particle initially placed at  $a_0$  (denoted by red circle).

region of islands connected through channels. Green, yellow, blue and black dots in Fig. 2.4(b) are generated with different sets of initial conditions. These intricate structures are the ones where a particle spend most of its time before coming out of the well and hence, causes the diffusion process to slow down.

The intricate island structures, seen in Fig. 2.4(b), decreases in size with increase in  $b$  and ultimately vanishes for barrier width  $\frac{b}{d} > 0.5$ . This can be explained by considering the stability of the fixed points at  $(x = 0, p = 0)$ . Fixed points can be obtained by setting  $x_{n+1} = x_n$  and  $p_{n+1} = p_n$ . There are two fixed points in the system, one is at  $(x = 0, p = 0)$  and the other is at  $(x = 2\pi, p = 0)$ . The fixed point at  $(x = 0, p = 0)$  is hyperbolic and the one at  $(x = 2\pi, p = 0)$  is elliptic in nature. Thus, by increasing the barrier width  $b$  and for  $\frac{b}{d} > 0.5$  and  $\phi = 0$ , we are just capturing the effect of this hyperbolic fixed point as the contribution from the elliptic point present at  $(x = 2\pi, p = 0)$  is inhibited.

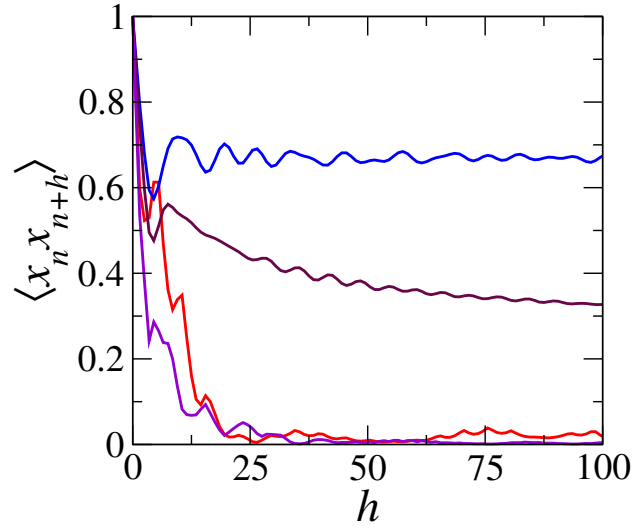


Figure 2.6: Decay of auto-correlation as a function of time for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $\phi = 0$  and for different barrier widths,  $b = 0.000002\pi$  (blue),  $b = 0.5\pi$  (maroon),  $b = \pi$  (red) and  $b = 1.5\pi$  (violet).

### 2.3.2 Effect of Change in Phase

In this section, we will concentrate on understanding the effect of the phase of the kicking field,  $\phi$ , on the dynamics of the kicked rotor in finite well system. In the stroboscopic sections shown in Fig. 2.7, we have fixed the parameters at  $V_0 = 1.5$ ,  $K = 0.25$ ,  $b = \pi$ ,  $\lambda = 2\pi$  and only phase  $\phi$  is varied. Variation of  $\phi \in [0, 2\pi]$  implies translating the system in position axis and the new position acquired is given by  $\tilde{x} = x + \phi$ . As a result of this shift in position, the qualitative nature of dynamics ranges from regular to chaotic depending on the value of  $\phi$ . In Fig. 2.7(a), since  $\phi = 0$ , we have captured the effect of hyperbolic fixed point at  $(x = 0, p = 0)$  inside the well region. The chaotic motion observed inside the well region in Fig. 2.7(a) is the result of tori hopping which we already discussed in the last section.

#### Emergence of KAM tori:

Interestingly, Fig. 2.7(b,d), corresponding to the parameter values  $V_0 = 1.5$ ,  $b = \pi$ ,  $K = 0.25$ , and  $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$  shows that regular structures are preserved inside the well region even though boundary encounters and tori hopping are taking place. The presence of regular structures for these particular choices of parameters can

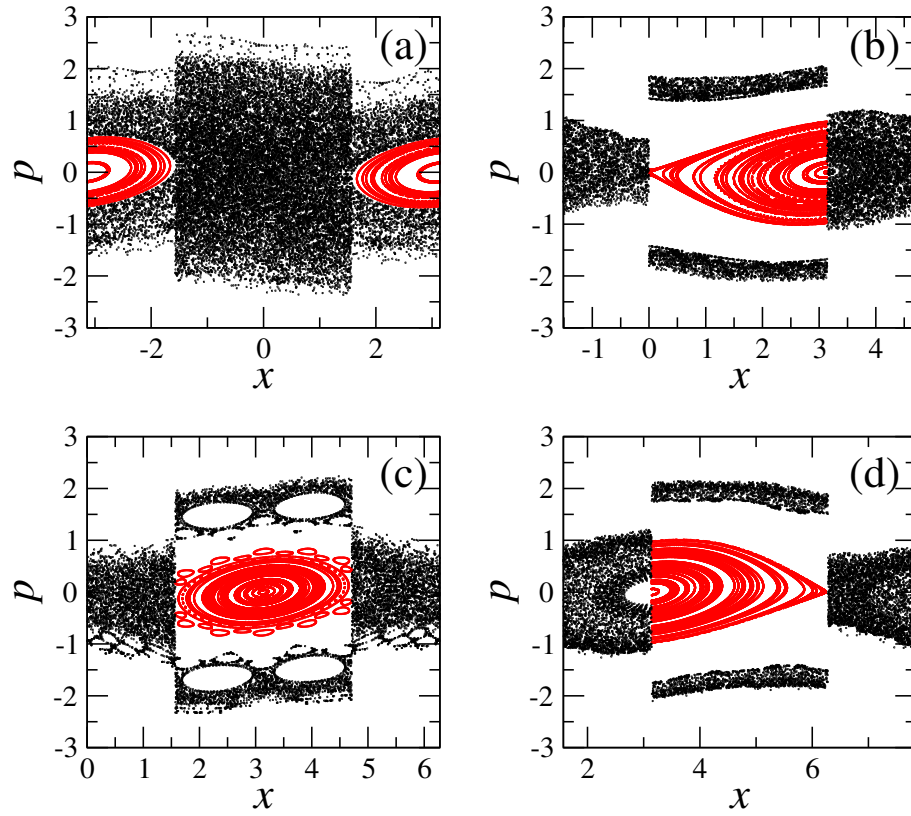


Figure 2.7: Stroboscopic sections of the kicked rotor in a periodic finite well potential for  $V_0 = 1.5$ ,  $K = 0.25$ ,  $b = \pi$  and different phases of the kicking field, a)  $\phi = 0$ , b)  $\phi = 0.5\pi$ , c)  $\phi = \pi$  and d)  $\phi = 1.5\pi$ .

be explained by considering the reflection symmetry of the system. The reflection symmetry only appears for the set of parameters that satisfy the KAM-like relation,

$$\pm \frac{R}{2}\pi + \phi = l\pi, \quad l \in \text{integer}. \quad (2.10)$$

If this condition in Eq. (2.10) is satisfied, the system preserves the KAM-like tori inside the well region in spite of the presence of barriers. To understand the mechanism more clearly, consider, if  $\omega_1$  and  $\omega_2$  denote the winding numbers of tori occupied by a particle before and after an encounter with the potential boundary, regular structures for  $K \ll 1$  can be expected if  $\omega_1 = \omega_2$ , i.e., a particle after encountering a boundary jumps onto a torus which has the same winding number as the one just before boundary encounter. This is clearly illustrated by the stroboscopic section plotted in Fig. 2.8 which corresponds to the same set of parameters used in Fig. 2.7(b). In Fig. 2.8, we can see that all the circles  $(a_0, \dots, a_7)$  lie on the same torus. The circles  $a_0, \dots, a_4$  ( $a_0, a_6, a_7$ ) represents the position of a particle after each kick before (after) colliding with the wall. As a result, regular structures are preserved. Thus, even in the non-KAM system, for certain choices of system parameters KAM-like tori can emerge.

In Fig. 2.7(c), the stroboscopic section presented for the parameters  $V_0 = 1.5$ ,  $K = 0.25$ ,  $b = \pi$  and  $\phi = \pi$  also has KAM tori inside the well region. In this case, the mechanism behind the origin of KAM tori is completely different from the one presented in Fig. 2.7(b, d) where the reflecting boundary condition plays the central role. Here, the KAM tori those are present never encountered the well boundary and thus, the breaking of tori never occurred. As a result, the KAM tori are preserved. Hence, we have noticed that changing phase  $\phi$  of an external field in the system in Eq. (2.3) have profound effects on the dynamics.

### 2.3.3 Effect of Change in Wavelength

In this section, we will focus on the effect of  $\lambda$ , the wavelength of the external kicking field. We consider two cases where the wavelength  $\lambda$  is compared with



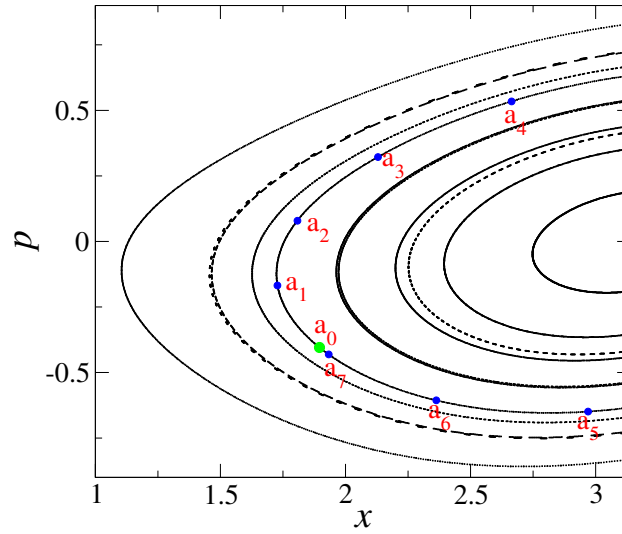


Figure 2.8: Stroboscopic sections of kicked particles in a periodic finite well potential for  $V_0 = 1.5$ ,  $K = 0.25$ ,  $\phi = 0.5\pi$ ,  $b = \pi$ . Blue circles describes the trajectory of a particle initially placed at  $a_0$  (denoted by green circle).

the length scale of the system. They are, (i) wavelength is equal to the well width,  $R\pi = \lambda$ , and (ii) wavelength does not match with the well width ( $\lambda \neq R\pi$ ) or the total width ( $\lambda \neq b + R\pi$ ). These two cases have been considered in order to understand the effect of length scales on the system dynamics.

Figure 2.9(a) displays the stroboscopic section for the situation described as case (i), where we have matched  $\lambda$  with the well width  $R\pi$ . Here, we have set  $R\pi = \lambda = 1.2\pi$  and thus, the barrier width is  $b = 0.8\pi$  which is a consequence of fixed periodicity of  $2\pi$ . In this instance, it is seen that all the tori inside the well are preserved. However, above the barrier height, the mismatch of length scale results in chaotic behavior.

On the other hand, if we do not match either of the two length scales of the system with  $\lambda$ , the chaotic motion arises which is shown in Fig. 2.9(b). Thus, observing the outcomes of varying wavelength,  $\lambda$  is a suitable parameter to control the qualitative nature, whether regular or chaotic, of system dynamics. An important point about wavelength,  $\lambda$ , is that from an experimental point of view it is one of the accessible parameters to control the dynamics.

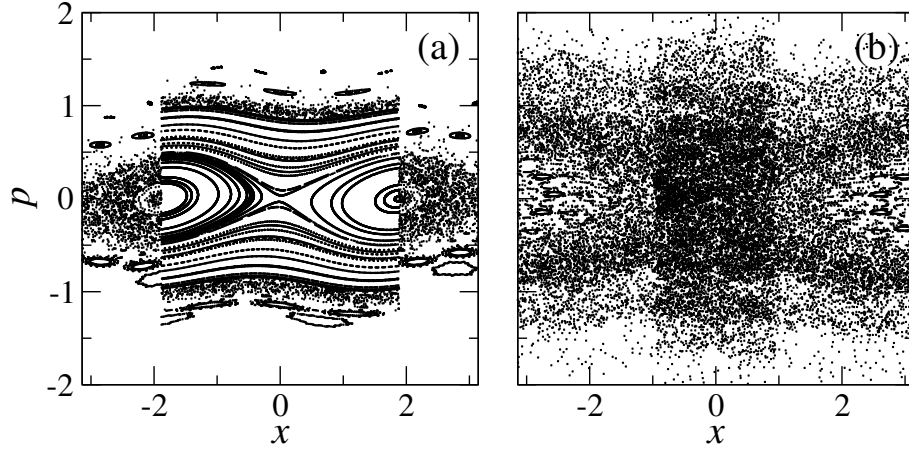


Figure 2.9: Stroboscopic sections of the system in Eq. (2.3) when the wavelength of the kicking field, a)  $\lambda = 1.2\pi$  matches with  $R = 1.2$  and b)  $\lambda = 2.81351\pi$  does not match either with  $R = 0.6\pi$  or with  $d = 2\pi$ . The other parameters are  $V_0 = 0.5$ ,  $K = 0.25$ ,  $\phi = 0$ .

## 2.4 The Classical Diffusion

In this section, we will focus on the energy growth of the system and quantify the diffusion in momentum space. Since the energy directly depends on the square of momentum ( $E \sim p^2$ ), so it is always positive and serves as a good candidate to measure diffusion. In order to avoid fluctuation in energy of a particle, we concentrate on average energy,  $\langle E \rangle$ , averaged over an ensemble of particles. We will discuss about the effects of these parameters on the energy growth in the subsequent subsections.

### 2.4.1 Effect of Change in Kick Strength

As kick strength varies from  $K = 0$  to  $K \gg 1$ , even as other parameters are kept constant at  $V_0 = 0.5$ ,  $b = \pi$ , and  $\phi = 0$ , the system goes from integrable to non-integrable limit. By increasing the kick strength, the system shows a transition from slow sub-diffusion to normal diffusion and finally to superdiffusion shown in Fig. 2.10. Figure 2.4 shows that even for a small value of kick strength,  $K = 0.25$ , the system shows diffusion and hence, a growth in the average energy is observed denoted by the black line in Fig. 2.10. This diffusion for  $K = 0.25$  can be attributed

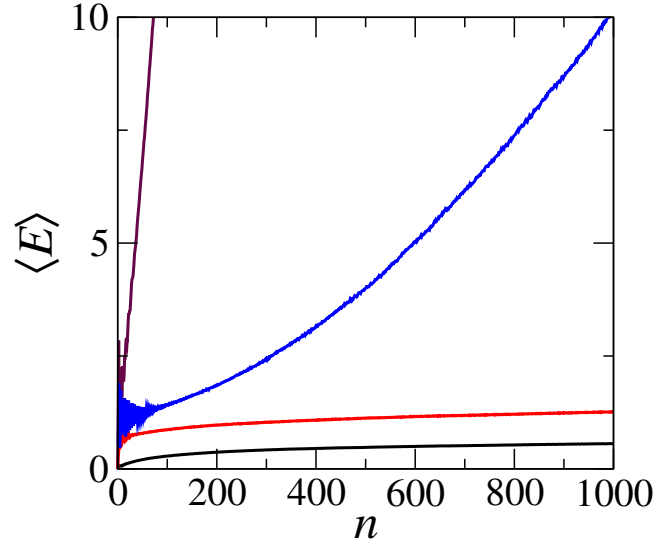


Figure 2.10: Classical average energy growth as a function of time for  $V_0 = 0.5$ ,  $b = \pi$ ,  $\phi = 0$  and different kick strengths,  $K = 0.25$  (black line),  $K = 1.0$  (red line), c)  $K = 1.5$  (blue line), and d)  $K = 2.0$  (maroon line).

to the presence of potential barriers and hence to the non-KAM nature of the system.

However, this diffusion is comparatively slower than normal diffusion. The average energy,  $\langle E \rangle$ , grows sub-diffusively with time, i.e.  $\langle E \rangle = D t^\mu$ , where,  $D$  is the diffusion coefficient and  $\mu < 1$  is the sub-diffusive exponent. This is an unusual feature not seen in any variants of kicked rotor earlier.

In fact, it turns out that sub-diffusion is the dominant mechanism of energy growth for wide range of system parameters and in that the average energy,  $\langle E \rangle$ , displays two sub-diffusive exponents,  $\mu_1$  and  $\mu_2$  which can be observed in the log-log plot presented in Fig. 2.11. The red and green lines are fit to the linear regimes of the average energy growth curve. The exponents  $\mu_1$  and  $\mu_2$  are the slopes of these linear regimes denoted by green and red lines respectively and the diffusion coefficients  $D_1$  and  $D_2$  are the intersection of the linear fits with the energy axis. We should note that transition from  $\mu_1$  to  $\mu_2$  regime is smooth and not abrupt. This transition takes place approximately at  $t = t_c = \left( \frac{D_2}{D_1} \right)^{\frac{1}{\mu_1 - \mu_2}}$ . The crossover time-scale,  $t_c$ , has been calculated by equating the two energy growths, i.e.,  $\langle E_1 \rangle = D_1 t_1^\mu$  and  $\langle E_2 \rangle = D_2 t_2^\mu$  at  $t = t_c$ . Thus, for  $t < t_c$ , the  $\langle E \rangle$  grows with sub-diffusive

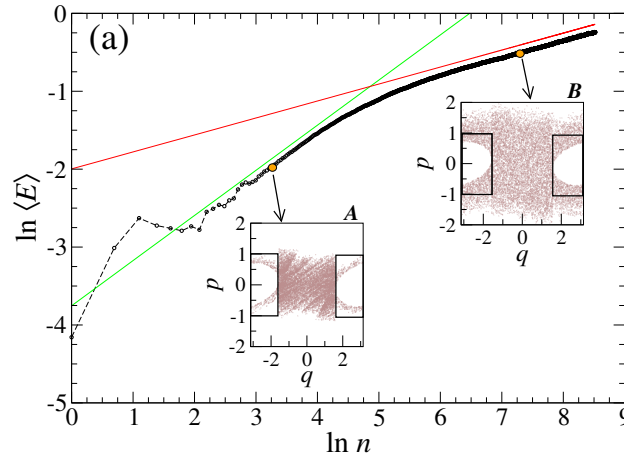


Figure 2.11: Log-log plot of classical average energy growth as a function of time for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $b = \pi$ ,  $\phi = 0$ . Inset represents the stroboscopic section of the system in Eq. (2.3) at two different times generated with same initial conditions.

exponent  $\mu_1$  and for  $t > t_c$ , it grows with exponent  $\mu_2$ .

Now, to understand the region of phase space where these sub-diffusive exponents govern the dynamics, we consider an ensemble of initial conditions  $p = 0$  and are distributed randomly inside the well region. Initially, the mean energy grows with sub-diffusion exponent  $\mu_1$  and later it grows with  $\mu_2$ . We have analyzed these sub-diffusive growth regimes carefully and noticed that  $\mu_1$  is dominant in the region of phase space when most of the particles are inside the well region while  $\mu_2$  takes over the growth rate when most of the particles are above the barrier height. These are shown as insets A and B in Fig. 2.11.

These sub-diffusive energy growths are a consequence of the non-KAM nature of the system. The mechanism that leads to the sub-diffusive energy growth characterized by  $\mu_1$  can be explained using Fig. 2.5. Figure 2.5 illustrates that a particle spends a long time (even after receiving few kicks), on the same torus before hopping onto another torus after encountering the wall. As a result, the average energy fluctuates about the mean energy of the torus for a few kicks and then increases when the particle jumps onto another torus after encountering the wall of the well. Thus, on an average over an ensemble of particles, the average energy,  $\langle E \rangle$ , increases but slowly and gives rise to sub-diffusion.

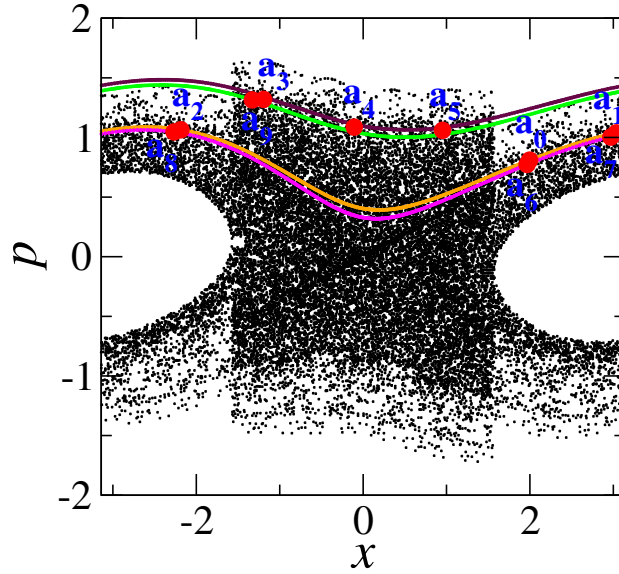


Figure 2.12: Stroboscopic sections of the kicked rotor in a finite well potential for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $b = \pi$ ,  $\phi = 0$  and of kicked rotor system for  $K = 0.25$ . Black points represents the trajectories of particles of the system in Eq. (2.3) and red points corresponds to the trajectory of a single particle. Maroon, orange, green and magenta lines correspond to kicked rotor for  $K = 0.25$ .

On the other hand, the mechanism behind sub-diffusive energy growth characterized by the exponent  $\mu_2$  can be explained using the Fig. 2.12. The energy growth that is observed in this regime is due to the punctures in the invariant tori at the potential boundaries. Figure 2.12 shows that a particle initially placed at  $a_0$ , upon receiving kicks follows the kicked rotor trajectory (orange line) till  $a_2$ . After that, the particle observes a puncture of torus at the potential boundary and an immediate kick pushes the particle to hop on to another torus (maroon line). It follows this torus until it reaches  $a_5$ , where it again finds a puncture in the torus and moves onto another torus (magenta) which is different from the torus denoted by orange line and this process continues. As a result, diffusion in momentum space takes place. But this diffusion is slow which is the result of long time spend by a particle on one torus before hopping on to other, and this effectively leads to sub-diffusive energy growth.

To understand the mechanism more clearly, we compare the change in position and momentum of the particle of the kicked rotor in periodic finite well potential

with that of standard kicked rotor system. When a particle with energy  $E_0$  enters the above-barrier region and assuming that it does not receive any kick while transiting this region, the net change in position between the last and the next kick is denoted by  $\Delta x$ . If  $\Delta x_{KR}$  represents a similar quantity for the kicked rotor ( $V_0 = 0$ ), we can show that  $\delta x = |\Delta x - \Delta x_{KR}| = b(\beta - 1)$ , where  $\beta = (1 - V_0/E_0)^{-1/2}$ . Clearly, if  $E_0 \gtrsim V_0$ , then  $\delta x \gg 0$  and if  $E_0 \gg V_0$  we have  $\delta x \rightarrow 0$ . This difference leads to torus hopping. Further, as  $\delta x \neq 0$ , it translates into momentum difference  $\delta p = |\Delta p - \Delta p_{KR}| = \varepsilon |\sin(x_k - (p/|p|)\delta x) - \sin x_k|$  at the position of next kick  $x_k$ . Thus, if  $E_0 \gtrsim V_0$ , the invariant tori do not survive due to large tori hopping induced by the discontinuities. This appears as chaotic dynamics in above-barrier region shown in inset B of Fig. 2.11. On the other hand, if  $E_0 \gg V_0$ , then invariant tori are largely preserved with minor dispersion and hence, the tori hoppings have only a negligible effect.

