A Study of Hyperbolic Geometry

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled A Study of Hyperbolic Geometry towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Anay Jain at Indian Institute of Science Education and Research under the supervision of Prof. Tejas Kalelkar, Associate Professor, Department of Mathematics, during the academic year 2025-2026.

Prof. Tejas Kalelkar

Committee:
Prof. Tejas Kalelkar
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Declaration

I hereby declare that the matter embodied in the report entitled A Study of Hyperbolic Geometry are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. Tejas Kalelkar and the same has not been submitted elsewhere for any other degree.

Anay Jain

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Abstract

We provide an expansive treatment of the fundamentals of hyperbolic geometry. The explicit computation of hyperbolic isometries enables us to find geodesics, calculate the metric, define the boundary, and compute the sectional curvatures of the canonical hyperbolic space in n dimensions. We also explore the theory of geometric structures on manifolds and state the rigidity theorems for hyperbolic manifolds. Lastly, we see an application of this theory in the triangulation of knot complements and work out a complete hyperbolic structure on the decomposition of the figure-8 knot complement into two ideal tetrahedra.

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Preface

This thesis is a meticulous treatment of existing work and none of the stated results are claimed to be original. The first three chapters only assume the knowledge of a first course in Riemannian geometry and some basic complex analysis. Some standard covering space theory ([Hat02]) is required for the next chapter. The last two chapters make use of most of the theory developed in the earlier chapters.

Chapter 0 provides some motivation for the study of hyperbolic geometry through results of Thurston and through knot theory, which [Pur20] elaborates on.

Chapters 1-4 form the bulk of the material and [BP92] is the main reference for the theory. Most proofs have been expanded for the sake of clarity and completeness. *Chapter 1* introduces the reader to three models of hyperbolic space and finds the isometries of the hyperboloid model. *Chapter 2* digresses into conformal geometry in order to explicitly compute the self-isometries of the other two models. This also allows for a calculation of their Riemannian metrics. *Chapter 3* obtains the geodesics in hyperbolic space and defines a canonical boundary of $\mathbb{H}^n$. It also classifies hyperbolic isometries based on the number of fixed points in the boundary and introduces the concept of horospheres. Lastly, it contains rather surprising results related to the curvature of hyperbolic space. *Chapter 4* introduces (X, G)-structures on manifolds and proves the existence of a developing map. Important results are proved for complete geometric structures and weaker generalizations are made incomplete structures. A proof of Mostow's rigidity theorem is briefly sketched.

Chapters 5-6 provide concrete applications of the theory by introducing polyhedral decompositions of knot complements. We start with a simple decomposition of the figure-8 knot complement into two topological polyhedra and sketch a generalization to all knots. We show how this can be subdivided into an ideal triangulation and discuss when this triangulation admits a complete hyperbolic structure. [Pur20] is the primary reference.

Chapter 0

Motivation

Introduction

A thesis spanning over a hundred pages of hyperbolic geometry begs the question "why study hyperbolic geometry in the first place?" The answer, for a low-dimensional topologist, stems from the work of William Thurston, who conjectured in 1982 that every closed 3-manifold can be canonically decomposed into pieces that each have one of eight types of geometries. Six of these geometries are accounted for by Seifert fiber spaces, which are relatively well understood. A seventh one, the *Sol* geometry is somewhat niche and also mostly classified. This leaves us with hyperbolic geometry, the most diverse and commonly occurring geometry. Mostow-Prasad rigidity establishes that if a connected 3-manifold admits a finite volume hyperbolic structure on it, then this structure is unique, making the hyperbolic geometry of such a manifold a topological invariant.

Additionally, hyperbolic geometry is crucially used in fields of physics such as special relativity in order to model spacetime. Unlike in dimension ≥ 3, two dimensional manifolds admit an infinite family of distinct hyperbolic structures which form the Teichmüller space. The study of this space is a central pillar of modern low-dimensional topology and string theory. The rest of this chapter showcases the occurrence of hyperbolic geometry in knot theory.

