

Simplicial Complexes on Seifert Surfaces of Links

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by

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Certificate

This is to certify that this dissertation entitled Simplicial Complexes on Seifert Surfaces of Links towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Ipsa Bezbarua at Indian Institute of Science Education and Research under the supervision of Prof Jennifer Schultens, Professor, Department of Mathematics, University of California, Davis, during the academic year 2022-2023.

Prof Jennifer Schultens



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Prof Jennifer Schultens

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This thesis is dedicated to the members of my family, both human and feline.

Declaration

I hereby declare that the matter embodied in the report entitled Simplicial Complexes on Seifert Surfaces of Links are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof Jennifer Schultens and the same has not been submitted elsewhere for any other degree.

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Abstract

In this thesis project, we study various properties related to two well-known simplicial complexes on the Seifert surfaces of links - the incompressible complex and the Kakimizu complex (named after Osamu Kakimizu). We first look at Kakimizu's original paper from 1992, in which he defines these complexes and describes a metric coming from the cyclic covering space of the knot complement, which is the same as the graph metric on the complexes. Then we study various properties of the Kakimizu complex, like connectedness, contractibility, local infiniteness and its coarse geometry. Finally, we try to extend some of these properties to the incompressible complex, and find a restriction on the types of knots that can have Kakimizu complex homeomorphic to $\mathbb{R}$.

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Introduction

Over the last century, knot theory has developed into an active area of research in topology. This is partly due to its connections with several other areas of science and technology. For instance, it has applications to DNA cutting in biology and to the motion of particles in fluid dynamics. In the domain of mathematics, low dimensional topologists look to results and techniques from knot theory to gain a better understanding of 3-manifolds. One of the fundamental structures studied in knot theory is a compact connected surface whose boundary is the knot under consideration, called a Seifert surface.

In 1992, Osamu Kakimizu introduced the notion of two simplicial complexes constructed using isotopy classes of Seifert surfaces - the incompressible complex (for isotopy classes of incompressible Seifert surfaces) and the Kakimizu complex (for isotopy classes of minimal genus Seifert surfaces) [Kak92]. These complexes are inspired by Harvey's curve complexes on simple closed curves of a surface, which is an important tool for the study of Teichmüller spaces, mapping class groups and Kleinian groups.

This thesis is an attempt to study the existing literature on these complexes, and extend a few of the known results.

The first half of this thesis (Chapters 1 - 7) is of expository nature, and is a study of present literature on the incompressible complex and the Kakimizu complex (mostly the latter as it is more widely studied due to the ease of performing cut-and-paste operations along minimal genus Seifert surfaces). The latter half (Chapters 8 and 9) is an attempt to generalize some of the results to the incompressible complex, and to find restrictions on knots that can have Kakimizu complexes of a specific form.

Outline of the Chapters

Chapter 1 is an introduction to useful concepts in 3-manifold theory and knot theory which will be used throughout the rest of the thesis.

Chapter 2 introduces the incompressible complex $\mathcal{IS}(L)$ and the Kakimizu complex $\mathcal{MS}(L)$ for a non-split oriented link L . We also study a distance metric on the set of isotopy classes of Seifert surfaces which is associated to the cyclic covering space of the knot complement and is equivalent to the graph metric on the complexes. This covering space is also useful in proving that the complexes are connected.

Chapter 3 is an explicit calculation of $\mathcal{IS}(L)$ and $\mathcal{MS}(L)$ for the composite of non-fibred knots.

In Chapters 4 and 5, we study the proof of how $\mathcal{MS}(K)$ is simply connected and contractible for a knot K .

In Chapter 6, we study links with locally infinite Kakimizu complexes. We see that the twisted Whitehead double for a trefoil knot has a locally infinite Kakimizu complex. We then learn about the types of links that can have locally infinite Kakimizu complex.

Chapter 7 discusses the coarse geometry of the Kakimizu complex and shows that it is quasi-isometric to the integer lattice $\mathbb{Z}^n$ for some $n \in \mathbb{Z}$.

In Chapter 8, we generalize the proof of contractibility of the Kakimizu complex to the incompressible complex.

In Chapter 9, we find a restriction on the types of knots that can have Kakimizu complex homeomorphic to $\mathbb{R}$.

Original Contribution

No claims to originality are made for the results in Chapters 1-7. These chapters are purely expository and meant to be a collection of the known literature on the subject. We have tried to write the proofs in a way that makes them easily accessible, with all required results

from other sources compiled alongside. Rather than focusing on the details of the proofs, we have instead tried to convey the main ideas behind them, especially for the long proofs; an interested reader may look up the references for the full proofs. We have also added several diagrams in an attempt to aid the reader's visual intuition for the topic.

The main results in Chapters 8 and 9 are original, though they largely draw from existing techniques in the previous chapters.

Chapter 1

Basics of 3-Manifold Theory and Knot Theory

Knot theory is a rapidly evolving sub-topic in topology. It first came into prominence in the 19th century, when chemists believed atoms existed in the form of knots. Throughout the 20th century, it has been studied using various tools from group theory, homology theory, and 3-manifold theory. Knot theory boasts of several applications to other areas of study; popular examples being its uses in string theory and understanding the topology of DNA.

In this chapter, we review the basic definitions and theorems from 3-manifold theory and basic knot theory that we shall use throughout the remainder of the thesis. The content of this chapter is taken from [Sch14], [Sco17], [Ada94] and [Lic97].

1.1 Manifolds

In this section, we define the types of manifolds commonly studied in topology (topological, triangulated and smooth).

Definition 1.1.1. *A second countable Hausdorff space M which has a family of pairs $\{(M_\alpha, \phi_\alpha)\}$ satisfying the following properties is called a **topological n -manifold**.*

- $\forall \alpha$, M_α is an open subset of M and $M = \bigcup_{\alpha} M_\alpha$.

- $\forall \alpha, \phi_\alpha$ is a homeomorphism from M_α to an open subset of $\mathbb{R}^n$.

A topological manifold is also referred to as a **manifold**.

The **dimension** of an n -manifold is defined to be n .

Popular examples of manifolds include the circle (a 1-manifold), the torus (a 2-manifold) and the 3-ball (a 3-manifold).

Definition 1.1.2. The pair (M_α, ϕ_α) , as defined above, is called a **chart** for M . The collection $\{(M_\alpha, \phi_\alpha)\}$ of all charts is called an **atlas** for M .

Definition 1.1.3. A map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called **C^q** if it has continuous partial derivatives of order q . If such a map has continuous partial derivatives of all possible orders, then it is called **smooth** or **C^∞** .

Definition 1.1.4. A **C^q** manifold (q is a natural number) is a topological manifold which has an atlas such that if $(M_\alpha, \phi_\alpha), (M_\beta, \phi_\beta)$ is a pair of charts in the atlas, then $\phi_\beta \circ \phi_\alpha^{-1}$ is a C^q map.

A C^∞ -manifold is also called a **differentiable/smooth manifold**.

Definition 1.1.5. Let v_i be the $(k+1)$ -tuple in $\mathbb{R}^{k+1}$ whose i^{th} entry is 1 and all other entries are 0. The **standard (closed) k -simplex**, denoted as $[v_0, \dots, v_k]$ or $[s]$, is defined to be the set

$$\{a_0v_0 + \dots + a_kv_k \mid a_0, \dots, a_k \geq 0, \sum_{i=0}^k a_i = 1\}.$$

Similarly, the **standard open k -simplex**, denoted as $(v_0, \dots, v_k)$ or (s) , is defined to be the set

$$\{a_0v_0 + \dots + a_kv_k \mid a_0, \dots, a_k > 0, \sum_{i=0}^k a_i = 1\}.$$

It is also called the **interior** of $[s]$.

Definition 1.1.6. A continuous map $f : [s] \rightarrow X$, where $[s]$ is the standard k -simplex and $f|_{(s)}$ is a homeomorphism onto its image, is called a **k -simplex** in a topological space.

Restricting a k -simplex to its interior gives an **open k -simplex**.

Remark 1.1.7. We will often abuse notation and call the image of the above map f as a k -simplex.

Definition 1.1.8. A subset of the standard k -simplex $[s]$ of the form

$$\{a_0v_0 + \dots + a_kv_k \mid a_{i_1} = 0, \dots, a_{i_j} = 0\}$$

for $j = 0, \dots, k$ is called a **face**. Such a face is of dimension $k-j$.

A **face** of a k -simplex $f : [s] \rightarrow X$ is the restriction map $f|_{[t]}$, where $[t]$ is a face of the standard k -simplex $[s]$.

A 0-dimensional face is called a **vertex**, and a 1-dimensional face is called an **edge**.

Definition 1.1.9. For the topological space X , the set of simplices

$$K = \{f : [s] \rightarrow X\}$$

in X is called a **simplicial complex** if it satisfies the following:

1. for any simplex f in K , all the faces of f are also in K .
2. for any simplices f_1, f_2 in K , if the images of $f_1|_{(s_1)}$ and $f_2|_{(s_2)}$ meet, then they must be equal.

The **dimension** of the simplicial complex is the maximum value of the dimensions of the simplices in K .

The union of the images of the simplices in K , denoted by $|K|$, is called the **underlying space** of K .

Definition 1.1.10. The pair (M, K) for a topological n -manifold M and simplicial complex K based on M is called a **triangulated n -manifold** if it satisfies:

- $|K| = M$.
- K is locally finite, ie, if C is a compact subset of M , then $\{f \in K \mid C \cap \text{im}(f) \neq \emptyset\}$ is finite.

- for $f, g \in K$, restricted to open simplices, the map $g^{-1} \circ f$ is affine on its domain.

K is called the **triangulation** of M .

Topological manifolds up to dimension 3 are also PL and smooth.

1.2 Surfaces in 3-Manifolds

In this section, we define some important concepts related to the study of 3-manifolds, which are used throughout knot theory.

Definition 1.2.1. A 2-manifold is also called a **surface**.

Definition 1.2.2. A 3-manifold M is called **irreducible** if any 2-sphere embedded in M bounds a 3-ball in M .

$\mathbb{R}^3$ is an irreducible 3-manifold.

Definition 1.2.3. A connected surface F properly embedded in a 3-manifold M is called **incompressible** in M if the following conditions hold:

1. Whenever a simple closed curve C on F bounds a 2-disc D in M such that $D \cap F = C$, then C also bounds a 2-disc in F .
2. If F is S^2 or D^2 , then in addition to (1), which is automatic, F does not separate M into two pieces, one of which is a 3-ball.

If F is disconnected, then it is **incompressible** iff each of its components is incompressible.

Definition 1.2.4. Let F be a properly embedded surface in a 3-manifold M . A **compressing disc** for F is a 2-disc D in M such that $D \cap F = \partial D$ is an essential curve on F .

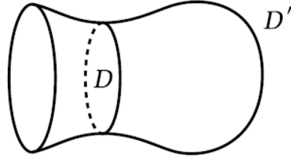


Figure 1.1: Schematic of an incompressible surface (courtesy of Allen Hatcher, used with permission)

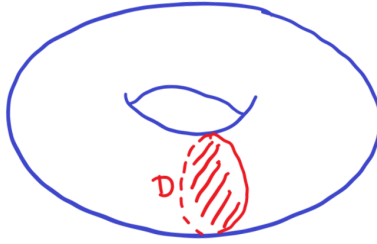


Figure 1.2: Schematic of a compressing disc

Definition 1.2.5. A connected surface F properly embedded in a 3-manifold M is **∂ -incompressible** in M if, whenever there is a disc D in M with boundary consisting of an arc λ in F and an arc μ in ∂M such that $D \cap F = \lambda$, then there is an arc μ' in ∂F such that $\mu' \cup \lambda$ bounds a disc in F , ie, λ is inessential in F .

If F is disconnected, then F is **∂ -incompressible** iff each component of F is ∂ -incompressible.

Definition 1.2.6. Let F be a properly embedded surface in a 3-manifold M . Then a **∂ -compressing disc** for F is a 2-disc D in M with boundary consisting of an arc λ in F and an arc μ in ∂M such that $D \cap F = \lambda$ and λ is an essential arc on F .

1.3 Basics of Knot Theory

In this section, we define some basic terms used in knot theory.

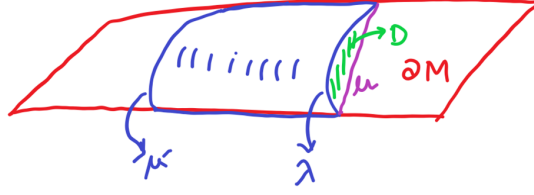


Figure 1.3: Schematic of a ∂ -incompressible surface

Definition 1.3.1. A **knot** is an embedding of $\mathbb{S}^1$ in $\mathbb{R}^3$.

Definition 1.3.2. One or more disjointly embedded circles in $\mathbb{R}^3$ is called a **link**.

Definition 1.3.3. Let L be an oriented link in $\mathbb{S}^3$ and $\mathcal{N}(L)$ be a fixed tubular neighbourhood for L . The **exterior of the link** or **link complement**, denoted by $E(L)$ or sometimes by E , is defined as

$$E(L) = \mathbb{S}^3 \setminus \mathcal{N}(L).$$

Definition 1.3.4. Let L be an oriented link, $\mathcal{N}(L)$ be a fixed tubular neighbourhood for L and $E(L)$ be the exterior of L . Let Σ be an oriented surface in $\mathbb{S}^3$ such that:

- $\partial\Sigma = L$.
- It has no closed component.
- It is possibly disconnected.
- $\Sigma \cap \mathcal{N}(L)$ is a collar of $\partial\Sigma = L$ in Σ .

A **spanning surface** for L is a surface $S = \Sigma \cap E(L)$.

A connected spanning surface is called a **Seifert surface**.

There are three types of knots - torus knots, satellite knots and hyperbolic knots.

Definition 1.3.5. Torus knots can be made to lie on an unknotted torus, without crossing over or under themselves.

The trefoil is a torus knot.

Definition 1.3.6. Let K be a knot in the solid torus T . K is called a **pattern**. $e : T \rightarrow \mathbb{S}^3$ is an embedding such that eT is a regular neighbourhood of a knot C in $\mathbb{S}^3$. Then eK is defined to be a **satellite** of C , and C is defined as the **companion** of eK .

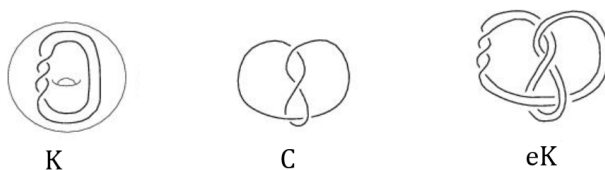


Figure 1.4: Schematic of a satellite knot (courtesy of WBR Lickorish, used with permission)

Definition 1.3.7. A **hyperbolic knot** is a knot whose complement can be endowed with a hyperbolic metric (metric with constant curvature -1).

The figure 8 knot is a hyperbolic knot.

Chapter 2

Simplicial Complexes on Links

In this chapter, we define the simplicial complexes $\mathcal{IS}(L)$ and $\mathcal{MS}(L)$ for an oriented non-split link. We also prove some basic results related to these complexes, like the fact that they are connected, as well as a useful observation about the graph metric of the simplicial complexes.

The content of this chapter has been taken from [Kak92], which one can look at for detailed proofs of the results.

2.1 Equivalence classes of spanning surfaces for a link

In this section, we define the notion of equivalence classes on the spanning surfaces for a given link.

Definition 2.1.1. *Spanning surfaces that are ambient isotopic to each other in $E(L)$ are said to be **equivalent**. The equivalence class of a spanning surface S is denoted by $[S]$.*

We use $\mathcal{S}(L)$ to denote the set of equivalence classes of spanning surfaces for the link L .

Definition 2.1.2. *A spanning surface S is said to have **minimal genus** if it has maximum Euler characteristic $\chi(S)$ amongst all the possible spanning surfaces for L .*

Definition 2.1.3. *The subset of $\mathcal{S}(L)$ consisting of equivalence classes of incompressible spanning surfaces for L is denoted by $\mathcal{IS}(L)$.*

The subset of $\mathcal{S}(L)$ consisting of equivalence classes of minimal genus spanning surfaces for L is denoted by $\mathcal{MS}(L)$.

2.2 Simplicial complexes on spanning surfaces

In this section we will define the notion of the incompressible complex $\mathcal{IS}(L)$ and the Kakimizu complex $\mathcal{MS}(L)$ for a non-split oriented link L .

Definition 2.2.1. The *incompressible complex* $\mathcal{IS}(L)$ is defined such that:

- its vertex set is $\mathcal{SS}(L)$.
- the vertices $\sigma_0, \dots, \sigma_k \in \mathcal{SS}(L)$ span a k -simplex if they have representatives $S_i \in \sigma_i$, $0 \leq i \leq k$ with $S_i \cap S_j = \emptyset$ for every $j > i$.

Remark 2.2.2. The name *incompressible complex* for $\mathcal{IS}(L)$ has not been used anywhere in [Kak92] or any other literature. However, we shall study this complex extensively throughout this thesis, hence it is useful to give it a name.

Definition 2.2.3. Replacing $\mathcal{SS}(L)$ with $\mathcal{MS}(L)$ in the definition of $\mathcal{IS}(L)$ gives a new complex, which we denote by $\mathcal{MS}(L)$. This is the **Kakimizu complex**.

Minimal genus Seifert surfaces are incompressible, so $\mathcal{MS}(L)$ is a full subcomplex of $\mathcal{IS}(L)$.

2.3 Distances on $\mathcal{S}(L)$

We will now define a notion of distance on the equivalence classes of spanning surfaces. First, we need the concept of an augmentation subgroup to define a cyclic covering space of $E(L)$.

Definition 2.3.1. L is an oriented link in $\mathbb{S}^3$ with μ components, $L_1, \dots, L_\mu$. We fix a basepoint q_0 and let $G = \pi_1(\mathbb{S}^3 \setminus L, q_0)$. We define a homomorphism $\epsilon : G \rightarrow \mathbb{Z}$ as follows: if l is a loop in $\mathbb{S}^3 \setminus L$ with basepoint q_0 and $[l]$ denotes its homotopy class, then

$$\epsilon([l]) = \sum_{i=1}^{\mu} \lambda(l, L_i),$$

where $\lambda(l, L_i)$ is the linking number of l and L_i in $\mathbb{S}^3$.

The kernel of ϵ is called the **augmentation subgroup** of L , denoted as E . Further

$$1 \rightarrow E \xrightarrow{i} G \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

is an exact sequence.

If L is a knot, then the augmentation subgroup is the commutator subgroup G' .

Let $p : (\tilde{E}, a_0) \rightarrow (E, a)$ be an infinite cyclic covering such that $p_*\pi_1(\tilde{E}, a_0)$ is the augmentation subgroup of $\pi_1(E, a)$ for some basepoint a . Let τ be a generator for the group of deck transformations, $S \subset E$ be a spanning surface for L and E_0 be the closure of a lift of $E \setminus S$ to $\tilde{E}$.

Let $E_j = \tau^j(E_0)$ and $S_j = E_{j-1} \cap E_j$ for every $j \in \mathbb{Z}$.

Then

$$\tilde{E} = \bigcup_{j \in \mathbb{Z}} E_j$$

$$p^{-1}(S) = \bigcup_{j \in \mathbb{Z}} S_j$$

$p|_{S_j} : S_j \rightarrow S$ is a homeomorphism.

See Figure 2.1.

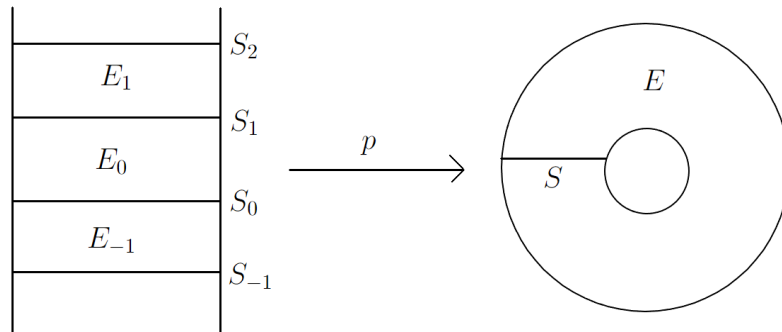


Figure 2.1: Schematic for the cyclic covering space

If $S' \subset E$ is another spanning surface for L , then:

$$\begin{aligned}\tilde{E} &= \bigcup_{k \in \mathbb{Z}} E'_k \\ S'_k &= E'_{k-1} \cap E'_k \\ p^{-1}(S') &= \bigcup_{k \in \mathbb{Z}} S_k \\ E'_k &= \tau^k(E'_0)\end{aligned}$$

We set:

$$\begin{aligned}m &= \min \{k \in \mathbb{Z} \mid E_0 \cap E'_k \neq \emptyset\} \\ r &= \max \{k \in \mathbb{Z} \mid E_0 \cap E'_k \neq \emptyset\} \\ d(S, S') &= r - m\end{aligned}$$

Definition 2.3.2. For $\sigma, \sigma' \in \mathcal{S}(L)$, we define $d(\sigma, \sigma') \in \mathbb{Z}^+$ as

$$d(\sigma, \sigma') = \begin{cases} 0 & \text{if } \sigma = \sigma' \\ \min_{S \in \sigma, S' \in \sigma'} d(S, S') & \text{if } \sigma \neq \sigma' \end{cases}$$

It satisfies the requirements of non-negativity, symmetry and triangle inequality to be considered a **distance** on $\mathcal{S}(L)$.

Some properties of this distance are:

1. $d(S, S') \geq 1$.
2. $d(S, S') = 1 \iff S \cap S' = \emptyset$.
3. $E_j \cap E'_k \neq \emptyset \iff m \leq k - j \leq r$.
4. $E_0 \subset \bigcup_{m \leq k \leq r} E'_k$ and $S_1 \subset \bigcup_{m+1 \leq k \leq r} E'_k$.

2.4 Connectedness of $\mathcal{IS}(L)$ and $\mathcal{MS}(L)$

In this section, we will prove that the incompressible complex $\mathcal{IS}(L)$ and the Kakimizu complex $\mathcal{MS}(L)$ are connected.

For the case when L is a knot, an alternate proof of connectedness of $\mathcal{MS}(L)$ can be found in [ST88].