Now, if we increase the kick strength to  $K = 1.5$ , the system initially shows normal diffusion when most of the particles are inside the well. Once particles reach the region much above the potential height, they start showing superdiffusion. At this energy, the effect of the potential is negligible ( $E \sim \frac{p^2}{2}$ , if,  $E \gg V_0$ ) and the system behaves as a standard kicked rotor system. For this kick strength, the standard kicked rotor also shows superdiffusive energy growth and hence, the super-diffusion in this system is expected.

If the kick strength  $K$  is further increased to a large value such as  $K = 2.0$  as shown in Fig. 2.10, the system shows normal diffusion. For this choice of parameters, the auto-correlation decays very rapidly. Thus, we can conclude that similar to standard kicked rotor, by increasing the kick strength  $K$  to a large value,  $K = 2.0$ , the system shows diffusive energy growth, but for small kick strength  $K = 0.25$ , the system due to its non-KAM nature shows slow diffusion in momentum space and hence gives rise to sub-diffusive energy growth which is absent in the standard kicked rotor.

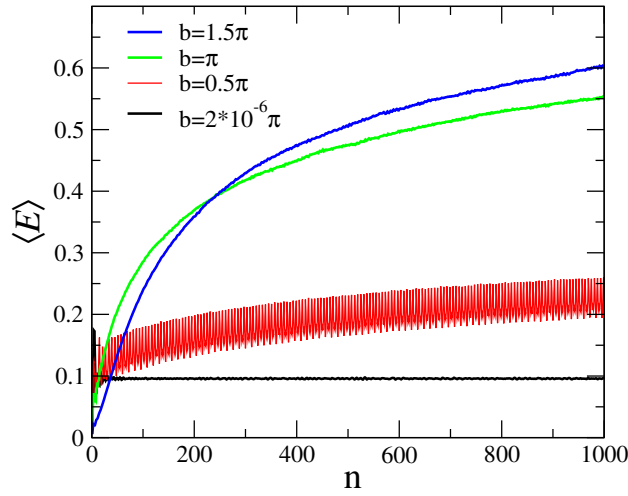


Figure 2.13: Classical average energy growth of the kicked rotor in a periodic finite well potential for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $\phi = 0$ , and for different barrier widths.

### 2.4.2 Effect of change in Barrier Width

From the point of view of theory, the barrier width of the stationary potential is most useful. This can be experimentally controlled as well. As discussed earlier, increasing the barrier width, the system shows a transition from regular to chaotic motion as shown in Fig. 2.4. Figure 2.13 displays the average energy growth corresponding to the same sets of parameters used to plot Fig. 2.4. Figure 2.4(a) shows that for small barrier width,  $b = 0.000002\pi$ , the phase space is filled with regular structures which implies that the diffusion in momentum space is suppressed. As a result, the average energy,  $\langle E \rangle$ , of the system is almost constant over time shown as black line in Fig. 2.13. For barrier width  $b = 0.5\pi$ , the phase is covered by chaos and intricate island-like structures which is shown in Fig. 2.4(b). The average energy growth,  $\langle E \rangle$ , for the same set of parameters shows a sub-diffusive energy growth shown by the red line in Fig. 2.13. The reason behind this sub-diffusive energy growth in addition to tori hopping process is the presence of sticky nature of intricate structures present inside the well region where the particles spend a long time before coming out of it.

For the situation presented in Fig. 2.4(c), the phase space is fully chaotic inside the well region which leads to diffusion in momentum space. The average energy

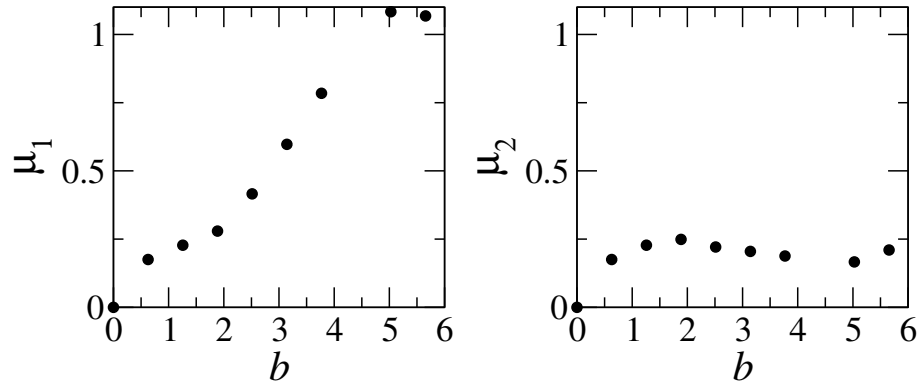


Figure 2.14: Effect of change in barrier width  $b$  on sub-diffusion exponents,  $\mu_1$  and  $\mu_2$  for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $\phi = 0$ ,  $b = 0.5\pi$ .

growth for  $b = \pi$  shown in Fig. 2.13 with green line corresponds to the same set of parameter values as in Fig. 2.4(c), shows energy growth, which is much slower than the normal diffusion,  $\langle E \rangle \sim t$ . The average energy grows sub-diffusively  $\langle E \rangle \sim t^\mu$ , where  $\mu < 1$  which is also the manifestation of tori hopping discussed in the last section.

Now, if we further increase the barrier width to a large value, for instance,  $b = 1.8\pi$ , as shown in Fig. 2.13 with the blue line, we can notice that after the initial transient period, the energy grows at a faster rate compared to the other energy growth for smaller barrier width presented in the same plot. This is because, for large barrier width, the region inside the well width decreases and hence the collision rate of a particle with the wall increase in each kick. As a result, after each kick the particle hops from one torus to the other, leading to a large change in the energy growth. Thus, the energy on an average over an ensemble of particles grows rapidly over time in the  $\mu_1$  regime. While above the barrier height, the energy growth is the result of leaking of particles through the puncture in the invariant tori. As there are only two such punctures for any barrier widths, so the diffusion is slow and hence the average energy,  $\langle E \rangle$  shows sub-diffusive growth.

To summarize the effect of barrier width on the average energy growth of the system, we have plotted the sub-diffusion exponents  $\mu_1$  and  $\mu_2$  as a function of barrier width,  $b$ , which is shown in Fig 2.14. In Fig. 2.14(a), it can be noticed



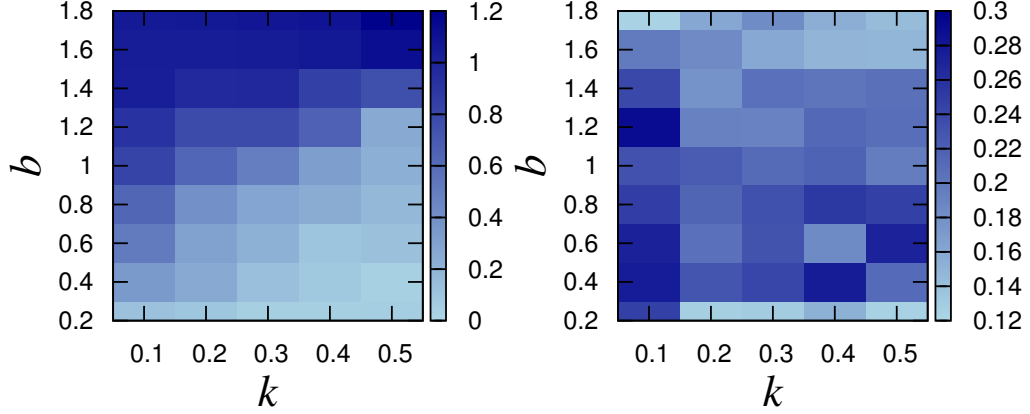


Figure 2.15: Plot to summarize the effects of barrier width and kick strength for a particular  $V_0 = 0.5$  and  $\phi = 0.0$  on sub-diffusive exponents,  $\mu_1$  (left) and  $\mu_2$  (right).

that depending on barrier width, the value of  $\mu_1$  varies from 0 to 1 which implies a transition from absence of diffusion to normal diffusion and in between for a wide range of values of  $b$ , the system shows sub-diffusive behavior. While Fig. 2.14(b) reveals that  $\mu_2$  barely depends on the change in barrier width,  $b$ . As there are only two punctures in the tori for each barrier width above the barrier height, effectively there is almost no change in the value of  $\mu_2$  as a function of barrier width.

## 2.5 Conclusions

In conclusion, we have studied a non-KAM system composed of a kicked rotor placed in a periodic finite well potential. The presence of this additional finite well potential provides us with several controllable parameters such as potential height, barrier width in addition to the kick strength already present in the kicked rotor system. We have found that each of these parameters has a profound effect on the qualitative nature of the dynamics of the system. For instance, by changing the barrier width, the system behavior ranges from absence of diffusion to sub-diffusion and finally, normal diffusion. The most important and central result of this study is that the system we considered shows sub-diffusive energy growth,  $\langle E \rangle \sim t^\mu$ , where  $\mu < 1$ , and surprisingly, the energy grows with two sub-diffusive exponents,  $\mu_1$  and  $\mu_2$ . The sub-diffusion observed is basically a manifestation of the non-KAM nature

of the system, and the underlying mechanism is the tori hopping.

To summarize the diffusion process and its dependency on the barrier width,  $b$  and the kick strength,  $K$ , we have plotted the sub-diffusive exponents,  $\mu_1$  and  $\mu_2$ , as a function of  $b$  and  $K$  in Fig. 2.15. Figure 2.15(a) illustrates that  $\mu_1$  has a strong dependence on both the parameters and also infers that the system could show sub-diffusive behavior for a wide range of  $b$  and  $K$  values. On the other hand, Fig. 2.15(b) displays that  $\mu_2$  weakly depends on these two parameters for the range of kick strengths used for plotting. Apart from this sub-diffusive energy growth, the system shows another surprising effect, the emergence of KAM-like tori (in a non-KAM system) which is due to the reflection symmetry that appears for certain choices of the system parameters.

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# Kicked Rotor in a Periodic Finite Well Potential : Quantum Dynamics

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In chapter 2, we have discussed the classical dynamics of a kicked rotor placed within a square well potential, and in this chapter, we will discuss the quantum dynamics of this system.

As discussed in the introduction, it is well-known that the quantum kicked rotor shows dynamical localization effect which is in contrast to its classical counterpart, a KAM system, that displays normal diffusion in the classically chaotic regime. In the standard kicked rotor, even as the classical mean energy grows approximately in a linear fashion, the quantum mean energy saturates after a certain time-scale. This localization in momentum representation observed in the kicked rotor is due to the quantum interference effects. Quantum kicked rotor is exponentially localized in momentum space and can be represented as  $\phi(l) \sim \exp(-l/l_s)$ , where  $l_s$  is the localization length [Reichl 2004]. The dynamical localization observed in the kicked rotor has been shown to be analogous to Anderson localization for electronic transport in crystalline solids which localizes in position space [Fishman 1982]. In the case of non-KAM systems, such as the kicked rotor in an infinite well potential

[Hu 1999] and kicked rotor with non-analytic kicking [Garcia 2005], eigenstates display power-law type localization. Thus, it can be noted that the standard quantum kicked rotor shows exponential localization, which is in contrast to the kicked rotor with non-analytic potential that exhibits power-law localization.

We have considered a non-KAM system, a kicked rotor in a finite well potential with potential height  $V_0$ . For large potential height, i.e.,  $V_0 \rightarrow \infty$ , this system tends towards the kicked rotor in an infinite well limit where we expect to observe power-law localization. On the other hand, as  $V_0 \rightarrow 0$ , this system behaves like a standard kicked rotor and exponential localization can be expected. Thus, by taking into account the results of these two limits, the question that we would like to address is whether it is possible to observe a transition from power-law to exponential localization by tuning the system parameters. More generally, we would also like to understand the quantum manifestation of a classically chaotic non-KAM system, especially for the results presented in chapter 2.

### 3.1 The System

The Schrödinger equation, corresponding to the Hamiltonian in Eq. (2.3), for a kicked rotor in a finite well potential, is

$$i\hbar_s \frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi, \quad (3.1)$$

where,  $\hbar_s = \frac{2\pi^2 \hbar T}{md^2}$  is the scaled Planck's constant,  $\Psi$  is the eigenstate of the Hamiltonian,  $\hat{H}(t)$ . In this, the total Hamiltonian,  $\hat{H}$ , can be considered as a superposition of unperturbed Hamiltonian,  $\hat{H}_0$ , and perturbation,  $\hat{H}_1$  which is

$$\begin{aligned} \hat{H}(t) &= \frac{\hat{p}^2}{2} + V_{sq}(\hat{x}) + K \cos(\hat{x}) \sum_n \delta(t - n) \\ &= \hat{H}_0 + \hat{H}_1(t). \end{aligned} \quad (3.2)$$

The unperturbed Hamiltonian,  $\hat{H}_0 = \frac{\hat{p}^2}{2} + V_{sq}(\hat{x})$  is time independent and the perturbed Hamiltonian,  $\hat{H}_1(t) = K \cos(\hat{x}) \sum_n \delta(t - n)$ , is periodic in time due to the presence of the delta function. Consider  $|\psi_n\rangle$  as the eigenstate of the unperturbed Hamiltonian,  $\hat{H}_0$ , with eigenenergy,  $E_n$ , satisfying the eigenvalue equation

$$\hat{H}_0|\psi_n\rangle = E_n|\psi_n\rangle. \quad (3.3)$$

We have considered  $|\psi_n\rangle$  as a superposition of all plane wave basis states

$$|\psi_n\rangle = \frac{1}{\sqrt{2\pi}} \sum_p a_{np} |p\rangle, \quad (3.4)$$

where,  $a_{np}$  are the expansion coefficients with the normalization condition,  $\frac{1}{2\pi} \sum_p |a_{np}|^2 = 1$ , and  $|p\rangle$  represents the momentum states. In this work, both of the quantity  $a_{np}$  and the values of momentum that will contribute to  $|\psi_n\rangle$  are unknown. But, as we have already mentioned in Chapter 2 that the dynamics of the system in Eq. (3.2) is considered on a cylinder which implies that the motion is unbounded in momentum axis and bounded in position axis with spatial periodicity  $2\pi$ . Since the system is spatially periodic, the momentum eigenstates need to have integer eigenvalues. Now, the problem boils down to finding the superposition coefficients,  $a_{np}$ , which can be obtained by calculating the eigenvectors of the Hamiltonian matrix. Thus, it is necessary to first calculate the Hamiltonian matrix elements which can be achieved by taking an inner product with respect to  $\langle\psi_n|$  on both sides of Eq. 3.3, we have

$$\begin{aligned} E_n &= \langle\psi_n|\hat{H}_0|\psi_n\rangle \\ &= \sum_{l,q} a_{nl}^* a_{nq} \frac{1}{2\pi} \langle l|\hat{H}_0|q\rangle \\ &= \sum_{l,q} a_{nl}^* a_{nq} H_{lq}, \end{aligned} \quad (3.5)$$

where,  $H_{lq}$  is the unperturbed Hamiltonian matrix element which on expanding in momentum basis takes the form

$$\begin{aligned}
 H_{lq} &= \frac{1}{2\pi} \langle l | \hat{H}_0 | q \rangle \\
 &= \frac{1}{2\pi} \langle l | \left( \frac{\hat{p}^2}{2} + V_{sq}(\hat{x}) \right) | q \rangle \\
 &= \frac{\hbar_s^2 l^2}{2} \delta_{lq} + \frac{1}{2\pi} \langle l | V_{sq}(\hat{x}) | q \rangle \\
 &= \frac{\hbar_s^2 l^2}{2} \delta_{lq} + \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ilx} V_{sq}(\hat{x}) e^{-iqx} dx}_I,
 \end{aligned} \tag{3.6}$$

where,

$$\begin{aligned}
 I &= \frac{1}{2\pi} \int_{-\pi}^{-\pi+\frac{b}{2}} e^{ilx} V_0 e^{-iqx} dx + \frac{1}{2\pi} \int_{\pi-\frac{b}{2}}^{\pi} e^{ilx} V_0 e^{-iqx} dx \\
 &= \frac{2V_0}{(l-q)\pi} \cos \left( (l-q) \left( \pi - \frac{b}{4} \right) \right) \sin \left( (l-q) \frac{b}{4} \right).
 \end{aligned}$$

Substituting the value of  $I$  in Eq. (3.6), we obtain

$$H_{lq} = \frac{\hbar_s^2 l^2}{2} \delta_{lq} + \frac{2V_0}{(l-q)\pi} \cos \left( (l-q) \left( \pi - \frac{b}{4} \right) \right) \sin \left( (l-q) \frac{b}{4} \right). \tag{3.7}$$

Thus, using  $H_{lq}$  obtained above, we diagonalize the unperturbed Hamiltonian matrix to obtain the eigenvalues and eigenvectors of the matrix. The eigenvalues represent the unperturbed energies,  $E$  and the elements of each eigenvector represents the values of the superposition coefficients  $a_{nl}$  of the corresponding wave vector. Thus, by plugging the values of  $a_{nl}$  in Eq. (3.4), we obtain the unperturbed energy states.

The perturbed Hamiltonian,  $\hat{H}_1(t)$  in Eq. (3.2), is time-periodic, and hence it enables us to use the well-known Floquet theorem which is an analogue of the Bloch theorem that holds for a spatially periodic system. The Floquet theorem ensures that there exists a Floquet operator,  $\hat{U}$ , which governs the dynamics of the quantum system. It can evolve some general state at time  $t$  to some later time  $t + T$ , where  $T$

is the time periodicity of  $\hat{H}_1(t)$ . In this system,  $T$  is considered to be 1. The Floquet operator can be represented as

$$\hat{U} = e^{-i\frac{\hat{H}_0}{\hbar s}} e^{-i\frac{K}{\hbar s} \cos(\hat{x})}. \quad (3.8)$$

The Floquet operator is a unitary operator, and it satisfies the condition  $U^\dagger U = 1$ . Being a unitary operator, it imposes the condition on its eigenvalues which needs to be of unit modulus. The eigenvectors of the Floquet operator are also very important as it plays a determining role in the behavior of the quantum system to let it show either localization or delocalization phenomenon. We will discuss more about the properties of the Floquet operator in the later part of this chapter.

The Floquet operator,  $\hat{U}$ , operates on any general state,  $|\Phi\rangle = \sum_n b_n |\psi_n\rangle$ , to evolve the state over time. The map that represents the evolution is

$$|\Phi(t + N)\rangle = \hat{U}^N |\Phi(t)\rangle, \quad (3.9)$$

where,  $\hat{U}^N = \hat{U}_N \cdots \hat{U}_1$ . The Floquet operator,  $\hat{U}$ , in the unperturbed energy basis is represented as

$$\begin{aligned} U_{mn} &= \langle \psi_m | \hat{U} | \psi_n \rangle \\ &= \langle \psi_m | e^{-i\frac{\hat{H}_0}{\hbar s}} e^{-i\frac{K}{\hbar s} \cos(\hat{x})} | \psi_n \rangle \\ &= e^{-i\frac{E_m}{\hbar s}} \langle \psi_m | e^{-i\frac{K}{\hbar s} \cos(\hat{x})} | \psi_n \rangle \\ &= \sum_{p,q} a_{mq}^* a_{np} e^{-i\frac{E_m}{\hbar s}} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iqx} e^{-i\frac{K}{\hbar s} \cos(\hat{x})} e^{-ipx} dx \\ &= \sum_{p,q} a_{mq}^* a_{np} e^{-i\frac{E_m}{\hbar s}} (-i)^{|p-q|} J_{|p-q|} \left( \frac{K}{\hbar s} \right), \end{aligned} \quad (3.10)$$

where,  $J_{|p-q|}(\cdot)$  is the Bessel function of order  $|p-q|$ . We notice from Eq. (3.10) that the decay of Floquet matrix elements away from the diagonal is not only determined by the Bessel function, which was the case of the standard kicked rotor, but also by the superposition coefficients,  $a_{np}$  of the eigenvectors of unperturbed states.

The Floquet matrix elements, thus obtained is used to evolve any general state,

$|\Phi(t)\rangle = \sum_n b_n |\psi_n\rangle$  over time. The evolution of an arbitrary state can be written as,

$$\begin{aligned}
 |\Phi(t+1)\rangle &= \hat{U}|\Phi(t)\rangle \\
 &= \sum_{m,n} |\psi_m\rangle \langle \psi_m | \hat{U} | \psi_n \rangle \langle \psi_n | \Phi \rangle \\
 &= \sum_{m,n} b_n U_{mn} |\psi_m\rangle.
 \end{aligned} \tag{3.11}$$

With the evolved state,  $|\Phi(t)\rangle$ , the average energy,  $\langle E \rangle$  of the system can be calculated as

$$\begin{aligned}
 \langle E \rangle &= \langle \Phi(t) | \hat{H}_0 | \Phi(t) \rangle \\
 &= \sum_{m,n} b_m^* b_n \langle \psi_m | \hat{H}_0 | \psi_n \rangle \\
 &= \sum_{m,n} b_m^* b_n E_n \delta_{mn} \\
 &= \sum_m |b_m|^2 E_m.
 \end{aligned} \tag{3.12}$$

From Eq. (3.12), it can be noticed that the average energy,  $\langle E \rangle$ , depends on the superposition coefficients  $b_m$ . Consider a general state,  $\Phi$ , which is just the ground state of the unperturbed Hamiltonian, i.e.,  $\Phi = \psi_0$ . After time evolutions, if the state tends towards a localized state where only a few  $b_m$  contributes, the average energy growth saturates. On the other hand, if the state is delocalized which implies more and more  $b_m$  contributes over time, as a result, there is an increase in the average energy of the system.

## 3.2 Numerical Calculations

We have calculated the quantum average energy growth of the system numerically. The first step in our calculation of quantum average energy is to calculate the eigenenergies,  $E_n$  of the unperturbed Hamiltonian and also the superposition coefficients,  $a_{np}$  to obtain the unperturbed eigenstates,  $|\psi_n\rangle$ . For that purpose, we have to calculate the unperturbed Hamiltonian matrix elements using the expres-



sion Eq. (3.7). The size of the unperturbed energy basis considered is in the range 1001 to 50,001 depending on values of scaled Planck's constant,  $\hbar_s$  and the potential height,  $V_0$ . For instance, for large scaled Planck's constant and small potential height, the size of the unperturbed energy basis can be taken to be small, say, approximately 1001 while for very small  $\hbar_s$ , the size of the unperturbed energy basis should be large. The wide range of unperturbed energy basis has been chosen to avoid convergence issues and also to decrease the computation time. After calculating the unperturbed Hamiltonian matrix,  $H_{lq}$ , we diagonalize the  $H_{lq}$  using LAPACK subroutine ZHEEV [Lapack]. After diagonalization, the eigenvalues and the eigenvectors of  $H_{lq}$  are obtained. Thus, we have calculated the unperturbed energies,  $E_n$  and the unperturbed energy states,  $|\psi_n\rangle$ , as the values of the superposition coefficients  $a_{np}$  are basically the elements of the eigenvectors.

Using the values of  $a_{np}$ ,  $E_m$  and the ordinary Bessel function  $J_{|p-q|}\left(\frac{K}{\hbar_s}\right)$  (calculated using Mathematica), we calculate the Floquet matrix elements,  $U_{mn}$  given in Eq. (3.10). After obtaining the Floquet matrix elements, the calculation of average energy simply depends on defining the initial arbitrary state,  $\Phi(0)$ . In our case, we have used the ground state of the unperturbed Hamiltonian as initial state, i.e., we have assumed  $b_0 = 1$  and  $b_n = 0$  if  $n \neq 0$ . We have time evolved the arbitrary state  $\Phi(0)$  using Eq. (3.11) and calculated the average energy of the system over time using Eq. (3.12).

Owing to understand the localization properties of the system, we have analyzed the Floquet states which are obtained by diagonalizing the Floquet matrix whose elements are represented by Eq. (3.10). To diagonalize the Floquet matrix, we have used ZGEEV subroutine of LAPACK [Lapack].