Knot Theoretic Definitions

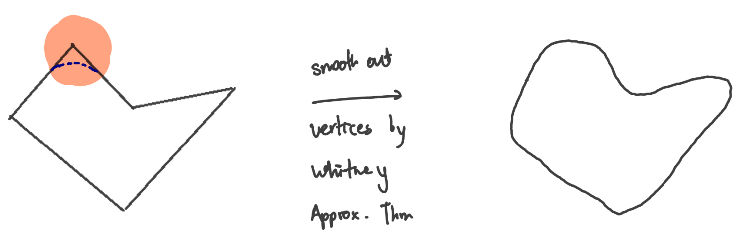
Definition 0.1 A **knot** $K \subseteq S^3$ is the homeomorphic image of S^1 under a PL (piecewise-linear) map $f : S^1 \longrightarrow S^3$, where S^3 has the PL structure of $\mathbb{R}^3 \cup \infty$.

We often use the symbol K to denote both the map f and its image $f(S^1)$.

Remark 0.1 We usually get the same results while working with smooth embeddings: $S^1 \longrightarrow S^3$ instead of PL ones. We sketch a proof for the equivalence of PL and smooth embeddings.

PL $\Rightarrow$ smooth

At each vertex, this modification within a ball can be undertaken via ambient isotopy fixing everything outside the ball. Doing this for each vertex and composing, we get an ambient isotopy from the left loop to right loop.



[Lee12]

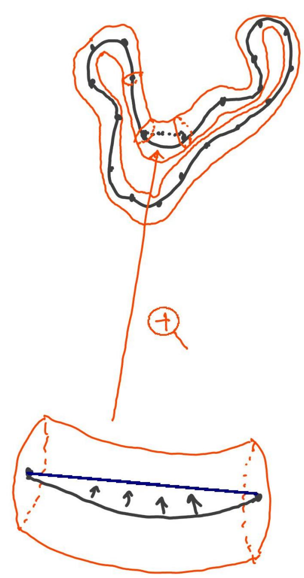
Smooth $\Rightarrow$ PL

Any smooth loop γ in a C^∞ manifold M admits a tubular neighbourhood in M. Assume $M = \mathbb{R}^3 \cup \infty$ and tr $\gamma \subseteq \mathbb{R}^3$.

- **Tubular neighbourhood theorem:** Any C^∞ embedded curve in a C^∞ manifold M has a tubular neighbourhood.

  Proof. Restrict exp : $TM \to M$ which is a local diffeomorphism, by the inverse function thoerem, to the normal bundle N. $\exists\, \varepsilon > 0$ such that $\forall$ vectors of length $< \varepsilon$, exp is a diffeomorphism, giving the required neighbourhood.

- Take a tubular neighbourhood around the knot and find finitely many points on the knot, such that the line seg-



ment between consecutive points lies entirely within the tubular neighbourhood.

- Now, construct an isotopy from each segment on the knot to the corresponding line segment between consecutive points within a slice of the tubular neighbourhood.

Then the C^∞ knot is ambient isotopic to a PL one.

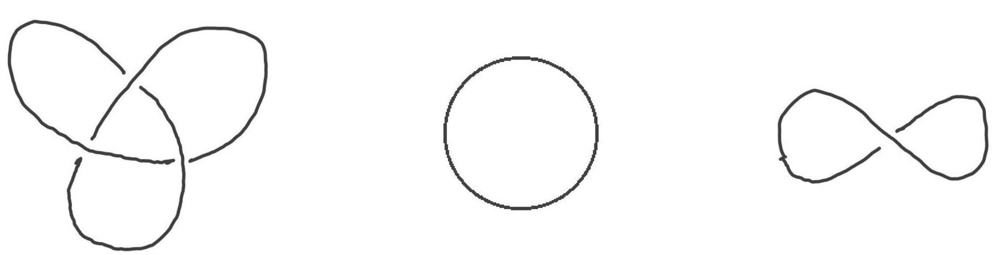
Definition 0.2 We say that 2 knots K_1 and K_2 are equivalent if they are ambient isotopic, i.e., there is a PL or C^∞ homotopy $H : S^3 \times [0,1] \to S^3$ such that $H_t := H_{|S^3 \times \{t\}}$ is a diffeomorphism $\forall \in [0,1]$, $H_0 = \mathrm{id}$ and $H_1(K_1) = K_2$.