Theorem 2.4.1. *Let $L \subset \mathbb{S}^3$ be a non-split link and $S, S' \subset E(L)$ be two incompressible (resp. minimal genus) spanning surfaces for L . We assume $n = d([S], [S']) \geq 1$. Then there exists a sequence of incompressible (resp. minimal genus) spanning surfaces $S = F_0, F_1, \dots, F_n$ satisfying the following:*

1. $[F_n] = [S']$.
2. $F_{i-1} \cap F_i = \emptyset$.
3. $d([S], [F_i]) = i \ \forall 0 \leq i \leq n$.

Proof. The idea is to use induction on $n = d([S], [S'])$.

For the case when $n = 1$, we already know that $S' \cap S = \emptyset$. We take $F_1 = S'$ and we are done.

Let us assume that the theorem holds for $n \leq q-1$ (where $q \geq 2$). We will find an incompressible (resp. minimal genus) surface $S'' \subset E$ such that $S' \cap S'' = \emptyset$ and $d([S], [S'']) = q-1$. Then by using the induction hypothesis on S and S'' , we get the desired result in the induction step.

We take X to be a regular neighbourhood of $S'_r \cup (E_0 \cap E'_r)$ in E'_r disjoint from E_q . Y is the closure of the component of $E'_r \setminus X$ containing S'_{r+1} and R is defined as $X \cap Y$ (see Figure 2.2). This gives us a spanning surface, $p(R)$.

Case I: Both S and S' are minimal genus spanning surfaces.

In this case, by arguing using the Euler characteristic, we can show that $p(R)$ is a minimal genus spanning surface. Thus, we take $S'' := p(R)$.

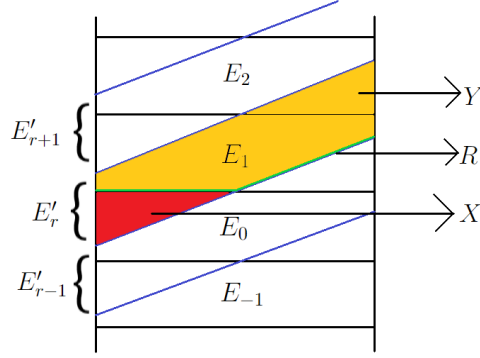


Figure 2.2: Setup for Theorem 2.4.1

Case II: Both S and S'' are incompressible.

In this case, we use a finite sequence of moves of the types mentioned in [DR 69] to obtain an incompressible surface F such that $p(F)$ is disjoint from S . We take this as S'' .

By arguing using the triangle inequality law of distances, we see that $d([S], [S'']) = q - 1$. Then applying the induction hypothesis gives us a sequence of incompressible spanning surfaces, $S = F_0, \dots, F_{q-1}$ satisfying the conditions in the induction hypothesis. We take the isotopy, $\{h_t\}$ such that $h_0 = Id$, $h_1(S'') = F_{q-1}$, and define $F_q := h_1(S')$. This completes the induction step, as well as the proof. $\square$

The above theorem shows that $\mathcal{IS}(L)$ and $\mathcal{MS}(L)$ are connected.

2.5 The Kakimizu distance and the graph metric

For $\sigma, \sigma' \in \mathcal{IS}(L)$, we denote the minimum length of edge paths in $\mathcal{IS}(L)$ connecting σ, σ' by $l_I(\sigma, \sigma')$.

Similarly for $\sigma, \sigma' \in \mathcal{MS}(L)$, we denote the minimum length of edge paths in $\mathcal{MS}(L)$ connecting σ, σ' by $l_M(\sigma, \sigma')$.

In this section, we will show that these minimum lengths of edge paths is the same as the Kakimizu distance between the vertices in the covering space.

Proposition 2.5.1. 1. For $\sigma, \sigma' \in \mathcal{IS}(L)$, $l_I(\sigma, \sigma') = d(\sigma, \sigma')$.

2. For $\sigma, \sigma' \in \mathcal{MS}(L)$, $l_M(\sigma, \sigma') = d(\sigma, \sigma')$.

Proof. Here, the proof for (1) is given. The proof for (2) is along the same lines.

$$d(\sigma, \sigma') = 1 \iff l_I(\sigma, \sigma') = 1.$$

Say $d(\sigma, \sigma') \geq 1$. By Theorem 2.4.1, we know that there is a path of length $d(\sigma, \sigma')$ joining σ and σ' . As the LHS is the minimum possible path length, we must have $l_I(\sigma, \sigma') \leq d(\sigma, \sigma')$.

Conversely, if $l_I(\sigma, \sigma') = n$, we have a (finite) sequence $\sigma = \sigma_0, \sigma_1, \dots, \sigma_n = \sigma' \in \mathcal{IS}(L)$ such that $l_I(\sigma_{i-1}, \sigma_i) = 1, \forall 1 \leq i \leq n$.

$$\begin{aligned} \therefore l_I(\sigma, \sigma') &= l_I(\sigma_0, \sigma_1) + \dots + l_I(\sigma_{n-1}, \sigma_n) = d(\sigma_0, \sigma_1) + \dots + d(\sigma_{n-1}, \sigma_n) \geq d(\sigma_0, \sigma_n) = d(\sigma, \sigma') \\ &\Rightarrow l_I(\sigma, \sigma') \geq d(\sigma, \sigma') \end{aligned}$$

Comparing the two inequalities gives $l_I(\sigma, \sigma') = d(\sigma, \sigma')$. □

Chapter 3

The Simplicial complexes of the composite of non-fibred knots

In this chapter, we show that the simplicial complexes $\mathcal{IS}(\mathcal{K})$ and $\mathcal{MS}(\mathcal{K})$ for the composite of two non-fibred knots is homeomorphic to $\mathbb{R}$.

The content of this chapter has been taken from [Kak92], which one can look at for detailed proofs of the results.

3.1 Eisner's construction

Throughout this chapter, we will assume that $K = K_1 \# K_2$, where K_1 and K_2 are non-fibred knots. In addition, the incompressible spanning surfaces for K_1 and K_2 are unique. This is not a very strong assumption as there are many non-fibred 2-bridge knots and non-fibred 2-bridge prime knots with at most 10 crossings which have unique incompressible spanning surfaces ([HT85], [Kak05]).

The technique used in this section is adopted from ([Eis77]).

We take $E(K) = E(K_1) \cup E(K_2)$, Then $A = E(K_1) \cap E(K_2) = \partial E(K_1) \cap \partial E(K_2)$ is an annulus.

S is a minimum genus spanning surface for K taken such that $R_i = S \cap E(K_i)$ is also a minimum genus spanning surface for K_i , $i = 1, 2$. Then $S = R_1 \cup R_2$. Further, $I = R_1 \cap R_2 = S \cap (E(K_1) \cap E(K_2)) = S \cap A$, which is an arc.

We identify

$$A = \{(e^{2\pi i\theta}, s) \mid 0 \leq \theta \leq 1, 0 \leq s \leq 1\}$$

$$I = \{(1, s) \mid 0 \leq s \leq 1\}$$

Then

$$m : [0, 1] \rightarrow E(K)$$

$$\theta \mapsto (e^{2\pi i\theta}, 1)$$

is a loop representing a meridian element $\mu \in \pi_1(E(K), a)$ for $a = (1, 1) \in \partial I$. (See Figure 3.1)

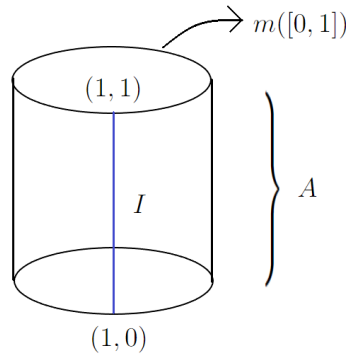


Figure 3.1:

This gives an embedding

$$A \times [0, 1] \subset E(K_1)$$

such that $A = A \times \{1\}$ and $(A \times [0, 1]) \cap \partial E(K) = \partial A \times [0, 1]$.

We construct a “twist” homeomorphism

$$f : E(K) \rightarrow E(K)$$

defined by

$$\begin{aligned} f|_{E(K_2)} &= Id \\ f|_{E(K_1) \setminus (A \times [0,1])} &= Id \\ f(e^{2\pi i \theta}, s, t) &= (e^{2\pi i(\theta+t)}, s, t) \text{ on } A \times [0, 1] \end{aligned}$$

We define $S^{(n)} := f^n(S) \forall n \in \mathbb{Z}$. Then $S^{(n)}$ is a minimal genus spanning surface satisfying:

1. $S^{(n)} \cap A = I$.
2. $S^{(n)} \cap E(K_2) = R_2$.
3. $S^{(n)} \cap E(K_1)$ is a minimal genus spanning surface for K_1 which is equivalent to R_1 .
4. $S^{(k)} = f^{k-n}(S^{(n)}) \forall k \in \mathbb{Z}$.

Proposition 3.1.1. $S^{(k)}$ is not equivalent to $S^{(n)} \forall k \neq n$.

The proof of the above can be found in ([Eis77]).

Thus, the construction gives us a way to obtain an infinite number of non-isotopic minimal genus Seifert surfaces for K .

3.2 The simplicial complexes for K

Proposition 3.2.1. K is the composite knot of two non-fibred knots K_1 and K_2 , and $\{S^{(n)}\}_{n \in \mathbb{Z}}$ is the infinite sequence of spanning surfaces for K as constructed previously. In addition, we assume that the incompressible spanning surfaces for K_1 and K_2 are unique. Then:

1. any incompressible spanning surface for K is equivalent to $S^{(n)}$ for some $n \in \mathbb{Z}$.
2. $d([S^{(n)}], [S^{(k)}]) = n - k \forall n \geq k$.

Proof. We move $S^{(k+1)}$ by a small isotopy of $E(K)$ to make $S^{(k+1)}$ disjoint from $S^{(k)}$.

$$\therefore d([S^{(k+1)}], [S^{(k)}]) = 1$$

This, along with Proposition 3.1.1, show that the subcomplex in Figure 3.2 lies in $\mathcal{IS}(\mathcal{K})$:

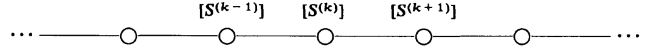


Figure 3.2: (courtesy of Osamu Kakimizu, used with permission)

We will show that this subcomplex is the whole of $\mathcal{IS}(\mathcal{K})$ and $\mathcal{MS}(\mathcal{K})$.

In Eisner's construction, we had denoted the annulus $E(K_1) \cap E(K_2)$ as A , and the arc $R_1 \cap R_2$ as I . Suppose there is an incompressible spanning surface that is not equivalent to any $S^{(k)}$. As $\mathcal{IS}(K)$ is connected, there is an incompressible surface disjoint from some $S^{(n)}$. We claim that F must be equivalent to one of $S^{(n-1)}$, $S^{(n)}$ or $S^{(n+1)}$. Without loss of generality, we can take $n = 0$. The idea of the proof is to denote $F \cap E(K_i)$ as F_i and $F \cap A$ as J , and then construct an isotopy of J to I . This isotopy is either a single twist in some direction (corresponding to $f^{(-1)}$ and $f^{(1)}$) or just a parallel movement (corresponding to $f^{(0)}$). Further, the isotopy extends to an isotopy of F to either of $S^{(-1)}$, $S^{(0)}$ or $S^{(1)}$, completing part (1) of the proposition.

Next we are to show that $d([S^{(n)}], [S^{(k)}]) = n - k \forall n \geq k$.

For $n = k$, this is trivial, as then $[S^{(n)}] = [S^{(k)}]$ and then $d([S^{(n)}], [S^{(k)}]) = 0 = n - k$.

If possible, say $d([S^{(n)}], [S^{(k)}]) < n - k$. Then for some i, j such that $j - i \geq 2$, $d([S^{(i)}], [S^{(j)}]) = 1$. We simply need to show that $d([S^{(n)}], [S^{(k)}]) \geq 2 \forall n, k$ such that $n - k \geq 2$. WLOG, we can assume that $k = 0$, and thus we need to show that $d([S], [S^{(n)}]) \geq 2 \forall n \geq 2$.

The idea of the proof is that when $d([S], [S^{(n)}]) = 1$, there are disjoint representatives of $[S]$ and $[S^{(n)}]$. Thus there is an isotopy $h : E(K) \times [0, 1] \rightarrow E(K)$ such that h_0 is identity and $h_1(S^{(n)}) \cap S = \emptyset$. However considering the lift of this isotopy to the covering space shows

that K_1 or K_2 is fibred, which is a contradiction to fact that they are not. $\square$

Thus, we can conclude that the incompressible complex and Kakimizu complex for the composite of non-fibred knots is homeomorphic to $\mathbb{R}$.

Chapter 4

The Kakimizu complex is simply connected

In this chapter, we show that the Kakimizu complex is a flag complex and is simply connected. We also show a quick proof that a 2-dimensional Kakimizu complex is contractible. The next chapter will discuss the contractibility of the Kakimizu complex in a more general context.

The content of this chapter has been taken from [Sch10], which one can look at for detailed proofs of the results.

4.1 Least area surfaces

In this section, we will provide a quick introduction to some relevant topics from minimal surface theory.

Definition 4.1.1. *Let M be a compact irreducible smooth manifold with boundary. A **relative Riemannian metric** on M is a Riemannian metric for which ∂M is strictly convex.*

Definition 4.1.2. *Let M be a compact irreducible smooth manifold with boundary such that ∂M consists of tori and let $\mathcal{J}$ be a smooth foliation of ∂M by closed curves. Then a properly embedded surface F in M is said to have **relative least area** if:*

- $\partial F \subset \mathcal{J}$.
- The surface F has minimum area amongst all surfaces in its proper isotopy class which satisfy the constraint $\partial F \subset \mathcal{J}$.

Let S be a compact connected surface, not necessarily with boundary. Let $f : (S, \partial S) \rightarrow (M, \partial M)$ be a smooth proper embedding and $[f]$ denote the (proper) isotopy class of f (not fixing the boundary value $f|_{\partial S}$). For $[f]$, $f^{-1}(\partial_i M)$ is assumed to be a single component of ∂S for each boundary component of $\partial_i M$ of M , and $f(S)$ is assumed to be an incompressible surface. Then Theorem 6.12 in [HS88] says that there is a (parameterized) surface in $\{g \in [f] \mid g|_{\partial S} = f|_{\partial S}\}$ having least area and it must be stable minimal.

The ‘moduli space’ $\mathcal{M}([f])$ is the space of stable minimal surfaces in the given proper isotopy class $[f]$ such that $f|_{\partial S}$ is a parameterized multi-curve in $\mathcal{J}$ (parameterized surfaces differing by a reparameterization of S are identified). $\mathcal{M} = \mathcal{M}(S) = \bigcup_{[f]} \mathcal{M}([f])$ is the space of all stable embedded minimal surfaces having the given topological type. It is given the C^1 -topology. For a given number a , $\mathcal{M}_a$ is the subset of $\mathcal{M}$ that consists of surfaces that have area not greater than a .

Proposition 4.1.3. *If S has non-empty boundary, then $\mathcal{M}_a$ is compact.*

Proof. Say we have a sequence of minimal surfaces $f_i : S \rightarrow M$ with boundary values in $\mathcal{J}$ and maximum area a . Then by Theorem 3.1 in [And85], there is a subsequence f_{i_j} such that the sequence $f_{i_j}(S)$ converges to a minimal surface $\Sigma \subset M$ with boundary in $\mathcal{J}$. Applying Schoen’s curvature estimates [Sch83] shows that $\{f_{i_j} : S \rightarrow M\}$ converges smoothly to the covering map $f : S \rightarrow \Sigma$, which must be one-one. $\square$

Corollary 4.1.4. *The set $\mathcal{M}_a$ is a disjoint union of a finite number of open and closed subsets $\mathcal{M}_a([g])$, each of consisting of isotopic surfaces.*

Proof. g is be a stable minimal surface in $\mathcal{M}_a$, and $\{f_k : S \rightarrow M\}$ is a sequence of surfaces in $\mathcal{M}_a \setminus \mathcal{M}_a([g])$. Say $\{f_k\}$ converges to $f : S \rightarrow M$. Then for sufficiently large k , f is isotopic to each f_k . Further, $f \notin \mathcal{M}_a([g])$, implying that $\mathcal{M}_a \setminus \mathcal{M}_a([g])$ is closed. Thus, the subsets $\mathcal{M}_a([g])$ are open. They also cover $\mathcal{M}_a$ by definition. So by compactness of $\mathcal{M}_a$, $\mathcal{M}_a([g])$ is finite. $\square$

Corollary 4.1.5. *Each $\mathcal{M}([f])$, and in particular, each class $\{g \in [f] : g|_{\partial S} \in \mathcal{J}\}$ contains an area minimizer.*

Proof. Area is known to be a continuous function and $\mathcal{M}_a([f])$ is compact. So the result follows from the extreme value theorem and the finiteness of $\mathcal{M}_a([f])$. $\square$

Next, ∂M is assumed to be made up of tori.

Claim 4.1.6. *M always admits a relative Riemannian metric.*

We consider a metric on $T^2 \times [0, 1]$ which is strictly convex on $T^2 \times \{0\}$. We identify $T^2 \times [0, 1]$ with neighbourhoods of boundary tori $\partial_i M$ of M , where $\partial_i M$ is identified with $T^2 \times \{0\}$. This metric is then extended arbitrarily on a neighbourhood of ∂M to the rest of M . It is clearly relative Riemannian.

Let $\mathcal{J}$ be a C^1 -foliation of ∂M by closed curves. [FHS83] uses a free boundary condition. But as curves in $\mathcal{J}$ are either disjoint or equal, so we can use the same cut-and-paste arguments without distorting our boundary conditions. This gives the following theorem.

Theorem 4.1.7. *$f_i : S_i \rightarrow M$ ($i = 1, \dots, n$) are pairwise non-isotopic and pairwise disjoint incompressible surfaces, and $g_i : S_i \rightarrow M$ ($i = 1, \dots, n$) are relative least area minimizers in the isotopy classes of f_i . Then $g_1(S_1), \dots, g_n(S_n)$ are also pairwise disjoint.*

Next, we take M to be diffeomorphic to the exterior of a knot in $\mathbb{S}^3$, $\mathcal{J}$ to be a foliation of ∂M by preferred longitudes, and S to be a surface with a single boundary component.

We fix $n \in \mathbb{N}$. Let $\mathcal{L} \subset \mathcal{M}^n$ be the subset of n -tuples represented by parameterized minimal genus Seifert surfaces $(f_1, \dots, f_n)$ such that:

- $d_K([f_i(S)], [f_{i+1}(S)]) = 1$ for i modulo n , d_K is the Kakimizu distance and $[f_k(S)] = v_k$ represents a vertex in the Kakimizu complex.
- A loop in the 1-skeleton of the Kakimizu complex which can be represented by the vertices $v_1, \dots, v_n$ is homotopically non-trivial in the Kakimizu complex.

From Corollary 4.1.4, we can conclude that $\mathcal{L}$ is a closed subset of $\mathcal{M}^n$.

The area functional $\mathcal{A} : \mathcal{M}^n \rightarrow \mathbb{R}_+$ is set as:

$$\mathcal{A}(f_1, \dots, f_n) = \sum_{i=1}^n A(f_i(S))$$

Corollary 4.1.8. $\mathcal{A}|_{\mathcal{L}}$ attains some minimum value.

Proof. $\mathcal{A}$ is continuous and positive as each $A : \mathcal{M} \rightarrow \mathbb{R}_+$ is. Proposition 4.1.3 shows that $\mathcal{A}$ is proper. $\mathcal{L}$ is closed in $\mathcal{M}^n$, so $\mathcal{A}|_{\mathcal{L}}$ is proper as well. Hence, $\mathcal{A}|_{\mathcal{L}}$ attains its minimum by the extreme value theorem. $\square$

Now we introduce complexity for a path in the Kakimizu complex of a knot.

Definition 4.1.9. For a path $v^1, \dots, v^n$ in the Kakimizu complex, its **complexity** is defined as (n, a) , where

$$a = A([S^1]) + \dots + A([S^n])$$

for representatives $S^1, \dots, S^n$ of $v^1, \dots, v^n$ respectively.

When $n = 1$, the complexity of the vertex v^1 is denoted by $c(v^1)$.

This allows us to restate Corollary 4.1.8 in the following more useful way.

Theorem 4.1.10. Let there be a homotopically non-trivial loop of length $n > 0$ in the Kakimizu complex of a knot. Then there must be a homotopically non-trivial loop of the same length n having smallest complexity.

Lastly, we describe a standard trick used in minimal surface theory, called the “exchange-round-off”. It refers to cut-and-paste along a pair of transverse least area surfaces to yield a pair of lower least area surfaces. See Figure 4.1.

4.2 Key lemmas

In this section, we will prove some useful lemmas that are used throughout this chapter.

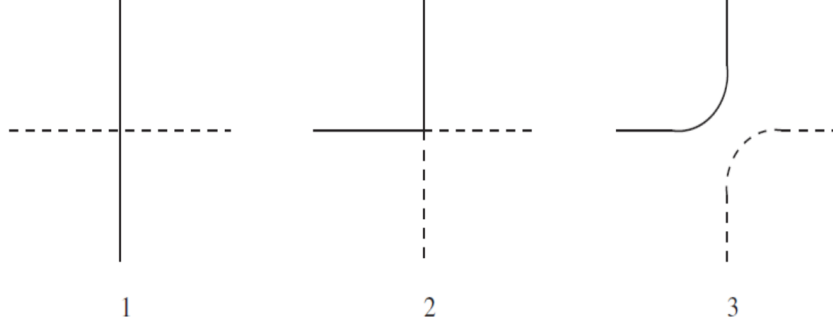


Figure 4.1: (1) Cross-section of the surfaces before exchange. (2) Cross-section of the surfaces after exchange. (3) Cross-section of the surfaces after round-off.

Lemma 4.2.1. *Let K be a knot in $\mathbb{S}^3$, and S, S^{+1}, S^{-1} be minimal genus Seifert surfaces for K satisfying the following:*

1. $S \cap (S^{+1} \cup S^{-1}) = \emptyset$.
2. S^{-1}, S^{+1} are not disjoint and their intersection is transverse, but their boundaries are disjoint.
3. There are no disc components in $S^{-1} \Delta S^{+1}$.