In the remaining part of this chapter, we will discuss the results of the quantum system in Eq. 3.2. We will mainly focus on the average energy growth of the system and will also analyze the effect of singularity in the potential on the quantum localization phenomenon. We will also look at the effects of the classical non-KAM features on the quantum system.

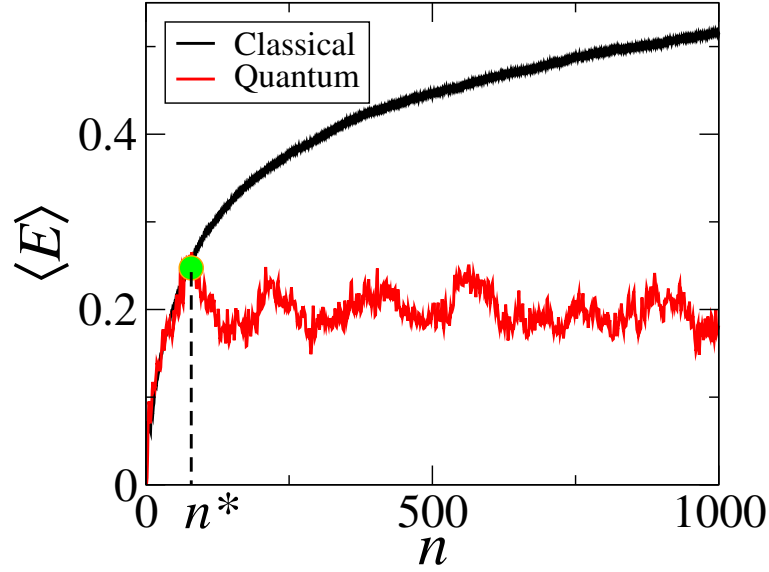


Figure 3.1: Average energy growths of the system in Eq. (3.2) for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $b = \pi$ , and  $\hbar_s = 0.0067$ . Vertical dotted line represents the break-time,  $n^*$ .

### 3.3 Quantum Dynamics

The average energy of the system has been calculated numerically using the procedure mentioned in the last section. The average energy growth quantifies the diffusion in the energy space since the unperturbed basis state is the energy state. As already mentioned in Sec. (3.1), to calculate the average energy of the system, we have considered the ground state of the unperturbed system,  $\hat{H}_0$ , as the initial state. After the evolution, this initial states spread over the unperturbed energy states. One quantity to quantify this spreading is the average energy of the system which depends on the absolute value of  $b_m$  (superposition coefficients of the evolved state) following the relation is shown in Eq. (3.12). Figure 3.1 displays both classical and quantum average energy growth of the system for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $\hbar_s = 0.0067$  and  $b = \pi$ . Figure 3.1 shows that the quantum average energy initially follows the classical energy up to some timescale, after which, the quantum average energy saturates, i.e., no more increase in the average energy. This is a manifestation of the well-known dynamical localization phenomenon. The time at which the quantum and classical average energy growth deviate from one another is called the break-time. This time-scale is denoted by  $n^*$  in Fig. 3.1. The break-time can be

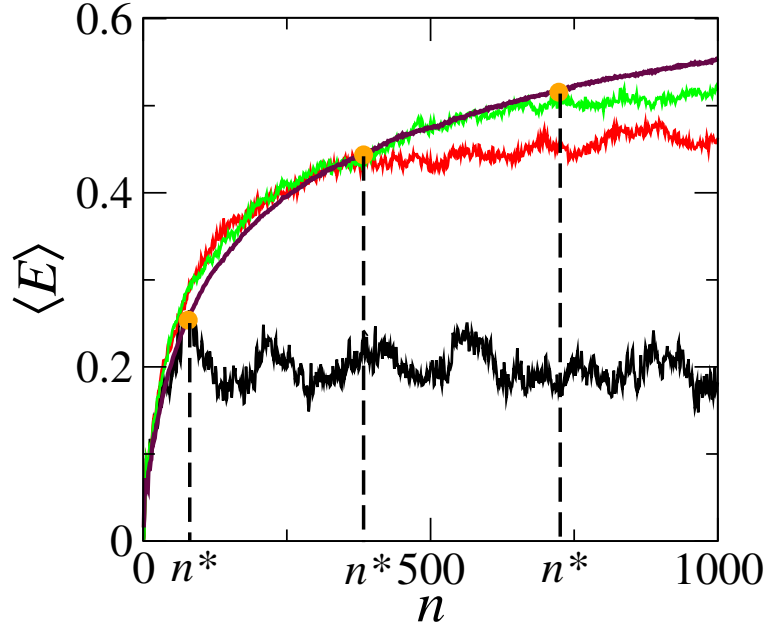


Figure 3.2: Average energy growth for  $V_0 = 0.5$ ,  $K = 0.25$ ,  $b = \pi$ , and different scaled Planck's constant,  $\hbar_s = 0.00057$  (green line),  $\hbar_s = 0.00097$  (red line),  $\hbar_s = 0.0067$  (black line). Maroon line represents classical average energy growth. Vertical black dashed lines represents break-times for different  $\hbar_s$ .

controlled by varying the scaled Planck's constant,  $\hbar_s$ , which is shown in Fig. 3.2. Figure 3.2 illustrates that by decreasing the scaled Planck's constant,  $\hbar_s$ , or taking the system towards the classical limit, the quantum average energy growth follows the classical one for a long time after which the quantum energy saturates.

To understand this dependence quantitatively, we calculated analytically the expression that describes the relation between the break-time and the scaled Planck's constant. It has been obtained by considering the mean unperturbed energy density as  $E^{-\frac{1}{2}}$  or  $\langle E \rangle_{ql} \sim n^2 \hbar_s^2$ . When the quantum energy saturates after break-time, the energy is of the order of maximum quantum number,  $n^*$ , that participated to form the evolved state  $|\Phi\rangle$ . On the other hand, the classical average energy grows as  $\langle E \rangle \sim D_i n^{\mu_i}$ . Until the break-time,  $n = n^*$ , both the classical and quantum energy grows with the same rate. So, at break-time, we can equate the two energy growths (both quantum and classical), leading to,

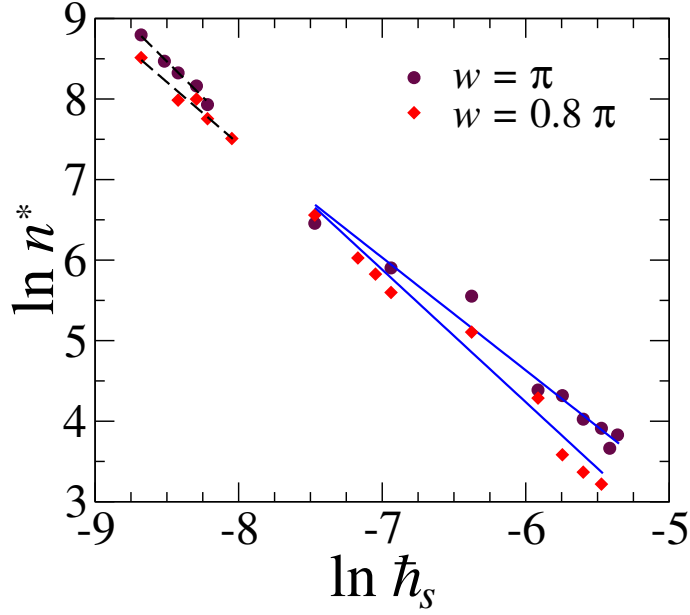


Figure 3.3: Break-time,  $n^*$ , as a function of scaled Planck's constant,  $\hbar_s$  for  $V_0 = 0.5$ ,  $K = 0.25$ , and different well widths a)  $w = \pi$  (purple) and b)  $w = 0.8\pi$  (red). Symbols represents numerical results and dashed and solid lines represents analytical results.

$$\begin{aligned}
 \langle E \rangle_{ql} &= \langle E \rangle_{cl} \\
 n^{*2} \hbar_s^2 &\sim D_i n^{*\mu_i} \\
 n^* &\sim \left( \frac{D_i}{\hbar_s^2} \right)^{\frac{1}{2-\mu_i}}.
 \end{aligned} \tag{3.13}$$

In this,  $\mu_i$  ( $i = 1, 2$ ) are the sub-diffusive exponents and  $D_i$  ( $i = 1, 2$ ) are the classical diffusion coefficients. Equation (3.13) relates the system's quantum parameters with that of its classical counterpart. Equation (3.13) also exhibits the dependence of break-time,  $n^*$  on classical sub-diffusion exponents,  $\mu_i$ . The break-time in addition shows a non-integral  $\hbar_s$  scaling, since,  $\mu_i \leq 1$  for  $i = 1, 2$ , which is an unusual feature displayed by this system.

Figure 3.3 shows the dependence of break-time on  $\hbar_s$  for two different choices of well widths,  $w = \pi$  and  $w = 0.8\pi$  represented by purple and red colour symbols, respectively. The plot shows that the numerical results are in good agreement with the analytical estimates which confirms that the break-time actually shows non-

integral  $\hbar_s$  scaling. From the plot, it can be noticed that there are two regimes present for each of the two well widths: i) for large  $\hbar_s$  and ii) for small  $\hbar_s$ . For the large  $\hbar_s$  regime, the analytical results are calculated by replacing  $\mu_i$  in Eq. (3.13) using  $\mu_1$  while for the small  $\hbar_s$  regime, it is calculated by considering  $\mu_i$  in Eq. (3.13) as  $\mu_2$ . In the intermediate regime, there is no specific value of  $\mu_i$ . This is because in the classical average energy growth shown in Fig. 2.11 in Chapter 2, the crossover from  $\mu_1$  regime to the  $\mu_2$  regime is not sharp instead shows a very smooth transition.

### 3.4 Husimi Distribution

In 1940, Kôdi Husimi introduced the Husimi distribution in order to study the correspondence between quantum and classical dynamics [Husimi 1940]. This distribution can be used to study the analogues of classical phase space for quantum systems. Although Wigner distribution [Wigner 1932], another popular quasi-probability distribution can also be used for this purpose, Husimi distribution is preferred in the present context since it is positive definite [Reichl 2004, Takahashi 1985] in contrast to Wigner which can take negative values as well.

Husimi distribution of some evolved state  $\Phi(t)$  is calculated by taking expectation value of the density operator constructed using  $\Phi(t)$  in the basis of minimum uncertainty wave packets, i.e., the coherent state,  $|Z\rangle$ , represented as

$$W(z) = |\langle \Phi | Z \rangle|^2, \quad (3.14)$$

where,  $|Z\rangle = |x_0, p_0\rangle$ . The coherent state has the minimum uncertainty product, i.e.,  $\Delta x_0 \Delta p_0 = \frac{\hbar_s}{2}$ . The uncertainty of the coherent state in the position basis is  $\sqrt{\frac{\hbar_s}{2}}$  and that in momentum is  $\sqrt{\frac{\hbar_s}{2}}$ . The coherent state expanded in position basis is represented as

$$\langle x | Z \rangle = \left( \frac{1}{\pi \hbar_s} \right)^{\frac{1}{4}} \exp \left( -\frac{1}{2\hbar_s} (x - x_0)^2 + i \frac{p_0}{\hbar_s} (x - x_0) \right). \quad (3.15)$$

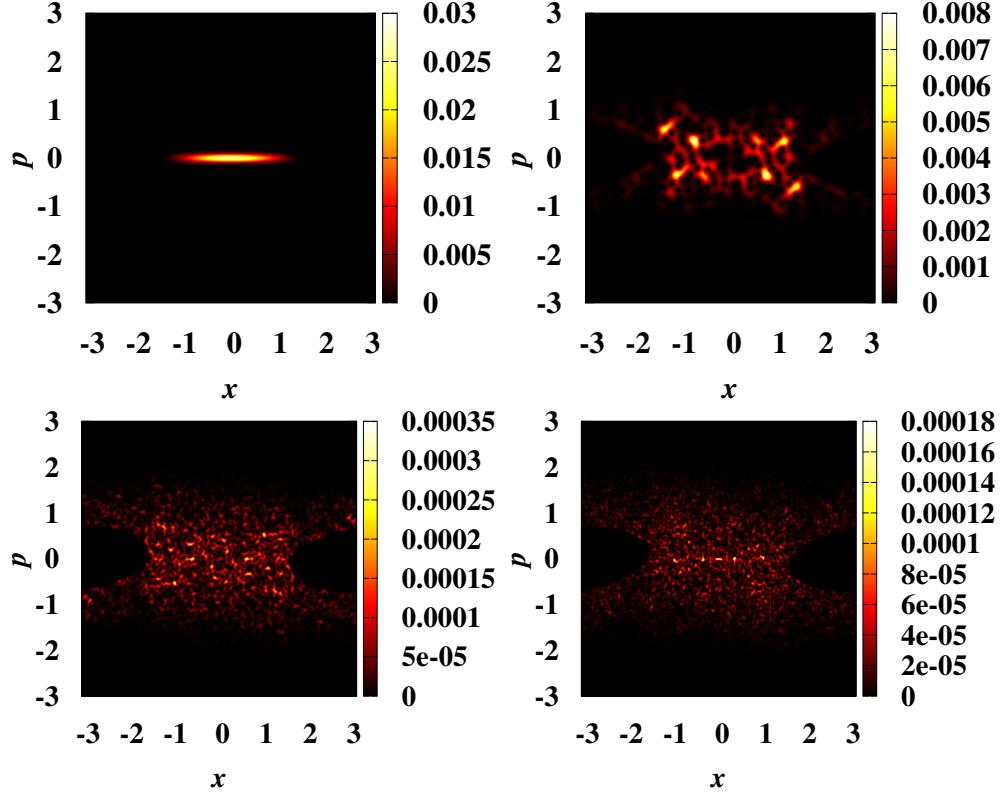


Figure 3.4: Husimi distribution for initial state (top left), and evolved states after  $n = 5000$  for  $\hbar_s = 0.0067$  (top right),  $\hbar_s = 0.00097$  (bottom left), and  $\hbar_s = 0.00057$  (bottom right). Other parameters are  $b = \pi$ ,  $V_0 = 0.5$ ,  $K = 0.25$ .

Substituting Eq. (3.15) in Eq. (3.14), Husimi distribution  $W(Z)$  can be calculated as

$$W(Z) = \left| \int dx \langle Z|x \rangle \langle x|\Phi \rangle \right|^2, \quad (3.16)$$

where,  $W(Z)$  gives the probability to find the state  $\Phi(t)$  in the phase space centered at  $(x_0, p_0)$  within a cell size  $\hbar_s$ .

Figure 3.4 shows the Husimi distribution for different scaled Planck's constant. Figure 3.4(a,b,c) are generated using evolved states which are obtained by evolving the ground states of a system in Eq. (3.2) for different scaled Planck's constants for  $n = 5000$ . This long time evolution is to ensure that the system has reached its localization phase and hence, there is no more spreading in phase space over time. The basis size for different values of scaled Planck's constants,  $\hbar_s = 0.0067$ ,  $\hbar_s = 0.00097$  and  $\hbar_s = 0.00057$  are 2001, 5001 and 10001, respectively.

Figure 3.4 displays that with decreasing scaled Planck's constant, there is spreading in probability distribution in the quantum phase space which gives rise to the quantum average energy growth that is observed in Fig. 3.2. We can also notice that for  $\hbar_s = 0.0067$ , maximum probability distribution is confined inside the well region. Hence, the quantum average energy saturates in the classical  $\mu_1$  regime which is evident from Fig. 3.3. While for small  $\hbar_s = 0.00097$  and  $\hbar_s = 0.00057$ , there is a partial spreading of probability distribution above the well region. As a result, the quantum average energy saturates in the intermediate regime between  $\mu_1$  and  $\mu_2$ . If we further decrease the scaled Planck's constant, we could go to the  $\mu_2$  regime. Because of computational difficulty of handling larger basis states that one has to take while tending towards very small scaled Planck's constant values, is not discussed in this thesis.

### 3.5 Localization Phenomenon and Its Type

In order to characterize and understand the nature of quantum localization, we will analyze the decay of Floquet states over the unperturbed basis. The Floquet states or quasi-energy states are the eigenstates of the Floquet operator. The eigenvalue equation for the Floquet operator,  $\hat{U}$ , is

$$\hat{U}|\phi\rangle = e^{i\omega}|\phi\rangle, \quad (3.17)$$

where  $\omega$  is the quasi-energy corresponding to the floquet state  $|\phi\rangle$ . Since the eigenvalues of  $\hat{U}$  should be of modulus 1 as  $\hat{U}$  is a unitary the operator, so  $\omega$  should lie on a unit circle, i.e.,  $\omega \in [0, 2\pi]$ .

The Floquet states,  $|\phi\rangle$ , can be expanded in terms of the unperturbed energy states as

$$|\phi\rangle = \sum_n c_n |\psi_n\rangle, \quad (3.18)$$

where,  $c_n$  represent the expansion coefficient which describes the overlap of the Floquet state with the unperturbed energy state,  $|\psi_n\rangle$ . The nature of localization,

if it exists, can be inferred from the profile of  $|c_n|^2$ . Consider a particular  $|c_n|^2 = |c_{n_{max}}|^2$  in  $\{c_1, c_2, \dots, c_n, \dots, c_N\}$  which has a maximum value and the value of  $|c_{n'}|^2$  decay as we move away from  $|c_{n_{max}}|^2$ . The nature of decay of these values determines the nature of localization. If the values decays exponentially fast as we are moving away from  $|c_{n_{max}}|^2$ , then the localization is of exponential in nature and power-law localization is observed if the values decay in a power-law fashion. However, if the value of  $|c_{n'}|^2$  does not show a decay profile as we move away from  $|c_{n_{max}}|^2$  instead all the  $|c_n|^2$  contributes more or less same, then the system will not localize instead will show a diffusive behavior. The corresponding states are said to be of extended type.

Since the system in Eq. (3.2) display localization phenomenon, we are interested to characterize the nature of localization. Figure (3.5) shows the nature of decay of averaged Floquet states over the unperturbed energy states,  $\psi_n$ . The averaging has been done by shifting the index,  $n$ , of  $c_n$  of each Floquet state by its corresponding  $n_{max}$ , i.e.,  $v = n - n_{max}$ . Thus the new index for  $c_n$  is  $c_v$ . This has been done in order to match the maxima of all the Floquet states to do the averaging as we are interested in finding the overall nature of decay of the Floquet states.

Figure (3.5a) shows the profile of averaged Floquet states when it is in the kicked rotor in an infinite well limit. To get to this limiting case, the potential height is given a large value,  $V_0 = 5000.0$ . As a result, most of the Floquet states are confined inside the well region which means a Floquet state have a maximum  $|c_n|^2$  at  $n = n_{max}$  for which the corresponding unperturbed energy  $E_{n_{max}} < V_0$ . These sets of Floquet states are defined in this thesis as Floquet states residing inside the well (FSIW). We notice that in this limit, the system shows power-law localization which is consistent with the results in Ref. [Hu 1999]. Now, we decrease the potential height,  $V_0$ , to a small value, i.e.,  $V_0 = 0.5$ , so that the system is in the kicked rotor limit. In this case, most of the Floquet states are confined above the barrier height,  $V_0$ . A Floquet state is said to be confined much above the barrier height (FSMAB) if the Floquet state have a maximum  $|c_n|^2$  at  $n = n_{max}$  for which the corresponding unperturbed energy  $E_{n_{max}} \gg V_0$ . The system in this limit exhibits exponential



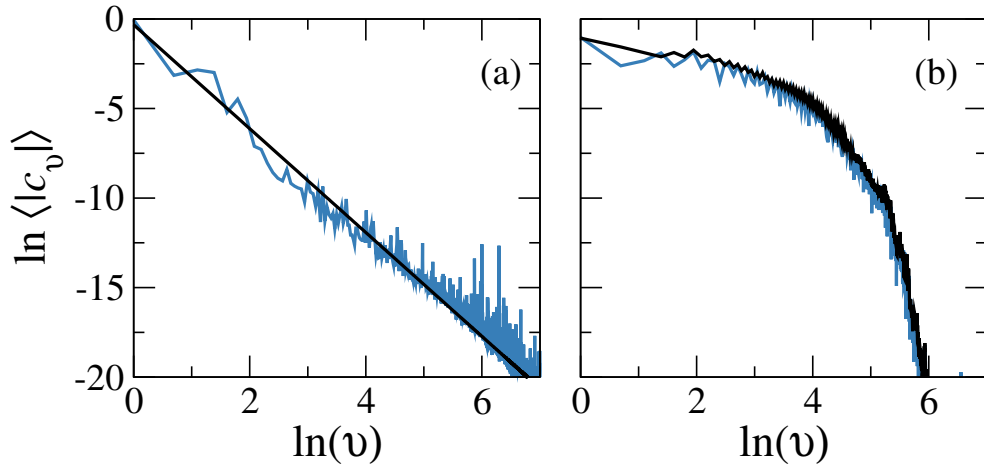


Figure 3.5: Averaged Floquet states decay over unperturbed energy states for  $b = 1.4\pi$ ,  $\hbar_s = 1.0$ , a)  $V_0 = 5000.5$ ,  $K = 0.25$  which is kicked rotor in an infinite well limit, and b)  $V_0 = 0.5$ ,  $K = 4.25$  which is kicked rotor limit.

localization which is shown in Fig. 3.5(b). Thus, the system in Eq. (3.2) in two different limiting cases exhibits results which are consistent with the results already known.

Now, we have tuned the system parameters in such a way in order to keep the system in the intermediate regime, i.e., in between kicked rotor in an infinite well limit and kicked rotor limit. This case is interesting as there is a sufficient number of FSIW as well as FSMAB. This situation is presented in Fig. 3.6. It can be noticed from Fig. 3.6 that initially, the Floquet states show an exponential decay followed by a power-law tail. Upon careful analysis, it is observed that FSIW display a power-law localization as seen in Fig. 3.6(b) and FSMAB show exponential localization as displayed in Fig. 3.6(c). Thus, in conclusion, the Floquet states which “feel” the presence of a singularity in the stationary potential well show a power-law localization and the Floquet states which effectively does not “feel” the presence of singularity shows exponential localization.