Definition 0.3 $K \to$ knot in S^3, $N(K) \to$ an open tubular neighbourhood of K

- $S^3 \setminus N(K)$ is called the **knot exterior**
- $S^3 \setminus K$ is called the **knot complement**

Note that any knot exterior is a compact 3-manifold with boundary homeomorphic to the torus. The knot complement is homeomorphic to the interior of the knot exterior. For many years, it was an open question whether knots with homeomorphic complements are homeomorphic. Gordon and Luecke gave an affirmative answer to this in 1989. This fact is not true for links. We remark that the required homeomorphism must be orientation-preserving. However, to avoid this technicality, we often consider knots up to reflection in this thesis.

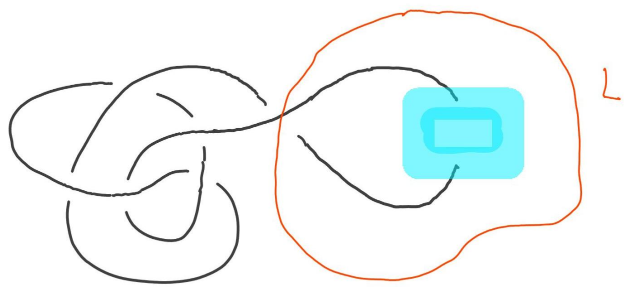
Definition 0.4 A **knot diagram** is a graph comprising only degree 4 vertices (4-valent), along with overcrossing / undercrossing information at each vertex.



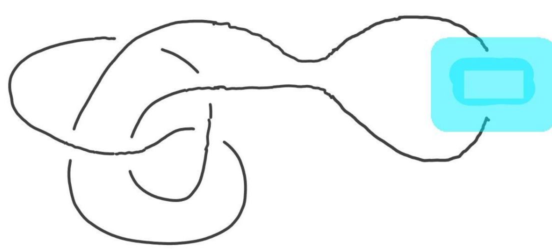
The Classification Problem

The main problem in knot theory is finding out when two different descriptions (like diagrams) of knots correspond to equivalent knots.
Are the complements of two given knots homeomorphic?
This is a difficult question, as the set of homeomorphisms can be huge. However, for a certain class of knots (hyperbolic knots), Mostow-Prasad Rigidity (Theorem 4.16) tells us that homeomorphisms are homotopic to isometries, and as the set of isometries is smaller than the set of homeomorphisms, this makes our job easier. Since we usually work with knot diagrams, it is worth exploring some basic simplification of diagrams.

Definition 0.5 A **reducible crossing** is a crossing such that, in the planar diagram, there exists a C^∞ loop L through the crossing that does not intersect the knot anywhere else.

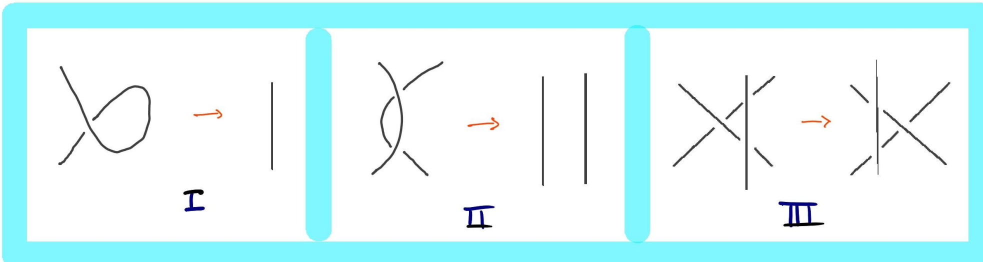


Note that this crossing can be removed by simply twisting the part of the knot enclosed in the loop.