Then there are two minimal genus Seifert surfaces S^{up} and S^{down} for K such that:

- a. $S \cap S^{up} = \emptyset$; $S \cap S^{down} = \emptyset$; $(S^{+1} \cup S^{-1}) \cap S^{up} = \emptyset$; $(S^{+1} \cup S^{-1}) \cap S^{down} = \emptyset$; $S^{down} \cap S^{up} = \emptyset$.
- b. If F is a surface disjoint from $S^{+1} \cup S^{-1}$, then it must be disjoint from S^{down} and S^{up} .

In addition, if $E(K)$ is endowed with a relative Riemannian metric and S^{+1}, S^{-1} are relative least area surfaces, then:

- c. $A(S^{-1}) + A(S^{+1}) > A([S^{down}]) + A([S^{up}])$.

Proof. We consider the lift of $E(K)$ to its infinite cyclic covering space $M(K)$, primarily in the component C that lies between S_0 and S_1 , as shown in Figure (4.2). We denote the lifts

of S^{+1} , S^{-1} in C as S_0^{+1} , S_0^{-1} respectively. The components of $M(K) \setminus S_0^{-1}$ are called M_-^{-1} , M_+^{-1} , where M_+^{-1} is the component containing S_1 . The components of $M(K) \setminus S_0^{+1}$ are called M_-^{+1} , M_+^{+1} , where M_+^{+1} is the component containing S_1 . We define $\tilde{B}^{down}$ as $fr(M_-^{-1} \cap M_-^{+1})$, and $\tilde{B}^{up}$ as $fr(M_+^{-1} \cap M_+^{+1})$ (where $fr(H) := \overline{H} \setminus int H$ denotes the frontier of a subset H). $\tilde{T}^{down}$ and $\tilde{T}^{up}$ are respectively small pushoffs of $\tilde{B}^{down}$ into $M_-^{-1} \cap M_-^{+1}$ and of $\tilde{B}^{up}$ into $M_+^{-1} \cap M_+^{+1}$; T^{down} and T^{up} are their homeomorphic projections to $E(K)$ respectively. See Figure 4.3. These contain a Seifert surface each, which we call S^{down} and S^{up} . These Seifert surfaces satisfy the properties required of them in the lemma.

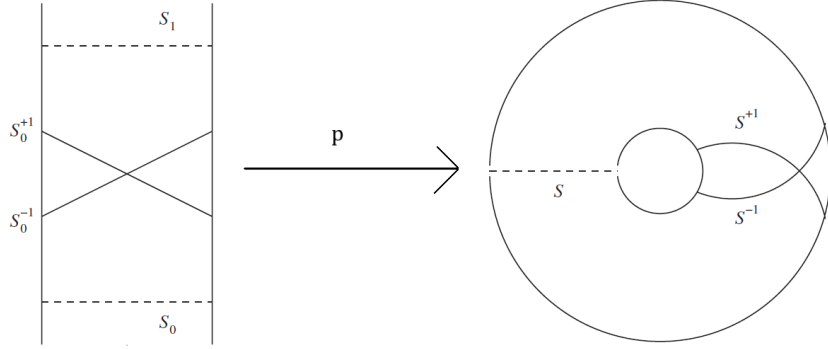


Figure 4.2: Covering space (courtesy of Jennifer Schultens, used with permission)

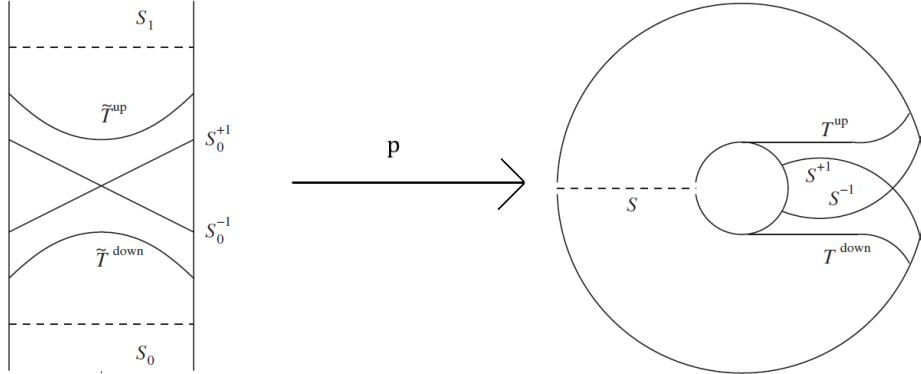


Figure 4.3: (courtesy of Jennifer Schultens, used with permission)

□

Definition 4.2.2. We say the surfaces S_1 and S_2 are in **general position** if their intersection is transverse and their boundaries in $\partial E(K)$ are disjoint.

Lemma 4.2.3. *Let K be a knot in $\mathbb{S}^3$. Let v , v^{+1} and v^{-1} be vertices in the Kakimizu complex of K such that:*

1. $d_K(v, v^{+1}) = 1$ and $d_K(v, v^{-1}) = 1$
2. $d_K(v^{-1}, v^{+1}) = 2$
3. *The complexity of v is not less than the complexity of v^{+1} , v^{-1} .*

Then there are vertices v^{down} , v^{up} in the Kakimizu complex of K satisfying:

- a. $d_K(v, v^{down}) \leq 1$; $d_K(v, v^{up}) \leq 1$; $d_K(v^{+1}, v^{down}) \leq 1$; $d_K(v^{-1}, v^{down}) \leq 1$; $d_K(v^{+1}, v^{up}) \leq 1$; $d_K(v^{-1}, v^{up}) \leq 1$; $d_K(v^{up}, v^{down}) \leq 1$.
- b. *If we have a vertex w such that*

$$d_K(w, v^{+1}) = 1; d_K(w, v^{-1}) = 1,$$

then

$$d_K(w, v^{down}) \leq 1; d_K(w, v^{up}) \leq 1.$$

- c. *The complexity of either v^{up} or v^{down} is strictly less than that of v .*

Proof. Say S^{-1} and S^{+1} are representatives of v^{-1} and v^{+1} respectively. We use the Lemma 4.2.1 to obtain the Seifert surfaces S^{up} and S^{down} , which correspond to vertices v^{up} and v^{down} in the Kakimizu complex and satisfy the required properties in the statement of Lemma 4.2.3. $\square$

4.3 The Kakimizu complex is flag

In this section, we define the notion of a flag simplicial complex, and then show that the Kakimizu complex is flag.

Definition 4.3.1. *A simplicial complex is said to be **flag** if it contains no empty simplices, ie, it contains an n -simplex whenever it contains its $(n - 1)$ -skeleton.*

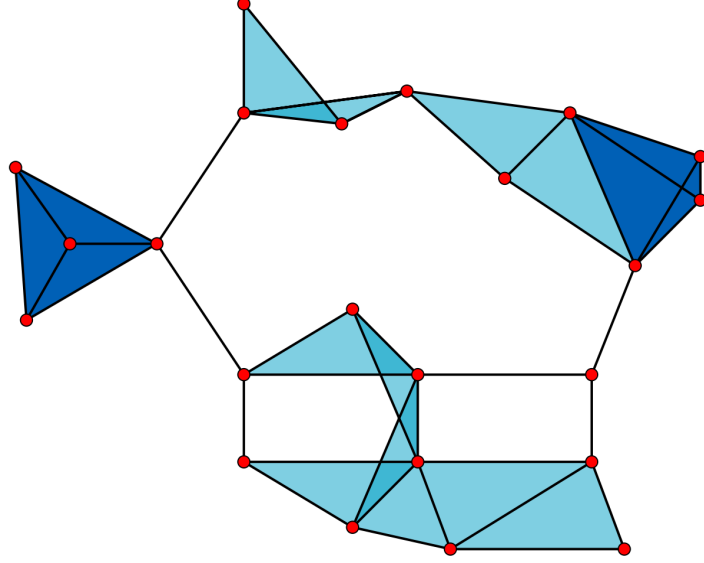


Figure 4.4: Flag simplicial complex (sourced from the internet)

Theorem 4.3.2. *The Kakimizu complex of a knot is flag.*

Proof. Let K be a knot. We take $E(K)$ to be endowed with a relative Riemannian metric.

Say the Kakimizu complex for K contains the 1-skeleton of a simplex σ . Then any pair of vertices in σ has disjoint minimal genus Seifert surfaces representing them. Least area representatives are chosen for the vertices. From Theorem 4.1.7, we know that these representatives are all simultaneously disjoint and span the simplex. Hence, σ is in the Kakimizu complex. $\square$

4.4 The Kakimizu complex is simply connected

In this section, we show that the Kakimizu complex is simply connected.

Remark 4.4.1. *Paths and loops in the Kakimizu complex can traverse vertices and edges, but not higher dimensional simplices. A path is specified by mentioning the vertices it traverses.*

Theorem 4.4.2. *Let K be a knot in $\mathbb{S}^3$. The Kakimizu complex of K is simply connected.*

Proof. Let $v^1, \dots, v^n$ be the vertices in a loop in the Kakimizu complex.

If $d_K(v^{i-1}, v^{i+1}) = 1$ for some i , then $v^1, \dots, v^n$ is homotopic to a shorter loop, as v^{i-1}, v^i, v^{i+1} will bound a 2-simplex and be homotopic to a single vertex. This gives a shorter loop $v^1, \dots, v^{i-1}, v^{i+1}, \dots, v^n$. Hence, we always assume that $d_K(v^{i-1}, v^{i+1}) = 2$.

Further, if $c(v^i) \geq \max\{c(v^{i-1}), c(v^{i+1})\}$, then by Lemma 4.2.3, the complexity of the loop $v^1, \dots, v^n$ is not minimal among the loops homotopic to $v^1, \dots, v^n$.

If there is a homotopically non-trivial loop in the Kakimizu complex, there must be one with lowest complexity for the same length. We call this $v^1, \dots, v^n$.

Then

$$c(v^i) < \max\{c(v^{i-1}), c(v^{i+1})\} \forall i.$$

This is not possible in a loop. Hence, there cannot be a homotopically non-trivial loop in the Kakimizu complex of K , ie, the Kakimizu complex of K must be simply connected. $\square$

4.5 Contractibility of 2-dimensional Kakimizu complexes

In this section, we prove that a 2-dimensional Kakimizu complex is contractible. This proof uses the fact that the Kakimizu complex is flag. A general proof of contractibility, which uses more sophisticated tools, is the subject of the next chapter.

Definition 4.5.1. *The union of all those simplices disjoint from the vertex v in the simplicial complex X which together with v span a simplex is called the **link** of v and is denoted by X_v .*

Definition 4.5.2. *The union of all those simplices in a simplicial complex X that contain the vertex v is called the **star** of v .*

Definition 4.5.3. *A 2-dimensional simplicial complex X is said to be **locally k -large** if for every vertex v in X , every homotopically non-trivial loop in X_v has length at least k .*

Propositions II.4.1 and II.5.25 in [BH99] give the following theorem.

Theorem 4.5.4. *The universal cover of a 2-dimensional connected locally 6-large simplicial complex is contractible.*

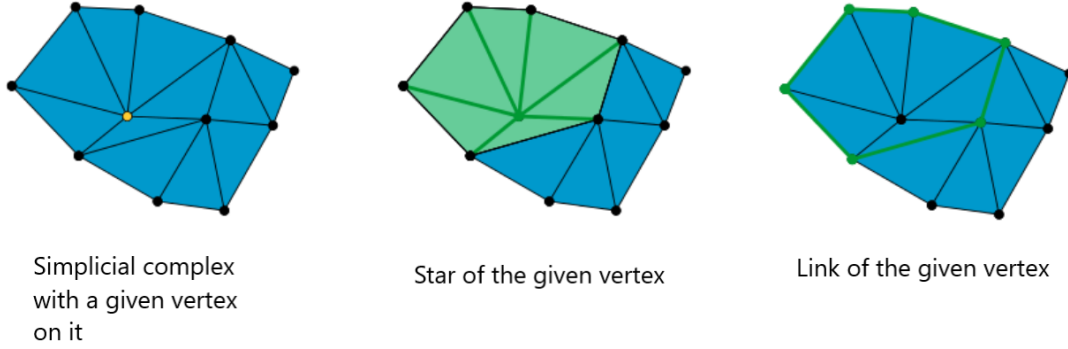


Figure 4.5: Star and link of a vertex (sourced from the internet)

Lemma 4.5.5. *A homotopically non-trivial loop in the link of a vertex in the Kakimizu complex must have length at least 5.*

Proof. As the Kakimizu complex is flag, so loops of length 3 bound 2-simplices. Hence, we only need to check the lemma for loops of length 4.

Let $v^1 v^2 v^3 v^4$ be a loop. Then from the proof of Theorem 4.4.2, we know that $d_K(v^{i-1}, v^{i+1}) = 2 \forall 1 \leq i \leq 4$.

The proof of Theorem 4.4.2 also implies that there is a loop homotopic to $v^1 v^2 v^3 v^4$ with lesser complexity. Applying Lemma 4.2.3 to the vertices v^i, v^{i+1}, v^{i-1} gives the vertices v^{up}, v^{down} . Say v^{up} replaces one of $v^1, \dots, v^4$ to obtain the loop homotopic to $v^1 v^2 v^3 v^4$ with lower complexity. Then it must lie in the star of v , ie, v^{up} either lies in the link of v or it is equal to v .

When v^{up} is in the link of v , we get the situation in Figure 4.6 and the loop is homotopically trivial.

So we analyse the cases when $v^{up} = v$.

Case I: $v^{up} = v, v^{down} \neq v$.

If v^{down} is one of $v^1, \dots, v^4$, say v^j , we have the situation in Figure 4.7.

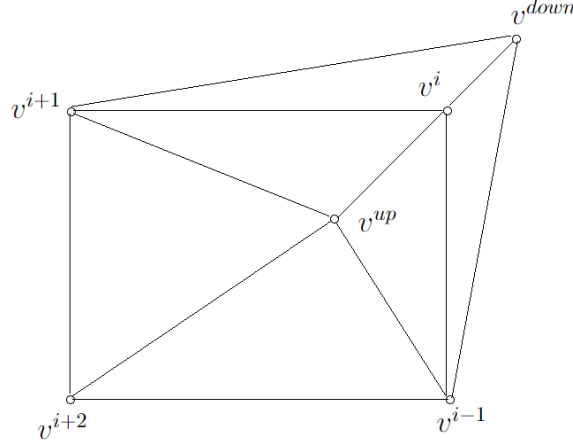


Figure 4.6:

In this, $v^{j-1}v^{down}v^{j+1}v^{j+2}$ is homotopic to $v^{j-1}v^{up}v^{j+1}v^{j+2}$, which is 2 loops of length 3. These must bound 2-simplices as the Kakimizu complex is flag. Thus, the loop is homotopically trivial.

Next, we assume that $v^{down} \neq v^j \forall 1 \leq j \leq 4$. This gives the situation in Figure 4.8.

As in the previous case, $v^{i-1}v^i v^{down}$ and $v^i v^{down} v^{i+1}$ are loops of length 3, bounding 2-simplices. So, $v^{i-1}v^i v^{i+1} v^{i+2}$ is homotopic to $v^{i-1}v^{down} v^{i+1} v^{i+2}$ in X_v . But $v^{down} v^{i+1} v^{i+2}$ and $v^{i+2} v^{i-1} v^{down}$ also bound 2-simplices. So $v^{i-1}v^{down} v^{i+1} v^{i+2}$ is homotopically trivial in X_v , and so is $v^{i-1}v^i v^{i+1} v^{i+2}$.

Case II: $v^{up} = v^{down} = v$.

F is chosen to be a representative of v , F_0 is a lift of F to $M(K)$ and $F_1 = \tau(F_0)$. C is the component of $M(K) \setminus (F_0 \cup F_1)$ that lies between F_0 and F_1 , and S_0^j is the lift of S^j to C . $\tilde{S}^{up}, \tilde{S}^{down}$ are the lifts of S^{up}, S^{down} to C . Further F, F_0, F_1 and each S_0^j are assumed to all be least area surfaces in their respective isotopy classes.

We denote the components of $M(K) \setminus S_0^j$ as:

- $M_-^j :=$ component containing F_0 .
- $M_+^j :=$ component containing F_1 .

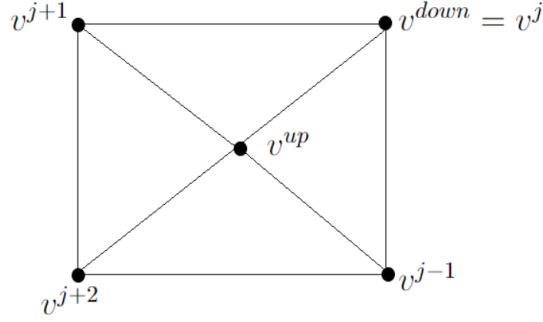


Figure 4.7: (courtesy of Jennifer Schultens, used with permission)

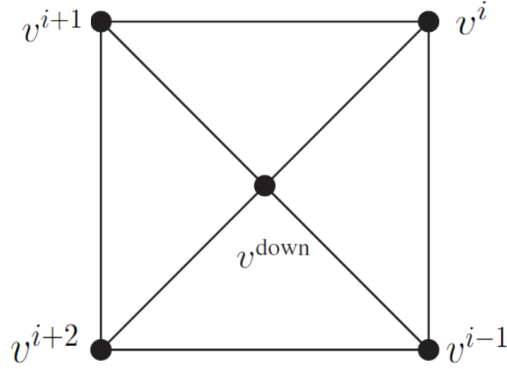


Figure 4.8: (courtesy of Jennifer Schultens, used with permission)

From construction, we get that $\tilde{S}^{up} \subset \tilde{T}^{up}$ and $\tilde{S}^{down} \subset \tilde{T}^{down}$. $\tilde{T}^{up}$ lies above S_0^{i-1} and S_0^{i+1} , ie, $\tilde{T}^{up} \in \text{int } M_+^{i-1}, \text{int } M_+^{i+1}$. On the other hand, $\tilde{T}^{down}$ lies below S_0^{i-1} and S_0^{i+1} , ie, $\tilde{T}^{down} \in \text{int } M_-^{i-1}, \text{int } M_-^{i+1}$.

By Theorem 4.1.7, S_0^i, S_0^{i+2} are disjoint from $S_0^{i-1} \cup S_0^{i+1}$. Hence, after isotopy, they are also disjoint from $\tilde{S}^{up} \cup \tilde{S}^{down}$, if necessary. See Figure 4.9.

Claim: Either S_0^i and S_0^{i+2} both lie below $\tilde{S}^{down}$ or both lie above $\tilde{S}^{up}$.

As $S_0^i \cup S_0^{i+2} \neq \emptyset$, we only need to exclude the case where $S_0^i \cup S_0^{i+2}$ lies above $\tilde{S}^{down}$ but below $\tilde{S}^{up}$.

$S_0^{i-1} \cap S_0^{i+1} \neq \emptyset$, so they both either lie in M_-^i or in M_+^i .

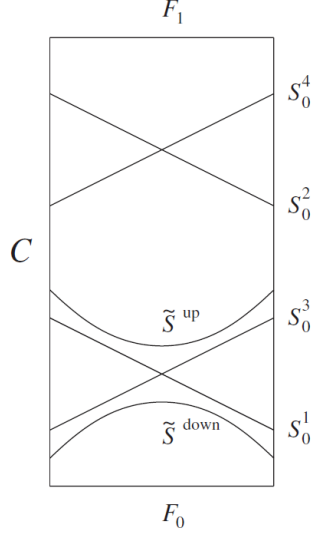


Figure 4.9: (courtesy of Jennifer Schultens, used with permission)

$S_0^{i-1} \cup S_0^{i+1} \subset M_-^i \Rightarrow S_0^i$ lies above $S_0^{i-1} \cup S_0^{i+1}$ and $\tilde{S}^{up}$.
 $S_0^{i-1} \cup S_0^{i+1} \subset M_+^i \Rightarrow S_0^i$ lies below $S_0^{i-1} \cup S_0^{i+1}$ and $\tilde{S}^{down}$.

As $v^{up} = v^{down} = v$, $\tilde{S}^{up}$ and $\tilde{S}^{down}$ must be isotopic to F_0 or F_1 . Hence, S^{i-1} and S^{i+1} lie in a product region, and S^{i-1} is isotopic to S^{i+1} . This is a contradiction.

Thus, we cannot have a homotopically non-trivial loop of length 4. $\square$

Lemma 4.5.6. *A homotopically non-trivial loop in the link of a vertex in the Kakimizu complex must have length at least 6.*

Proof. Lemma 4.5.5 says that we only need to show that there are no homotopically non-trivial loops of length 5.

We assume $d_K(v^{i-1}, v^{i+1}) = 2 \forall i$, otherwise the loop is homotopic to a loop of length 4 in X_v , which is trivial by Lemma 4.5.5.

We endow $E(K)$ with a relative Riemannian metric. Let $S^1, \dots, S^5$ be relative least area representatives of $v^1, \dots, v^5$ respectively. Let $S_0^1, \dots, S_0^5$ be lifts of $S^1, \dots, S^5$ to C respectively. Then $S_0^i \cap (S_0^{i+1} \cup S_0^{i-1}) = \emptyset \forall i$ and $S_0^i \cap S_0^j \neq \emptyset \forall j \neq i, i \pm 1$. Further, each S_0^i is separating in C . We also assume that $i = 2$. See Figure 4.10.

$S_0^2 \cap S_0^5 \neq \emptyset$, so they must both either lie above or below S_0^1 . Suppose they lie above S_0^1 .