### 3.6 Participation Ratio

We have calculated the participation ratio of the Floquet states in order to obtain information about the localization properties of Floquet states. The participation

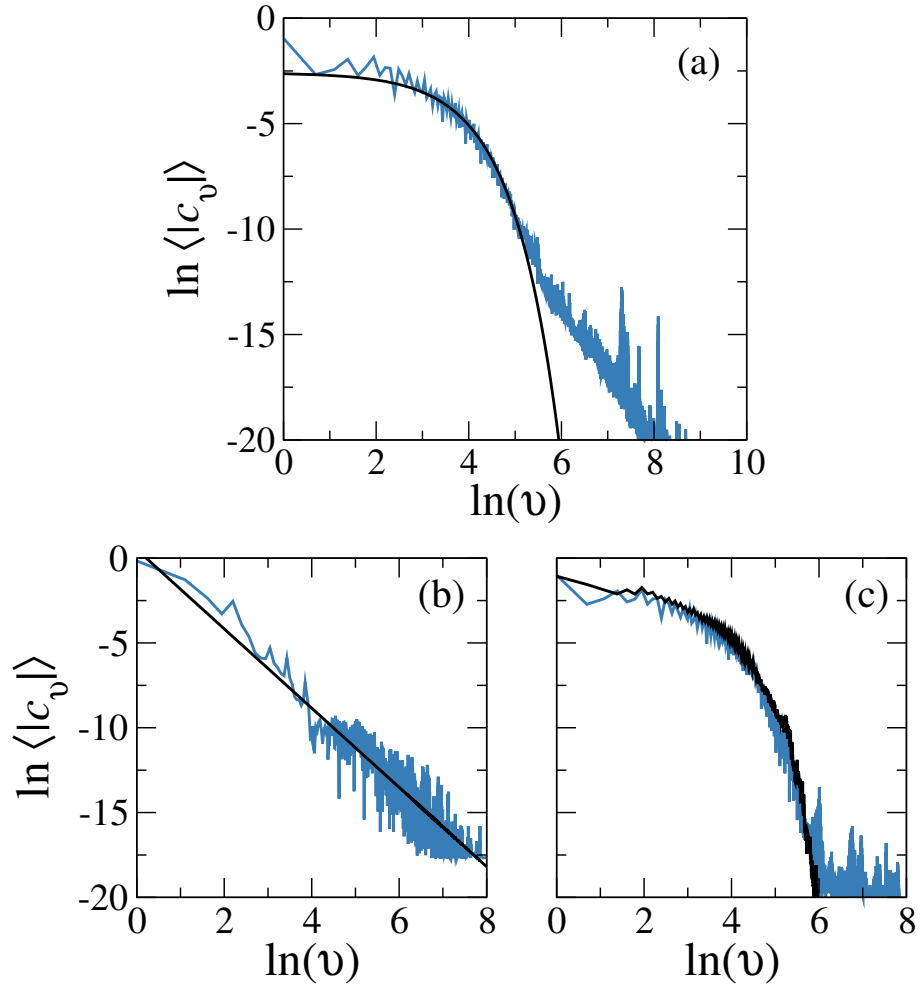


Figure 3.6: Averaged Floquet states decay for  $V_0 = 5000.5$ ,  $K = 4.25$ ,  $b = 1.4\pi$ ,  $\hbar_s = 1.0$ . a) Averaged over all Floquet states (blue line), b) averaged over all FSIW (blue line), and c) averaged over all FSMAB (blue line). In (a) and (c) black line corresponds to kicked rotor and in (b) black line represents a linear fit.

ratio is defined as [Reichl 2004]

$$P = \sum_{i=1}^N |c_i|^4, \quad (3.19)$$

where,  $|c_i|$  are components of a Floquet state. If participation ratio  $\sim 1$ , then the Floquet states are strongly localized and only a few components of the Floquet states contribute significantly while the contribution from others are almost negligible. On the other hand, if the participation ratio  $\sim \frac{1}{N}$ , Floquet states is of extended nature and all of the components of the Floquet states make nearly equal contribution.

The participation ratio of all the Floquet states of the system in Eq. (3.2) are shown in Fig. 3.7. Figure 3.7(a) illustrates that the participation ratio of a FSIW tends toward 1 exhibiting strong localization compared to FSMAB. Thus, the system can clearly distinguish its two regimes, one inside the well and other above the barrier height. This feature becomes more prominent as we increase the barrier height and is evident from Fig. 3.7(a,b). Even though the classical phase space for large barrier width is fully chaotic, but the FSIW shows strong localization. And this happens because, with the increase in barrier width, the number of bound states inside the well region decreases. As a result, the Floquet states residing on these bound states have a very small overlap with the states above the barrier height and hence, exhibits strong localization. On the other hand, it is reflected from Fig. 3.7(c) that the nature of localization above the barrier height weakly depends on the change in barrier width.

### 3.7 Floquet Matrix Element : Its Decay Behavior

In this section, we will look at the decay of the Floquet matrix elements and will try to understand how the decay of Floquet matrix elements away from the diagonal determines the localization properties. It is known that exponential localization observed in the kicked rotor is associated with the exponential decay of the matrix elements of the corresponding Floquet operator. Further, from Ref. [Sankaranarayanan

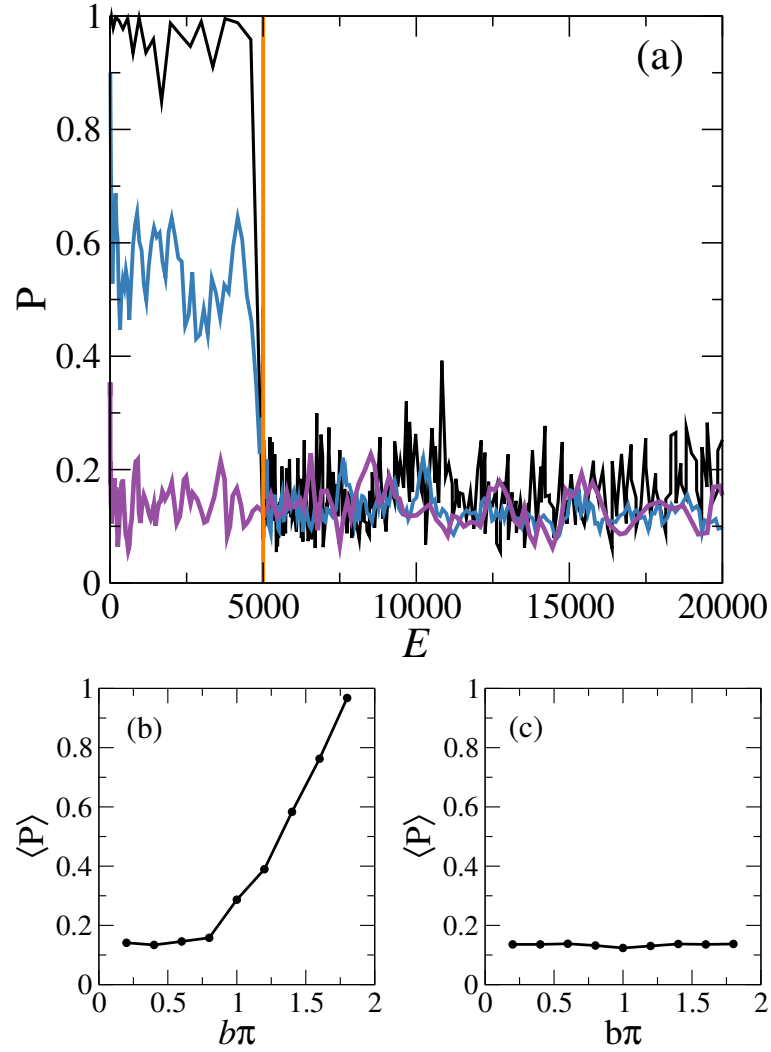


Figure 3.7: a) Participation ratio of Floquet states as a function of  $E_{n_{max}}$  for  $V_0 = 5000.5$ ,  $K = 4.25$ ,  $\hbar_s = 1.0$  nd for different barrier widths,  $b = 1.8\pi$  (black),  $b = 1.4\pi$  (blue), and  $b = \pi$  (purple). Orange vertical line marks for energy,  $E_n = 5000.5$ . b) Averaged participation ratio of all the FSIW as a function of barrier width. b) Averaged participation ratio of all the FSMAB as a function of barrier width.

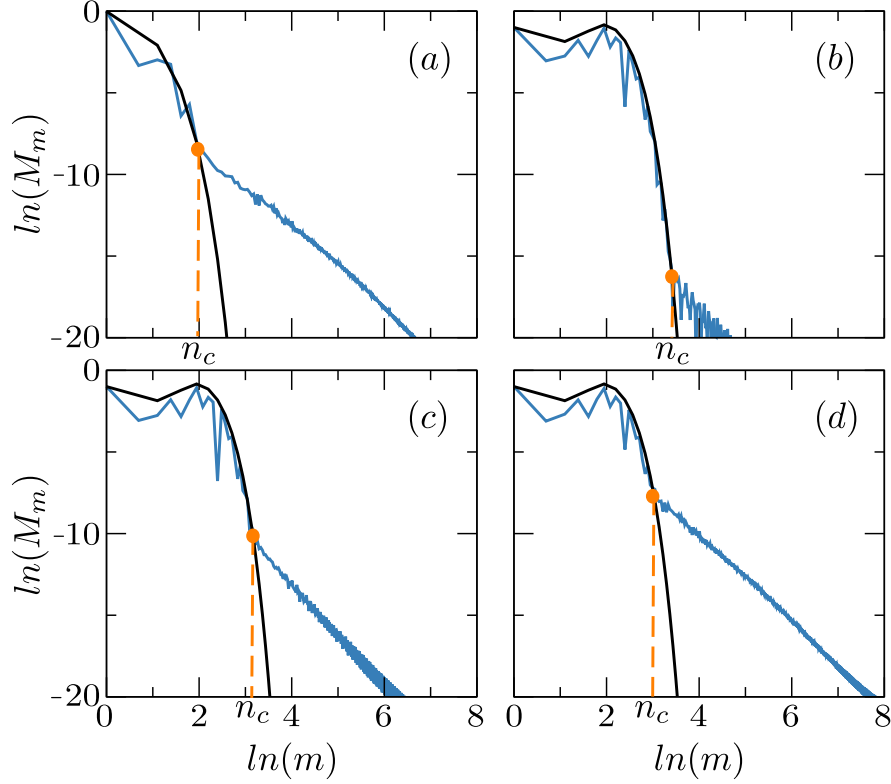


Figure 3.8: Averaged decay of the matrix elements of the Floquet operator  $\hat{U}$  as a function of  $m$ . The parameters are  $b = 1.4\pi$ ,  $\hbar_s = 1.0$ , (a) (kicked rotor in an infinite well limit)  $V_0 = 5000.0$ ,  $K = 0.25$ , and (b) (kicked rotor limit)  $V_0 = 0.5$ ,  $K = 4.25$ , (c)  $V_0 = 100.0$ ,  $K = 4.25$ , (d)  $V_0 = 5000.0$ ,  $K = 4.25$ .  $n_c$  represents the crossover point from exponential to power-law profile. All black curves corresponds to kicked rotor.

2001, Hu 1999], it is also known that in the case of the kicked rotor in an infinite well, a non-KAM system, the matrix elements of  $\hat{U}$  display a power-law decay after a bandwidth  $\eta \propto K$ . Here, we inquire how the decay of matrix elements changes its character as  $V_0 \gg 1$  approaches the limit  $V_0 \rightarrow 0$ . In the unperturbed basis, the matrix elements are  $U_{nm}$  as given by Eq. (3.10). This is illustrated in Fig. 3.8 which shows  $M_m = \langle |U_{nm}| \rangle_n$  as a function of  $m$ , with  $m > n$ , in log-log plot.

Figure 3.8(a,b) shows  $M_n$  as a log-log plot for the same choice of parameters as in Fig. 3.6(a,b). Figure 3.8(a) corresponds to kicked rotor in an infinite well limit and shows a short regime of exponential decay followed by an asymptotic power-law decay. In Fig. 3.8(b),  $V_0 = 0.5$  corresponding to the kicked rotor limit and the decay of  $M_n$  largely follows that of kicked rotor except for  $n \gg 1$  where it decays

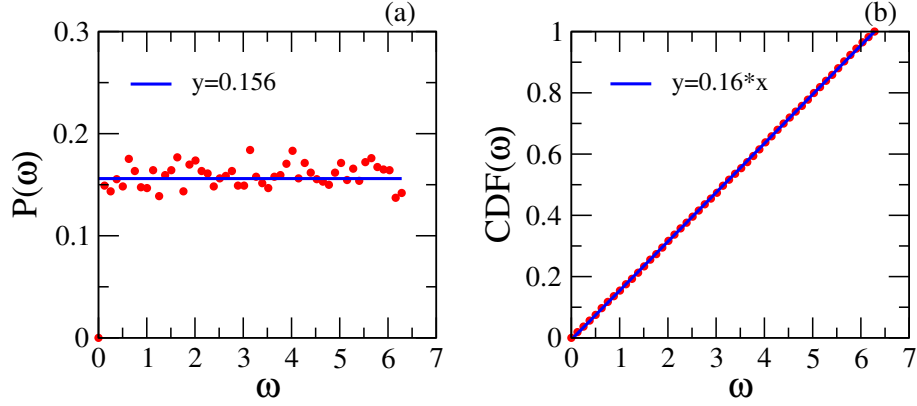


Figure 3.9: a) Distribution of Floquet eigenvalues,  $\omega$  of the system in Eq. (3.2) for  $V = 5000.5$ ,  $K = 4.25$ ,  $\hbar_s = 1.0$ , and  $b = 1.4\pi$ . b) Cumulative distribution of  $\omega$ .

as a power-law. In general, the following features are observed. In the limit as  $V_0 \rightarrow \infty$ , the decay is of power-law form. In the opposite limit of  $V_0 \rightarrow 0$ , the decay is exponential in nature. In general, for any intermediate  $V_0$ , i.e.,  $0 < V_0 < \infty$ , an initial exponential decay is followed by an asymptotic power-law decay whose slope is approximately 2.7. If  $V_0 < \infty$ , the initial exponential decay is always present. The exponential decay sharply changes over to a power-law decay at  $n = n_c$  as shown by dotted vertical lines in Fig. 3.8. For any fixed value of kick strength  $K$ , as  $V_0$  varies from 0 to  $\infty$ , then  $n_c$  changes from  $\infty$  to 0. It is also to be noted that for fixed  $V_0$ , as  $K$  increases,  $n_c$  also increases.

In the next section, we look at the level spacing distribution of the system and would like to address the question whether the Floquet eigenvalues also show a dependence on the effective presence or absence of singularity.

### 3.8 Spacing Distribution

In this section, we concentrate on the level statistics of the Floquet eigenvalues. Since the Floquet operator,  $\hat{U}$ , is a unitary operator all eigenvalues will lie on a unit circle and are uniformly distributed in the interval  $[0, 2\pi]$  which is shown in Fig. 3.9. In Fig. 3.9(a) distribution of quasi-energies of the system in Eq. (3.2) with parameter values  $V = 5000.5$ ,  $K = 4.25$ ,  $\hbar_s = 1.0$ , and  $b = 1.4\pi$  are plotted. It

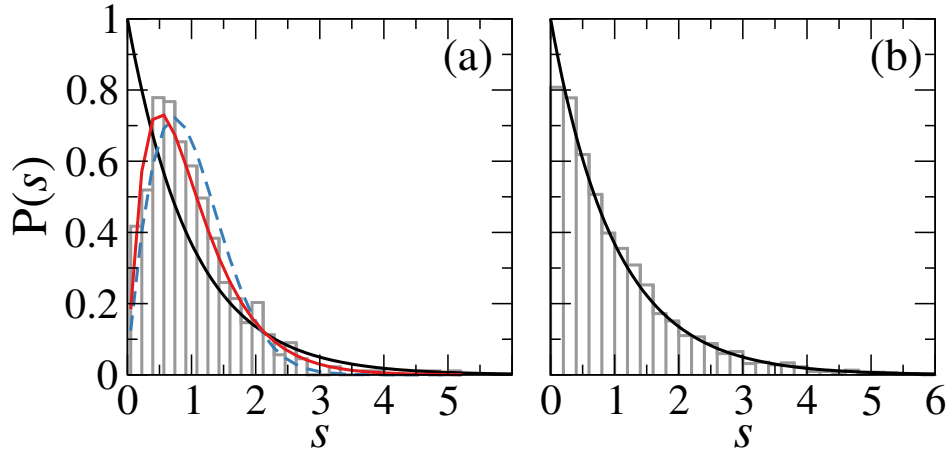


Figure 3.10: a) Level spacing distribution of Floquet eigenvalues corresponding to a) FSIW and b) FSMAB for the parameter values  $V_0 = 1500000$ ,  $K = 4.25$ ,  $\hbar_s = 1.0$  and  $b = 1.4\pi$ . Black solid line, red solid line and blues dashed line represents Poisson, semi-Poisson and Wigner distributions.

shows that the distribution is uniform. The cumulative distribution function (CDF) of the same quasi-energies are displayed in Fig. 3.9(b) and the linear growth in CDF again confirms that the distribution is uniform. Thus, the level density, in this case, is constant ( $\frac{N}{2\pi}$ ) and hence, the unfolding of energy levels is not necessary.

To calculate the spacing distribution, we have treated the eigenvalues of even and odd parity states separately. Since our Hamiltonian is symmetric,  $H(-\theta) = H(\theta)$ , parity is conserved. The spacing between nearest neighbor quasi-energies can be calculated as  $S_n = \frac{N}{4\pi}(\omega_{n+1} - \omega_n)$ , where  $n = 1, 3, \dots, N$  or  $n = 2, 4, \dots, N$ . The factor ( $\frac{N}{4\pi}$ ) ensures that the mean level spacing is 1. The histograms presented in Fig. 3.10 has been obtained using spacings  $S_n$  for the parameter values  $V_0 = 1500000$ ,  $K = 4.25$ ,  $\hbar_s = 1.0$  and  $b = 1.4\pi$ . The reason for taking this large value of potential height,  $V_0$  is to get sufficient numbers of FSIW as well as FSMAB (and hence the quasi-energies).

In Fig. 3.10, we have considered the two cases separately: i) the spacing distribution for the quasi-energies corresponding to the FSIW as shown in Fig. 3.10(a), and ii) also for the quasi-energies corresponding to the FSMAB which is shown in Fig. 3.10(b). Figure 3.10(b) shows a Poissonian spacing distribution,  $P(s) = e^{-s}$  which is expected as the Floquet states corresponding to these quasienergies are

confined much above the barrier height and they exhibit an exponential decay over unperturbed energy states which is similar to the standard kicked rotor. Poissonian spacing distribution implies that the Floquet states are placed far apart and hence the level repulsion is inhibited. On the other hand, Fig. 3.10(a) shows a non-Poissonian spacing distribution which implies Floquet states are densely situated inside the well region and thus give rise to level repulsion. Black curve represents Poisson distribution and the blue curve is the semi-poisson distribution [Wiersig 2002],  $P(S) = 4Se^{-2S}$  which shows approximately good fit to the spacing distribution. Thus, the kicked rotor in finite well potential exhibits an unusual feature that is, its spacing distribution is not unique but depends on the energy range from which the levels are drawn.

### 3.9 Mapping to the Lattice Model

For a better understanding of the localization phenomenon, we will map the kicked rotor in finite well potential to the Anderson-like model. Anderson model describes the motion of a particle on a lattice with a random disorder in the onsite energy. The Hamiltonian of the Anderson model with a nearest neighbor hopping term is

$$H = T \sum_i (|i+1\rangle\langle i| + |i\rangle\langle i+1|) + \sum_i U_i |i\rangle\langle i|, \quad (3.20)$$

where,  $i$  represents the lattice site and  $T_i$  is the nearest neighbor hopping term to hop from lattice site  $i$  to  $i+1$  or vice versa.  $U_i$  is the onsite energy which is uniformly drawn from the interval  $[-W, W]$ . The particle, in this case, localizes exponentially in position space. The standard kicked rotor can be exactly mapped into this lattice model which is already reported in Ref. [Fishman 1982, Grempel 1984] and kicked rotor also shows exponential localization. The major difference between the kicked rotor and Anderson model is that the kicked rotor system shows exponential localization in momentum space while in the Anderson model, it happens in position space.



Now, we will map the system in Eq. (3.2) to Anderson model and investigate the reasons that lead the system in Eq. (3.2) to behave differently from the 1D Anderson Model. Since Eq. (3.2) represents a time periodic system and using Floquet-Bloch theorem, we can write the quasi-energy state as

$$\phi(\theta, t) = e^{-i\omega t} u(\theta, t), \quad (3.21)$$

where,  $u(\theta, t) = u(\theta, t+1)$ . The system in between two consecutive kicks behaves as a free particle and thus the evolution is

$$\phi_n^-(t+1) = e^{-i\hat{H}_0} \phi_n^+(t), \quad (3.22)$$

where,  $\phi_n^-(t+1)$  and  $\phi_n^+(t)$  are the quasi-energy states just before the  $(t+1)$ th kick and just after  $t$ th kick. During the free evolution it gathers an extra phase  $e^{-i\hat{H}_0}$ . Now by substituting the expression of  $\phi(\theta, t)$  from Eq. (3.21) in Eq. (3.22) and utilizing the time periodic property of  $u(\theta, t)$ , we can rewrite Eq. (3.22) in terms of  $u(\theta, t)$  as

$$u_n^-(\theta, t+1) = e^{i\omega} e^{-i\hat{H}_0} u_n^+(\theta, t). \quad (3.23)$$

Now the quasi-energy state just after  $t$ -th kick can be obtained using a map

$$\phi^+(\theta, t) = e^{-iV(\theta)} \phi^-(\theta, t). \quad (3.24)$$

Again, by using Eq. (3.21), Eq. (3.24) can be written in terms of  $u(\theta, t)$  as,

$$u^+(\theta, t) = e^{-iV(\theta)} u^-(\theta, t). \quad (3.25)$$

Rewriting  $e^{-iV(\theta)}$  in terms of trigonometric function as

$$e^{-iV(\theta)} = \frac{1 + iW(\theta)}{1 - iW(\theta)}, \quad (3.26)$$

where,  $W(\theta) = -\tan(\frac{V(\theta)}{2})$ . Substituting Eq. (3.26) in Eq. (3.23), we obtain

$$\frac{u^+(\theta)}{1+iW(\theta)} = \bar{u} = \frac{u^-(\theta)}{1-iW(\theta)}, \quad (3.27)$$

where,  $\bar{u}$  is defined as

$$\bar{u} = [u^+(\theta) + u^-(\theta)]/2. \quad (3.28)$$

Now the evolution of the quasi-energy state after one period can be obtained using Eq. (3.23) and Eq. (3.26)

$$u^+(\theta) = e^{-iV(\theta)} e^{i(\omega - \hat{H}_0)} u^+(\theta), \quad (3.29)$$

$$\frac{1-iW(\theta)}{1+iW(\theta)} u^+ = e^{i(\omega - \hat{H}_0)} u^+, \quad (3.30)$$

$$(1-iW(\theta))\bar{u} = e^{i(\omega - \hat{H}_0)} \bar{u} (1+iW(\theta)), \quad (3.31)$$

where,  $\bar{u} = \frac{u^+}{1+iW(\theta)}$ . Now rearranging all the terms we can rewrite the equation as

$$\tan\left(\frac{\omega - \hat{H}_0}{2}\right) \bar{u} + W(\theta) \bar{u} = 0, \quad (3.32)$$

where,  $\tan\left(\frac{\omega - \hat{H}_0}{2}\right) = \frac{1-e^{i(\omega - \hat{H}_0)}}{1+e^{i(\omega - \hat{H}_0)}}$ .

Now, the quasienergy state can be expanded as

$$|\bar{u}\rangle = \sum_m u_m |\psi_m\rangle, \quad (3.33)$$

where,  $|\psi_m\rangle$ 's are the wavefunctions of the unperturbed Hamiltonian and the superposition coefficient  $u_m$  can be calculated as

$$\begin{aligned} u_m &= \int \bar{u} \psi_m(\theta) d\theta \\ &= \int \frac{1}{2} [u^+(\theta) + u^-(\theta)] \psi_m(\theta) d\theta. \end{aligned} \quad (3.34)$$

Now in the unperturbed energy basis states Eq. (3.32) reduces to

$$T_m u_m + \sum_l W_{ml} u_l = 0. \quad (3.35)$$

Here,  $T_m = \tan\left(\frac{\omega - E_m}{2}\right)$  represents the onsite energy and  $W_{ml}$  is the hopping strength to hop from  $m$ -th site to  $l$ -th site. Equation (3.35) represents the tight binding version for the kicked rotor in finite well potential and it is similar to the Anderson model represented by Eq. (3.20).