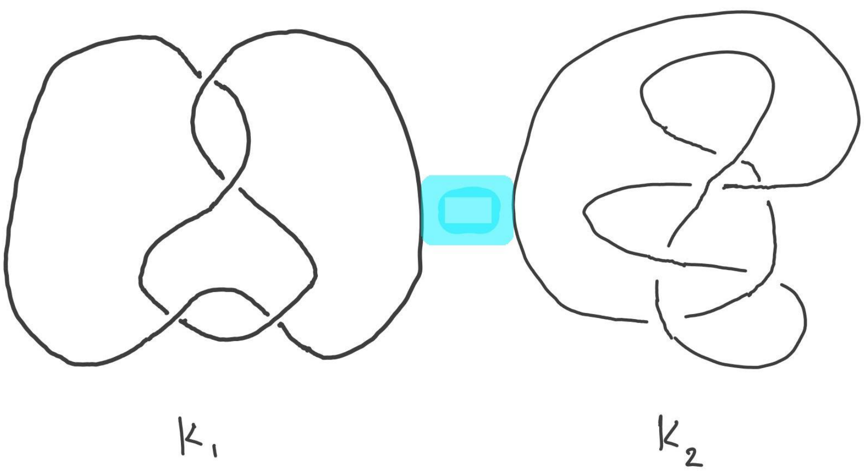


A diagram is **reduced** if it has no reducible crossings. We usually assume our knot diagrams to be reduced.

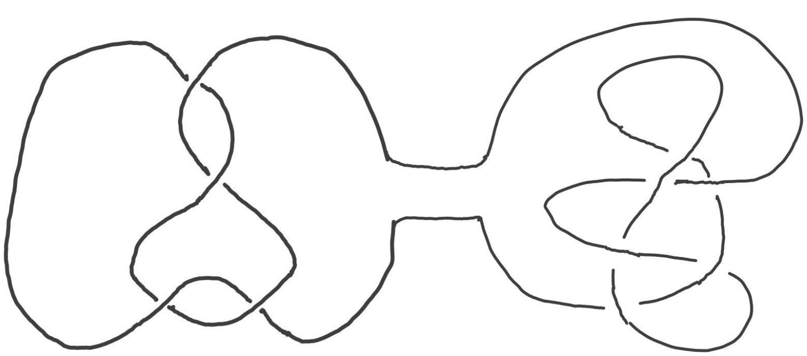
There are three Reidemeister moves on diagrams, which can, in theory distinguish between any two knot diagrams, i.e. any equivalent knots are related by a sequence of Reidemeister moves ([Rei27]).



We now describe an operation that can be performed on two knots, resulting in a new knot.



Starting with planar diagrams K_1 and K_2, draw a rectangle such that one edge is a segment of K_1 avoiding crossings, its opposite edge is a segment of K_2 avoiding crossings, and the rest of the rectangle does not intersect K_1 or K_2. We remove the two vertical arcs in the rectangle and join the resultant endpoints with the horizontal edges of the rectangle.

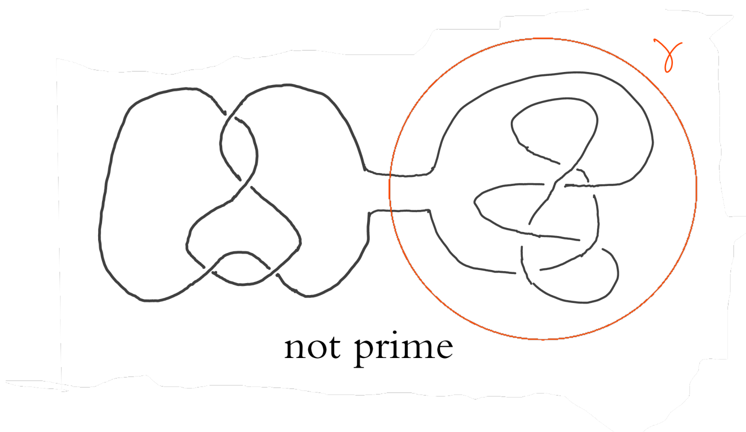


$K_1 \# K_2$

Definition 0.6

i) The result of the operation described above is called the **connected sum** or **knot sum** of K_1 and K_2, denoted by $\mathbf{K_1 \# K_2}$.

ii) If a knot K cannot be written as $K = K_1 \# K_2$ for any knots K_1 and K_2, then K is said to be **prime**. An equivalent condition is that if γ is an embedded, planar, simple closed curve intersecting K transversally at exactly two points, then γ bounds a disc containing no crossings.