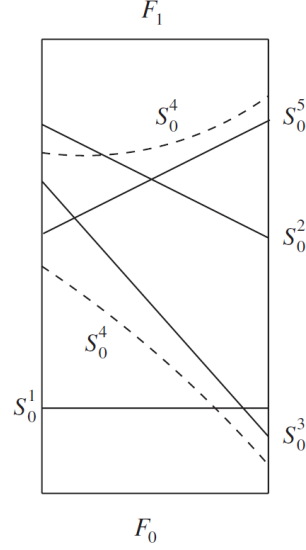


Figure 4.10: (courtesy of Jennifer Schultens, used with permission)

$S_0^3 \cap S_0^1 \neq \emptyset$, so S_0^3 lies below S_0^2 . We must have $S_0^4 \cap S_0^2 \neq \emptyset$, $S_0^4 \cap S_0^1 \neq \emptyset$, $S_0^4 \cap S_0^3 = \emptyset$, $S_0^4 \cap S_0^5 = \emptyset$. So S_0^4 must lie above S_0^3 to have intersection with S_0^2 , and must lie below S_0^5 to have intersection with S_0^1 . But $S_0^5 \cap S_0^3 \neq \emptyset$, leading to a contradiction. Hence, we cannot have a homotopically non-trivial loop of length 5. $\square$

Theorem 4.5.7. *If the Kakimizu complex of a knot is two dimensional, then it is contractible.*

Proof. Lemmas 4.5.5 and 4.5.6 together imply that the Kakimizu complex satisfies the conditions in Theorem 4.5.4. Hence, the universal cover of a two-dimensional Kakimizu complex is contractible. But the Kakimizu complex is simply connected, so its universal cover is itself. Thus, a two-dimensional Kakimizu complex is contractible. $\square$

Chapter 5

Contractibility of the Kakimizu complex

In this chapter, we present a general proof that shows the Kakimizu complex for a knot is contractible. The content of this chapter is taken from [PS12], which one can look at for detailed proofs of the results.

5.1 Redefining the Kakimizu complex

Definition 5.1.1. *Let M be a differential manifold and $c \in H_2(M)$. Then c can be represented by a smooth embedding $S \rightarrow M$, where S is a compact (but not necessarily connected) surface without boundary. We define $\chi_-(S) = \max \{0, -\chi(S)\}$. The **Thurston norm** of c is defined as*

$$||c||_T = \min_S \sum_{i=1}^n \chi_-(S_i)$$

where the minimum is taken over all embedded surfaces $S = \bigcup_i S_i$ (S_i being the connected components) representing c .

We now redefine the Kakimizu complex in a way that makes the proof of contractibility easier.

Definition 5.1.2. *Let E be a compact, connected, orientable, irreducible and ∂ -irreducible 3-manifold, γ be a union of oriented, disjoint simple closed curves on ∂E which do not separate any component of ∂E , and $\alpha \in H_2(E, \partial E, \mathbb{Z})$ which satisfies $\partial\alpha = [\gamma]$. Then $\mathbf{MS}(E, \gamma, \alpha)$ is defined as follows:*

- *The vertex set of $\mathbf{MS}(E, \gamma, \alpha)$ is $\mathcal{MS}(E, \gamma, \alpha)$, the set of isotopy classes of spanning surfaces with minimal Thurston norm.*
- *We span an edge on $\sigma, \sigma' \in \mathcal{MS}(E, \gamma, \alpha)$ only if they have representatives $S \in \sigma, S' \in \sigma'$ such that the (connected) lift of $E \setminus S'$ to the infinite cyclic cover associated with α intersects exactly two lifts of $E \setminus S$, ie, the Kakimizu distance between σ, σ' is exactly 1.*

The second point may not always hold for disjoint $S \in \sigma, S' \in \sigma'$ as they may be disconnected.

Unless stated otherwise, we will use the following notation throughout the rest of this chapter.

$p : \tilde{E} \rightarrow E$ is the infinite cyclic covering map associated with the kernel of the element of $H^1(E, \mathbb{Z})$ dual to α .

τ is the generator of the group of deck transformations of $\tilde{E}$. For some Thurston norm minimizing spanning surface $S \subseteq E$, E_0 is a lift of $E \setminus S$ to $\tilde{E}$ and $E_j = \tau^j(E_0)$ for $j \in \mathbb{Z}$. We set $S_j = \overline{E_{j-1}} \cap \overline{E_j}$.

For some other Thurston norm minimizing surface R , we take E^R to be some lift of $E \setminus R$ to $\tilde{E}$.

5.2 Simplified Intersection

In this section, we talk about simplified intersections between surfaces and a few other related concepts. This will help us in picking the right representatives to proceed with our proof of contractibility. The terms in this section are taken from [Oer88].

Unless stated otherwise, we will assume that S, R are compact surfaces properly embedded in a 3-manifold M without boundary.

Definition 5.2.1. Let β be an arc and $\mathfrak{J} = \beta \times I$ such that for any $x \in \partial\beta$, $\{x\} \times I$ is collapsed to a point. A **product region in ∂M** is an embedded copy of $\mathfrak{J}$ with $\beta \times \{0\} \subseteq S$, $\beta \times \{1\} \subseteq R$ and $\mathfrak{J} \cap (S \cup R) = \partial\mathfrak{J}$.

Definition 5.2.2. Let W be a compact surface with boundary, δ be a closed 1-submanifold of ∂W and $\mathfrak{J}$ be the product $W \times I$ with intervals $\{x\} \times I$ collapsed to points for $x \in \delta$. A **product region in M** (also called **blister**) is an embedded copy of $\mathfrak{J}$ with $W \times \{0\} \subseteq S$, $W \times \{1\} \subseteq R$ and $\mathfrak{J} \cap (S \cup R) = \partial\mathfrak{J} \setminus \text{int}\mathfrak{J} \cap \partial M$.

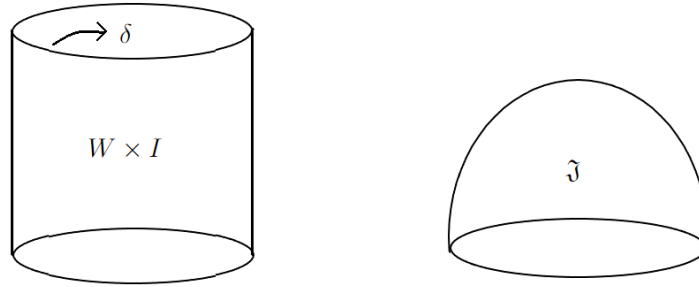


Figure 5.1: Product Region

Definition 5.2.3. The surfaces S , R are said to have **simplified intersection** if they do not bound any product region. In particular, if a component $\dot{S}$ of S is isotopic to a component $\dot{R}$ of R , then we must have $\dot{S} = \dot{R}$.

Definition 5.2.4. We say the surfaces S , R are **almost transverse** if for any component $\dot{S}$ of S and $\dot{R}$ of R , either $\dot{S} = \dot{R}$ or they intersect transversely.

If $S = R$, they are almost transverse.

Definition 5.2.5. S , R are almost disjoint if for intersecting components $\dot{S}$ of S and $\dot{R}$ of R , we have $\dot{S} = \dot{R}$.

S is almost disjoint from itself.

The following proposition is stated without proof from [Sak94].

Proposition 5.2.6. *M is a (possibly non-compact) orientable, irreducible and ∂ -irreducible 3-manifold, and W, N are (possibly non-compact) proper 3-submanifolds of M such that $\partial W, \partial N$ are almost transverse, incompressible and ∂ -incompressible surfaces with simplified intersection. If N is isotopic to a submanifold N' such that $\text{int}(N'') \cap W = \emptyset$, then $\text{int}(N) \cap W = \emptyset$ also.*

Corollary 5.2.7. *Let W, N be proper 3-manifolds of $\tilde{E}$ such that $\partial W, \partial N$ are unions of lifts of Thurston norm minimizing almost transverse spanning surfaces having simplified intersection. If N is isotopic to N' such that $\text{int}(N'') \cap W = \emptyset$, then $\text{int}(N) \cap W = \emptyset$ also.*

Using the above, we prove a useful proposition about representatives for isotopy classes of spanning surfaces.

Proposition 5.2.8. *Let S, R be almost transverse spanning surfaces with simplified intersection in E , representing $\sigma, \rho \in \mathcal{MS}(E, \gamma, \alpha)$. Then they satisfy*

$$d(\sigma, \rho) = d(S, R).$$

Proof. Let R' be a spanning surface in $[R] = \rho$ which minimizes distance from S . Then $d(\sigma, \rho) = d(S, R') = r' - m'$, where r' and m' are respectively the maximum and minimum values of k such that E_k intersects $E^{R'}$. This means $E^{R'}$ is disjoint from all E_j with $j \geq r' + 1$ or $j \leq m' - 1$.

Now we consider E^R , the lift of $E \setminus R$ to $\tilde{E}$; it is isotopic to $E^{R'}$. Further, R and S have simplified intersection, so the lifts of R have simplified intersections with those of S . Then by Corollary 5.2.7, $E^R \cap E_j = \emptyset \forall j \geq r' + 1$ or $j \leq m' - 1$, which means we must have $r' + 1 \geq r + 1$ and $m' - 1 \leq m - 1$, ie, $r' \geq r$ and $m' \leq m$. This means $r - m \leq r' - m'$, ie, $d(S, R) \leq d(S, R')$, ie, $d(S, R) = d(S, R')$ and $R = R'$ as R' minimizes the Kakimizu distance. $\square$

5.3 Projection Map

In this section, we define the notion of a projection map, which takes each vertex to an adjacent vertex.

Definition 5.3.1. We fix a basepoint $\sigma \in \mathcal{MS}(E, \gamma, \alpha)$. The **projection map** π_σ maps every $\rho \in \mathcal{MS}(E, \gamma, \alpha)$ at a distance $n > 0$ from σ to a vertex $\pi_\sigma(\rho) \in \mathcal{MS}(E, \gamma, \alpha)$ which is adjacent to ρ and is at a distance $n - 1$ away from σ .

Definition 5.3.2. S, R have **almost disjoint boundaries** if their boundary components are either disjoint or they are equal.

The following theorem is stated without proof from [Oer88].

Theorem 5.3.3. Let M be an orientable, irreducible, ∂ -irreducible 3-manifold and S, R be orientable, incompressible, ∂ -incompressible surfaces properly embedded in M . We assume that S, R are almost transverse with simplified intersection and are isotopic to S', R' respectively, which are also almost transverse with simplified intersection. If cut-and-paste on S, R yields an orientable, incompressible, ∂ -incompressible surface P , then a corresponding cut-and-paste operation exists on S', R' to yield a surface P' isotopic to P .

Now we will try to obtain a surface that satisfies the requirements for being a representative of the image of a vertex under the projection map.

Let $\sigma \neq \rho$ be vertices of $\mathcal{MS}(E, \gamma, \alpha)$. We fix a spanning surface $S \in \sigma$ and choose $R \in \rho$ so that S, R are almost transverse with simplified intersection. If $d(\sigma, \rho) = n$, then by Proposition 5.2.8, $d(S, R) = n$.

r is defined as the largest integer such that $E_r \cap E^R \neq \emptyset$. We define $\tilde{R} = \overline{E}^R \cap \tau(\overline{E}^R)$. Let $\tilde{P}_* \subseteq S_r \cup \tilde{R}$ be the surface obtained by performing cut-and-paste on S_r and $\tilde{R}$.

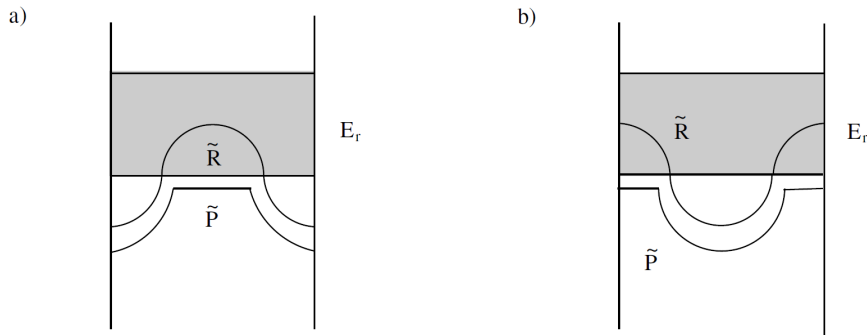


Figure 5.2: Constructing $\tilde{P}$ (courtesy of Jennifer Schultens, used with permission)

We set $P_* = p(\tilde{P}_*)$. Removing any torus components from $\tilde{P}_*$ gives us the surface $\tilde{P}$, which

projects down to $P = p(\tilde{P})$. We can show that this surface P satisfies all the required conditions to be a representative of $\pi_\sigma(\rho)$.

Theorem 5.3.4. $[P]$ is not dependent on the choice of S and R .

Proof. We fix $S \in \sigma$ and choose $R, R' \in \rho$ which are almost transverse to S with simplified intersection.

Say $\tilde{P}_*$ is obtained by cut-and-paste on $\tilde{R}$ and S_r , where r is the largest integer such that E^R intersects E_r . Then Corollary 5.2.7 says that r is also the largest integer for which $E^{R'} \cap E_r \neq \emptyset$. We take $\tilde{P}'$ to be the surface obtained by cut-and-paste along $\tilde{R}'$ and S_r . Theorem 5.3.3 says that $\tilde{P}'$ is isotopic to $\tilde{P}_*$. This implies that $\tilde{P}'$ is isotopic to $\tilde{P}$, and so are their images under the projection map. $\square$

5.4 Order on the vertices

In this section, we develop a notion of order on the vertices of the Kakimizu complex.

We fix $\sigma \in \mathcal{MS}(E, \gamma, \alpha)$. Let $\rho, \rho' \in \mathcal{MS}(E, \gamma, \alpha)$ be two adjacent vertices. We choose the almost disjoint surfaces $R \in \rho, R' \in \rho'$ that are almost transverse to a fixed $S \in \sigma$ and have simplified intersection with S and with each other.

E_0 is the lift of $E \setminus S$ to $\tilde{E}$ and $E_k = \tau^k(E_0)$. $E^{R'}$ is a lift of $E \setminus R'$ to $\tilde{E}$. r' is the largest integer such that $E_{r'} \cap E^{R'} \neq \emptyset$. E^R is the lift of $E \setminus R$ contained in $\overline{E}^{R'} \cup \tau^{-1}(\overline{E}^{R'})$.

Definition 5.4.1. If E^R intersects $E_{r'}$, then we say that $\rho <_\sigma \rho'$ (see Figure 5.3). We write $\rho \leq_\sigma \rho'$ if $\rho <_\sigma \rho'$ or $\rho' = \rho$.

Lemma 5.4.2. Let $\rho \neq \rho'$ be adjacent vertices of $\mathcal{MS}(E, \gamma, \alpha)$ and consider any $\sigma \in \mathcal{MS}(E, \gamma, \alpha)$. Then either $\rho' <_\sigma \rho$ or $\rho <_\sigma \rho'$.

Proof. We assume that $\rho \not<_\sigma \rho'$, ie, $E^R \cap E_{r'} = \emptyset$. Taking ρ' in place of ρ , we see that $r - 1$ is the largest integer such that E_{r-1} intersects E^R . $\tau^{-1}(E^{R'}) \subset \overline{E}^R \cup \tau^{-1}(\overline{E}^R)$. As $E^{R'}$ intersects $E_{r'}$, so $\tau^{-1}(E^{R'})$ intersects E_{r-1} . This gives $\rho' <_\sigma \rho$. $\square$

Lemma 5.4.3. If the vertex ρ is farther from σ than ρ' , then $\rho <_\sigma \rho'$.

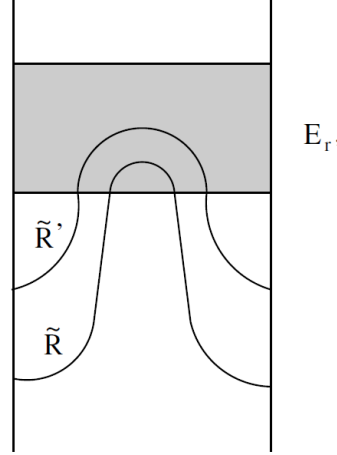


Figure 5.3: Ordering on vertices: $[\rho] <_{\sigma} [\rho']$ (courtesy of Jennifer Schultens, used with permission)

Proof.

$$E^R \subseteq \overline{E}^{R'} \cup \tau^{-1}(\overline{E}^{R'})$$

$$\therefore E^R \cap E_k \neq \emptyset$$

$$\Rightarrow m' - 1 \leq k \leq r'$$

$$\Rightarrow d(S, R) = d(S, R') + 1$$

Thus, E^R must intersect all the above E_k , ie, $E^R \cap E_{r'} \neq \emptyset$. This proves the lemma. $\square$

Lemma 5.4.4. *There are no $\rho^1, \rho^2, \dots, \rho^k$ for $k \geq 2$ satisfying*

$$\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^k <_{\sigma} \rho^1$$

Proof. We inductively choose $R^k \in \rho^k$, $R^{k-1} \in \rho^{k-1}$, $\dots$, $R^1 \in \rho^1$ such that:

- Each R^i is almost transverse to S and has simplified intersection with S .
- $\forall k > i$, R^i is almost disjoint from R^{i+1} with simplified intersection. We take $r := \max \{n \in \mathbb{Z} \mid E_n \cap E^{R^k} \neq \emptyset\}$.

We inductively define E^{R^i} as the lift of $E \setminus R^i$ which is contained in $(\overline{E}^{R^{i+1}}) \cup \tau^{-1}(\overline{E}^{R^{i+1}})$. Then all the E^{R^i} intersect E_r as $\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^k$. $R^* \in \rho^k$ is chosen to be almost

transverse to S with simplified intersection, and almost disjoint from R^1 with simplified intersection. E^{R^*} is the lift of $E \setminus R^*$ such that $E^{R^*} \subseteq \overline{E}^{R^1} \cup \tau^{-1}(\overline{E}^{R^1})$. $\rho^k <_\sigma \rho^1$, so E^{R^*} intersects E_r . Then Corollary 5.2.7 says that E^{R^*} and E^{R^1} are in the same isotopy class. This means $\overline{E}^{R^*} \cap \tau(\overline{E}^{R^*})$ and $\overline{E}^{R^1} \cap \tau(\overline{E}^{R^1})$ are almost disjoint, bounding a product region that contains all the $\overline{E}^{R^i} \cap \tau(\overline{E}^{R^i})$. This is a contradiction, as it implies all the ρ^i coincide, which is not true. $\square$

Corollary 5.4.5. *The relation $<_\sigma$ can be extended to give a linear order on $\mathcal{MS}(E, \gamma, \alpha)$.*

5.5 Properties of the Projection Map

In this section, we state and prove some useful properties of the projection map.

Lemma 5.5.1. *Let ρ, ρ' be adjacent vertices of $\mathcal{MS}(E, \gamma, \alpha)$ such that ρ is different from some $\sigma \in \mathcal{MS}(E, \gamma, \alpha)$. We assume that $\rho \leq_\sigma \rho'$. Then we have $\rho' \leq_\sigma \pi_\sigma(\rho)$. In particular, $\pi_\sigma(\rho)$ and ρ' are adjacent or equal.*

Proof. We take $S \in \sigma$, $R \in \rho$, $R' \in \rho'$ in $\mathcal{MS}(E, \gamma, \alpha)$. Let E^P be the lift of $E \setminus P$ to $\tilde{E}$ such that $E^P \subseteq \overline{E}^R \cup \tau^{-1}(\overline{E}^R)$. Then $E^{R'} \subseteq \overline{E}^P \cup \tau(\overline{E}^P)$, ie, $\pi_\sigma(\rho)$ and ρ' are either equal or adjacent. See Figure 5.4.

We can isotope P via the isotopy i which makes $i(P)$ almost transverse to S with simplified intersection and almost disjoint from R' with simplified intersection. Further, $E^P \cap E_r = \emptyset$, so after applying Corollary 5.2.7, we can conclude that the lift of $E \setminus i(P)$ in the isotopy class of E^P does not meet E_r . Hence, $\pi_\sigma(\rho) \not\leq_\sigma \rho'$, ie, $\rho' \leq \pi_\sigma(\rho)$. $\square$

Corollary 5.5.2. *Let ρ, ρ' be adjacent vertices of $\mathcal{MS}(E, \gamma, \alpha)$ distinct from some $\sigma \in \mathcal{MS}(E, \gamma, \alpha)$. If $\rho \leq_\sigma \rho'$, then we must have $\pi_\sigma(\rho) \leq_\sigma \pi_\sigma(\rho')$.*

Lemma 5.5.3. *Assume that there are vertices $\rho^1, \dots, \rho^k$ at some non-zero distance from $\sigma \in \mathcal{MS}(E, \gamma, \alpha)$ satisfying*

$$\rho^1 <_\sigma \rho^2 <_\sigma \dots <_\sigma \rho^k$$

and

$$\pi_\sigma(\rho^1) = \pi_\sigma(\rho^k).$$

Then all the $\pi_\sigma(\rho^i)$ are equal and all the ρ^i are pairwise disjoint.

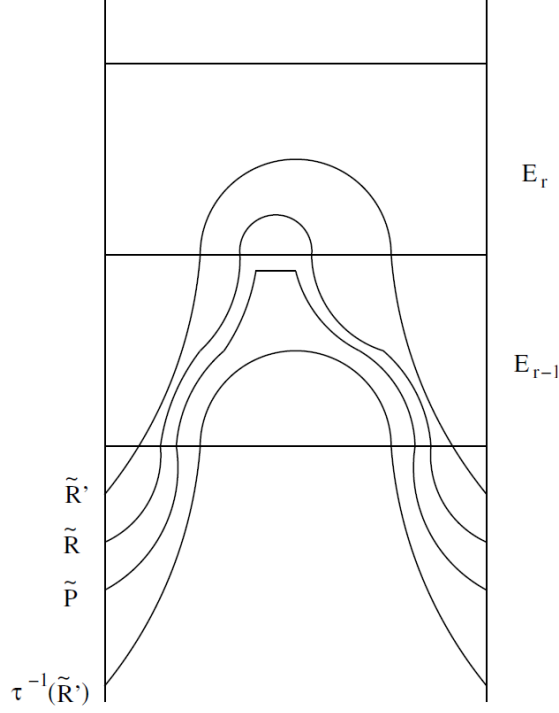


Figure 5.4: Setup for Lemma 5.5.1 (courtesy of Jennifer Schultens, used with permission)

Proof. We are given that

$$\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^k$$

Then from Corollary 5.5.2

$$\pi_{\sigma}(\rho^1) \leq_{\sigma} \pi_{\sigma}(\rho^2) \leq_{\sigma} \dots \leq_{\sigma} \pi_{\sigma}(\rho^k)$$

But

$$\pi_{\sigma}(\rho^1) = \pi_{\sigma}(\rho^k) \Rightarrow \pi_{\sigma}(\rho^1) = \pi_{\sigma}(\rho^2) = \dots = \pi_{\sigma}(\rho^k) \text{ (from Lemma 5.4.4.)}$$

It only remains to prove that all the ρ^i 's are adjacent. For this, we can simply show that ρ^1 and ρ^k are adjacent.