However, the kicked rotor in finite potential well is a fully deterministic system. The randomness in the system is absent which is in sharp contrast to the Anderson model, where randomness in the on-site energies leads to exponential localization. Now, only if  $E_m (\equiv \frac{\hbar_s}{\pi})$  become irrational, then  $T_m$  will behave as quasirandom. In this situation, Eq. (3.35) will be analogous to Anderson model. In this work, we have chosen the values of  $\hbar_s$  in such that  $\frac{\hbar_s}{\pi}$  irrational.

Now, we will express the hopping term  $W_{ml}$  in the momentum basis leading to

$$\begin{aligned} W_{ml} &= \langle \psi_m | W(\theta) | \psi_l \rangle \\ &= \int \sum_{p,q} a_{mp}^* e^{-ip\theta} W(\theta) a_{lq} e^{iq\theta} d\theta \\ &= \sum_{p,q} a_{mp}^* a_{lq} \int W(\theta) e^{-i(p-q)\theta} d\theta \\ &= \sum_{p,q} a_{mp}^* a_{lq} W_{p-q}. \end{aligned} \quad (3.36)$$

In this,  $W_n = \frac{1}{2\pi} \int_0^{2\pi} W(\theta) e^{-in\theta} d\theta$  is the fourier transform of  $W(\theta)$ . Incorporating the expression of the hopping term,  $W_{ml}$ , from Eq. (3.36) in Eq. (3.35), the tight binding model of the system in Eq. (3.2) get modifies to

$$\left( T_m + \sum_{p,q} a_{mp}^* a_{mq} W_{p-q} \right) u_m + \sum_{p,q,l \neq m} a_{mp}^* a_{lq} W_{p-q} u_l = 0, \quad (3.37)$$

where,  $(T_m + \sum_{p,q} a_{mp}^* a_{mq} W_{p-q})$  represents the diagonal elements and  $a_{mp}^* a_{lq} W_{p-q}$  forms the off-diagonal terms of the transfer matrix. It does not appear straightfor-

ward to analytically prove power-law profile of Floquet states starting from Eq. (3.37), though it appears fair to expect that in this case the decay of Floquet state profile will be different from exponential form. As numerical results show, we obtain power-law localization. Similar results has also been reported in [Fishman 2002].

In the high energy limit where  $E_n \gg V_0$ , the system behaves similar to a kicked rotor system. As in this limit, the system does not “feel” the presence of the stationary potential. The unperturbed energy become simply a free particle energy  $E_n = \frac{\hbar_s^2 n^2}{2}$  and the wavefunction of an unperturbed Hamiltonian,  $H_0$  in Eq. (3.2) collapses into a single momentum state,  $|\psi_n\rangle = a_{nn}e^{in\theta}$ , with  $a_{nn} = 1$  and  $a_{mn} = 0$ . Thus, if we consider a Floquet state which is confined in this limit,  $E_{n_{max}} \gg V_0$ , then Eq. (3.37) reduces to

$$(T_m + W_0)u_m + \sum_l W_{m-l}u_l = 0. \quad (3.38)$$

Equation (3.38) is analogous to 1D Anderson model and hence, explains the reason behind the origin of the exponential localization that is observed in this limit which is shown in Fig. 3.8(c).

### 3.10 KAM-like System : Its Quantum Localization Phenomenon

We have already mentioned that we have considered a non-KAM system, the stationary potential present in the system in Eq. (3.2) is non-analytic and hence, violates one of the KAM conditions. But, under certain conditions, when the system follows the relation  $\pm R\pi + \lambda = l\pi$ , where  $l$  is an integer and  $\lambda$  is the phase of the external kicking field, it behaves as a KAM-like system. The reason behind the origin of the KAM-like property is due to the emergence of reflection symmetry of the system in Eq. (3.2) for the set of system parameters satisfying the KAM-like relation. By making a proper choice of parameters such that it maintains the KAM-like

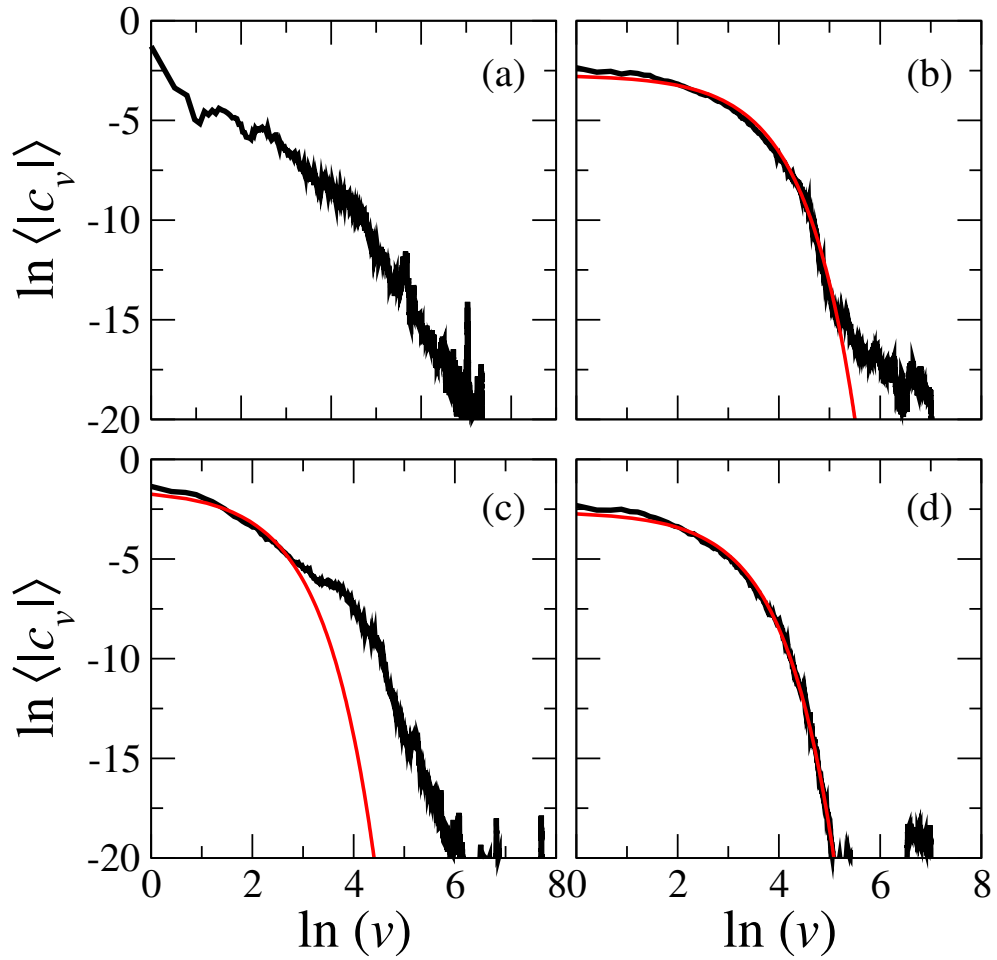


Figure 3.11: Plots (a) and (c) represents averaged decay profile of all the FSIW and (b) and (d) are for FSMAB. The system parameters are  $K = 4.25$ ,  $V_0 = 5000.5$ ,  $b = \pi$ , and  $\phi = 0$  (for (a) and (b)), and  $\phi = \pi/2$  (for (c) and (d)). All the red solid lines are exponential fits.

relation, we would recover the KAM-like tori inside the well region. Thus, under this condition, the system behaves as a KAM-like system even though the potential is non-analytic. For more details look at Chapter 2 of this thesis.

In the present work, we are interested in looking at the quantum signatures of the classically chaotic system when it fulfills the KAM-like relation. In order to distinguish its signature from non-KAM-like behavior, we plotted in Fig. 3.11, the decay profiles of the averaged Floquet states for both the cases. Figure 3.11(a) and Fig. 3.11(b) are plotted for the system when it does not fulfill the KAM-like relation. Figure 3.11(a) illustrates that on an average FSIW decay as a power-law

while FSMAB shows an exponential decay followed by a power-law tail shown in Fig. 3.11(b). These decay profiles have already been observed in the previous section. But, the instances we are interested in, are plotted in Fig. 3.11(c) and Fig. 3.11(d). Here, the classical counterpart of the quantum system in Eq. (3.2) follows the KAM-like relation described above. Figure 3.11(c) displays that Floquet states which are confined inside the well, i.e., those Floquet states which are mostly spread over the unperturbed energy states those are inside the well region and less over the unperturbed states above the well, tries to maintain the exponential decay profile over the unperturbed energy states but are disturbed by the higher energy states as the contribution from them decays as a power-law. As a result, it shows deviation from exponential decay, after first few energy states. But, if we consider FSMAB, we can notice from Fig. 3.11(d) that the averaged Floquet state shows exponential decay which is not surprising as Eq. (3.38) infers that at this energy range the system in Eq. (3.2) behaves similar to the kicked rotor.

An interesting fact that we have observed is that the quantum system can distinguish the KAM-like and non-KAM-like behavior of its classical counterpart. In the KAM-like situation, KAM tori are present inside the well region only, while above the barrier height, there is a puncture in tori at the potential boundary. This behavior has been beautifully demonstrated by the quantum system as there is an initial exponential decay followed by power-law decay and the origin of this power-law decay can be attributed to the non-KAM nature of the system.

### 3.11 Experimental realization

To verify the results of the system which are discussed above, one can perform experiment on the ultra-cold atom setup. For this, first a static periodic potential needs to be created [Łański 2016, Wang 2018] and then place a cloud of atom in this periodic potential. Once the cloud is placed, a pulsed laser is to be applied to kick the system.

### 3.12 Conclusions

In summary, we have studied the quantum dynamics of a classically chaotic non-KAM system composed of a kicked rotor system placed in a finite well potential. The system shows the well-known dynamical localization phenomenon. Initially, the quantum energy grows up to certain time called break-time,  $n^*$ , after which the energy saturates. We have found that break-time depends on the classical sub-diffusive exponents,  $\mu_1$  and  $\mu_2$ , and shows a non-integral  $\hbar_s$  (scaled Planck's constant) scaling. We have also looked at the Husimi distribution which gives the probabilistic view of wavefunction spreading and observed that with decreasing scaled Planck's constant, the wavefunction spreads over the higher momentum states.

We have also studied the nature of localization by analyzing the decay of the Floquet states over the unperturbed basis states. We have considered two limiting cases by : i) increasing the potential height,  $V_0$  to a large value which is the kicked rotor in an infinite well limit and ii) decreasing the potential height,  $V_0$ , to a small value, a standard kicked rotor limit. For the kicked rotor in an infinite potential limit, the Floquet state decays as a power-law while for kicked rotor limit, exponential decay is observed.

For the intermediate case, the Floquet states decay as exponential with a power-law tail. It has been observed that FSIW shows a power-law localization while exponential localization is displayed by FSMAB. Thus, the Floquet states that “feel” the presence of singularity of potential show power-law localization and those that do not “feel” the presence of singularity shows exponential localization. We have also mapped the system in Eq. (3.2) to Anderson model in order to get a clear understanding of the localization phenomenon. We observed that considering the Floquet states which “feels” the presence of singularity, the mapping is not exact and deviation from exponential localization is found. While considering the Floquet states those who do not “feel” the presence of singularity, the system exactly maps onto Anderson model and hence exponential localization is achieved.

We have also plotted the participation ratio of the Floquet states which gives the

information about the localization property of a state. We have found that FSIW shows a strong localization effect and strongness increases with increase in barrier width. While FSMAB also show localization effect, but the localization property does not change with changing the barrier width.

The presence of these two contrasting Floquet state profiles in Hamiltonian in Eq. (3.2) leaves its signature in the spectral correlations as well. The level spacing distribution of the Floquet eigenvalues corresponding to either FSIW or FSMAB turns out to be semi-Poisson distribution or a Poisson distribution. Typically, the spacing distribution is taken to characterize quantum chaos in a system, and it is generally independent of the energies being considered provided it is in the semi-classical limit. Quite surprisingly, the semi-classical limit of the system in Eq. (3.2) lacks a unique spacing distribution as it depends on the Floquet eigenvalues being considered.



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### Kicked Rotor with Lévy Distributed Kicks

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In the last few chapters, we have discussed about the classical and quantum dynamics of a single particle system. Such a quantum particle is well described by a single wavefunction. But, in nature, maintaining a quantum system, which may be single-particle or many-particle systems, isolated from the environment is generally difficult. The quantum system easily interacts with the environment, either during measurement using classical apparatus or through the environmental noise, and ultimately, the quantum system loses its coherence property.

In the context of the kicked rotor and its experimental realization using cold atoms in optical lattices, the process of interaction with the environment can be modelled through the introduction of noise in the system parameters. The effects of interaction with the environment can be studied by treating the kick strength [Klaupauf 1998, Steck 2000] or the phase [White 2014] or the kicking time-period, [Oskay 2003] as a stochastic process or it could be due to spontaneous emissions [Ammann 1998, Lepers 2010] at every kick. In all of these cases, the quantum system decoheres. It has been found that the rate of decay of coherence is generally known to be exponentially fast,  $D(t) \sim \exp(-t/\tau_c)$ , where  $D(t)$  is the decoherence factor and  $\tau_c$  is the coherence time. Although, the rate can be slowed down by properly control-

ling the interaction between the environment and the system, but it only modifies the mean coherence time,  $\tau_c$ , and still the loss of coherence remains exponentially fast. Thus, interaction with the environment and the consequent decoherence are a hindrance in the development of quantum-mechanical applications such as, for instance, the quantum computers. Hence, it is desirable to obtain some level of control to possibly slow down the decoherence process.

One such scheme of manipulating the environment to obtain slower decoherence was introduced in Ref. [Schomerus 2007, Schomerus 2008]. It was shown theoretically that if there is noise in the kicks that appears probabilistically after intervals of  $\tau$  kicks, with  $\tau$  itself drawn from a Lévy waiting time distribution,  $w(\tau) \sim c\tau^{-1-\alpha}$  ( $\alpha$  is the Levy exponent), then the system shows a loss of coherence, which is of power-law nature,  $D \sim t^{-\alpha}$ , for  $\alpha < 1$ . For  $\alpha < 1$ , the mean waiting time,  $\bar{\tau}$ , diverges and the noise is non-stationary.

In this thesis, we also addressed a question whether there is some other form of environmental influence, for which the system decoheres in a non-exponential fashion. In this regard, we consider an atom-optics kicked rotor system which interacts with the environment in such a way that it misses a few kicks. An experiment using atop-optics test-bed has been performed to realize this system. The Hamiltonian representing this system is

$$\tilde{H} = \tilde{p}^2/2M + V \cos(2k_L \tilde{x}) \sum_{n=1}^N (1 - g_n) \phi_{sq}(\tilde{t} - nT), \quad (4.1)$$

where,  $g_n$  is a stochastic variable which takes values either 0 or 1. Equation (4.1) represents the Hamiltonian of a center of mass of the atom having momentum  $\tilde{p}$  and mass  $M$  receiving kicks when  $g_n = 0$ . The waiting time between the successive occurrences of 0 are taken from Lévy waiting time distribution  $w(\tau) = \alpha\Gamma(\tau)\Gamma(\alpha + 1)/\Gamma(\tau + \alpha + 1)$  which asymptotically decays as  $c\tau^{-\alpha-1}$ , where,  $\alpha$  is the Lévy exponent and  $c$  is a constant [Simon 1955]. For  $\alpha < 1$ , the mean time interval,  $\bar{\tau}$ , diverges while for  $\alpha > 1$ ,  $\bar{\tau}$  is  $\alpha/(\alpha - 1)$ . The kicking sequence representing the position of kicks in time axis as a function of Lévy exponent

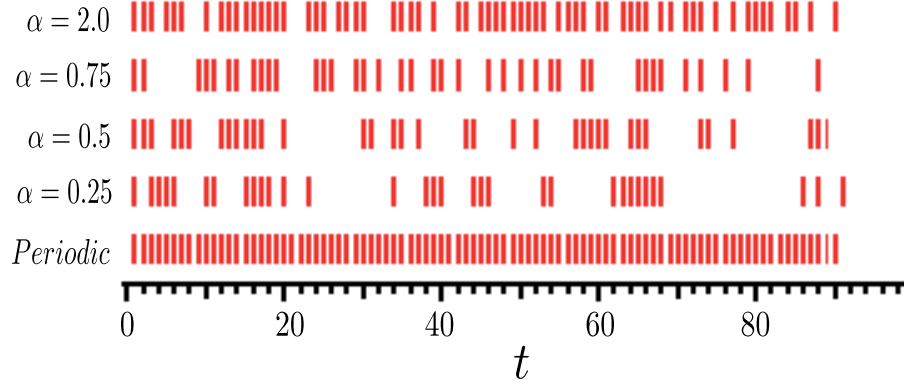


Figure 4.1: Sequence of kicks for different values of  $\alpha$  over time.

$\alpha$  is shown schematically in Fig. 4.1.

If  $\omega_0$  is the transition frequency of the two level atoms, the detuning is  $\delta = \omega_0 - \omega_L$ , where  $\omega_L$  is the frequency of the pulsed standing wave. In this,  $k_L$  is the wave number,  $T$  is the period of pulsing,  $\phi_{sq}(\cdot)$  represents the square pulse of duration  $t_p$  centered at  $\tilde{t} = 0$  and  $V = \hbar\Omega^2/8\delta_L$  is the amplitude of kick with  $\Omega$  being the resonant Rabi frequency. By appropriate rescaling, we obtain a dimensionless Hamiltonian given by

$$H = p^2/2 + \varepsilon \cos x \sum_{n=1}^N (1 - g_n) \phi_{sq}(t - n), \quad (4.2)$$

where, the scaled variables are,

$$x = 2k_L \tilde{x}, \quad p = 2k_L T \tilde{p}/M, \quad t = \tilde{t}/T. \quad (4.3)$$

Further,  $\phi_{sq}(\cdot)$  is the square pulse of unit amplitude and duration  $t_p/T$ , and scaled kick strength is  $\varepsilon = (8V/\hbar)\omega_r T^2$ , recoil frequency  $\omega_r = \hbar k_L^2/2M$  such that  $H = (4k_L^2 T^2/M)\tilde{H}$ .

The Hamiltonian in Eq. (4.2) represents the atom-optics realization of kicked rotor with kicks imparted by rectangular pulses at few selected times taken from Lévy waiting time distribution in contrast to the standard kicked rotor with  $\delta$ -kicks [Reichl 2004]. This distinction leads to effective chaos parameter becoming  $K =$

$\beta\varepsilon$ , ( $\beta = t_p/T$ ). For  $\beta \rightarrow 0$  and  $\varepsilon \rightarrow \infty$ , such that  $K$  is constant, Eq. (4.2) can be reduced to the delta kicked rotor Hamiltonian. The qualitative classical dynamics of the system represented by Hamiltonian in Eq. (4.2), *i.e.*, whether it is chaotic or regular or mixed, is determined by the scaled chaos parameter  $K$ .

In the next section, we will discuss the classical dynamics of the system in Eq. (4.2). We will try to understand how the diffusion process is affected by variation in the Lévy exponent  $\alpha$ .

## 4.1 The Classical System

In this section, we will discuss about the classical dynamics of the system described by the Hamiltonian in Eq. (4.2). We will rewrite Eq. (4.2) by Fourier expanding the infinite train of rectangular pulses [Ammann 1998] in the form,

$$H = p^2/2 + K \sum_{m=-\infty}^{\infty} \text{sinc}(\pi\beta m) \cos(x - 2\pi m\omega_\tau t). \quad (4.4)$$

We are studying a noisy system in which kicks do not apply at certain times. Hence to incorporate this timing noise, we have introduced a kick frequency  $\omega_\tau = 1/\tau_i$  in Eq. (4.4), where,  $\tau_i$  are the waiting times between two consecutive kicks. Thus,  $\omega(\tau)$  takes care of the waiting time between occurrences of two consecutive kicks. We notice from Eq. (4.4) that the fundamental frequencies are located at  $p_m = 2\pi m\omega_\tau$ . However, these fundamental frequencies  $\omega_\tau$  are a function of the waiting time  $\tau$ . Since,  $\tau$  is a varying entity,  $\omega(\tau)$  is not a constant. These fundamental frequencies for different  $\omega_\tau$  merge and hence regular orbits corresponding to these fundamental frequencies vanishes to give rise to chaos.

With the Hamiltonian in Eq. (4.4), we calculate the Hamilton's equations of motion which take the form,

$$\dot{p} = K \sum_{m=-\infty}^{\infty} \text{sinc}(\pi\beta m) \sin(x - 2\pi m\omega_{\tau}t), \quad (4.5a)$$

$$\dot{x} = p. \quad (4.5b)$$

Equation (4.5) governs the dynamics of the system. First of these, Eq. (4.5a) accounts for the change in momentum of an atom after receiving a kick. With this new momentum, the atom translates in position space described by Eq. (4.5b) until it receives the next pulse. We have applied periodic boundary conditions on the position axis and left the momentum axis unbounded to keep the dynamics on a cylinder instead on a torus. This is done in order to study the diffusion process in momentum space.

#### Numerical calculations:

To generate the time sequence for kicking with Lévy waiting time distribution numerically from the Yule-Simon distribution given by  $w(\tau) = \alpha \Gamma(\tau) \Gamma(\alpha + 1) / \Gamma(\tau + \alpha + 1)$ , we used the standard inverse transformation method. For the Yule-Simon distribution, the cumulative distribution function is given by,

$$I(\tau; \alpha) = 1 - \tau \Gamma(\tau) \Gamma(\alpha + 1) / \Gamma(\tau + \alpha + 1).$$

Starting from uniform random deviates generated using `rand() / RAND_MAX()` functionality provided by the C compiler, gcc 4.8.4, we use  $I(\tau; \alpha)$  to generate long time series of random waiting times  $\tau$ . We have also verified that using the asymptotic form of Yule-Simon distribution, i.e.,  $\tau^{-1-\alpha}$  instead of  $w(\tau)$  does not change the results presented in this thesis. Equation (4.5) represents a flow, and hence to compute the stroboscopic section numerically, the fourth order Runge-Kutta method [William 1997] has been used.

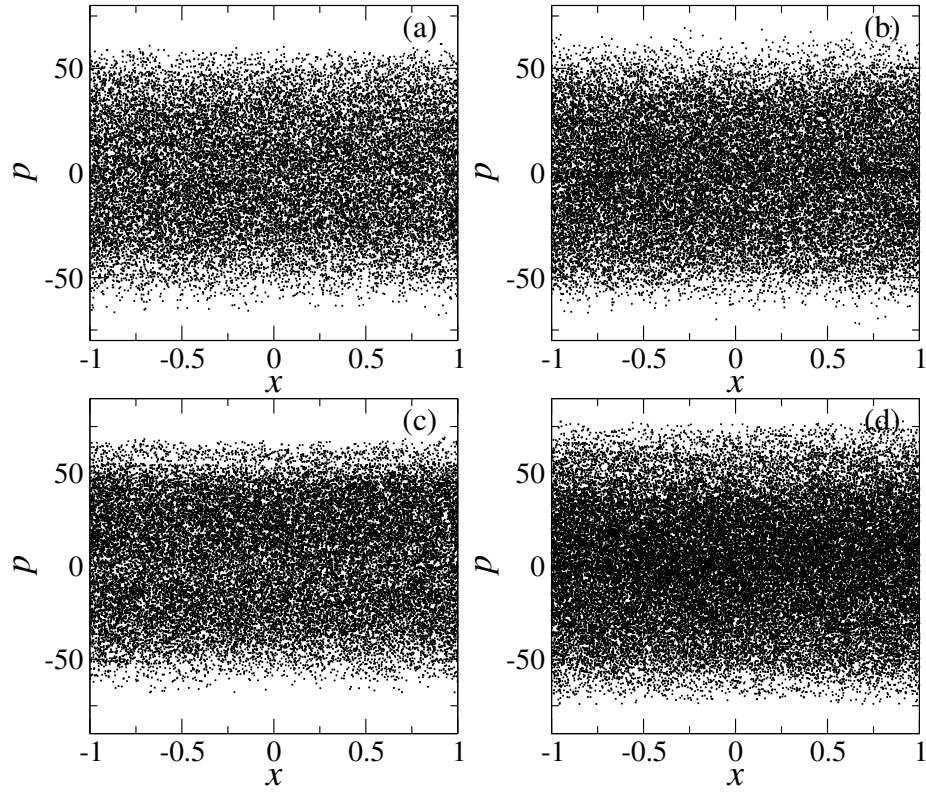


Figure 4.2: Stroboscopic sections of the system in Eq. (4.4) for (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$  (c)  $\alpha = 0.75$  and (d)  $\alpha = 2.0$ . The kick strength,  $K = 5.8$ , and  $\beta = 0.02$ , are same for all the plots.