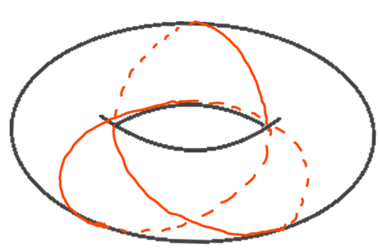


For the sake of classification, we usually consider prime knot diagrams.

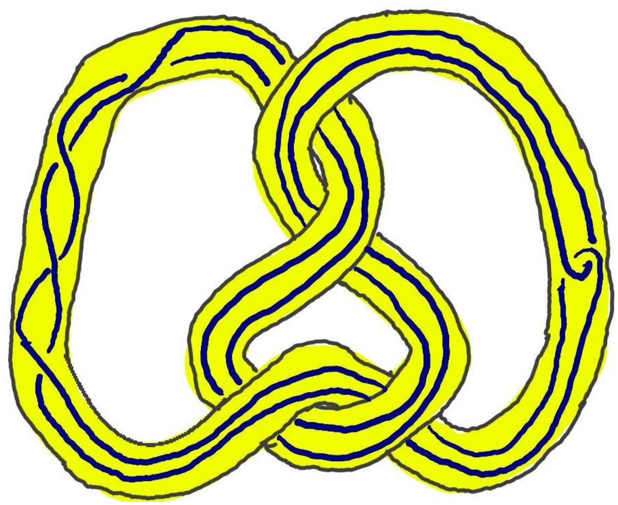
Using certain geometric techniques, Thurston ([Thu82]) showed that there are precisely three different categories of knots in S^3.

Definition 0.7

i) Torus knots are the knots that can be embedded in a torus $\subseteq S^3$.



The trefoil : a torus knot



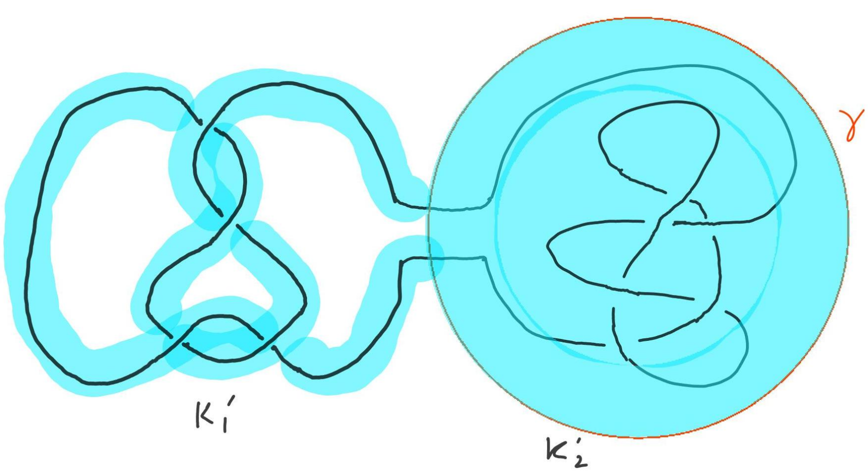
ii) Let N be a regular neighbourhood of a non-trivial knot K. A satellite knot is a knot $\neq K$, embedded non-trivially in N.

iii) Hyperbolic knots are knots whose complement in S^3 admits a hyperbolic structure (see chapter 4).

From Chapter 5 onward, we will see how to determine a hyperbolic metric on a knot complement and find when this metric is complete.

Lemma 0.8 Any knot sum of non-trivial knots is a satellite knot.