We choose $E^{R^1}, \dots, E^{R^k}$ as in the proof of Lemma 5.4.4, and define $\tilde{P}^1, \tilde{P}^k$ as the surfaces obtained by cut-and-paste, as described previously. See Figure 5.5.

So $\tilde{R}^1$ and $\tilde{R}^k$ are isotopic to $i(\tilde{R}^1)$ and $i(\tilde{R}^k)$ which are almost disjoint and are in the closure

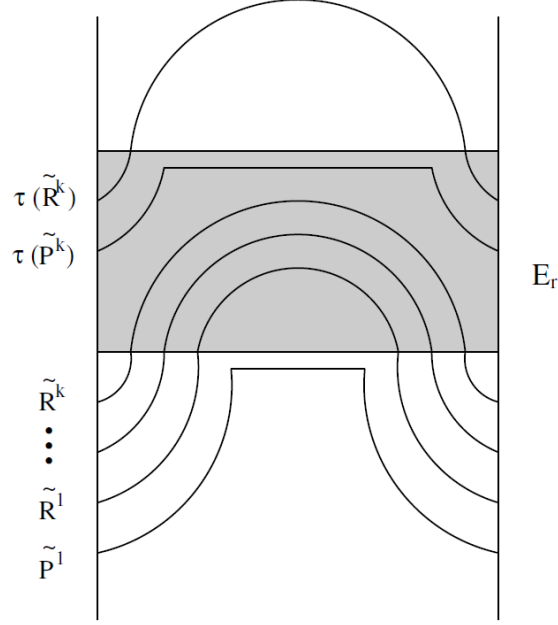


Figure 5.5: Setup for Lemma 5.5.3 (courtesy of Jennifer Schultens, used with permission)

of the lift of $E \setminus P$ bounded by $\tilde{P}_1$ and $\tau(\tilde{P}_1)$. Then

$$d(p(i(\tilde{R}^1)), p(i(\tilde{R}^k))) = 1.$$

□

Lemma 5.5.4. $\forall n > 0$, there is a constant l_n satisfying the following:

Let σ be any vertex of $\mathcal{MS}(E, \gamma, \alpha)$ and let $\rho^1, \dots, \rho^l$ be at a distance n from σ satisfying

$$\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^l.$$

Then $l \leq l_n$.

Proof. Let L denote an upper bound on the dimension of the simplices in $MS(E, \gamma, \alpha)$. We will use induction on n and show that $l_n = L^n$.

For $n = 1$, the result follows from Lemma 5.5.3. We assume that it holds for some $n \geq 1$.

Let $\rho^1, \dots, \rho^l$ be vertices located at a distance $n + 1$ from the vertex σ which satisfy $\rho^1 <_{\sigma} \dots <_{\sigma} \rho^l$. We set $i_0 = 0$, and for $k \geq 1$, we inductively define i_k to be the maximal value

satisfying $\pi_\sigma(\rho^{i_{k-1}+1})$ until $i_m = l$ for some m . Lemma 5.5.3 says that $\forall 1 \leq k \leq m$, $i_k - i_{k-1} \leq L$. On adding, this gives $l = i_m - i_0 \leq mL$. This implies that we need to find a bound on m . Then from Corollary 5.5.2, $\pi_\sigma(\rho^{i_k}) <_\sigma \pi_\sigma(\rho^{i_{k+1}}) \forall 1 \leq k \leq m$.

$$\therefore \pi_\sigma(\rho^{i_1}) <_\sigma (\rho^{i_2}) <_\sigma \dots <_\sigma \pi_\sigma(\rho^{i_m}).$$

All the $\pi_\sigma(\rho^{i_k})$ are at a distance n from σ , so they satisfy the induction hypothesis and $m \leq l_n$. $\square$

Lemma 5.5.5. *For $\sigma \neq \rho$, $\sigma' \in \mathcal{MS}(E, \gamma, \alpha)$ with $d(\sigma', \rho) \leq d$ and $d(\sigma', \sigma) \leq d$, we have $d(\sigma', \pi_\sigma(\rho)) \leq d$.*

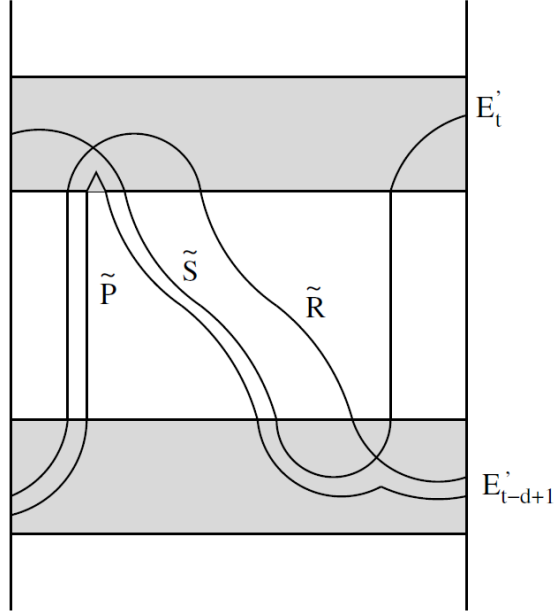


Figure 5.6: Setup for Lemma 5.5.5 (courtesy of Jennifer Schultens, used with permission)

Proof. Let $S \in \sigma$, $R \in \rho$ and $S' \in \sigma'$ be pairwise almost transverse surfaces with simplified intersection. $r, \tilde{P}, P$ are as in Sections 5.1 and 5.2. Let

$$t = \max \{k \in \mathbb{Z} \mid E'_k \cap (E^R \cap E_{r-1}) \neq \emptyset\}.$$

See Figure 5.6. We also have

$$\tilde{P} \subseteq [\tilde{R} \cap (\bigcup_{k \leq r-1} \overline{E}_k)] \cup [\tilde{S} \cap (\bigcup_{i \leq 0} \tau^i(\overline{E}^R))].$$

Thus, E^P is contained in $E^R \cap E_{r-1}$ such that the $\overline{E}'_k$'s satisfy $k \leq t$. $d(S', R) \leq d$ and $d(S', S) \leq d$, so the k 's must satisfy $t - k \leq d$. $\square$

Lemma 5.5.6. *Let $\sigma, \sigma' \in \mathcal{MS}(E, \gamma, \alpha)$ be adjacent. Let $\rho, \rho' \in \mathcal{MS}(E, \gamma, \alpha)$ also be adjacent, satisfying $\rho' <_{\sigma'} \rho$ and $\rho <_{\sigma} \rho'$. If $\sigma' \neq \rho'$, then:*

1. $\rho \leq_{\sigma} \pi_{\sigma'}(\rho')$.
2. If $\sigma \neq \rho'$, then $d(\sigma, \pi_{\sigma'}(\rho')) \leq d(\sigma, \rho')$.

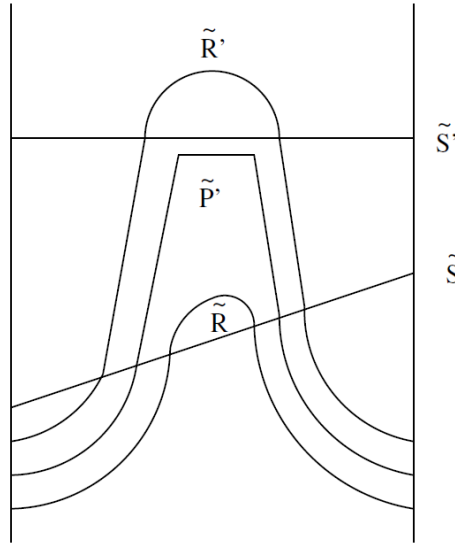


Figure 5.7: Setup for Lemma 5.5.6 (courtesy of Jennifer Schultens, used with permission)

Proof. $S \in \sigma$, $S' \in \sigma'$, $R \in \rho$, $R' \in \rho'$ are taken to be pairwise almost disjoint with simplified intersection. E_0 is a lift of $E \setminus S$ and E'_0 is the lift of $E \setminus S'$ in $\overline{E}_0 \cup \overline{E}_1$. Let r' be the largest integer such that $E'_{r'} = \tau^{r'}(E'_0)$ intersects a lift $E^{R'}$ of $E \setminus R'$. Further, E^R is the lift of $E \setminus R$ contained in $\overline{E}^{R'} \cup \tau^{-1}(\overline{E}^{R'})$. See Figure 5.7.

$\rho' <_{\sigma'} \rho$ and $\rho <_{\sigma} \rho'$. So $E^R \cap E'_{r'} = \emptyset$, but $E^R \cap E_{r'} \neq \emptyset$.

$P' = p(\tilde{P}) \in \pi_{\sigma'}(\rho')$ is as obtained in Section 5.3. $E^{P'}$ is the lift of $E \setminus P'$ bounded by $\tilde{P}'$ and $\tau^{-1}(\tilde{P}')$. $E^R \cap E'_{r'} = \emptyset$, so $\tilde{P}' \subseteq \tau(\overline{E}^R)$. This means that ρ and $\pi_{\sigma'}(\rho')$ must be equal or adjacent.

i is an isotopy of P' such that $i(P')$ is almost transverse to S with simplified intersection, and almost disjoint from R with simplified intersection.

$E^{P'} \cap E_{r'+1} = \emptyset$, so the lift of $E \setminus i(P')$ (containing $\tilde{R}$) in the isotopy class of $E^{P'}$ is disjoint from $E_{r'+1}$. Hence $\tilde{R} \cap E_{r'} \neq \emptyset$, giving (1).

$E^{P'}$ intersects the same E_k as $E^{R'}$, giving (2). $\square$

5.6 Contractibility

Whitehead's theorem implies that to prove contractibility of $\mathcal{MS}(E, \gamma, \alpha)$, it suffices to prove that all finite subcomplexes of $\mathcal{MS}(E, \gamma, \alpha)$ exist in contractible subcomplexes of $\mathcal{MS}(E, \gamma, \alpha)$.

Definition 5.6.1. *A finite graph is called **dismantlable** if its vertices can be linearly ordered $x_0, \dots, x_m$ such that $\forall i \neq m, \exists j > i$ satisfying*

- x_j is adjacent to x_i .
- $\forall x_k$ adjacent to x_i with $k > i$, the vertex x_j is adjacent to or equal to x_k .

The following theorem is taken from [CO15].

Theorem 5.6.2. *Finite flag complexes whose 1-skeleta are dismantlable are contractible.*

Thus to prove contractibility of $\mathcal{MS}(E, \gamma, \alpha)$, it suffices to prove the following theorem.

Theorem 5.6.3. *Finite σ -complexes of $\mathcal{MS}(E, \gamma, \alpha)$ have dismantlable 1-skeleta.*

Proof. We extend the relation $<_\sigma$ to all the vertices. From Lemma 5.5.1, $\forall \rho \neq \sigma, \rho <_\sigma \pi_\sigma(\rho)$. Further, σ is the largest in this order.

For any non-largest x_i , we set $x_j := \pi_\sigma(x_i)$. Then $x_i <_\sigma x_j$ and $j > i$, so we satisfy condition (1).

If x_k be adjacent to x_i for $i < k$, then $x_k <_\sigma x_i$ or $x_i <_\sigma x_k$. But $k > i$, so by construction, $x_i <_\sigma x_k$. Lemma 5.5.1 implies that x_j and x_k are either adjacent or equal. $\square$

Chapter 6

Locally infinite Kakimizu complexes

In [Kak91], Osamu Kakimizu showed that the incompressible complex can be locally infinite. A natural follow-up to this would be whether the Kakimizu complex can be locally infinite.

In this chapter, we will show that the Kakimizu complex for a knot can be locally infinite. We also give a restriction on the types of knots that can have locally infinite Kakimizu complexes.

The content of this chapter is taken from [Ban11], which one can look at for detailed proofs of the results.

6.1 A knot with a locally infinite Kakimizu complex

In this section, we will show that the twisted Whitehead double of the trefoil knot is locally infinite.

Definition 6.1.1. *A **locally finite** simplicial complex is one where each vertex belongs to finitely many simplices.*

*A simplicial complex that is not locally finite is called **locally infinite**.*

We need the concept of sutured manifolds throughout this section.

Definition 6.1.2. A **sutured manifold** (M, γ) is a compact oriented 3-manifold M together with a partition of ∂M into $R_+ \cup_\gamma R_-$, ie, the union of two surfaces R_+ and R_- along their common boundary, a collection of simple closed curves γ called **sutures**. R_+ is oriented so that its normal vector points outward and R_- is such that its normal vector points inwards.

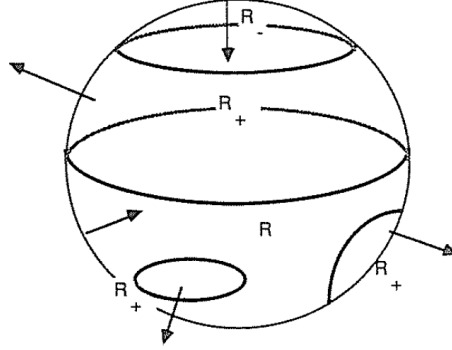


Figure 6.1: An example of a sutured manifold (courtesy of Martin Scharlemann, used with permission)

The following proposition is stated without proof from [Sak94].

Proposition 6.1.3. *Let M be a ∂ -irreducible Haken manifold, and let S, S' be incompressible, ∂ -incompressible surfaces properly embedded in M in general position. If S can be isotoped to be disjoint from S' , then there is a product region between S and S' .*

Using the above concepts, we will prove the first major result of this chapter.

Theorem 6.1.4. *The Kakimizu complex for the twisted Whitehead double of the trefoil knot is not locally finite.*

Proof. We take K_α to be the twisted Whitehead double of the trefoil knot, and R to be a genus 1 spanning surface for K_α as in Figure 6.2.

We take M to be $\mathbb{S}^3 \setminus \text{int}\mathcal{N}(K_\alpha)$, T to be the torus bounding the trefoil knot companion of K_α such that K_α lies in the solid torus bounded by T , M_1 to be the part of M outside T , M_0 to be the part of M inside T and μ to be a meridian of T .

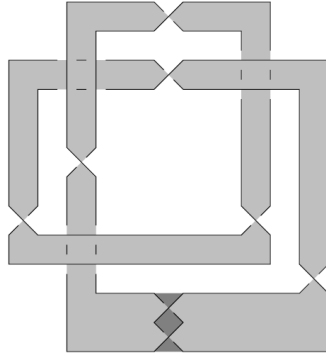


Figure 6.2: Twisted Whitehead double of the trefoil knot and one of its spanning surfaces (courtesy of Jessica Banks, used with permission)

M_1 contains a properly embedded Mobius band (see Figure 6.3), whose boundary is a longitude λ of the solid torus bounded by T .

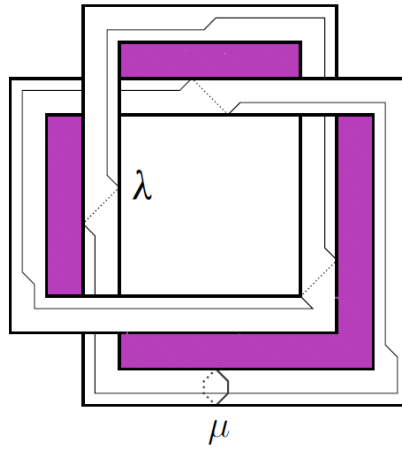


Figure 6.3: Mobius band embedded in M_1

We denote the annulus in the boundary of the regular neighbourhood of this Mobius band as S_1 . ∂S_1 is two copies of λ , which divide T into two annuli, one of which we call S_T . Then R is a plumbing of two annuli S_0 and S'_0 in M_0 , where S_0 is parallel to S_T (see Figure 6.4).

We isotope R in M so that $R \cap T = S_T$, while fixing ∂R . R_0 is the Seifert surface for K_α obtained by replacing S_T in R with S_1 . We define R_n as $\Psi^n(R_0)$, where each Ψ performs a Dehn twist in a neighbourhood of μ .

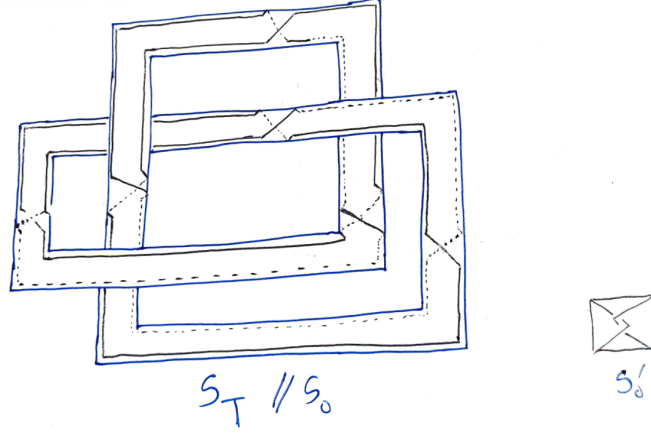


Figure 6.4: Plumbing of two annuli

To show that $[R_n] \neq [R]$ for a fixed n , it suffices to show that R_n cannot be isotoped to be disjoint from T . None of the components in the complement of $R_n \cup T$ are product manifolds, so by Proposition 6.1.3, this holds true.

The same argument also works to show that $[R_m] \neq [R_n]$ for some fixed m, n such that $m \neq n$, as none of the components of $M \setminus (R_n \cup R'_m)$ are product manifolds. This proves the theorem.

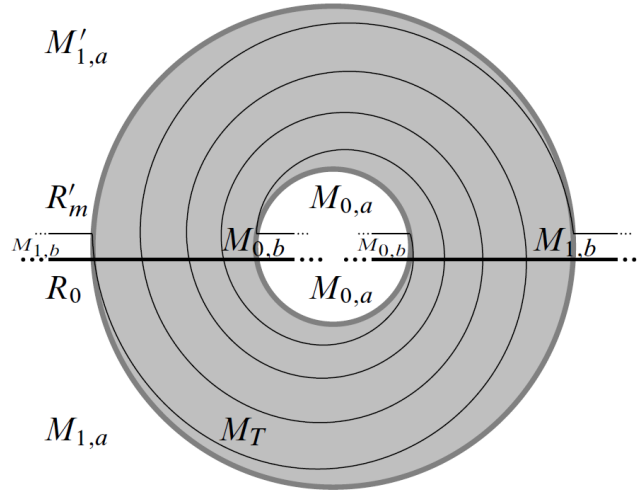


Figure 6.5: Components of $M \setminus (R_0 \cup R'_m)$ (courtesy of Jessica Banks, used with permission)

□

6.2 Links that can have locally infinite Kakimizu complex

In this section, we give a characterization of the kinds of links that can have a locally infinite Kakimizu complexes. We will use the following proposition stated without proof from [BZ85].

Proposition 6.2.1. *Let K be a knot and M denote $\mathbb{S}^3 \setminus \text{int } \mathcal{N}(K)$. Suppose there exists an annulus S properly embedded in M that is not ∂ -parallel. If neither component of ∂S bounds a disc in ∂M , then K is a torus knot, a cable knot or a connected sum.*

We also need some concepts from normal surface theory, which are stated below.

Definition 6.2.2. *The **weight** of a normal surface S in a triangulated 3-manifold is defined as the number of times it intersects the 1-skeleton of the triangulation.*

Definition 6.2.3. *S is called **minimal** if, among normal surfaces isotopic to S by an isotopy fixing ∂S , it has minimal weight.*

Definition 6.2.4. *Let S_1, S_2 be normal surfaces, and $S = S_1 + S_2$. S_1, S_2 are said to be in **reduced form** if they have been isotoped to minimize $|S_1 \cap S_2|$ while maintaining $S = S_1 + S_2$.*

The following theorem is stated without proof from [Wil08].

Theorem 6.2.5. *Let K be a non-trivial knot and M denote $\mathbb{S}^3 \setminus \mathcal{N}(K)$. Then there is a finite set $\{R_1, \dots, R_m\}$ of incompressible Seifert surfaces for K and a finite set $\{S_1, \dots, S_n\}$ of closed surfaces in M that are not ∂ -parallel, such that any incompressible Seifert surface R is isotopic to a Haken sum $R = R_i + a_1 S_1 + \dots + a_n S_n$, for non-negative integers $a_1, \dots, a_n$.*

The nature of this isotopy is described in the following lemma.

Lemma 6.2.6. *Let K be a knot, M denote $\mathbb{S}^3 \setminus \mathcal{N}(K)$ and R be an incompressible Seifert surface for K in M . If M is triangulated and ∂R meets each 2-simplex of the triangulation in at most one normal arc, then R can be put into normal form by an isotopy that fixes ∂R .*

If, in the proof of Theorem 6.2.5, we take M to be the complement of a minimal genus Seifert surface for a link, we get the following version of the theorem.

Theorem 6.2.7. *Let L be an oriented link such that every minimal genus Seifert surface for L is connected, R be a minimal genus Seifert surface for L and M denote $\mathbb{S}^3 \setminus \text{int } \mathcal{N}(R)$. $\rho_1, \dots, \rho_k$ is a fixed set of core curves of the annuli $\partial M \cap \partial \mathcal{N}(L)$, one for each link component.*

Then there is a triangulation of M such that every Seifert surface R' for L disjoint from R can be put into normal form, such that $\partial R' = \bigcup_{i=1}^k \rho_i$.

Further, there is a finite set $\{R_1, \dots, R_m\}$ of surfaces in M with non-empty boundary contained in $\bigcup_{i=1}^k \rho_i$, and a finite set $\{S_1, \dots, S_n\}$ of closed surfaces in M , such that all these surfaces are incompressible and in normal form, and the following holds:

Any minimal genus Seifert surface R' for L in M with $\partial R' = \bigcup_{i=1}^k \rho_i$ and in minimal normal form can be expressed as

$$a_1 R_1 + \dots + a_m R_m + b_1 S_1 + \dots + b_n S_n$$

for some $a_i, b_i \in \mathbb{Z}_{\geq 0}$.

The following lemma from [JO84] also comes handy in our proof.