### 4.1.1 Stroboscopic Sections and Energy Diffusion

The stroboscopic sections of the system in Eq. (4.4) for different values of Lévy exponents,  $\alpha$ , are presented in Fig. 4.2. The system parameters used to generate these plots are chosen in such a way to closely match with the experimental parameters. We can see from Fig. 4.2 that for all the four values of  $\alpha = \{0.25, 0.5, 0.75, 2.0\}$  and keeping the kick strength constant at  $K = 5.8$ , phase space obtained shows complete chaos for the experimentally accessible range of momenta. These plots are generated using an ensemble of initial conditions and time sequence corresponding to each value of Lévy exponent  $\alpha$ . The initial positions of the particles are drawn uniformly from  $[-\pi, \pi]$  and momenta are drawn uniformly from  $[-5, 5]$ . It is evident from Fig. 4.2 that the particles are diffusing in momentum space. To quantify the rate of diffusion, we calculate the average energy growth of the system with the new momenta obtained using Eq. (4.5a). The average energy is represented as

$$\langle \bar{E} \rangle = \frac{1}{N} \sum_i \frac{p_i^2}{2}, \quad (4.6)$$

where,  $\langle \bar{E} \rangle$  represents averaging over an ensemble of particles as well as over different sample sets of kicking times. In this,  $N$  is the total number of particles and  $p_i$  is the momentum of the  $i$ -th particle.

Figure (4.3) displays the average energy growth of the system for  $\alpha = 0.75$  which is generated using the same initial conditions that have been used to obtain the phase space shown in Fig. 4.2(c). It is seen that even at this large value of kick strength, the system shows classical sub-diffusive energy growth. This sub-diffusive energy growth is the manifestation of the ratio,  $\beta$  of the pulse width and the total time period of the rectangular pulse present as the argument of the sinc function in Eq. (4.4). If we diminish  $\beta$ , the system displays normal diffusive energy growth. Thus, we can notice that pulse width has a profound impact along with the sequence of pulse timing in governing the classical dynamics.

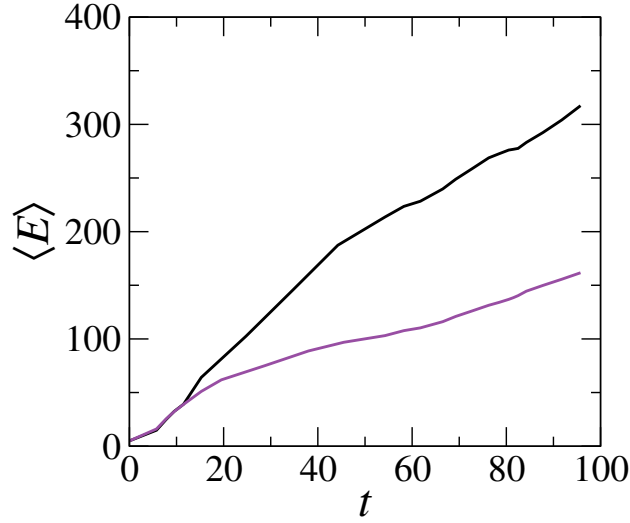


Figure 4.3: Classical mean energy growths of the system in Eq. (4.4) for Lévy exponent,  $\alpha = 0.75$  and kick strength,  $K = 5.8$ , and for different  $\beta$ ,  $\beta = 0.02$  (purple line) and  $\beta = 0.0002$  (black line).

## 4.2 The Quantum System

In the last section, we discussed about the classical dynamics of the noisy kicked rotor system described by the Hamiltonian presented in Eq. (4.2). We found that the sets of parameters used in Fig. 4.2 are suitable to study the quantum chaotic properties as the classical counterparts are fully chaotic. Here, in this section, we endeavor to understand the quantum dynamics of the same system.

The system represented by Eq. (4.2) can be thought of as a standard kicked rotor system which is perturbed by a noisy kicking sequence whose kick strength is the same as that of the original kicked rotor system. The perturbation acts in such a way that the system misses few kicks as illustrated schematically in the Fig. 4.4.

In the limit of  $\beta \rightarrow 0$  and  $\varepsilon \rightarrow \infty$ , such that  $K$  is constant, the square wave pulse in Eq. (4.2) can be replaced by a delta function pulse. The modified form of the Hamiltonian in Eq. (4.2) is

$$\hat{H} = \hat{p}^2/2 + K \cos \hat{x} \sum_n (1 - g_n) \delta(t - n). \quad (4.7)$$

We will remove the hat from the operators in further calculations. Using delta func-



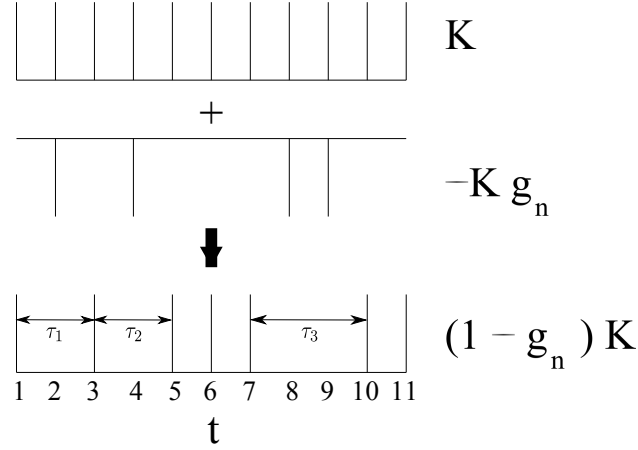


Figure 4.4: Kicked rotor model along with perturbation results in missing of kicks.  $K$  is the kick strength,  $g_n$  is the stochastic variable that takes value only 0 and 1, and  $\tau_1, \tau_2, \tau_3$  are the waiting times.

tion in Eq. (4.7), it become easy to study the evolution of any arbitrary state under the action of the Hamiltonian in Eq. (4.7). The Floquet operator of the system in Eq. (4.7) is,

$$F = e^{-ip^2/2\hbar_s} e^{-iK \cos x/\hbar_s} e^{-iK'_n \cos x/\hbar_s}. \quad (4.8)$$

Equation (4.8) can be written as a product of two Floquet operators,

$$F(K_n) = F(K) F(K'_n), \quad (4.9)$$

where,  $K'_n = -K g_n$ . Further,  $F(K)$  represents the Floquet operator of a standard kicked rotor system and  $F(K'_n)$  is for the noisy part. Note that if  $g_n = 0$  for all the times, then Eq. (4.9) reduces to the Floquet operator of the standard kicked rotor and is given by

$$F(K) = e^{-ip^2/2\hbar_s} e^{-iK \cos x/\hbar_s}. \quad (4.10)$$

The eigenvalue equation of the Floquet operator in Eq. (4.10) for the quasi-energy state  $|r\rangle$  is

$$F(K)|r\rangle = e^{-i\eta_r}|r\rangle,$$

with  $\eta_r$  representing the quasi-energy of the state  $|r\rangle$ . Now, we are interested in looking at the effects of external noise on the the quasi-energy states and its role

in attenuating the dynamical localization, a well-known effect observed in quantum kicked rotor system.

To understand the effect of noise described in Eq. (4.9) on the quasi-energy state  $|r\rangle$ , we calculate the survival probability amplitude,  $A_r(t', t'')$ , in the state  $|r\rangle$  after the operation of the noisy events on the same state,  $|r\rangle$ , over a larger time interval  $[t'', t']$  [Schomerus 2007, Schomerus 2008]

$$A_r(t', t'') = \left\langle r \left| \mathcal{T} \prod_{n=t''}^{t'-1} F(K_n) \right| r \right\rangle e^{-i\eta_r(t'-t'')}, \quad (4.11)$$

where,  $\mathcal{T}$  represents time ordering and  $e^{-i\eta_r(t'-t'')}$  is introduced to normalize the survival probability amplitude for the noiseless system.

In order to calculate the survival probability amplitude,  $A_r(t', t'')$ , it is necessary to first obtain the number of noisy events,  $G(t', t'')$  in the time interval  $[t'', t']$ . This can be calculated if we know the actual number of kicks,  $N(t', t'')$ , that imparted in the time interval  $[t'', t']$ , which is governed by the Lévy waiting time distribution. Then,  $G(t', t'')$  can be obtained as

$$G(t', t'') = (t' - t'') - N(t', t''), \quad (4.12)$$

where,  $(t' - t'')$  is the total time in the interval  $[t'', t']$ . The number of actual kicks imparted can be calculated from the sprinkling distribution,  $f(t)$ , of the Lévy waiting time distribution. The relation between the sprinkling distribution,  $f(t)$ , and the number of kicks,  $N(t, 0)$  is

$$f(t) = \partial_t \overline{N(t, 0)}. \quad (4.13)$$

In Eq. (4.13), the bar over  $N(t, 0)$  represents the time averaging over different time samples. The sprinkling distribution is further related to the waiting time distribu-

tion via Laplace transformation as

$$\tilde{f}(u) = \int_0^\infty dt e^{-ut} f(t) = \frac{\tilde{w}(u)}{1 - \tilde{w}(u)} \quad (4.14)$$

in which tilde represents the Laplace transformed functions. We will remove tilde from Laplace transformed variables in further calculations.

The Laplace transform of Lévy waiting distribution for small values of  $u$ ,

$$w(u) \sim 1 - u\bar{\tau} \quad (\alpha > 1), \quad (4.15a)$$

$$w(u) \sim 1 - u^\alpha \frac{c\pi}{\Gamma(\alpha + 1) \sin \pi\alpha} \quad (\alpha < 1). \quad (4.15b)$$

Inserting the expression of  $w(u)$  for different  $\alpha$  into Eq. (4.14) leads to

$$f(u) \sim \frac{1}{u\bar{\tau}} - 1 \quad \alpha > 1, \quad (4.16a)$$

$$f(u) \sim \frac{1}{u^\alpha c_1} - 1 \quad \alpha < 1, \quad (4.16b)$$

and  $c_1 = \frac{c\pi}{\Gamma(\alpha+1) \sin \pi\alpha}$ . Now, taking the inverse Laplace transform of  $f(u)$ , the sprinkling distribution,  $f(t)$ , for different  $\alpha$  regimes, is obtained as,

$$f(t) \sim \frac{1}{\bar{\tau}} \quad \alpha > 1, \quad (4.17a)$$

$$f(t) \sim \frac{t^{\alpha-1}}{c_1} \quad \alpha < 1. \quad (4.17b)$$

Now, substituting  $f(t)$  and  $c_1$  in Eq. (4.13) and at large time asymptotes, for  $\alpha > 1$  gives

$$\overline{N(t, 0)} \sim t/\bar{\tau}, \quad (4.18)$$

while for  $\alpha < 1$ ,

$$\overline{N(t, 0)} \sim t^\alpha (\sin(\pi \alpha)/\pi \alpha), \quad (4.19)$$

in which  $\bar{\tau}$  is the mean waiting time (See Ref. [Schomerus 2007, Schomerus 2008] for details).

Once the number of kicks,  $N(t, 0)$ , is obtained, and the number of noisy events,

$G(t, 0)$  are extracted using Eq. (4.12), the calculation of survival probability amplitude in the quasi-energy state  $|r\rangle$  become straightforward. Now, we perform an average over all the quasi-energy states and also over the random phases of the initial state. These averages become straightforward by using random-phase approximations. The averaged modulus of the diagonal elements of  $F(K'_N)$  is

$$\overline{e^{-i(K'_n/\hbar_s)\cos(x)}} = 1 - i(K'_n/\hbar_s) \overline{\cos(x)} - (K_n'^2/2! \hbar_s^2) \overline{\cos^2(x)} + i(K_n'^3/3! \hbar_s^3) \overline{\cos^3(x)} + \dots \quad (4.20)$$

Over a period of  $2\pi$ , the averages of all odd powers of  $\cos(x)$  vanishes, and hence

$$\begin{aligned} \overline{e^{-i(K'_n/\hbar_s)\cos(x)}} &= 1 - (K_n'^2/2! \hbar_s^2) \overline{\cos^2(x)} + (K_n'^4/4! \hbar_s^4) \overline{\cos^4(x)} + \dots \\ &\equiv q(K'_n/\hbar_s). \end{aligned} \quad (4.21)$$

Here,  $q(K'_n/\hbar_s)$  is constant because  $|K'_n| = |K|$  as  $g_n = 1$  for noisy events. Incorporating Eq. (4.21) and the number of noisy events  $G(t', t'')$  in the expression for the survival probability amplitude in Eq. (4.11) results in

$$A_r(t', t'') = q(K'_n/\hbar_s)^{G(t', t'')}. \quad (4.22)$$

Thus, we have obtained the survival probability which depends on the kick strength of the noisy part and also on the number of noisy events that have occurred,  $G(t', t'')$ . Consider, there is no noisy event, which can happen if  $K'_n = 0$  or  $G(t', t'') = 0$ , then the survival probability is 1, which implies no transition from one quasi-energy state to another. Hence, the system will display dynamical localization and coherences will be preserved. On the other hand, Eq. (4.22) illustrates that if there is a noisy event, then the survival probability in any quasi-energy state decays as  $0 < q(K'_n/\hbar_s) < 1$ . As a result, the system shows a transition from one state to the other. Hence, the system starts absorbing energy and loses its coherence property. Thus, the rate of decay of coherence or the decoherence factor,

$D(t', t'')$ , can be calculated using the survival probability amplitude and is defined as [Schomerus 2007, Schomerus 2008]

$$D(t', t'') = \overline{A_r^*(t', t'') A_s(t', t'')}, \quad (4.23)$$

where the average is over both quasi-energy indices,  $r \neq s$ , as well as over the number of noisy events,  $G(t', t'')$ . By inserting the expression of survival probability amplitude into Eq. (4.23), the decoherence factor can be written as

$$D(t', t'') = \overline{q (K'_n / \hbar_s)^{2G(t', t'')}}. \quad (4.24)$$

Here, the averaging over the quasi-energy index has been taken care of by the random phase approximation. Now, the only task left is to do the averaging over the timing noise, which we shall do next.

Re-designating  $q (K'_n / \hbar_s) \equiv a$  to make the notation simple, Eq. (4.24) can be rewritten as

$$\begin{aligned} D(t', t'') &= \overline{a^{2G(t', t'')}} \\ &= \overline{a^{2(t' - t'') - 2N(t', t'')}} \\ &= a^{2(t' - t'')} \overline{a^{-2N(t', t'')}} \\ &= D_0(t', t'') D_1(t', t''). \end{aligned} \quad (4.25)$$

In the third step of the calculation of  $D(t', t'')$ , the averaging over time on  $a^{2(t' - t'')}$  was removed because  $(t' - t'')$  is constant in the interval  $[t', t'']$  as  $a$  is constant for a fixed kick strength,  $K$ . Now, the two expressions,  $D_0(t', t'')$  and  $D_1(t', t'')$  for two different  $\alpha$  regimes, ( $\alpha < 1$ ) and ( $\alpha > 1$ ) are separately calculated. Since  $D_0(t', t'') = a^{2(t' - t'')}$  is independent of  $\alpha$ ,  $D_0(t', t'')$  is same for all  $\alpha$  regimes. The simplified form of  $D_0(t', t'')$  is

$$\begin{aligned} D_0(t', t'') &= a^{2(t' - t'')} \\ &= e^{-(1-a^2)(t' - t'')}. \end{aligned} \quad (4.26)$$

In Eq. (4.26), we kept only the first order of the expansion of  $\ln a^2$  and neglected the higher orders, since,  $|(a^2 - 1)| < 1$ .

Now, we will calculate the other term  $D_1(t', t'')$ , which depends on the value of  $\alpha$  and hence the behavior of  $D_1(t', t'')$  will be different for two different  $\alpha$  regimes. Thus, it should be calculated separately for different  $\alpha$  regimes.

In order to obtain  $D_1(t', t'')$  for  $\alpha > 1$ , firstly  $D_1(t, 0)$  needs to be calculated. This is because the decoherence factor,  $D_1(t', t'')$ , for general time argument  $t'$  and  $t''$  is related to that of initial and final time arguments,  $D_1(t, 0)$  [Schomerus 2008, Schrieffer 2005] by the relation

$$D_1(t', t'') = D_1(t, 0) - (a^{-1} - 1) \int_0^{t''} f(s) D_1(t' - s, 0) ds. \quad (4.27)$$

Equation (4.27) illustrates that once  $D_1(t, 0)$  is obtained,  $D_1(t', t'')$  can be calculated for general time arguments since  $a$  is already known. So, to calculate  $D_1(t, 0)$ , firstly the Laplace transform of  $D_1(t, 0)$  is obtained as  $D_1(u, 0)$ . Note that it is easier to calculate  $D_1(u, 0)$  instead of  $D_1(t, 0)$  in this case. After that, inverse Laplace transform is performed to obtain  $D_1(t, 0)$ . So,  $D_1(u, 0)$  using the expression given in Ref. [Schomerus 2008] can be calculated as

$$D_1(u, 0) = \frac{1}{u} \frac{1}{1 - f(u)(a^{-2} - 1)}. \quad (4.28)$$

Here, the Laplace transformed sprinkling distribution  $f(u)$  of  $\overline{N(t, 0)}$  have been used to calculate the time averaging because  $D_1(t, 0)$  depends on the actual number of kicks,  $\overline{N(t, 0)}$ , imparted within the time interval  $[0, t)$ . The value of  $f(u)$  for  $\alpha > 1$  is taken from Eq. (4.28), and leads to

$$\begin{aligned}
D_1(u, 0) &= \frac{1}{u} \frac{1}{1 - (\frac{1}{u\bar{\tau}} - 1)(a^{-2} - 1)} \\
&= \frac{1}{u} \frac{u\bar{\tau}}{u\bar{\tau} - (a^{-2} - 1) + u\bar{\tau}(a^{-2} - 1)} \\
&= \frac{\bar{\tau}}{u\bar{\tau}a^{-2} - (a^{-2} - 1)} \\
&= \frac{a^2}{u - \frac{1-a^2}{\bar{\tau}}}.
\end{aligned} \tag{4.29}$$

Now, the inverse Laplace transform of  $D_1(u, 0)$  (obtained in Eq. (4.29)) is performed to get

$$D_1(t, 0) = a^2 e^{(1-a^2)t/\bar{\tau}}. \tag{4.30}$$

Inserting the expression of  $D_0(t, 0)$  from Eq. (4.26) and  $D_1(t, 0)$  from Eq. (4.30) in Eq. (4.25), the expression of  $D(t, 0)$  for  $\alpha > 1$  is obtained as

$$D(t, 0) = a^2 e^{-(1-a^2)t(1-\frac{1}{\bar{\tau}})}. \tag{4.31}$$

Thus, having obtained  $D(t, 0)$ , now  $D_1(t', t'')$  can be calculated for any general time argument  $t'$  and  $t''$  using Eq. (4.27) and substituting the value of  $D_1(t, 0)$  obtained in Eq. (4.30).

$$\begin{aligned}
D_1(t', t'') &= a^2 e^{(1-a^2)t'/\bar{\tau}} - \frac{1-a^2}{a^2} \int_0^{t''} \frac{1}{\bar{\tau}} a^2 e^{(1-a^2)(t'-s)/\bar{\tau}} ds \\
&= e^{(1-a^2)t'/\bar{\tau}} \left( (a^2 - 1) + e^{(a^2-1)t''/\bar{\tau}} \right) \\
&= e^{(1-a^2)t'/\bar{\tau}} \left[ 1 + (a^2 - 1) \left( \frac{t''}{\bar{\tau}} + 1 \right) + \frac{(a^2 - 1)^2 t''^2}{2! \bar{\tau}^2} + \dots \right].
\end{aligned} \tag{4.32}$$

For  $t'' \gg \bar{\tau}$ , Eq. (4.32) simplifies to

$$\begin{aligned}
D_1(t', t'') &= e^{(1-a^2)t'/\bar{\tau}} e^{(a^2-1)t''/\bar{\tau}} \\
&= e^{(1-a^2)(t'-t'')/\bar{\tau}}.
\end{aligned} \tag{4.33}$$

Substituting Eq. (4.33) and Eq. (4.26) in Eq. (4.25), we have the decoherence factor,  $D(t', t'')$ , for  $(\alpha > 1)$  for any two general time arguments  $[t', t'']$  as

$$D(t', t'') \sim e^{-(1-a^2)(1-1/\bar{\tau})(t'-t'')}. \quad (4.34)$$

Equation (4.34) illustrates that for  $\alpha > 1$  (stationary noise case), the decoherence factor is exponential in nature where the coherence time has undergone modification.

Now, the decoherence factor for  $\alpha < 1$  (non-stationary noise case) will be calculated. We will follow the same steps as we did to calculate  $D_1(t', t'')$  for  $\alpha > 1$ . First, we will calculate  $D_1(u, 0)$  in Eq. (4.28) for  $\alpha < 1$  leading to

$$D_1(u, 0) = \frac{1}{u} \frac{1}{1 - f(u)(a^{-2} - 1)}. \quad (4.35)$$

Substituting  $f(u) = \frac{1}{u^{\alpha c_1}} - 1$  for  $\alpha < 1$  where,  $c_1 = \frac{c\pi}{\Gamma(\alpha+1)\sin\pi\alpha}$ , we have  $D_1(u, 0)$  as

$$\begin{aligned} D_1(u, 0) &= \frac{1}{u} \frac{1}{1 - \left( \frac{1}{u^{\alpha c_1}} - 1 \right) (a^{-2} - 1)} \\ &= a^2 \frac{u^{\alpha-1}}{u^{\alpha} - \frac{(1-a^2)}{c_1}} \\ &= a^2 \frac{u^{\alpha-1}}{u^{\alpha} - z}, \end{aligned} \quad (4.36)$$

where  $z = \frac{(1-a^2)}{c_1}$ . Now, by taking the inverse Laplace transform of  $D_1(u, 0)$ , we obtain  $D_1(t, 0)$  as follows :

$$D_1(t, 0) \sim \frac{a^2}{2\pi i} \int_{\Gamma} \frac{u^{\alpha-1}}{u^{\alpha} - z} e^{ut} du, \quad (4.37)$$

Consider,  $ut = y$ , then

$$\begin{aligned} D_1(t, 0) &\sim \frac{1}{2\pi i} \int_{\Gamma} \frac{y^{\alpha-1} e^y}{y^{\alpha} - z t^{\alpha}} dy \\ &\sim E_{\alpha}[z t^{\alpha}]. \end{aligned} \quad (4.38)$$



Now, substituting the value of  $z = \frac{(1-a^2)}{c_1}$  and  $c_1$  in Eq. (4.38), the result is

$$D_1(t, 0) \sim E_\alpha \left[ (1 - a^2) (\sin(\pi\alpha)/\pi\alpha) t^\alpha \right], \quad (4.39)$$

where  $E_\alpha[\cdot]$  is the Mittag-Leffler function [Erdélyi 1995, Haubold 2011].