Proof. Suppose that we have the knot sum $K_1 \# K_2$. Let K_1' and K_2' be the parts of K_1 and K_2 in $K_1 \# K_2$, respectively. There exists a loop γ intersecting $K_1 \# K_2$ twice and bounding K_2', as shown below.



This loop can be made a sphere S bounding K_2' in S^3. We fill it up to form a ball B. Now, we take a regular neighbourhood N of K_1' and notice that $N \cup B$ is a regular neighbourhood of K_1. However, $K_1 \# K_2$ is embedded non-trivially in $N \cup B$, and as K_2 is non-trivial, $K_1 \# K_2 \neq K_1$. Thus, $K_1 \# K_2$ is a satellite knot orbiting around K_1. $\square$

The torus constructed above is called a *swallow-follow torus* as it swallows K_2 and follows K_1.

By Gordon-Luecke's theorem and Mostow Prasad rigidity, a complete hyperbolic structure on a knot complement completely determines the knot. An active area of research is finding hyperbolic invariants such as the volume and systole length of a knot complement from the diagram of the knot. Popular computer applications to study hyperbolic knots such as SnapPy rely on computing hyperbolic ideal triangulations of the knot complements to study such manifolds, which we elaborate on in chapters 5 and 6.

Hyperbolic geometry is therefore one of the most commonly found geometries in 3-manifolds after decomposing the manifold into its geometric pieces and it has applications in the study of knots and surfaces.

Chapter 1

Models of Hyperbolic Space and Linear Isometries

The Models of Hyperbolic Space

We present three models of hyperbolic n-space and prove an important theorem on the rigidity of local isometries. We usually only talk about $n \in \mathbb{N} \setminus \{1\}$ to avoid trivialities.

Definition 1.1 In $\mathbb{R}^{n+1}$ we define

$$\langle x \mid y \rangle_{(n,1)} = \left(\sum_{i=1}^{n} x_i y_i \right) - x_{n+1} y_{n+1}$$

where x_i, y_i are the coordinates of x and y, respectively.

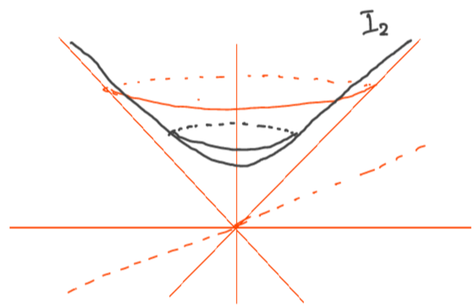
$C^\infty(\mathbb{R}^{2n+2})$ is a ring, i.e., the sums and products of smooth real-valued functions are smooth, and thus

$$\begin{aligned} \langle \cdot \mid \cdot \rangle_{(n,1)} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} &\longrightarrow \mathbb{R} \\ (x,\ y) &\longmapsto \left(\sum_{i=1}^{n} x_i y_i \right) - x_{n+1} y_{n+1} \end{aligned}$$

is smooth.

Now we define

$$I_n := \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \langle x \mid x \rangle_{(n,1)} = -1,\ x_{n+1} > 0 \right\}$$



Theorem 1.2

i) I_n is an oriented regular C^∞ submanifold of $\mathbb{R}^{n+1}$.

ii) $T_x I_n = \left\{ y \in \mathbb{R}^{n+1} \;\middle|\; \langle x \mid y \rangle_{(n,1)} = 0 \right\}$.

iii) $\langle \cdot \mid \cdot \rangle_{(n,1)}$ restricted to $T_x I_n$ is an inner product $\forall \;\; x \in I_n$ (Riemannian metric).

This makes I_n an n-dimensional Riemannian manifold, denoted I_n and called the **hyperboloid model** of the hyperbolic n-space.