Lemma 6.2.8. *Let $\mathcal{H}$ be a handle-decomposition of the closed irreducible 3-manifold M . Suppose that F is a minimal normal surface in $\mathcal{H}$ and $F = F_1 + F_2$ in reduced form. If F is incompressible, then each patch of $F_1 + F_2$ is incompressible and no patch of $F_1 + F_2$ is a disc.*

Now we will use the above mentioned theorems to prove the following result on links that can have locally infinite Kakimizu complexes.

Theorem 6.2.9. *Let L be an oriented link such that every minimal genus Seifert surface for L is connected. If $\mathcal{MS}(L)$ is locally infinite, then L is a satellite of either a torus knot, a cable knot or a connected sum, with winding number zero.*

Proof. We take R to be a minimal genus Seifert surface for L such that $\mathcal{MS}(L)$ is locally infinite at R , and M to be $\mathbb{S}^3 \setminus \mathcal{N}(R)$. We fix core curves $\rho_1, \dots, \rho_k$ of the annuli $\partial M \cap \partial \mathcal{N}(L)$ and apply Theorem 9.2.2. Say R' is a minimal genus Seifert surface in minimal normal form

such that

$$R' = R_1 + b_1 S_1 + \dots + b_n S_n,$$

where $b_1 > 0$ after suitable rearrangement of the terms. We define $R^- := R_1 + (b_1 - 1)S_1 + b_2 S_2 + \dots + b_n S_n$ and $T := S_1$, so that $R' = R^- + T$. As R' is minimal genus, so $\chi(T) \geq 0$, but it cannot be a sphere, as then R^- and T can be isotoped to be disjoint; hence it must be a torus.

Using the concept of algebraic intersection number, we can argue that there are an even number of curves in $|R^- \cap T|$, half of which have one orientation and half have the other. We perform surgery on R^- along the subannulus of T between adjacent curves with opposite orientations, removing all the curves of $R^- \cap T$. This gives a new Seifert surface R'' and a closed surface S'' (which might be disconnected). Due to constraints on the Euler characteristic of S'' , we conclude that the components of $(R^- \cup T) \setminus (R^- \cap T)$ which give S'' must be annuli.

Let S is one such annulus parallel to a subannulus S_T of T . If possible, say it is ∂ -parallel (it cannot bound a disc by Lemma 6.2.8). Then cut-and-paste along the boundary curves of S_T gives the following:

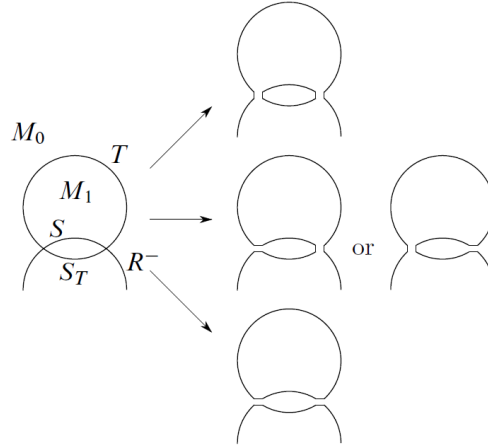


Figure 6.6: Cut-and-paste operation (courtesy of Jessica Banks, used with permission)

- If both the operations join S and S_T , we get a torus component of R' , which contradicts its connectedness.
- If both the operations do not join the annuli, then there is an isotopy of R^- and T

that reduces $R^- \cap T$ without changing R' , which contradicts the fact that R^- and T are in minimal normal form.

- If only one of the operations joins the annuli, we can isotope along the product region to reduce the weight of R' . This contradicts the fact that R' is in minimal normal form.

Thus, S cannot be ∂ -parallel in M_1 .

The component of $\mathbb{S}^3 \setminus \text{int}\mathcal{N}(T)$ containing L is a solid torus V , with core curve K . R is in V and T is incompressible, so L must be a satellite of K having winding number 0. In addition, S is not parallel to T , so it satisfies the hypotheses of Proposition 6.2.1. This proves the theorem. $\square$

Chapter 7

The coarse geometry of the Kakimizu complex

In this chapter, we show that the Kakimizu complex is quasi-isometric to $\mathbb{Z}^n$, and give an upper bound for n .

The content of this chapter is taken from [JPW14], which one can look at for detailed proofs of the results.

7.1 Quasi-isometry

In this section, we define some basic terms related to the concept of quasi-isometry.

Definition 7.1.1. A *quasi-isometry* is a map $f : X \rightarrow Y$ between metric spaces X, Y such that

$$\frac{1}{L}d_Y(f(x), f(y)) - L \leq d_X(x, y) \leq Ld_Y(f(x), f(y)) + L$$

for some constant L and every point of Y within an L -neighbourhood of the image $f(X)$.

For example, the inclusion map $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry:

$$d_{\mathbb{Z}}(x, y) = |y - x| = d_{\mathbb{R}}(x, y)$$

$$\Rightarrow \frac{1}{1}d_{\mathbb{R}}(x, y) - 1 \leq d_{\mathbb{Z}}(x, y) \leq 1 \cdot d_{\mathbb{R}}(x, y) + 1$$

Definition 7.1.2. A metric space (X, d) is **quasi-Euclidean** if it is quasi-isometric to a Euclidean space $\mathbb{R}^n$ for some n .

Thus, $\mathbb{Z}$ is quasi-Euclidean.

7.2 Jaco-Shalen-Johannson decomposition of a 3-manifold

In this section, we discuss some concepts related to Seifert-fibred spaces and the JSJ decomposition. These ideas will be used to describe the coarse geometry of the Kakimizu complex in later sections. These definitions are taken from [Sch14].

Definition 7.2.1. A **fibred solid torus of type (m, n)** is a solid torus partitioned into circles as follows:

- We consider the solid cylinder $\mathbb{D}^2 \times [-1, 1]$. It is partitioned into intervals of the form $\{\text{point}\} \times [-1, 1]$.
- We identify $\mathbb{D}^2 \times \{-1\}$ with $\mathbb{D}^2 \times \{1\}$ by identifying $(re^{i2\pi\theta}, 1)$ with $(re^{i2\pi(\theta + \frac{m}{l})}, -1)$, where $m \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{N}$.

Definition 7.2.2. A circle formed by intervals of the form $\{\text{point}\} \times [-1, 1]$ is called a **fiber** of the fibred solid torus.

The **core** of a fibred solid torus is the fiber obtained from $\{0\} \times I$.

Definition 7.2.3. If $l > 1$, we call the fibred solid torus T of type (l, m) an **exceptionally fibred solid torus**. The core of T is called an **exceptional fiber** and all other fibers of T are called **regular fibers**.

If $l = 1$, then T is called a **regularly fibred solid torus** and all its fibers are called **regular fibers**.

Definition 7.2.4. A **fiber-preserving homeomorphism** between the fibred solid tori T_1 and T_2 is a homeomorphism $h : T_1 \rightarrow T_2$ that takes fibers to fibers.

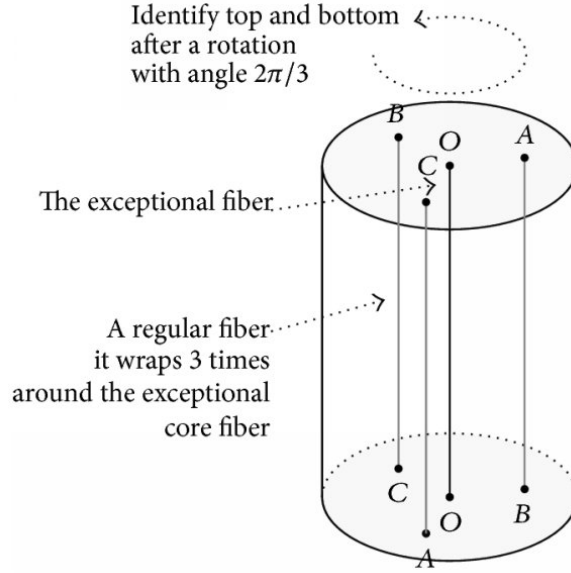


Figure 7.1: Fibered solid torus of type (m, n) (sourced from the internet)

Definition 7.2.5. A **fibered neighbourhood** of a simple closed curve f in a 3-manifold M that consists of simple closed curves is:

- a closed regular neighbourhood homeomorphic via a fiber-preserving homeomorphism to some fibered solid torus when f is in $\text{int } M$.
- a closed regular neighbourhood homeomorphic to $(\text{half-disc}) \times \mathbb{S}^1$ partitioned into fibres $\{\text{point}\} \times \mathbb{S}^1$ when $f \subset \partial M$.

Definition 7.2.6. A **Seifert fibered space** is an orientable 3-manifold such that it is a union of pairwise disjoint simple closed curves called **fibers**, each of which has a closed regular neighbourhood that is fibered.

The description of the 3-manifold M in terms of fibers is called the **Seifert fibration** of M .

Definition 7.2.7. A 3-manifold is **atoroidal** if it contains no essential torus submanifold.

Definition 7.2.8. A closed orientable irreducible 3-manifold M contains a collection of disjointly embedded incompressible tori $\mathcal{T}$ such that each component of the 3-manifold obtained by cutting M along $\mathcal{T}$ is either atoroidal or Seifert fibered. Moreover, such a collection of

tori is unique upto isotopy. The collection of tori $\mathcal{T}$ is called the **JSJ decomposition** of M .

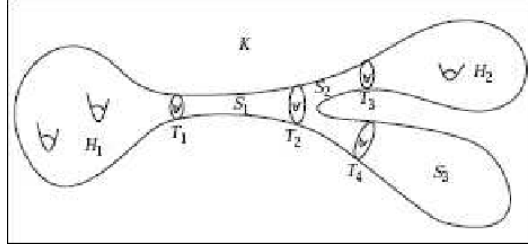


Figure 7.2: JSJ decomposition of a space (sourced from the internet)

7.3 Kakimizu complex and core blocks

Let K be a knot with more than two factors. Its complement is denoted by M_K . The JSJ decomposition is given by a minimal collection of pairwise disjoint, incompressible tori $T_1, \dots, T_n$ such that the complement their union is comprised of Seifert-fibred spaces and atoroidal (hyperbolic) spaces. Each T_i bounds a solid torus in $\mathbb{S}^3$ which can be oriented to contains K on the negative side. We denote the neighbourhood of T_i in M_K as $\mathcal{N}(T_i)$.

Definition 7.3.1. *Each compact connected component in $M_K \setminus \bigcup_{i=1}^n \text{int} \mathcal{N}(T_i)$ is called a **block**.*

Definition 7.3.2. *Let B be a block not containing K and M be a component of $M_K \setminus B$ that contains K . If K is not nullhomologous in M , then B is called a **core block**. B is also called a core block if it contains K .*

Every minimal genus Seifert surface for K must intersect ∂M as the boundary of a Seifert surface is K . Thus, every minimal genus Seifert surface for K must intersect every core block.

Definition 7.3.3. *The union of all the core blocks is called the **core** of the JSJ decomposition of M_K . It is a connected subset of M_K .*

Proposition 7.3.4. *Let T be a JSJ torus in the interior of the core. There is a fixed slope α of T such that every Seifert surface S for the knot must intersect T in one or more parallel loops with exactly this slope.*

Proof. T is a JSJ torus, so it separates $\mathbb{S}^3$ into two components - M (a solid torus containing the knot) and M' (on the opposite side, homeomorphic to the complement of some other knot in $\mathbb{S}^3$). So $H_1(T) \hookrightarrow H_1(M')$ has infinite cyclic image and infinite cyclic kernel. Let $\alpha \subseteq T$ be a loop representing the generator of this kernel.

$S \cap T$ is the collection of parallel loops whose orientations are induced from S and T . T is in the interior of the core and S is not a 2-cycle in M with boundary K , so $S \cap T$ must be homologically non-trivial in M . So the loops $S \cap T$ define a non-trivial element in $H_1(T)$. $S' = S \cap M'$ is a surface on the other side of T . Then the loops in $S \cap T$ determine a trivial element in $H_1(M')$. Hence, they represent a power of the kernel map, ie, $S \cap T$ is a collection of parallel copies of α . $\square$

Lemma 7.3.5. *Let S be a minimal genus Seifert surface for the knot K and $T = \bigcup_{i=1}^n T_i$ be the collection of all JSJ tori. If S can be isotoped to intersect T minimally, then $|S \cap T| \leq 6g - 4$, where g is the genus of S .*

Proof. We denote the blocks of the JSJ decomposition as $B_1, \dots, B_k$.

S is a once-punctured surface of genus g , so it has at most $3g - 2$ pairwise disjoint isotopy classes of essential loops. This gives an upper bound for isotopy classes of curves in $S \cap T$.

If it were possible for three adjacent pairwise disjoint curves in $S \cap T$ to be parallel in S , they would bound two adjacent essential annuli. In this case, there is an argument in [Hat] that allows us to remove the JSJ torus that the annuli lie on, which contradicts minimality of the decomposition.

$$\therefore |S \cap T| \leq 2(3g - 2) = 6g - 4.$$

$\square$

The following is a generalization of the main result in [Wil08].

Theorem 7.3.6. *Let M_L be the link complement for the link L and $\alpha_1, \dots, \alpha_k$ be a set of preferred longitudes. If there are an infinite collection of essential surfaces S_i in M_L which all have the same Euler characteristic, ∂S_i is isotopic to a subcollection of the α_i , and there is an integer N such that $|\partial S_i| \leq N \forall i$, then M_L contains a closed incompressible torus.*

Corollary 7.3.7. *Let M_L be the link complement for the link L and $\alpha_1, \dots, \alpha_k$ be a set of preferred longitudes. Then there are at most finitely many essential surfaces S_i in M_L*

having maximal Euler characteristic such that ∂S_i is isotopic to a subcollection of the α_i and $|\partial S_i| \leq N \forall i$.

The following result is a classification of Seifert fibered submanifolds of $\mathbb{S}^3$ given in [Bud06].

Lemma 7.3.8. *Let $V \neq \mathbb{S}^3$ be a Seifert fibered submanifold of $\mathbb{S}^3$. Then V is diffeomorphic to one of the following:*

- *A Seifert fibered space over an n -times punctured sphere containing two exceptional fibers, which appears as the complement of n regular fibers in a Seifert fibering of $\mathbb{S}^3$.*
- *A Seifert fibered space over an n -times punctured sphere containing one exceptional fiber, which appears as the complement of $(n - 1)$ regular fibers in a Seifert fibering of an embedded solid torus in $\mathbb{S}^3$.*
- *A Seifert fibered space over an n -times punctured sphere containing no exceptional fibers.*

The above lemma gives the following result which says that there can be at most finitely many incompressible surfaces in each Seifert fibered block of a knot complement.

Theorem 7.3.9. *Let K be a knot and B be a core block of the JSJ decomposition for M_K . Then there are finitely many essential surfaces $S_1, \dots, S_m$ such that for any minimal genus Seifert surface S for K in the core of M_K , every component of $S \cap B$ is isotopic to one of the S_i 's.*

Proof. If B is a hyperbolic block, then it is atoroidal. $\partial(S \cap B)$ is fixed and bounded by by Proposition 7.3.4 and Lemma 7.3.5. Then the conclusion follows directly from Corollary 7.3.7.

If B is a Seifert fibered block, we get three cases corresponding to Lemma 7.3.8.

Case I: B is Seifert fibered over an n -times punctured sphere containing two exceptional fibers, which appears as the complement of n regular fibers in a Seifert fibering of $\mathbb{S}^3$.

In this case, we can show that B is Seifert fibered over a once-punctured sphere with two critical fibres. Such spaces are atoroidal, so we arrive at the conclusion.

Case II: B is Seifert fibered over an n -times punctured sphere containing one exceptional fiber, which appears as the complement of $(n - 1)$ regular fibers in a Seifert fibering of an embedded solid torus in $\mathbb{S}^3$.

In this case too, we can show that the space is atoroidal and satisfies the conclusion.

Case III: B is Seifert fibered over an n -times punctured sphere containing no exceptional fibers.

In this case, B is a product and has a unique incompressible surface boundary of a specified homotopy class. $\square$

7.4 Core Kakimizu complex

In this section, we define the notion of the core Kakimizu complex, and look at some of its properties which will help us prove the main result of this chapter.

Claim 7.4.1. *Every automorphism of the knot complement induces an automorphism of the Kakimizu complex.*

We fix a JSJ torus T_i and consider a closed regular neighbourhood U of it homeomorphic to $I \times T_i$. The aim is to define an automorphism of the knot complement that is identity outside U and spins T_i in a given direction.

We choose a coordinate system on the universal cover $\mathbb{R}^2$ of T_i . $\forall (m, n) \in \mathbb{Z}^2$, φ_t ($t \in I$) is a family of automorphisms of T_i lifting to translations of the plane that take $(0, 0)$ to (m, n) , with φ_0, φ_1 as identity on T_i . The automorphisms on the knot complement are given as follows:

$$\Phi(x) = \begin{cases} (t, \varphi_t(z)) & \text{if } x = (t, z) \in I \times T_i \\ x & \text{otherwise} \end{cases}$$

Choosing some $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ for a fixed T_i gives one of the above-mentioned automorphisms of the knot complement. Their compositions correspond to addition in $\mathbb{Z} \times \mathbb{Z}$, forming an abelian group of rank 2. If T_i is a torus in the interior of the core, we take α to be a

loop in T_i which represents the slope on T given by Proposition 7.3.4. Spinning any Seifert surface parallel to α takes it to itself, which is the same as action by identity on $\mathcal{MS}(K)$. This generates an infinite cyclic subgroup, whose quotient by $\mathbb{Z}^2$ is an infinite cyclic group. A representative for the generator of this group is denoted by Φ_i . It is an element of the corresponding coset in $\mathbb{Z}^2$.

Definition 7.4.2. *The subcomplex of the Kakimizu complex that is spanned by Seifert surfaces which can be isotoped into the core C is called the **core Kakimizu complex**, denoted as $\mathcal{MS}(C)$.*

Let G' be the group of all automorphisms of the knot complement generated by the Φ_i 's. Each Φ_i takes a block to a block, so the action of G' restricts to an action on $\mathcal{MS}(C)$. If $i \neq j$, then $\text{supp}(\Phi_i) \cap \text{supp}(\Phi_j) = \emptyset$. So the Φ_i 's commute, and G' is abelian.

Let $N \trianglelefteq G'$ such that N acts trivially on $\mathcal{MS}(C)$. The group $G := G'/N$ is also finitely generated and abelian.

Claim 7.4.3. $\mathcal{MS}(C) \neq \emptyset$.

Proof. Let S be a Seifert surface for the knot such that $S \cap (\bigcup_i T_i)$ is minimal over all Seifert surfaces. If S remains inside the core, then $\mathcal{MS}(C)$ is trivially non-empty.

If S exits the core, it must intersect some JSJ torus T separating a core block from a non-core block. We assume $S \cap T$ is minimal. T is not in the interior of the core. So if X denoted the component of the complement of T that contains K , then K must be homologically trivial in X . Further, X is a solid torus by arguments in Section 7.3. S is orientable, so the curves of $S \cap T$ define the trivial element of $H_1(X)$. If these are non-meridional loops, each loop gives a non-trivial element of $H_1(X)$. So we must have an equal number for each type of orientation. For meridional loops, $S \setminus X$ gives a meridional surface for $\mathbb{S}^3 \setminus X$. We again get an equal number of loops with each orientation as the meridian is homologically non-trivial in $\mathbb{S}^3 \setminus X$.

Thus in both the cases, the loops in $S \cap T$ occur in pairs with opposite orientations. But then they bound an annulus and we can perform cut-and-paste to reduce $S \cap T$, which contradicts the assumption that $S \cap (\bigcup_i T_i)$ is minimal. $\square$

Claim 7.4.4. $\mathcal{MS}(C)$ is connected.

Proof. This proof is adopted from Scharlemann and Thompson's proof of connectedness of the Kakimizu complex [ST88].

Let S and S' be representatives of the vertices v_1 and v_2 respectively. We take double curve sums to get a sequence S_i of minimal genus Seifert surfaces for $0 \leq i \leq k$ satisfying $S_i \cap S_{i+1} = \emptyset$ for $1 \leq i \leq k$, with S_0 isotopic to S and S_k isotopic to S' . If $S, S' \subset C$, then all double curve sums are in C . So $S_i \subset C \forall 1 \leq i \leq k$, and the path lies in $\mathcal{MS}(C)$. $\square$

Claim 7.4.5. $\mathcal{MS}(C)$ is locally finite.

Proof. S is a minimal genus Seifert surface that represents $v \in \mathcal{MS}(C)$ and $S' \subseteq C$ is another minimal genus Seifert surface disjoint from it. Theorem 7.3.9 shows that there are only finitely many ways in which S' can intersect the core blocks of the JSJ decomposition. The annuli joining the subsurfaces of intersection cannot spin all the way around a torus in the interior of the core without intersecting S . So there are only finitely many possibilities for these subsurfaces, and thus for S' to be disjoint from S . $\square$

The following theorem is stated without proof from [Har00].