Thus, after averaging over timing noise, we obtain the decoherence factor  $D(t, 0)$  for  $\alpha < 1$  for the quantum evolution from an initial time until some final time,  $t$ , to be [Schomerus 2007, Schomerus 2008],

$$D(t, 0) \sim e^{-(1-a^2)t} E_\alpha \left[ (1 - a^2) (\sin(\pi\alpha)/\pi\alpha) t^\alpha \right]. \quad (4.40)$$

Equation (4.40) elucidates that the decoherence factor has been modified in form for  $\alpha < 1$  (non-stationary noise case) and is non-exponential in nature due to the presence of the special function called Mittag-Leffler function, which cannot be reduced to an exponential form. Physically, this result implies that rate of decay of coherence has become slower than exponential in time.

Until now, we have calculated the decoherence factor for the two  $\alpha$  regimes and found that the system is decohering under the effect of external noise with two different decoherence rate. Hence, the system will also absorb energy with two different rates and will no more show localization effect.

Now, the quantum average energy growth will be calculated for the system. This is done using the force-force correlation function,  $C(t', t'')$  as there are increments in momenta at each time step. Thus, the total change in momentum is obtained by integrating the force-force correlation function over time,

$$\text{var } p(t) = \sum_{t', t''=0}^{t-1} C(t', t''), \quad (4.41)$$

where, the correlation function is  $C(t', t'') = \overline{\langle K_{t'} K_{t''} \sin x_{t'} \sin x_{t''} \rangle}$ .

The change in momentum is the result of decoherence of the system. The effect of the decoherence factor can be observed if we expand the force-force correlation function,  $C(t', t'')$  in the quasi-energy basis and it takes the form [Schomerus 2008]

$$C(t', t'') = \frac{\sum_{r_n, s_n} \langle r_{t'} | \sin x | s_{t'} \rangle \langle s_{t''} | \sin x | r_{t''} \rangle K_{t'} K_{t''}}{\prod_{l, m=t''}^{t'-1} \langle r_l | F(K_l) | r_{l+1} \rangle^* \langle s_{m+1} | F(K_m) | s_m \rangle}. \quad (4.42)$$

Here, the averaging is done with respect to all the quasi-energy states. The random phase approximation is carried out on the matrix elements of the sine potential. In this averaging, due to symmetry of the standard kicked rotor,  $(x, p) \rightarrow (-x, -p)$ , the contribution from all the terms like, i)  $r_{t'} = s_{t'} = r_{t''} = s_{t''}$  and ii)  $r_{t'} = s_{t'}, r_{t''} = s_{t''}, r_{t'} \neq r_{t''}$ , which gives the expectation value of the sine potential vanishes. On the other hand, the terms like  $r_{t'} = r_{t''}, s_{t'} = s_{t''}, r_{t'} \neq s_{t'}$  contribute [Schomerus 2008]. Now, at  $t' = t''$ , if the noisy terms act, then the average over the noisy kicks gives rise to a force which is  $\overline{K_{t'}^2} \langle \sin x^2 \rangle = K^2/2 - (K^2/2)h(t')\delta_{t', t''}$ , where  $h(t')$  represents the distribution of the noisy events.

Thus, the total force-force correlation function,  $C(t', t'')$ , considering all the contributions, is obtained as

$$C(t', t'') = C_0(t', t'')D_1(t', t'') - (K^2/2)h(t')\delta_{t', t''}, \quad (4.43)$$

where,  $C_0(t', t'')$  is the force-force correlation function of the standard kicked rotor without any noise. Now, substituting the expression of the total force-force correlation function,  $C(t', t'')$  from Eq. (4.43) in Eq. (4.41), the variance of momentum can be written down as,

$$\text{var } p(t) = \sum_{t', t''=0}^{t-1} C_0(t', t'')D_1(t', t'') - (K^2/2)h(t')\delta_{t', t''}. \quad (4.44)$$

Now, we change over from discrete time argument to continuous time argument and write the sum in terms of integration. By performing an integration by parts, all the force-force correlation terms of the integration can be expressed in terms of the variance of momentum of the noise-free kicked rotor [Schomerus 2008].

$$\begin{aligned}
\text{var } p(t) &= \text{var } p_0(t) - (K^2/2)\overline{G(t,0)} \\
&+ \int_0^t ds \text{var } p_0(t-s) \partial_s D(t,s) - \int_0^t ds \text{var } p_0(t) \partial_s D(s,0) \\
&- \int_0^t ds' \int_0^{s'} ds'' \text{var } p_0(s'-s'') \partial_s' \partial_{s''} D(s',s'') \\
&= I_1 - I_2 + I_3 - I_4 - I_5,
\end{aligned} \tag{4.45}$$

with  $I_1, I_2, \dots$  representing the corresponding integration terms.

The variance of momentum for  $\alpha > 1$  is obtained by substituting the expression of  $D(t', t'')$ ,  $D(t, 0)$  from Eq. (4.34) and Eq. (4.31) respectively and the variance of momentum of the standard kicked rotor without noise,  $\text{var } p_0(t) \simeq \frac{K^2}{2} t^* \left[ 1 - e^{-\frac{t}{t^*}} \right]$ , where  $t^*$  is the break-time of standard kicked rotor. Integrating each term of Eq. (4.45), we have

$$\begin{aligned}
I_1 &\sim \frac{K^2}{2} t^* \left[ 1 - e^{-\frac{t}{t^*}} \right], \\
I_2 &\sim \frac{K^2}{2} t \left( 1 - \frac{1}{\bar{\tau}} \right), \\
I_3 - I_4 &\sim \frac{K^2}{2} t^* e^{-\frac{t}{t^*}} \left[ 4a^{2t(1-\frac{1}{\bar{\tau}})} (-1 + e^{\frac{t}{t^*}}) t^* \ln a^2 - 2(-(-1 + a^{2t(1-\frac{1}{\bar{\tau}})}) e^{\frac{t}{t^*}} \right. \\
&\quad \left. + 2a^{2t(1-\frac{1}{\bar{\tau}})} (-1 + e^{-\frac{t}{t^*}}) t^* \ln a^2 \right) \bar{\tau} \right] / [-2t^* \ln a^2 + (-1 + 2t^* \ln a^2) \bar{\tau}], \\
I_5 &\sim \frac{K^2 t^*}{2\bar{\tau}} (\ln a^2)^2 (-1 + \bar{\tau})^2 \left[ \frac{(-1 + a^{2t(1-\frac{1}{\bar{\tau}})}) \bar{\tau}}{(\ln a^2)^2 (-1 + \bar{\tau})^2} + \frac{(1 - a^{2t(1-\frac{1}{\bar{\tau}})}) e^{-\frac{t}{t^*}} t^{*2} \bar{\tau}}{(2t^* \ln a^2 - (-1 + 2t^* \ln a^2) \bar{\tau})^2} \right. \\
&\quad \left. + t \left( \frac{1}{\ln a^2 - \bar{\tau} \ln a^2} - \frac{t^*}{t^* \ln a^2 - (-1 + t^* \ln a^2) \bar{\tau}} \right) \right].
\end{aligned} \tag{4.46}$$

For,  $\frac{t}{t^*} > 1$  and  $a < 1$ , it can be shown that  $I_1, I_3$  and  $I_4$  becomes constant and only  $I_2$  and  $I_5$  contribute. The expression of  $I_2$  and  $I_5$  at the limit ( $\frac{t}{t^*} > 1$  and  $a < 1$ )

reduces to

$$\begin{aligned} I_2 &\sim \frac{K^2}{2}t\left(1 - \frac{1}{\bar{\tau}}\right), \\ I_5 &\sim \frac{K^2 t^*}{2\bar{\tau}}(\ln a^2)^2(-1 + \bar{\tau})^2 \left[ \frac{1}{\ln a^2 - \ln a^2 \bar{\tau}} - \frac{t^*}{t^* \ln a^2 - (-1 + t^* \ln a^2)\bar{\tau}} \right] t. \end{aligned} \quad (4.47)$$

Substituting Eq. (4.47) in Eq. (4.45), the variance of momentum for  $\alpha > 1$  is

$$\begin{aligned} \text{var } p(t) &\sim -\frac{K^2}{2}\left(1 - \frac{1}{\bar{\tau}}\right)t \\ &\quad - \frac{K^2 t^*}{2\bar{\tau}}(\ln a^2)^2(-1 + \bar{\tau})^2 \left[ \frac{1}{\ln a^2 - \ln a^2 \bar{\tau}} - \frac{t^*}{t^* \ln a^2 - (-1 + t^* \ln a^2)\bar{\tau}} \right] t \\ &\sim B_2 t. \end{aligned} \quad (4.48)$$

Thus, we notice from Eq. (4.48), the variance of momentum varies linearly with time and shows a normal diffusion. Here, we also notice that the diffusion coefficient has been modified and is found to be a function of break-time ( $t^*$ ), Lévy exponent ( $\alpha$ ), mean waiting time ( $\bar{\tau}$ ) and kick strength ( $K$ ).

Next, the variance of momentum for  $\alpha < 1$  will be calculated. At large time asymptotes, only the terms  $I_2$  and  $I_5$  of Eq. (4.45) dominates and the approximate expression of  $\text{var } p(t)$  is

$$\begin{aligned} \text{var } p(t) &\sim \left[ \frac{K^2 t^*}{2(1 - t^* \ln a^2)}(t^* - \alpha c_1 \ln a^2 \Gamma[\alpha]) - \frac{K^2}{2} \right] t + \frac{K^2}{2} \frac{\sin \pi \alpha}{\pi c} t^\alpha \\ &\sim B_0 t + B_1 t^\alpha. \end{aligned} \quad (4.49)$$

Here, we can see that  $\text{var } p(t)$  has two terms, one shows a normal diffusion over time and other exhibits a sub-diffusive growth over time. In the experimentally accessible time regime, the term with  $B_1$  dominates as  $B_0 \ll B_1$ , thus the system displays sub-diffusive energy growth.

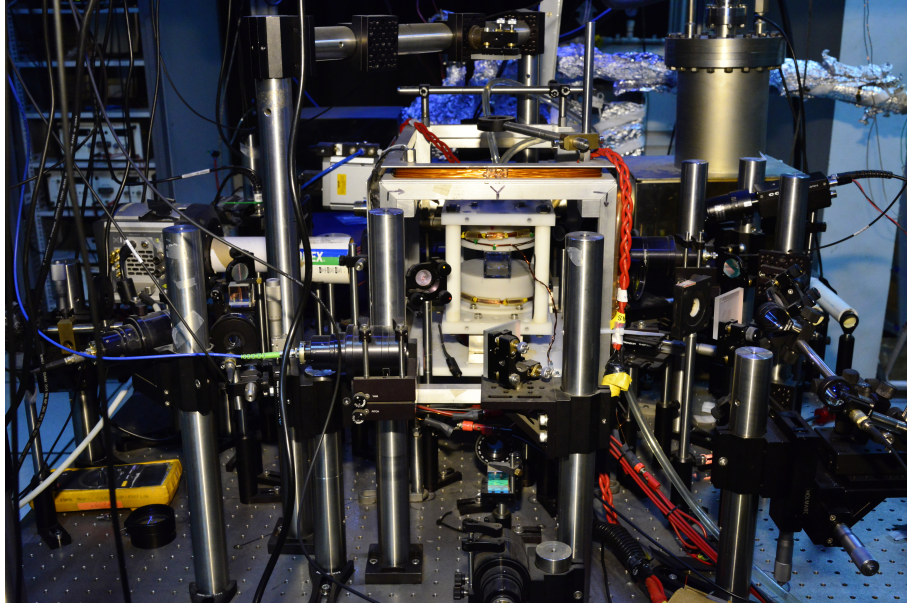


Figure 4.5: Experimental setup to perform the atom-optics kicked rotor experiment.

### 4.3 The Experimental Setup

The experimental realization of the atom optics kicked rotor with Lévy distributed kicks have been performed on ultra-cold atoms and the experimental setup is shown in Fig. 4.5. A collection of approximately  $10^7$  Rb atoms has been loaded in a Magneto-Optic Trap. The temperature of the cloud at this point is about  $\sim 250\mu K$ . Now, by using laser cooling at the MOT and also by compressing the MOT, the temperature gets reduced to  $30\mu K$ . This cloud is then moved to the crossed optical dipole trap which is formed by two crossed laser beams that intersect at the center of the trap. The wavelengths of these laser beam to form the dipole trap are  $\lambda \sim 1064nm$  each. Here, at this stage by using forced evaporative laser cooling, the temperature is further reduced to  $3\mu K$ . Now, the cloud is ready to carry out the experiment.

To impart kick to the cloud, we have used a pulsed laser which is detuned from  $5S_{1/2}F = 1 \rightarrow 5P_{3/2}F = 2$  transition by  $-6.8$  GHz. The wavelength of the pulsed laser is  $780$  nm. The on-time of this laser is  $220$  ns and off time is  $10.6\mu s$ . Now, in the system in Eq. (4.2), we have applied kicks at few selected times, which are dictated by the Lévy waiting time distribution. To implement this missing of kicks

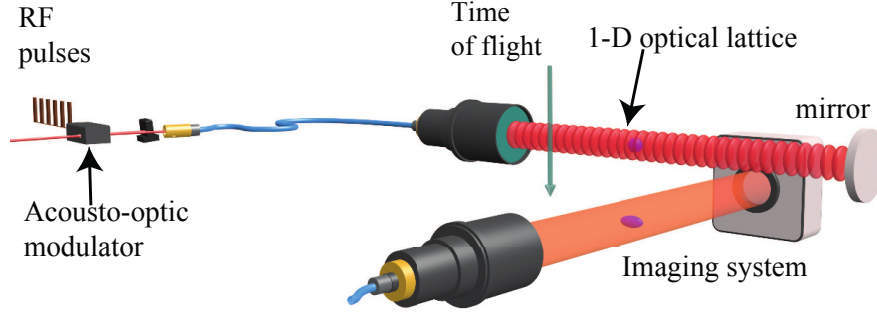


Figure 4.6: Schematic diagram of the atom-optics kicked rotor experimental setup.

in the experiment, the laser pulse is fed through the acousto-optic modulator (AOM) which is driven by an RF amplifier. The RF amplifier is further controlled by the computer attached to the system. In the computer, we load the information about the pulse width, total time period, laser intensity and the kicking sequence obtained from Lévy distribution. The intensity of the laser beam is kept to be 5.8 with 10% uncertainty. After applying kicks, the information about the momentum distribution of the cloud after a fixed time of flight is obtained from the absorption imaging spectra which are then analyzed. The schematic diagram of the experimental setup is shown in Fig. 4.6.

## 4.4 Decoherence and Diffusion

### 4.4.1 For Lévy Exponent, $\alpha < 1$

In this section, we will mainly discuss about the quantum dynamics of the system when the kicks are distributed according to the Lévy waiting time distribution for  $\alpha < 1$ . Figure 4.7 shows the experimental (symbols with standard error), numerical (solid lines) and analytical (dashed lines) results for the average energy growth corresponding to different values of  $\alpha$  and also for classical and quantum kicked rotor and kicked rotor with random timing noise. For the kicked rotor with random timing noise case, the kicks are imparted at every time with noise in the time-period, i.e.,  $T_n = T + t_n$ , where,  $T$  is the time-period and  $\Delta \leq t_n \leq \Delta$  is a uniformly

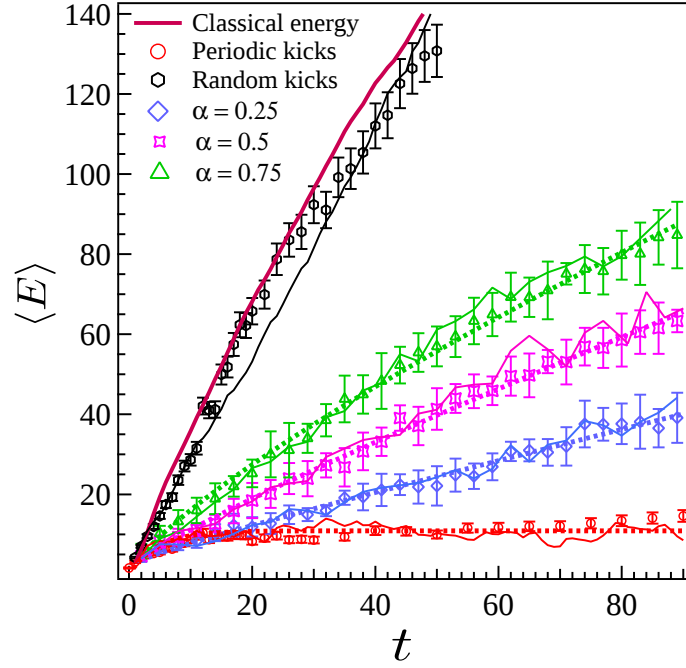


Figure 4.7: Mean energy growth for  $K = 5.8$  and  $\hbar_s = 2.0$ . Symbols, dotted lines and solid lines represents experimental, analytical and numerical results respectively.

distributed random variable with noise strength,  $\Delta 10\%$ . The experimental data presented in Fig. 4.7, for each value of  $\alpha$ , are averaged over 14 realizations of Lévy distributed waiting times while the numerical data are averaged over 1000 realizations of timing noise as well as over different initial phases.

Figure 4.7 shows that the experimental, numerical and analytical results are in good agreement with each other. We can notice from Fig. 4.7 that for the standard kicked rotor, the quantum average energy shows the localization effect while for Lévy distributed kicks the average energy grows sub-diffusively with time for  $\alpha < 1$ . Figure 4.7 also displays that for the random timing noise case, the quantum average energy grows at a rate which is approximately same as that for classical average energy for standard kicked rotor. Thus, Fig. 4.7 illustrates that for the random timing noise case, the quantum system becomes almost classical and hence, all the coherent superposition states are vanished. While, also for Lévy exponent  $\alpha < 1$ , the system is decohering but has not become fully classical. Thus, kicked rotor with noise shows decoherence effect.

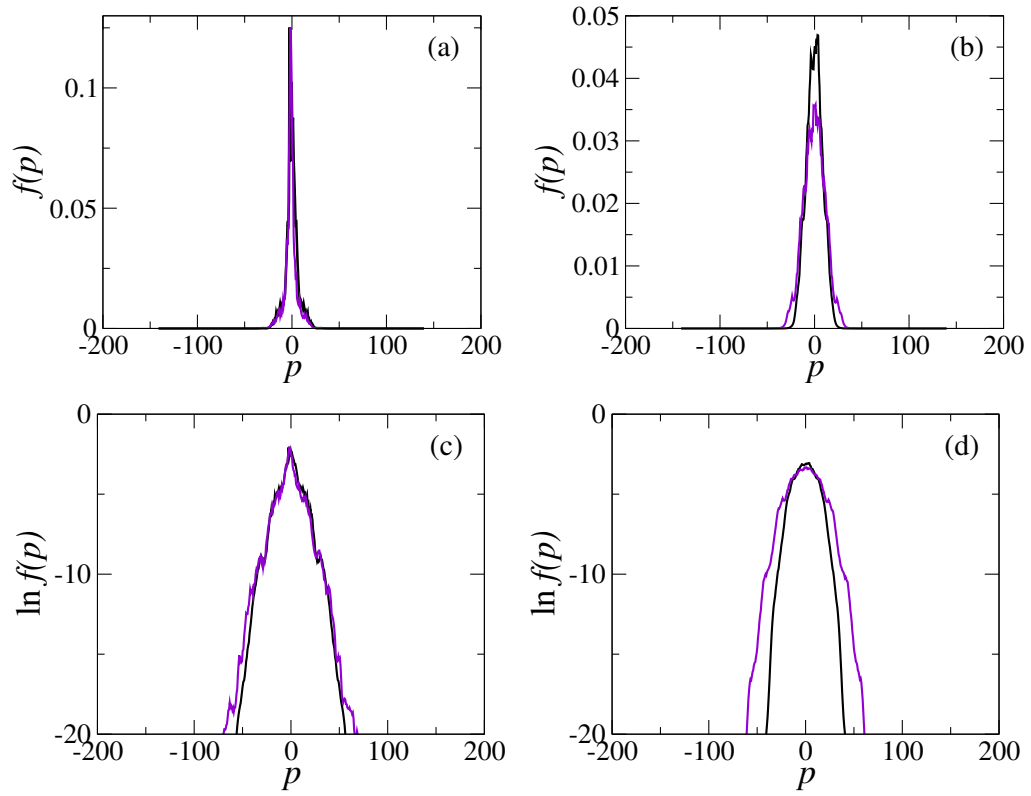


Figure 4.8: Plots of momentum distributions for the kicked rotor with (a) uniform kicks and (b) Lévy distributed kicks with  $\alpha = 0.75$  at two different kicking times,  $n = 33$  (black) and  $n = 70$  (violet) for  $K = 5.8$  and  $\hbar_s = 2.0$ . Plots (c) and (d) represents log-normal of (a) and (b) respectively.



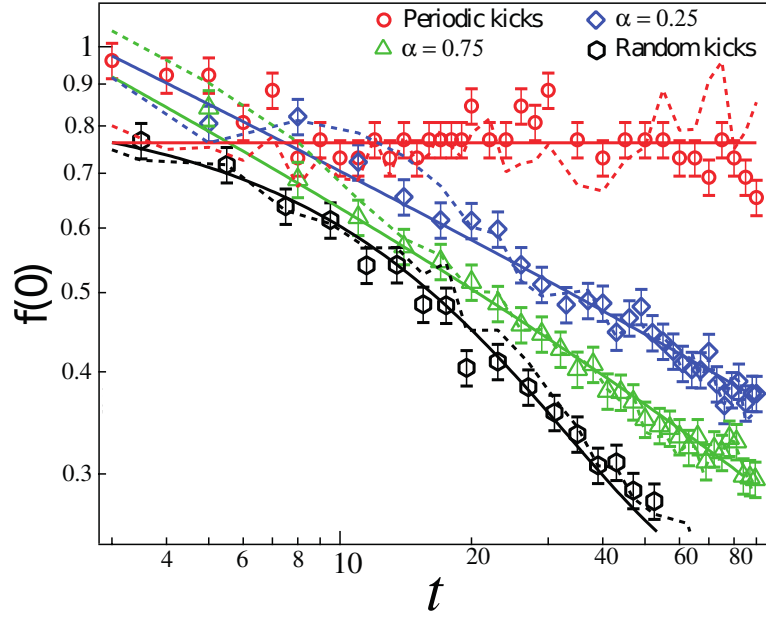


Figure 4.9: Log-log plot of number of populations in the zero momentum state as a function of time for  $K = 5.8$  and  $\hbar_s = 2.0$ . Symbols and dashed lines represents experimental and numerical results respectively. Black solid line is the exponential fit. Green, red and blue straight lines are guides to an eye.

Now, the rate of decay of coherence or the decoherence factor for different values of  $\alpha$  will be analyzed and also for random timing noise. The decoherence is the loss of phase coherence between the states. But, experimentally, with the present experimental setup, it is impossible to directly probe the phase of a particle. Hence, an indirect measure to study the decoherence factor is considered. Since due to decoherence, the system starts absorbing energy and the particles are occupying the higher momentum states. Thus, there is a spreading in the momentum distribution in each time step which can be seen in the momentum distribution plotted in Fig. 4.8(b,d). Figure 4.8 displays the momentum distribution of uniform kicking kicked rotor and kicked rotor with Lévy distributed kicks for two different kicking times  $n = 33$  and  $n = 70$ . Both are generated using same Gaussian distributed initial state with  $\sigma_p = 2.7$ . It can be inferred from Fig. 4.8(a) that for the uniform kicking case there is a negligible change in the momentum distribution value of the zero momentum for the two different kicking times and while for the kicked rotor with Lévy distributed kicks for  $\alpha = 0.75$ , there is a significant change observed which is

displayed by Fig. 4.8(b).