Proof: **i)** Let

$$\mathbb{R}^{n+1} \xrightarrow{C^\infty} \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \xrightarrow{\langle \cdot | \cdot \rangle_{(n,1)}} \mathbb{R}$$

$$x \longmapsto (x, x) \longmapsto \langle x \mid x \rangle_{(n,1)}.$$

Then

$$\text{Jac } f = [2x_1, 2x_2, \ldots, 2x_n, -2x_{n+1}]$$

$$= [0, 0, \ldots, 0] \quad \text{iff} \quad (x_1, \ldots, x_n) = \vec{0}.$$

But $\vec{0} \notin f^{-1}(-1)$.
So -1 is a regular value and I_n is an oriented regular C^∞ submanifold of $\mathbb{R}^{n+1}$.
ii) If $y \in \mathbb{R}^{n+1}$, then $y \in T_x I_n$ if and only if $\nabla f(x) \cdot y = 0$, i.e.,

$$2 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ -x_{n+1} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = 0$$

$$\text{or} \quad \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1} = 0.$$

iii) Say $y \in T_x I_n$ is non-zero.
To prove: $\langle y \mid y \rangle_{(n,1)} > 0$ or $\sum y_i^2 > y_{n+1}^2$.
We know $\langle x \mid x \rangle_{(n,1)} = -1$,

$$\Rightarrow x_{n+1}^2 > \sum x_i^2$$

Say $y_{n+1}^2 \geq \sum y_i^2$. Multiplying these inequalities, we get

$$x_{n+1}^2 y_{n+1}^2 > \left(\sum x_i^2\right)\left(\sum y_i^2\right) \geq \left(\sum x_i y_i\right)^2 \quad \text{(Cauchy-Schwarz).}$$

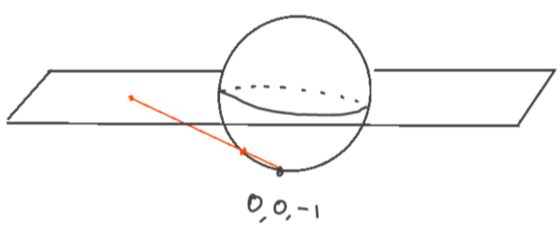
$$\Rightarrow x_{n+1} y_{n+1} > \sum x_i y_i,$$

which contradicts $\langle x \mid y \rangle_{(n,1)} = 0$. □

Now recall the stereographic projection of S^2 (minus a point) onto the plane. We perform the projection with respect to $(0, 0, -1)$ onto the plane $z = 0$, then the map is given by

$$f : S^2 \setminus \{(0, 0, -1)\} \longrightarrow \mathbb{R}^2$$

$$(x, y, z) \longmapsto \left(\frac{x}{1+z}, \frac{y}{1+z}, 0\right).$$



Similarly, we have a general stereographic projection

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$$

$$(x_1, x_2, \ldots, x_n, x_{n+1}) \longmapsto \frac{1}{1 + x_{n+1}} \cdot (x_1, x_2, \ldots, x_n, 0)$$

Let $\pi = f|_{\mathbb{I}^n}$ and omit the last coordinate so that $\operatorname{Im} \pi \subset \mathbb{R}^n$.

Observe $\forall \ x \in \mathbb{I}^n$,

$$\|\pi(x)\| = \frac{1}{1+x_{n+1}} \sqrt{\sum_{i=1}^{n} x_i^2}$$

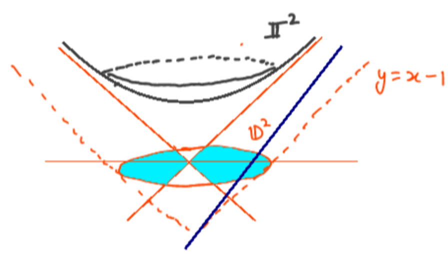
(no modulus required as $x_{n+1} \geq 1$).
But $\langle x \mid x \rangle_{(n,1)} = -1 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i^2 = x_{n+1}^2 - 1$
So

$$\|\pi(x)\| = \frac{\sqrt{x_{n+1}^2 - 1}}{x_{n+1}+1} = \left(\frac{x_{n+1}-1}{x_{n+1}+1}\right)^{1/2} < 1$$

In fact,

$$\operatorname{Im} \pi = D^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

and it can be shown $\pi : \mathbb{I}^n \to D^n$ is a diffeomorphism.
D^n with pullback Riemannian metric with respect to π^{-1} is the **disc model** of $\mathbb{H}^n$.