Theorem 7.4.6. *Let X be a metric space that is geodesic and proper, G be a group and $G \times X \rightarrow X$ be the action by isometries. We assume that the action is proper and that X/G is compact. Then G is finitely generated and quasi-isometric to X . More precisely, the mapping $G \rightarrow X$ given by $g \mapsto gx_0$ is a quasi-isometry.*

Lemma 7.4.7. *There are finitely many minimal genus Seifert surfaces in the core which are called **fundamental surfaces**, such that every Seifert surface for K in the core is either fundamental or can be obtained by spinning a fundamental surface around some number of JSJ tori in the interior of the core. In other words, there are finitely many isotopy classes of minimal genus Seifert surfaces $\sigma_1, \dots, \sigma_j$ in the core such that any other isotopy class $\sigma' \in \mathcal{MS}(C)$ of minimal genus Seifert surfaces is given by $\sigma' = g\sigma_k$ for some $g \in G$ and some fundamental surface σ_k .*

Proof. Let S be a minimal genus Seifert surface for K that meets the JSJ tori $T_1, \dots, T_n$ transversally and minimally. We denote the neighbourhood of T_i as $\mathcal{N}(T_i)$ and the components of the complement of the JSJ tori as B_k . For a given k , each component of $S \cap B_k$ is isotopic to one of the properly embedded surfaces in Theorem 7.3.9 and each component of

$S \cap \mathcal{N}(T_i)$ is an incompressible annulus. The theorem says that there are a finite number of ways to connect the incompressible surfaces on either side of each torus, up to spinning around the torus. The result follows from this observation. $\square$

Lemma 7.4.8. *The action of the group G on $\mathcal{MS}(C)$ is proper, which means that the stabilizer $G_v = \{g \in G \mid g(v)\}$ is finite for every vertex v of $\mathcal{MS}(C)$.*

Proof. We set $v \in \mathcal{MS}(C)$. The idea of the proof is that there is some monomorphism from G_v to some finite group. From Lemma 7.4.7, we know that G has finitely many orbits $O_1, \dots, O_r$; we take a representative $v_i \in O_i$ for each orbit. We set $V = \{v, v_1, \dots, v_r\}$ and denote its diameter as d . Then there is a homomorphism from G_v to the permutation group of the vertices of B . If g is an automorphism in its kernel, $g(v_i) = v_i \forall i$. As G is abelian, we can conclude that g fixes every element of $\mathcal{MS}(C)$ and must be the identity, as we have quotiented out all the elements that fix $\mathcal{MS}(C)$. Thus, the homomorphism is injective. $\square$

Lemma 7.4.9. *$\mathcal{MS}(C)$ is quasi-isometric to a finitely generated abelian group.*

Proof. $\mathcal{MS}(C)$ satisfies all the conditions in Theorem 7.4.6, so the conclusion follows. $\square$

7.5 Main Theorem

The aim of this section is to prove the following theorem, which is the main result of this chapter.

Theorem 7.5.1. *For any knot $K \subset \mathbb{S}^3$, the Kakimizu complex $\mathcal{MS}(K)$ is quasi-isometric to $\mathbb{Z}^n$ for some $n \geq 0$.*

The idea of the proof is to show that $\mathcal{MS}(K)$ is quasi-isometric to $\mathcal{MS}(C)$ which we know from Lemma 7.4.9 is quasi-isometric to a finitely generated abelian group.

Lemma 7.5.2. *The core Kakimizu complex $\mathcal{MS}(C)$ is quasi-isometric to the entire Kakimizu complex $\mathcal{MS}(K)$.*

Proof. We will show that the inclusion map $\mathcal{MS}(C) \hookrightarrow \mathcal{MS}(K)$ preserves distances and each vertex $\sigma \in \mathcal{MS}(K)$ is within a bounded distance from some vertex $\sigma' \in \mathcal{MS}(C)$.

In [ST88], double curve sums are used to produce geodesics in $\mathcal{MS}(K)$. Geodesics between the vertices of $\mathcal{MS}(C)$ will lie in $\mathcal{MS}(C)$ as double curve sums of surfaces in C produce new surfaces still lying in C . If we have two vertices in $\mathcal{MS}(C)$, measuring the distance between them is equivalent to measuring it in $\mathcal{MS}(K)$. In other words, the inclusion map preserves distances.

Let $S \in \sigma$ be a minimal genus Seifert surface for K . If S is in the core, then $\sigma \in \mathcal{MS}(C)$. So we assume that it is not. Then S intersects some JSJ torus T bounding a block B inside the core and B' outside it. S has minimal genus, so T cuts it into a compact surface S' inside the core and finitely many annuli A_i outside it. Lemma 7.3.5 says that a minimal genus Seifert surface intersects the JSJ tori in at most $6g - 4$ curves, so there are at most $3g - 2$ annuli. For each such annulus A_i , its boundaries $C_i^+ \cup C_i^-$ are joined by an annulus D_i inside B . We can attach D_i to S' to obtain the minimal genus Seifert surface $\tilde{S}$ that completely lies inside C . Say it is a representative of $\sigma' \in \mathcal{MS}(C)$. Surgering the surface this way at a single annulus gives a new surface disjoint from the previous one each time.

$\therefore$ there is a path between σ, σ' in $\mathcal{MS}(K)$ which has length not more than the number of annuli outside the core, ie, at most $3g - 2$.

$\therefore$ Every $\sigma \in \mathcal{MS}(K)$ is within a bounded distance from $\mathcal{MS}(C)$. □

Proof of Theorem 7.5.1. From Lemma 7.5.2, we know that $\mathcal{MS}(K)$ is quasi-isometric to $\mathcal{MS}(C)$. But we also know that quasi-isometry is an equivalence relation, so from Lemma 7.4.9, $\mathcal{MS}(K)$ is quasi-isometric to a finitely generated abelian group, ie, to $\mathbb{Z}^n$ for some n . □

The abelian group $\mathcal{MS}(K)$ quasi-isometric to is generated by spinning around the tori in the interior of the core C . So it has rank not more than the number of such tori. For each block that is homeomorphic to a fibered link complement, spinning around the torus closest to the knot is isotopically equivalent to spinning around the remaining boundary tori. Hence

$$\text{rank} = \text{number of JSJ tori} - \text{number of fibered core blocks}$$

Later, it was proved by Jessica Banks that this is in fact the correct value for the rank (see [Ban16]).

Chapter 8

Contractibility of the incompressible complex

In Chapter 5, we looked at a proof of how the Kakimizu complex of a link L is contractible. So a natural question to then ponder over is whether the incompressible complex $\mathcal{IS}(L)$ can be contractible.

Performing cut-and-paste along two surfaces preserves the sum of their Euler characteristics, ie, if S_1 and S_2 are two surfaces which give the surfaces R_1 and R_2 after cut-and-paste, then $\chi(S_1) + \chi(S_2) = \chi(R_1) + \chi(R_2)$. The Euler characteristic of minimal genus Seifert surfaces is maximum, so this ensures that the surfaces obtained after cut-and-paste will remain minimal genus. However, this is not the case with incompressible surfaces - if performing cut-and-paste on the incompressible surfaces S_1 and S_2 gives the surfaces R_1 and R_2 , then there is no guarantee that R_1 and R_2 are also incompressible.

One way to go about this problem is by using a set of moves suggested by DR McMillan Jr in [DR 69]. Kakimizu used these moves in his proof of connectedness of $\mathcal{IS}(L)$ in his paper [Kak92], which was discussed in detail in Chapter 2.

In this chapter, we prove that the incompressible complex $\mathcal{IS}(L)$ for the link L is indeed contractible. We will largely adopt the techniques used by Przytycki and Schultens in their paper [PS12] where they proved the contractibility of the Kakimizu complex.

8.1 McMillan's moves

Since DR McMillan Jr's moves form the central part of this proof, we start this chapter by describing them.

Let M^3 be an orientable non-closed PL 3-manifold, and Z^3 be a compact polyhedron in $\text{int } M^3$ such that each component of Z^3 is a 3-manifold with non-empty boundary.

Definition 8.1.1. Say $B^i \subset \text{int } M^3$ is a polyhedral i -cell such that $\partial B^i \subset \partial Z^3$ and $Z^3 \cap \text{int } B^i = \emptyset$. Then if $N(B^i)$ is a regular neighbourhood of B^i in $M \setminus \text{int } Z^3$, we say that $Z^3 \cup N(B^i)$ is obtained from Z^3 by **adding an i -handle** in M^3 .

While this definition of a handle is not exactly the same as the modern definition of the same which was popularized by Stephen Smale, there are similarities.

Definition 8.1.2. We say $Z^3 \cup N(B^i)$ is obtained from Z^3 by a **simple annexation** in M^3 of type i if:

- $i = 3$.
- $i = 2$ and ∂B^2 fails to bound a 2-cell in ∂Z^3 .
- $i = 1$ and ∂B^1 intersects at least one 3-cell component of Z^3 in exactly one point.

Definition 8.1.3. If there is a polyhedral 2-cell $B^2 \subset Z^3$ such that $B^2 \cap \partial Z^3 = \partial B^2$ and ∂B^2 fails to bound a 2-cell in ∂Z^3 , then if $N(\partial B^2)$ is a nice regular neighbourhood of B^2 in Z^3 , the closure of $Z^3 \setminus N(B^2)$ is said to be obtained from Z^3 by **simple reduction**.

Definition 8.1.4. A **simple move** in M^3 means either a simple annexation in M^3 or a simple reduction.

Define

$$c(Z^3) := \sum_{n=0}^{\infty} (n+1)^2 g(n)$$

where $g(n)$ is the number of surfaces of genus n in ∂Z^3 . Then if Z_0^3 is obtained from Z^3 by simple moves,

$$1 \leq c(Z_0^3) < c(Z)$$

Thus, the number of simple moves is finite and bounded.

8.2 Idea of the proof

The aim of this chapter is to prove the following theorem.

Theorem 8.2.1. *For a link L , the incompressible complex $\mathcal{IS}(L)$ is contractible.*

Since the proof of the theorem is long, it is useful to lay out a broad outline of it. This is the same as the proof for contractibility of $\mathcal{MS}(L)$.

- We define a projection map $\pi : \mathcal{IS}(L) \rightarrow \mathcal{IS}(L)$, which assigns to the vertices $\sigma, \rho \in \mathcal{IS}(L)$ at a distance $d > 0$ another vertex $\pi_\sigma(\rho)$ which is adjacent to ρ and at a distance $d - 1$ from σ .
- We show that amongst the vertices farthest from σ , there is a vertex ρ such that all the vertices adjacent to ρ are also adjacent to or equal to $\pi_\sigma(\rho)$.
- This shows that there is a homotopy retraction of $\mathcal{IS}(L)$ onto the subcomplex spanned by all the vertices except ρ .
- Thus, the entire complex retracts onto σ .

8.3 Redefining $\mathcal{IS}(L)$

We redefine the incompressible complex like we did with the Kakimizu complex.

Definition 8.3.1. *For the complex $\mathcal{IS}(E, \gamma, \alpha)$, E , γ and α stand for the following:*

- E is a compact, connected, orientable, irreducible and ∂ -irreducible 3-manifold.
- γ is the union of oriented disjoint simple closed curves on ∂E which do not separate any component of ∂E .
- $\alpha \in H_2(E, \partial E, \mathbb{Z})$ satisfies $\partial\alpha = [\gamma]$.

Definition 8.3.2. The vertex set of $\mathcal{IS}(E, \gamma, \alpha)$ is $\mathcal{IS}(E, \gamma, \alpha)$, the set of isotopy classes of incompressible spanning surfaces. There is an edge between $\sigma, \sigma' \in \mathcal{IS}(E, \gamma, \alpha)$ only if they have representatives $S \in \sigma, S' \in \sigma'$ such that the connected lift of $E \setminus S'$ to the infinite cyclic cover associated with α intersects exactly two lifts of $E \setminus S$, ie, the Kakimizu distance between σ and σ' is 1.

We also describe the covering space associated to $\mathcal{IS}(E, \gamma, \alpha)$. We will use this notation throughout the chapter, unless explicitly stated otherwise.

$p : \tilde{E} \rightarrow E$ is the infinite cyclic covering map associated with the kernel of the element of $H^1(E, \mathbb{Z})$ dual to α .

τ is the generator of the group of covering transformations of $\tilde{E}$. $S \subseteq E$ is an incompressible spanning surface. E_0 is a lift of $E \setminus S$ to $\tilde{E}$. E_j is defined to be $\tau^j(E_0) \forall j \in \mathbb{Z}$. S_j is defined as $\overline{E_{j-1}} \cap \overline{E_j}$.

If R is another incompressible spanning surface, E^R is defined to be a lift of $E \setminus R$ to $\tilde{E}$. We set $E_j^R := \tau^j(E^R)$.

r and m are respectively defined to be the maximum and minimum values of k for which E_k meets E^R . We set $d(S, R) = r - m$.

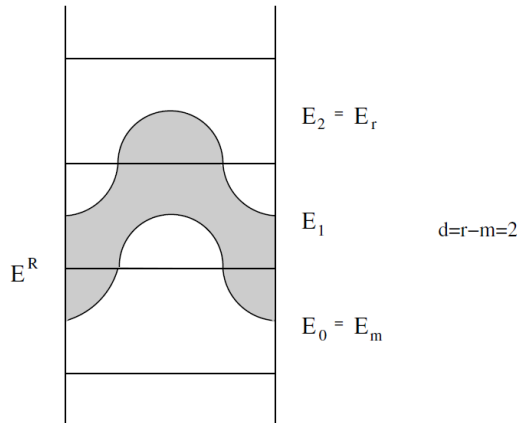


Figure 8.1: Infinite cyclic covering space (courtesy of Jennifer Schultens, used with permission)

The distance between the vertices σ, ρ is defined as

$$d(\sigma, \rho) := \min \{d(S, R) \mid S \in \sigma, R \in \rho\}$$

This definition of d yields a metric on $\mathcal{SS}(E, \gamma, \alpha)$. If l is the path metric on which all edges have length 1, the metric d coincides with l on $\mathcal{SS}(E, \gamma, \alpha)$.

We recall Proposition 5.2.8, which will be used throughout this chapter as well to obtain suitable representatives.

Proposition 8.3.3. *Let S, R be almost transverse spanning surfaces with simplified intersection in E , representing $\sigma, \rho \in \mathcal{MS}(E, \gamma, \alpha)$. Then they satisfy*

$$d(\sigma, \rho) = d(S, R).$$

8.4 Projection Map

In this section, we define a projection map on the vertices of the incompressible complex, which takes each vertex to an adjacent vertex.

Definition 8.4.1. *We fix a basepoint $\sigma \in \mathcal{SS}(E, \gamma, \alpha)$. The **projection map** π_σ maps every $\rho \in \mathcal{SS}(E, \gamma, \alpha)$ at a distance $n > 0$ from σ to a vertex $\pi_\sigma(\rho) \in \mathcal{SS}(E, \gamma, \alpha)$ which is adjacent to ρ and is at a distance $n - 1$ away from σ .*

Let $\sigma \neq \rho$ be vertices of $\mathcal{SS}(E, \gamma, \alpha)$. We fix a spanning surface $S \in \sigma$ and choose $R \in \rho$ so that S, R are almost transverse with simplified intersection. If $d(\sigma, \rho) = n$, then by Proposition 5.2.8, $d(S, R) = n$.

As previously, let r be the largest integer such that $E_r \cap E^R \neq \emptyset$. We define $\tilde{R} = \overline{E}^R \cap \tau(\overline{E}^R)$. Let $\tilde{P}_* \subseteq S_r \cup \tilde{R}$ be the surface obtained by cut-and-paste on S_r and $\tilde{R}$. Clearly $\tilde{P}_* = \partial \overline{E}^R \setminus E_r \cap (\partial \tau(\overline{E}^R) \cup \overline{E}_r)$, with orientation from that of $\tilde{R}$ and S_r .

We set $P_* = p(\tilde{P}_*)$. It is in the homology class of α . $\tilde{P}_*$ is inside $\overline{E}_{r-1}$, so p embeds it into E . ∂P_* is homotopic to a combination of curves in γ , meaning ∂P_* is homologous and homotopic

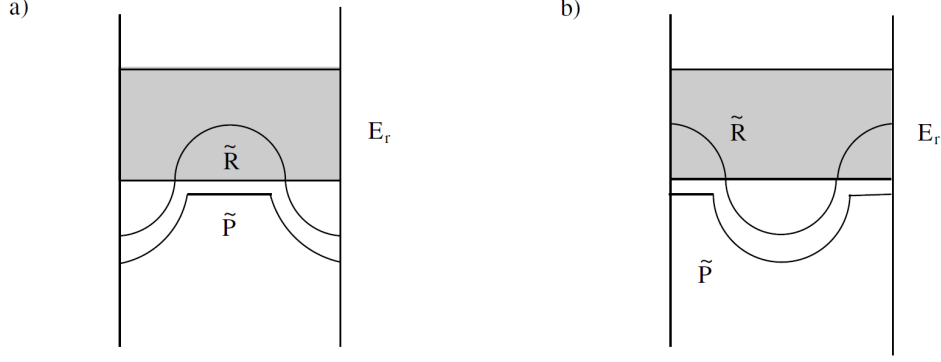


Figure 8.2: Projection map for vertices (courtesy of Jennifer Schultens, used with permission)

to γ . Further, γ does not separate the components of ∂E , so a non-trivial combination of curves in γ cannot be nullhomologous.

We define X to be $\overline{E}^R \cap \overline{E}_{r-1}$. We apply a finite sequence of moves to X in $\overline{E}_{r-1}$ and obtain a 3-submanifold W such that the components of $\overline{\partial W \cap \text{int } E_{r-1}}$ are incompressible in $\overline{E}_{r-1}$. This gives a finite sequence of 3-submanifolds of $\overline{E}_{r-1}$, $X = X_0, X_1, \dots, X_k = W$ such that $\forall 1 \leq i \leq k$, one of the following holds:

- X_i is obtained from X_{i-1} by adding a 2-handle whose core is a 2-disc $D \subseteq \text{int } E_{r-1}$ such that $D \cap X_{i-1} = \partial D \subseteq \overline{\partial X_{i-1} \cap \text{int } E_{r-1}}$ and ∂D is essential in $\overline{\partial X_{i-1} \cap \text{int } E_{r-1}}$.
- There is a 3-ball $C \subseteq \text{int } E_{r-1}$ such that $X_i = X_{i-1} \cup C$ and $X_{i-1} \cap C = \partial C = \overline{\partial X_{i-1} \cap \text{int } E_{r-1}}$.
- X_i is obtained from X_{i-1} by splitting a 2-disc $D \subseteq X_{i-1}$ such that $\partial D = D \cap \overline{\partial X_{i-1} \cap \text{int } E_{r-1}}$ and ∂D is essential in $\overline{\partial X_{i-1} \cap \text{int } E_{r-1}}$.
- There is a component C of X_{i-1} such that C is a 3-ball and $X_i = X_{i-1} - C$.

Claim: We can take W such that $W \cap E_n^R = \emptyset$ and $E^R \cap E_{r-1} \subseteq W$.

We will prove the claim by induction on i for X_i . We will show that each X_i can be taken such that $X_i \cap E_n^R = \emptyset$ and that $E^R \cap E_{r-1} \subseteq X_i$ inductively.

$X_0 = X$ trivially satisfies these requirements by construction. So we assume that X_{i-1} satisfies the hypothesis. Then for each of the construction methods:

- We modify D so that $D \cap E_n^R = \emptyset$. Each component of $S_n^R \cap E_{r-1}$ is incompressible in E_{r-1} , so the modification can be done by cut-and-paste.
- The added 3-ball C is disjoint from E_n^R as $C \subseteq \text{int } E_{r-1}$ and there is no component of $E_n^R \cap E_{r-1}$ in $\text{int } E_{r-1}$.
- The splitting disc D is taken to be disjoint from E_n^R .
- The removed 3-ball is disjoint from E_n^R .

In addition, each mode of construction implies that $E^R \cap E_{r-1} \subseteq X_i$.

We define Z to be the union of the components of W which contain S_{r-1} , and P to be $\overline{\partial Z \cap \text{int } E_{r-1}}$. $Z \cap E_n^R = \emptyset$ and $E^R \cap E_{r-1} \subseteq Z$, as there is no component of $E^R \cap E_{r-1}$ disjoint from S_{r-1} .

$\tilde{P}$ is incompressible in E_{r-1} , so $p(\tilde{P})$ is an incompressible spanning surface for K disjoint from R . We set $P = p(\tilde{P})$. Then

$$d(P, R) = 1.$$

Further, the largest k for which $E_k \cap \tilde{P} \neq \emptyset$ is $k = r - 1$. So

$$d(S, P) = n - 1.$$

We define

$$\pi_\sigma(\rho) := [P].$$

The existence of this $\pi_\sigma(\rho)$ is justified by Theorem 5.3.3, which we state here for recollection.

Theorem 8.4.2. *Let M be an orientable, irreducible, ∂ -irreducible 3-manifold and S, R be orientable, incompressible, ∂ -incompressible surfaces properly embedded in M . We assume that S, R are almost transverse with simplified intersection and are isotopic to S', R' respectively, which are also almost transverse with simplified intersection. If cut-and-paste on S, R yields an orientable, incompressible, ∂ -incompressible surface P , then a corresponding cut-and-paste operation exists on S', R' to yield a surface P' isotopic to P .*

This theorem implies the following.

Theorem 8.4.3. *The class $[P]$ is not dependent on the choice of S and R .*

8.5 Ordering the vertices in $\mathcal{IS}(E, \gamma, \alpha)$

In this section, we develop a notion of order on the vertices of the incompressible complex, akin to that on the vertices of the Kakimizu complex.

We fix $\sigma \in \mathcal{IS}(E, \gamma, \alpha)$. Let $\rho, \rho' \in \mathcal{IS}(E, \gamma, \alpha)$ be two adjacent vertices. We choose the almost disjoint surfaces $R \in \rho, R' \in \rho'$ that are almost transverse to a fixed $S \in \sigma$ and have simplified intersection with S and with each other.

E_0 is the lift of $E \setminus S$ to $\tilde{E}$ and $E_k = \tau^k(E_0)$. $E^{R'}$ is a lift of $E \setminus R'$ to $\tilde{E}$. r' is the largest integer such that $E_{r'} \cap E^{R'} \neq \emptyset$. E^R is the lift of $E \setminus R$ contained in $\overline{E}^{R'} \cup \tau^{-1}(\overline{E}^{R'})$.

Definition 8.5.1. *If $E^R \cap E_{r'} \neq \emptyset$, then we say that $\rho <_\sigma \rho'$. We write $\rho \leq_\sigma \rho'$ if $\rho <_\sigma \rho'$ or $\rho' = \rho$. See Figure 8.3.*

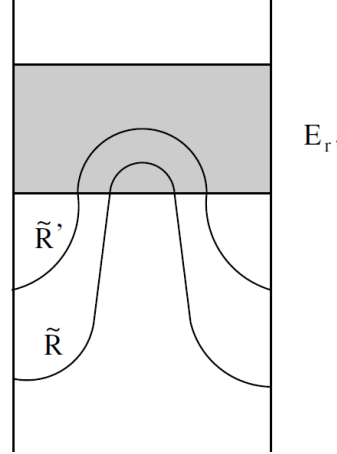


Figure 8.3: Ordering on vertices: $[\rho] <_\sigma [\rho']$ (courtesy of Jennifer Schultens, used with permission)

The following theorems have similar proofs as the corresponding ones in Chapter 5 on replacing $\mathcal{MS}(E, \gamma, \alpha)$ with $\mathcal{IS}(E, \gamma, \alpha)$.