Figure 4.8(c) illustrates that for the uniform kicking case, the system shows exponential localization. But for Lévy distributed kicks shown in Fig. 4.8(d), the system still maintains the Gaussian distribution near zero momentum value but shows exponential localization for the higher momentum states, i.e., the system is trying to maintain the coherence. Thus, Fig. 4.8 explains that there is a spreading in the momentum distribution when kicks are missed and hence, a decay in the number of particles in the initially occupied momentum states. We consider the decay in the number of population in the zero momentum state as a measure of decoherence which is an experimentally measurable quantity. In Fig. 4.9, log-log of the population in zero momentum state as a function of time is plotted. It is found that for the uniform kicking case (red symbols), the population does not decline as a function of time after break-time. This is because the quantum energy saturates after the break-time. As a result, the momentum profile does not change over time after break-time and the number of population in the zero momentum state remains constant. Thus, there is no decoherence for the uniform kicking case. On the other hand, for random timing case, it is known that the coherence decays exponentially fast. Figure 4.9 illustrates that the number of population decays exponentially as a function of time denoted by black line. Thus, this method of calculation of decoherence agrees with the known results for the random timing case.

To further check the correctness of the measure, we calculated numerically the quantum average energy growth of the kicked rotor with amplitude noise case as a function of time. Amplitude noise implies noise in the kick strength,  $K_n = K + k_n$ , at every time, where,  $\Delta \leq k_n \leq \Delta$  is a uniformly distributed random variable with noise strength,  $\Delta$ . It is observed from Fig. 4.10, the quantum average energy grows diffusively with time for the two noisy kicking cases: i) with noise strength,  $\Delta = 16\%$  and ii) with noise strength,  $\Delta = 25\%$ . In Fig. 4.11, we plot the number of population in the zero momentum state as a function of time for the same set of noises introduced in the kick strength to generate the plot in Fig. 4.10. Figure 4.11 displays that for these two cases also, the number has declined exponentially

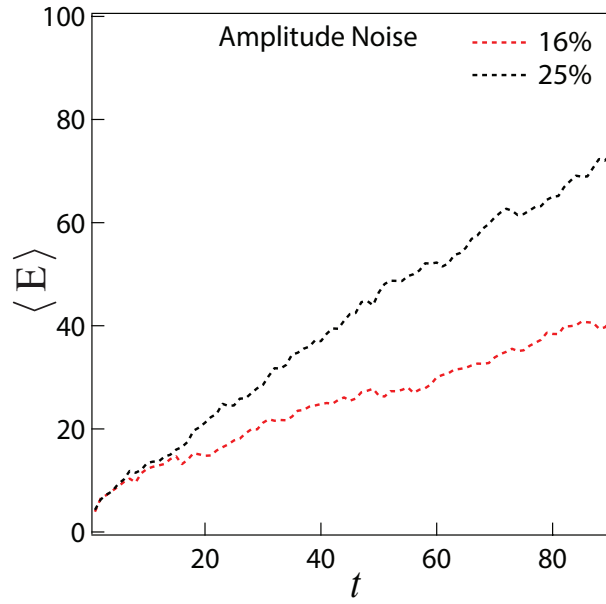


Figure 4.10: Quantum average energy growth of the kicked rotor with amplitude noise for  $\hbar_s = 2.0$ ,  $K = 5.8$  but with two different noise strengths,  $\Delta = 16\%$  (red) and  $\Delta = 25\%$  (black). Both black and red dotted lines are the numerical results only.

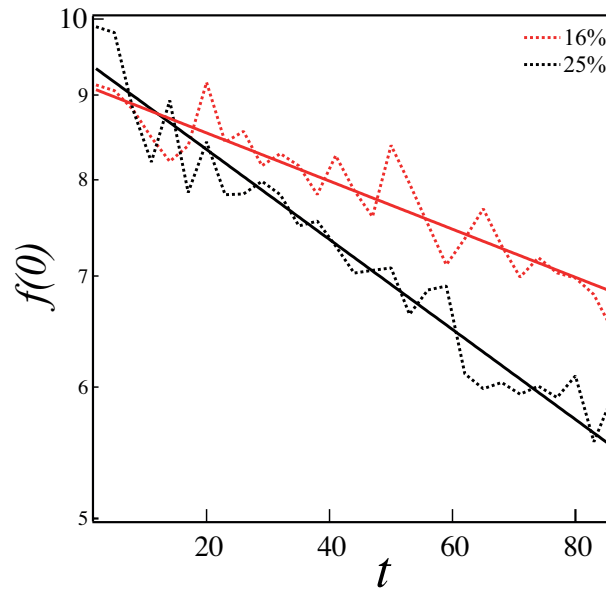


Figure 4.11: Log-normal plot of number of populations in the zero momentum state as a function of time for the kicked rotor with amplitude noise. The scaled Planck's constant,  $\hbar_s = 0.02$ , kick strength  $K = 5.8$ , and the noise strengths are  $\Delta = 16\%$  (red) and  $\Delta = 25\%$  (black). Solid lines are guide to an eye.

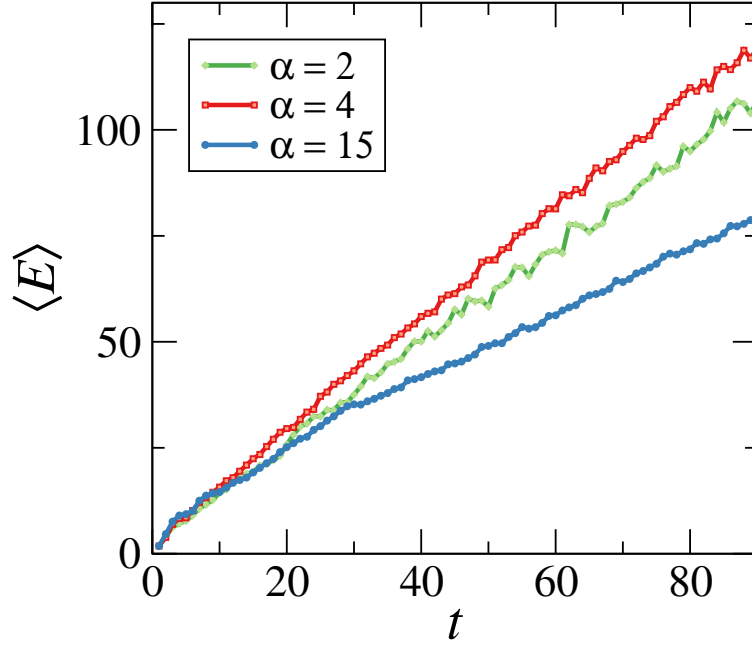


Figure 4.12: Quantum mean energy growths for  $\hbar_s = 2.0$ ,  $K = 6.0$ , and for different Lévy exponents,  $\alpha$ . All are the numerical results only.

as a function of time, since a straight line in a semi-log implies exponential decay. Now, the population in the zero momentum state as a function of time for the kicked rotor system with Lévy distributed kicks with Lévy exponents,  $\alpha = 0.25$  and  $\alpha = 0.75$  has been calculated. It is seen that the coherence decays as a non-exponential fashion which is shown by blue and green lines in Fig. 4.9. A straight line in a log-log plot implies power-law decay. Thus, we have successfully slowed down the rate of decay of coherence for  $\alpha < 1$ .

#### 4.4.2 For Lévy Exponent, $\alpha > 1$

In this section, we study the quantum average energy growth for the kicked rotor with Lévy distributed kicks with Lévy exponents  $\alpha > 1$ . It is found analytically from Eq. (4.48) that the quantum average energy grows diffusively with time at large time asymptotes for small  $\alpha$ . To study diffusion, in Fig. 4.12 the numerical results of quantum average energy growth for higher values of  $\alpha > 1$  are displayed and it is found that the numerical results show diffusive energy growth which is in agreement with the analytical results in Eq. (4.48). Surprisingly, it is found from

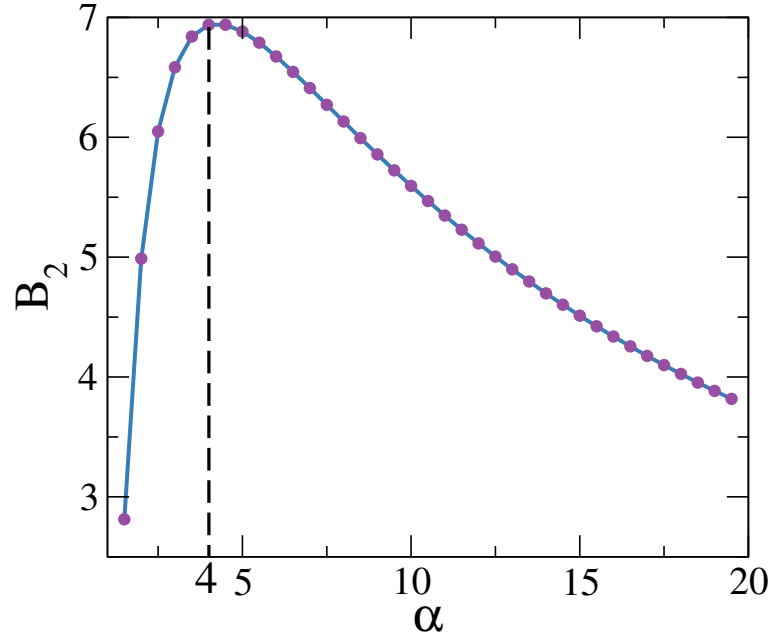


Figure 4.13: Diffusion coefficient,  $B_2$ , as a function of  $\alpha$  for  $\hbar_s = 2.0$ ,  $K = 6.0$ ,  $t^* = 6.5$  and  $a = 0.248$ .

Fig. 4.12 that with increase in  $\alpha$ , the rate of diffusion increases till a certain critical value of  $\alpha$  (in this case  $\alpha = 4$ ) and after that the rate of diffusion,  $B_2$  decreases. For very large value of  $\alpha$ , when  $B_2$  become very small, the other integration terms,  $I_1$ ,  $I_3$  and  $I_4$  of Eq. (4.45) start contributing. For extremely large values of  $\alpha$ , the diffusion gets attenuated. To understand this decay of diffusion rate after a critical value of  $\alpha$ , Fig. 4.13 shows the diffusion coefficient in Eq. (4.48) with the experimental parameters as a function of  $\alpha$ . Figure 4.13 displays that up to  $\alpha = 4$ , the diffusion rate increases and with further increase in  $\alpha$ , the diffusion rate decreases. To understand it more clearly, we should look at the timing sequence plot in Fig. 4.1. Figure 4.1 illustrates that with an increase in  $\alpha$ , the number of kicks in a given time interval increases and for large  $\alpha$ , the kicking sequence will look similar to that of the uniform kicking case where the energy saturates. Thus, it is clear that there should exist a critical value of  $\alpha$  after which the diffusion rate will decrease which in this case is  $\alpha = 4$  for the chosen set of experimental parameters.

## 4.5 Conclusions

We have studied decoherence phenomenon in a noisy kicked rotor system composed of the kicked rotor where kicks are occurring at selected times. The waiting time between two consecutive kicks is drawn from Lévy waiting time distribution,  $w(\tau) \sim c\tau^{-1-\alpha}$ , where  $\alpha$  is the Lévy exponent. We have found that in this system, depending on  $\alpha$ , the system shows sub-diffusive (for  $\alpha < 1$ ) and diffusive (for  $\alpha > 1$ ) energy growth. This implies a loss of coherence in the system. For  $\alpha < 1$ , the decoherence factor is found to be of non-exponential in nature. Hence, the rate of decoherence is slower than exponential.

For  $\alpha > 1$ , it is noticed that for an increase in the value of  $\alpha$ , the diffusion rate increases up to a certain critical value of  $\alpha$  after which the diffusion rate decreases and tends towards uniform kicking case (kicked rotor system). This initial increase and then, a decrease of the diffusion rate profile is well predicted by the analytical result derived in Eq. (4.48).

Thus, in conclusion, it is shown that the decoherence factor and the diffusion rate can be controlled by varying the Lévy exponent  $\alpha$ .

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## Future Plan

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- **Multifractality**

The term “fractal” was first introduced by Mandelbrot in 1960’s [Alligood 1996]. A fractal system has a self-similar property in all length scales, and its dynamics is characterized by a fractal dimension which is not an integer. Multifractality is the generalization of fractal statistics, where the single exponent is not enough to describe the dynamics of the system, but, requires a continuous set of exponents [Mandelbrot 1974]. The multifractal system includes the Sun’s magnetic field time series [Lawrence 1993], heartbeat dynamics [Ivanov 1999], strange attractors in chaotic dynamical systems. Recently, multifractal analysis has been extensively used to study the behavior of wavefunctions at Anderson transition [Evers 2000], quantum Hall transition [Huckestein 1995], etc. For a quantum system, multifractality analysis is carried out using the inverse participation ratio (P) which shows an anomalous scaling relation with the system size, N,

$$P_q \sim N^{-D_q(q-1)}, \quad (5.1)$$

where  $D_q$  are the multifractal exponents. A system, which exhibits the metallic behavior, i.e., the eigenfunctions are ergodic, the dynamics are described by a single exponent,  $P_q \sim N^{-d(q-1)}$ , where  $d$  is same for all  $q$ . The system is mono-fractal. While, the system which shows strong localization  $D_q \approx 0$  for all  $q$ . But, in the intermediate case, the system's dynamics is described by a set of critical exponents, and it exhibits a multifractal behavior.

In our system presented in Chapter 3, the eigenfunctions show power-law decay and the level spacing distribution of the Floquet eigenvalues in some unperturbed energy regime follows a semi-Poisson distribution which are the characteristics of a multifractal system. Thus, the system in Eq. (3.2) could be a multifractal system. But, carrying out the direct multifractal analysis, like, to calculate  $D_q$  as a function of  $q$ , is bit intricate because its dynamics is described on an infinite dimensional Hilbert space. Since we observed in Chapter 3 that the eigenfunctions decay very fast over the unperturbed basis states, so the contributions from most of the unperturbed basis are almost zero which creates a problem in multifractal analysis. Hence, the immediate work would be to find out a way to calculate the critical exponents,  $D_q$ , and the  $f(\alpha)$  statistics, which is another measure to check whether a system is multifractal or not.

### • Kicked Rotor with Missing Kicks

In Chapter 4, we studied the decoherence properties in a kicked rotor system with kicks at few selected times drawn from Lévy waiting distribution,  $w(\tau) \sim \tau^{-1-\alpha}$  and found that for  $\alpha < 1$  when mean waiting time,  $\bar{\tau}$  diverges, i.e., for non-stationary noise case, the system shows non-exponential decoherence. Now, we would like to extend our study by considering few other distributions whose either mean diverges such as Cauchy distribution, or the second moment diverges, and we will draw the times at which kicks are to be applied from these distributions. The aim of this study would be to find the form of the decoherence factor for all these non-stationary noise cases and



further, to ask if it is possible to make a statement that non-stationarity leads to non-exponential decoherence. Then, the slow decoherence will no longer be difficult to achieve and this will help to perform experiments and build quantum technologies.

### • Interacting Kicked Rotor Systems

Until now, we have studied two models, i) a periodically driven non-interacting single particle system and ii) a periodically driven single particle system which interacts only with the environment and the effects of interaction are introduced as classical noise to the system. Both the models considered are single particle systems. But, now, we would like to extend our study to a many-particle interacting system with periodic drives.

Periodically driven interacting systems are of interest in many areas of physics like condensed matter, quantum optics, quantum chaos, etc. It is known that interaction among particles plays a vital role in governing the dynamics. Periodic driving in addition to interactions may show a wide variety of dynamics which ranges from quantum localization to quantum diffusion for different types of interaction. These periodically driven interacting systems are of great importance because of their physical applications in spectroscopy, etc.

In this regard, we will be considering a coupled kicked rotor system as our starting point in the field of interacting systems. Later, we will extend this coupled kicked rotor to N-interacting kicked rotor system to study many-body quantum physics. The Hamiltonian of a coupled kicked rotor system that we would like to study is

$$H = \frac{L_1^2}{2m_1} + \frac{L_2^2}{2m_2} + U(\theta_1, \theta_2) + (\epsilon_1 \cos(\theta_1) + \epsilon_2 \cos(\theta_2)) \sum_n \delta(t - n), \quad (5.2)$$

where  $L_1$  and  $L_2$  represent the angular momentum of the particles with masses  $m_1$  and  $m_2$  respectively. Both the particles are receiving kicks of

kick strengths,  $\epsilon_1$  and  $\epsilon_2$ , at periodic intervals of time taken care of by the  $\delta$ -function in Eq. (5.2). In Eq. (5.2),  $\theta_1$  and  $\theta_2$  are the angular positions of the two particles and  $U(\theta_1, \theta_2)$  ( $= K \cos(\theta_1 - \theta_2)$ ) represents the interaction between them, and  $K$  is the interaction strength. When the particles are at a maximum angular distance from one another, i.e. the angular difference ( $\theta_1 - \theta_2$ ) is  $\pi$ , they experience an attractive potential of maximum strength  $K$ . On the other hand, when they are close to each other, i.e. the angular difference ( $\theta_1 - \theta_2$ ) is 0, they strongly repel each other.

The main motive of our study would be to understand the effect of this stationary interacting potential between the particles on the quantum properties of the system and also to explore novel classical dynamical features which will emerge upon interaction. We would also like to study the entanglement properties of this interacting system. The important point of considering this system is that it can be easily experimentally realized using ultra-cold atom set-up.

### • Kicked Bose Hubbard model

Now, after addressing all the above-mentioned problems, we will move on to work on a condensed matter problem. In this regard, we will consider a Bose-Hubbard model which will be periodically driven by an external field. The Hamiltonian of a 1-dimensional Bose-Hubbard model is

$$H_0 = -J \sum_{l=1}^L (a_{l+1}^\dagger a_l + h.c.) + \frac{U}{2} \sum_l n_l(n_l - 1), \quad (5.3)$$

where  $a_l^\dagger$  ( $a_l$ ),  $n_l$  are the creation (annihilation) and number operator for a spin-less particle at the lattice site  $l$ .  $L$  is the total number of lattice sites.  $J$  is the nearest neighbor hopping term and  $U$  is the on-site interaction energy which is repulsive in nature. If  $J \gg U$  then the particles will hop to adjacent lattice site and thus gives rise to the superfluid state. On the other hand, if  $J \ll U$ , then, the hopping will be suppressed giving rise to Mott insulating

phase. Many experiments have been performed on Bose-Einstein condensate in an optical lattice to realize this model which shows a clear signature of this phase transition [Jaksch 1998, Zwerger 2003, Jaksch 2005, Mintert 2016].

Now, the model that we would like to study is composed of interacting Bose-Hubbard model, as described in Eq. 5.3, with an additional space-dependent time-periodic drive. The Hamiltonian of the system is,

$$H = H_0 + \sum_l (-1)^j K \cos(2\pi l/L) n_l \sum_m \delta(t - m), \quad (5.4)$$

where,  $j$  is a stochastic variable which can randomly take either 0 or 1 and  $K$  is the strength of the kick which is site dependent.

The objective of my study would be to understand the effect of time-periodic external driving on phase transitions and explore if other phases like normal or glass phase exist. Briefly, I would like to understand the interplay between chaos and disorder and also many body localization in the context of external driving.

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## Appendix

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### Kicked Rotor with Random Kicks

We have already discussed in Chapter 4 of the thesis about the kicked rotor receiving kicks with a waiting time between two consecutive kicks. The waiting times were drawn from Lévy waiting time distribution,  $w(\tau) \sim c\tau^{-1-\alpha}$ , where  $\alpha$  is the Lévy exponent. But, now we are interested in studying, how the form of the decoherence factor will get modify if the waiting times are drawn uniformly from the interval  $[0, C]$ ? The uniform distribution is represented as  $g(\tau) = \frac{1}{C}$ . Here,  $C$  is introduced to normalize the distribution. In order to calculate the decoherence factor, we will follow the same procedure as we did in Chapter 4.

Firstly, the Laplace transform of the uniform waiting time distribution,  $g(\tau)$ , is to be calculated which leads to

$$\begin{aligned}
g(u) &= \int_0^C e^{-u\tau} g(\tau) d\tau \\
&= \frac{1}{uC} [1 - e^{-uC}] \\
&= \frac{1}{uC} [1 - 1 + uC - u^2 C^2/2 + \dots] \\
&\sim 1 - \frac{uC}{2}.
\end{aligned} \tag{6.1}$$

The Laplace transformed,  $g(u)$  is obtained by making an assumption that  $u$  is small. Next, by incorporating  $g(u)$  from Eq. (6.1) in Eq. (4.14), the Laplace transform of the sprinkling distribution is calculated which is

$$\begin{aligned}
f(u) &= \frac{g(u)}{1 - g(u)} \\
&\sim \frac{1 - uC/2}{uC/2} \\
&\sim \frac{2}{uC} - 1.
\end{aligned} \tag{6.2}$$

Now, the decoherence factor,  $D(t, 0)$ , can be written in terms of decoherence factor for uniform kicking  $D_0(t, 0)$  and decoherence factor for random kicking,  $D_1(t, 0)$  as

$$D(t, 0) = D_0(t, 0) D_1(t, 0). \tag{6.3}$$

This expression of  $D(t, 0)$  we have already seen in Chapter 4 in Eq. (4.25). Now, since  $D_0(t, 0)$  is for uniform kicking and does not depend on any form of waiting time distribution, so the analytical expression of  $D_0(t, 0)$  will remain the same as in Eq. (4.26), which is

$$D(t, 0) = e^{-(1-a^2)t}, \tag{6.4}$$

where,  $a \equiv q(K_n/\hbar_s)$  and the expression of  $q(K_n/\hbar_s)$  is given in Eq. (3.24).

Now, the final task remained is to calculate  $D_1(t, 0)$ , which can be obtained by taking a Laplace transform of  $D_1(u, 0)$  and leads to

$$\begin{aligned}
D_1(u, 0) &= \frac{1}{u} \frac{1}{1 - f(u)(a^{-2} - 1)} \\
&= \frac{1}{u} \frac{1}{1 - (\frac{2}{uC} - 1)(a^{-2} - 1)} \\
&= \frac{2a^2}{uC} \frac{uC/2}{u - (1 - a^2)2/C}.
\end{aligned} \tag{6.5}$$

Taking the inverse Laplace transform of  $D_1(u, 0)$ , we have

$$D_1(t, 0) \sim e^{(1-a^2)(2/C)t}. \tag{6.6}$$

Substituting both  $D_0(t, 0)$  from Eq. (6.4) and  $D_1(t, 0)$  from Eq. (6.6) in Eq. (6.3), we have

$$D(t, 0) \sim e^{-(1-a^2)(1-2/C)t}. \tag{6.7}$$

Now, for a random waiting time distribution case,  $C$  can in general be considered to be a large number, and hence,  $D(t, 0)$  reduces to

$$D(t, 0) \sim e^{-(1-a^2)t}. \tag{6.8}$$

From Eq. (6.8), it can be noticed that even for kicked rotor with kicks distributed over times which are taken from random waiting time distributions, the system shows exponential decoherence. Thus, for waiting time distributions with well defined mean: i) Lévy waiting time distribution with Lévy exponent,  $\alpha > 1$  and ii) uniform distribution, the system shows exponential decoherence.

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