Lemma 8.5.2. *Let $\rho \neq \rho'$ be adjacent vertices of $\mathcal{IS}(E, \gamma, \alpha)$ and consider any $\sigma \in \mathcal{IS}(E, \gamma, \alpha)$. Then either $\rho' <_\sigma \rho$ or $\rho <_\sigma \rho'$.*

Lemma 8.5.3. *If the vertex ρ is at a greater distance from σ than ρ' , then $\rho <_\sigma \rho'$.*

Lemma 8.5.4. *There are no $\rho^1, \rho^2, \dots, \rho^k$ for $k \geq 2$ satisfying*

$$\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^k <_{\sigma} \rho^1$$

Corollary 8.5.5. *The relation $<_{\sigma}$ can be extended to a linear order on $\mathcal{I}\mathcal{S}(E, \gamma, \alpha)$.*

8.6 Properties of the Projection Map

In this section, we state some useful properties of the projection map. Wherever the proofs are not given, they are the same as the proofs for the corresponding ones of the Kakimizu complex.

Lemma 8.6.1. *Let ρ, ρ' be adjacent vertices of $\mathcal{I}\mathcal{S}(E, \gamma, \alpha)$ such that ρ is different from some $\sigma \in \mathcal{I}\mathcal{S}(E, \gamma, \alpha)$. We assume that $\rho \leq_{\sigma} \rho'$. Then we have $\rho' \leq_{\sigma} \pi_{\sigma}(\rho)$. In particular, $\pi_{\sigma}(\rho)$ and ρ' are adjacent or equal.*

Corollary 8.6.2. *Let ρ, ρ' be adjacent vertices of $\mathcal{I}\mathcal{S}(E, \gamma, \alpha)$ distinct from some $\sigma \in \mathcal{I}\mathcal{S}(E, \gamma, \alpha)$. Assume that $\rho \leq_{\sigma} \rho'$. Then $\pi_{\sigma}(\rho) \leq_{\sigma} \pi_{\sigma}(\rho')$.*

Lemma 8.6.3. *Assume that there are vertices $\rho^1, \dots, \rho^k$ at some non-zero distance from $\sigma \in \mathcal{I}\mathcal{S}(E, \gamma, \alpha)$ satisfying*

$$\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^k$$

and

$$\pi_{\sigma}(\rho^1) = \pi_{\sigma}(\rho^k).$$

Then all the $\pi_{\sigma}(\rho^i)$ are equal and all the ρ^i are pairwise disjoint.

Proof. We are given that

$$\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^k$$

Then from Corollary 8.6.2

$$\pi_{\sigma}(\rho^1) \leq_{\sigma} \pi_{\sigma}(\rho^2) \leq_{\sigma} \dots \leq_{\sigma} \pi_{\sigma}(\rho^k)$$

But

$$\pi_{\sigma}(\rho^1) = \pi_{\sigma}(\rho^k) \Rightarrow \pi_{\sigma}(\rho^1) = \pi_{\sigma}(\rho^2) = \dots = \pi_{\sigma}(\rho^k) \text{ (from Lemma 8.5.4.)}$$

We are only left to show that all the ρ^i are adjacent. It suffices to show adjacency of ρ^1 and ρ^k .

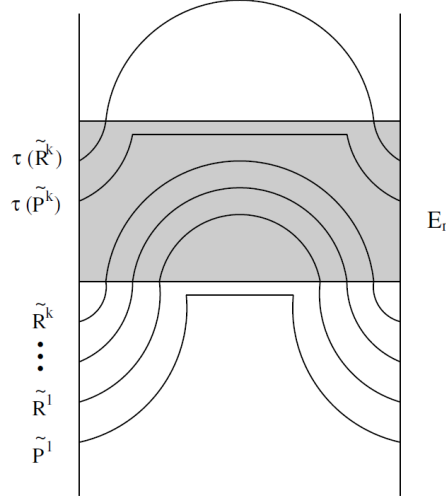


Figure 8.4: Setup for Lemma 8.6.3 (courtesy of Jennifer Schultens, used with permission)

We choose $E^{R^1}, \dots, E^{R^k}$ as in the proof of Lemma 8.5.4, ie, We inductively choose $R^k \in \rho^k$, $R^{k-1} \in \rho^{k-1}, \dots, R^1 \in \rho^1$ such that:

- Each R^i is almost transverse to S and has simplified intersection with S .
- $\forall k > i$, R^i is almost disjoint from R^{i+1} with simplified intersection. We take $r := \max \{n \in \mathbb{Z} \mid E_n \cap E^{R^k} \neq \emptyset\}$.

We inductively define E^{R^i} as the lift of $E \setminus R^i$ which is contained in $(\overline{E}^{R^{i+1}}) \cup \tau^{-1}(\overline{E}^{R^{i+1}})$. Then all the E^{R^i} intersect E_r as $\rho^1 <_\sigma \rho^2 <_\sigma \dots <_\sigma \rho^k$. $R^* \in \rho^k$ is chosen to be almost transverse to S with simplified intersection, and almost disjoint from R^1 with simplified intersection.

We define $\tilde{P}^1, \tilde{P}^k$ as the representative surfaces obtained by applying the projection map, as described previously. Then $\tau(\tilde{P}^k)$ is disjoint from E^{R^k} and is in the same isotopy class as $\tau(\tilde{P}^1)$. See Figure 8.4.

Hence, $\tilde{R}^1$ and $\tilde{R}^k$ are isotopic to $i(\tilde{R}^1)$ and $i(\tilde{R}^k)$, which are almost disjoint and contained in the closure of the lift of $E \setminus P$ bounded by $\tilde{P}_1$ and $\tau(\tilde{P}_1)$. Then

$$d(p(i(\tilde{R}^1)), p(i(\tilde{R}^k))) = 1.$$

□

Lemma 8.6.4. $\forall n > 0$, there is a constant l_n satisfying the following:

Let σ be any vertex of $\mathcal{SS}(E, \gamma, \alpha)$ and let $\rho^1, \dots, \rho^l$ be at a distance n from σ satisfying

$$\rho^1 <_{\sigma} \rho^2 <_{\sigma} \dots <_{\sigma} \rho^l.$$

Then $l \leq l_n$.

Lemma 8.6.5. For $\sigma \neq \rho$, $\sigma' \in \mathcal{SS}(E, \gamma, \alpha)$ with $d(\sigma', \rho) \leq d$ and $d(\sigma', \sigma) \leq d$, we have $d(\sigma', \pi_{\sigma}(\rho)) \leq d$.

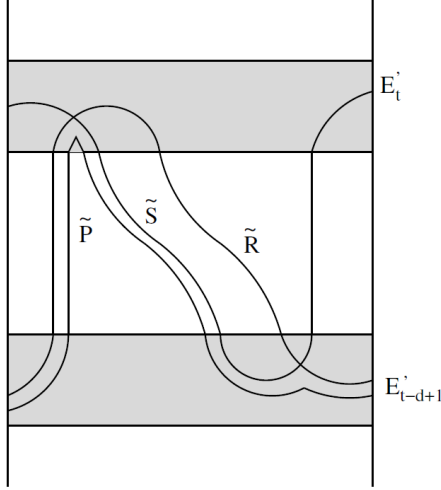


Figure 8.5: Setup for Lemma 8.6.5 (courtesy of Jennifer Schultens, used with permission)

Proof. Let $S \in \sigma$, $R \in \rho$ and $S' \in \sigma'$ be pairwise almost transverse surfaces with simplified intersection. $r, \tilde{P}, P$ are as in Sections 8.3 and 8.4. Let

$$t = \max \{k \in \mathbb{Z} \mid E'_k \cap (E^R \cap E_{r-1}) \neq \emptyset\}.$$

See Figure 8.5. We also have

$$\tilde{P} \subseteq [\tilde{R} \cap (\bigcup_{k \leq r-1} \overline{E}_k)] \cup [\tilde{S} \cap (\bigcup_{i \leq 0} \tau^i(\overline{E}^R))].$$

Thus, E^P is contained in $E^R \cap E_{r-1}$, with the $\overline{E}'_k$'s satisfying $t \geq k$. We are given that $d(S', R) \leq d$ and $d(S', S) \leq d$, so the k 's must satisfy $t - k \leq d$. □

Lemma 8.6.6. *Let $\sigma, \sigma' \in \mathcal{IS}(E, \gamma, \alpha)$ be adjacent. Let $\rho, \rho' \in \mathcal{IS}(E, \gamma, \alpha)$ also be adjacent, satisfying $\rho' <_{\sigma'} \rho$ and $\rho <_{\sigma} \rho'$. If $\sigma' \neq \rho'$, then:*

1. $\rho \leq_{\sigma} \pi_{\sigma'}(\rho')$.
2. If $\sigma \neq \rho'$, then $d(\sigma, \pi_{\sigma'}(\rho')) \leq d(\sigma, \rho')$.

Proof. Let $S \in \sigma$, $S' \in \sigma'$, $R \in \rho$, $R' \in \rho'$ be pairwise almost disjoint with simplified intersection. E_0 is a lift of $E \setminus S$ and E'_0 is the lift of $E \setminus S'$ contained in $\overline{E}_0 \cup \overline{E}_1$. Let r' be the largest integer such that $E'_{r'} = \tau^{r'}(E'_0)$ intersects a lift $E^{R'}$ of $E \setminus R'$. Further, E^R is the lift of $E \setminus R$ contained in $\overline{E}^{R'} \cup \tau^{-1}(\overline{E}^{R'})$. See Figure 5.7.

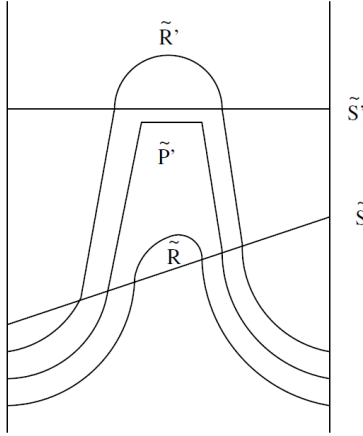


Figure 8.6: Setup for Lemma 8.6.6 (courtesy of Jennifer Schultens, used with permission)

$\rho' <_{\sigma'} \rho$ and $\rho <_{\sigma} \rho'$. So $E^R \cap E'_{r'} = \emptyset$, but $E^R \cap E_{r'} \neq \emptyset$.

$P' = p(\tilde{P}) \in \pi_{\sigma'}(\rho')$ is as obtained in Section 5.3. $E^{P'}$ is the lift of $E \setminus P'$ bounded by $\tilde{P}'$ and $\tau^{-1}(\tilde{P}')$. $E^R \cap E'_{r'} = \emptyset$, so $\tilde{P}' \subseteq \tau(\overline{E}^R)$. So ρ and $\pi_{\sigma'}(\rho')$ are equal or adjacent.

There is an isotopy i of P' such that $i(P')$ is almost transverse to S with simplified intersection, and almost disjoint from R with simplified intersection.

$E^{P'} \cap E_{r'+1} = \emptyset$, so the lift of $E \setminus i(P')$ in the isotopy class of $E^{P'}$ is also disjoint from $E_{r'+1}$. This lift contains $\tilde{R}$, which intersects $E_{r'}$, giving (1).

$E^{P'}$ intersects the same E_k as $E^{R'}$, giving (2). □

8.7 Contractibility

Whitehead's theorem implies that to prove contractibility of $\mathcal{IS}(E, \gamma, \alpha)$, it suffices to prove that all finite subcomplexes of $\mathcal{IS}(E, \gamma, \alpha)$ are contained in contractible subcomplexes of $\mathcal{IS}(E, \gamma, \alpha)$.

We recall the following definition of dismantlable graphs, as stated in Chapter 5

Definition 8.7.1. *A finite graph is called **dismantlable** if its vertices can be linearly ordered $x_0, \dots, x_m$ such that $\forall i \neq m, \exists j > i$ satisfying*

- x_j is adjacent to x_i .
- $\forall x_k$ adjacent to x_i with $k > i$, the vertex x_j is adjacent to or equal to x_k .

The following theorem, taken from [CO15], was also stated in Chapter 5.

Theorem 8.7.2. *Finite flag complexes with dismantlable 1-skeleta are contractible.*

Thus to prove contractibility of $\mathcal{IS}(E, \gamma, \alpha)$, it suffices to prove the following theorem.

Theorem 8.7.3. *The 1-skeleta of finite σ -complexes of $\mathcal{IS}(E, \gamma, \alpha)$ are dismantlable.*

The proof for this theorem is similar to that of Theorem 5.6.3.

This proves the claim of this chapter that the incompressible complex is contractible.

Chapter 9

Knots with Kakimizu complex homeomorphic to $\mathbb{R}$

In Chapter 3, we saw that the Kakimizu and incompressible complexes for the composite of two non-fibred knots are both homeomorphic to $\mathbb{R}$. In this chapter, we look at the converse to this result, ie, the types of knots that can have Kakimizu complex homeomorphic to $\mathbb{R}$. The main result of this chapter is original, though the technique used to obtain it is adapted from [Ban11].

9.1 Main Result

We are given a link whose Kakimizu complex is known to be homeomorphic to $\mathbb{R}$. We will prove the following theorem.

Theorem 9.1.1. *Let L be an oriented link such that every minimal genus Seifert surface for L is connected. If $\mathcal{MS}(L)$ is homeomorphic to $\mathbb{R}$, then L is a satellite of either a torus knot, a cable knot or a connected sum, such that the companion knot has a unique Seifert surface.*

We will roughly follow the proof given by Jessica Banks in her paper [Ban11].

9.2 Theorems used in the proof

The following theorem from [BZ85] will form the basis of our proof. It has already been stated Proposition 6.2.1 in Chapter 6.

Theorem 9.2.1. *Let K be a knot and M denote $\mathbb{S}^3 \setminus \text{int}\mathcal{N}(K)$. Suppose there exists an annulus S properly embedded in M that is not ∂ -parallel. If neither component of ∂S bounds a disc in ∂M , then K is a torus knot, a cable knot or a connected sum.*

We will also use the following theorem from [Wil08]. It has already been stated as Theorem 6.2.7 in Chapter 6.

Theorem 9.2.2. *Let L be an oriented link such that every minimal genus Seifert surface for L is connected, R be a minimal genus Seifert surface for L and M denote $\mathbb{S}^3 \setminus \text{int}\mathcal{N}(R)$. We fix a set $\rho_1, \dots, \rho_k$ of core curves of the annuli $\partial M \cap \partial\mathcal{N}(L)$, one for each link component.*

Then there is a triangulation of M such that every Seifert surface R' for L disjoint from R can be put into normal form, with $\partial R' = \bigcup_{i=1}^k \rho_i$.

Further, there is a finite set $\{R_1, \dots, R_m\}$ of surfaces in M with non-empty boundary contained in $\bigcup_{i=1}^k \rho_i$, and a finite set $\{S_1, \dots, S_n\}$ of closed surfaces in M , such that all these surfaces are incompressible and in normal form, and the following holds:

Any minimal genus Seifert surface R' for L in M with $\partial R' = \bigcup_{i=1}^k \rho_i$ and in minimal normal form can be expressed as

$$a_1 R_1 + \dots + a_m R_m + b_1 S_1 + \dots + b_n S_n$$

for some $a_i, b_i \in \mathbb{Z}_{\geq 0}$.

9.3 The Proof

Proof of Theorem 9.1.1. We are given that the Kakimizu complex has the following structure.

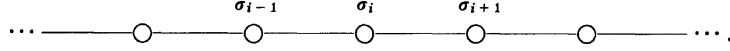


Figure 9.1: Given Kakimizu complex (courtesy of Osamu Kakimizu, used with permission)

Let $R \in \sigma_0$ be a minimal genus Seifert surface for L . Let $M = \mathbb{S}^3 \setminus \text{int}\mathcal{N}(R)$. We fix core curves $\rho_1, \dots, \rho_k$ of the annuli $\partial M \cap \partial\mathcal{N}(L)$, one for each link component, and apply Theorem 9.2.2. We assume that after a suitable reordering, S_1 has minimal genus out of all the S_i .

Let $R' \in \sigma_1$ be a minimal genus Seifert surface in minimal normal form that is disjoint from R . By Theorem 9.2.2, it is given by

$$R' = R_i + b_1 S_1 + \dots b_n S_n$$

for some $1 \leq i \leq m$ and $b_1 > 0$. We set $T := S_1$ and $R^- := R_i + (b_1 - 1)S_1 + b_2 S_2 + \dots + b_n S_n$.

$$\therefore R' = R^- + T$$

R^- and T are isotoped into minimal normal form, keeping $\partial R'$ fixed. As T is closed, $\partial R' = \partial R^-$, so the isotopy fixes ∂R^- . [JO84] says that no curve in $R^- \cap T$ can bound a disc in either R^- or T .

T cannot be a sphere, as in that case R^- and T can be isotoped to be disjoint, which contradicts the fact that R' is connected. $R' = R^- + T$, so $\chi(R') = \chi(R^-) + \chi(T)$. But both R' and R^- are minimal genus surfaces (ie, they have maximal Euler characteristic), so $\chi(T) = 0$, ie, T is a torus.

The component of $M \setminus \text{int}\mathcal{N}(T)$ that contains ∂M is denoted by M_0 and the other component by M_1 . R' inherits an orientation from L ; this gives an orientation on the components of $R' \cap M_0$ and, thus, on the curves of $R^- \cap T$. We take a curve ρ on T that intersects each curve of $R^- \cap T$ one time. As R' is disjoint from R , then so is T . If we denote the algebraic

intersection of ρ and R by $\rho.R$, then we must have $\rho.R = 0$. Algebraic intersection is preserved over homology classes, so

$$\rho.R' = 0, \rho.(R^- \cap T) = 0$$

on T . This means that there are an even number of curves in $R^- \cap T$, with half oriented in one direction and the other half in the other. We surger R^- along the subannulus of T between adjacent curves with opposite orientations to remove all the curves of $R^- \cap T$. This gives a new Seifert surface R'' and a closed, possibly disconnected, surface S'' .

R' is minimal genus, so

$$\chi(R') \geq \chi(R'') = \chi(R^-) - \chi(S'') = \chi(R') - \chi(T) - \chi(S'')$$

T is a torus, so its Euler characteristic is zero.

$$\therefore \chi(R') \geq \chi(R') - \chi(S'') \Rightarrow \chi(S'') \geq 0$$

Each of the components of $(R^- \cup T) \setminus (R^- \cap T)$ which are used to construct S'' have boundary and are not discs. So, they are annuli, including each component of $R^- \cap M_1$.

If possible, let S be an annulus in M_1 that is parallel to a sub-annulus S_T of T . Any other curves of $R^- \cap T$ in S_T bound annuli parallel to T , implying that $R^- \cap \text{int } S_T = \emptyset$. There are two boundary curves of S_T where we can perform cut-and-paste in either of two ways. This gives the following (see Figure 9.2):

- If both the operations join S and S_T , we get a torus component of R' , which contradicts its connectedness.
- If both the operations do not join the annuli, then there is an isotopy of R^- and T that reduces $R^- \cap T$ without changing R' , which contradicts the fact that R^- and T are in minimal normal form.
- If only one of the operations joins the annuli, we can isotope along the product region to reduce the weight of R' . This contradicts the fact that R' is in minimal normal form.

Thus, S cannot be ∂ -parallel in M_1 .

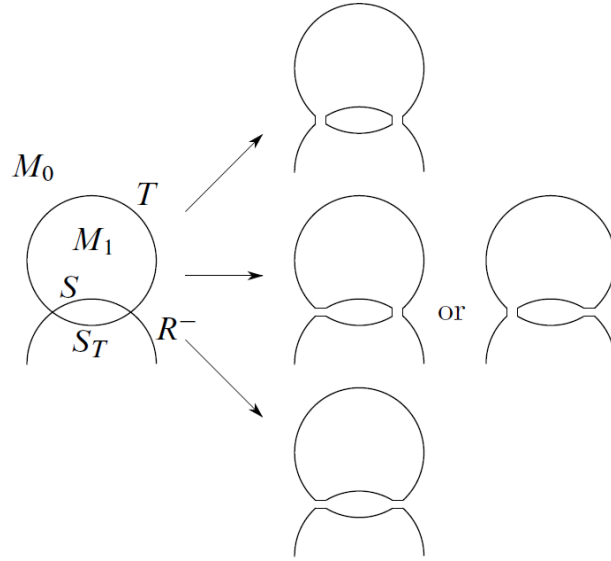


Figure 9.2: Cut-and-paste (courtesy of Jessica Banks, used with permission)

The part of $\mathbb{S}^3 \setminus \text{int} \mathcal{N}(T)$ containing L is a solid torus V , with core curve K . $R \subseteq V$ and T is incompressible, so L must be a satellite of K . In addition, S is not parallel to T , so it satisfies the hypotheses of Theorem 9.2.1.

If the companion knot has non-unique Seifert surface, we can spin one of them around the swallow-follow torus to give another, non-isotopic surface. Then the corresponding vertex in the Kakimizu complex becomes at least trivalent, which contradicts the fact that the Kakimizu complex is homeomorphic to $\mathbb{R}$. Thus, the companion knot must have unique Seifert surface.

This proves the theorem. □

Conclusions and Future Directions

In this thesis, we looked at the incompressible complex and the Kakimizu complex, which are constructed using isotopy classes of Seifert surfaces of a given link. These simplicial complexes have rich structural features (connectedness, contractibility, local infiniteness etc). We studied how some of these features can also give valuable information about the knot on which the complexes are constructed.

We also worked on extending the arguments for contractibility of the Kakimizu complex to show that the incompressible complex is also contractible. In addition, we also found a restriction on the kinds of links that can have Kakimizu complex homeomorphic to $\mathbb{R}$.

In the future, we would like to see if we can find restrictions on the kinds of links that have certain structures on the incompressible complex. Another question to look at is whether the incompressible complex is also quasi-Euclidean like the Kakimizu complex. Exploring these kinds of questions can give us interesting leads on which properties are common to both the complexes and where they are different. Ultimately, we would like to explore if and how these complexes could be useful in obtaining substantial information about the knots themselves, for example, in classifying knots.

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