The blue line intersects $\mathbb{I}^2$ if we go high enough, illustrating the surjectivity of π

Lastly, we have the **half-space model** derived from the C^∞ map

$$i : D^n \longrightarrow \mathbb{R}^n$$

$$x \longmapsto \frac{2(x+e_n)}{\|x+e_n\|^2} - e_n$$

where $e_n = (0, 0, \ldots, 0, 1)$. We will later define inversion with respect to a sphere, and i is precisely inversion with respect to the sphere of radius $\sqrt{2}$ centered at $-e_n$..
Then we can check i is a diffeomorphism of D^n onto the open half-space

$$\Pi^n := \{x \in \mathbb{R}^n \mid x_n > 0\}.$$

Give it the pullback metric with respect to i^{-1} and orient it canonically in $\mathbb{R}^n$.

Isometries of the Hyperboloid Model

Definition 1.3 $M \to$ Riemannian manifold

$$\mathcal{I}(M) := \text{set of isometric diffeomorphisms (isometries) of } M \text{ onto itself.}$$

$$\mathcal{I}^+(M) := \text{set of orientation preserving isometries if } M \text{ is oriented.}$$

Both $\mathcal{I}(M)$ and $\mathcal{I}^+(M)$ are groups, as isometries and orientation behave well with respect to composition.

$$f : M \to M \quad \text{preserves orientation if and only if}$$

$$(e_1, e_2, \ldots, e_n)_p \xrightarrow{\text{A (change of basis)}} (f_*(e_1), f_*(e_2), \ldots, f_*(e_n))_{f(p)}$$

is such that $\det A = +1 \quad \forall\, p \in M$.

Theorem 1.4 $M, N \to$ Riemannian manifolds of dimension n.
Assume M is connected and $\phi_1, \phi_2 : M \to N$ are local isometries onto their range. Suppose that

$$\phi_1(x_0) = \phi_2(x_0), \quad (\phi_1)_{*,x_0} = (\phi_2)_{*,x_0} \quad \text{for some } x_0 \in M.$$

Then $\phi_1 = \phi_2$.

Proof. Let

$$S := \{x \in M \mid \phi_1(x) = \phi_2(x),\ (\phi_1)_{*,x} = (\phi_2)_{*,x}\}.$$

Note

$$\phi : M \to N \times N$$

$$x \mapsto (\phi_1(x), \phi_2(x))$$

is continuous.
Since N is Hausdorff, Δ is closed in $N \times N$, and thus

$$\phi^{-1}(\Delta) = \{x \in X \mid \phi_1(x) = \phi_2(x)\} \subset M$$

is closed. The same applies for $(\phi_1)_{*,x} = (\phi_2)_{*,x}$, and thus S is closed.
Since $x_0 \in S$, S is nonempty.
We now show S is open in M as well.

Choose $x \in S$. As ϕ_1, ϕ_2 are local isometries, we can choose a neighborhood U of x such that $\phi_i : U \to \phi_i(U)$ is an isometry for $i \in \{1,2\}$, and $V_i := \phi_i(U) \overset{\text{open}}{\subset} N$.
By intersecting if necessary, without losing generality, we may assume that U is a normal neighborhood
(i.e. $\exp_x : W \overset{\text{open}}{\subset} T_xM \to U$ is a diffeomorphism).
$\phi_{1|U}, \phi_{2|U}$ are both isometries, let $f := \phi_{1|U}, g := \phi_{2|U}$.

$$\phi_1(x) = \phi_2(x), \quad (\phi_1)_{*,x} = (\phi_2)_{*,x}.$$

Let γ be any geodesic starting at x in some open set $\widetilde{U} \subset U$. Then $f \circ \gamma, g \circ \gamma$ are geodesics in V_1 and V_2.
In fact $(f \circ \gamma)_{*,0} = f_{*,x} \circ \gamma'(0) = g_{*,x} \circ \gamma'(0) = (g \circ \gamma)_{*,0}$, so that, due to the uniqueness of geodesics with initial conditions, $f \circ \gamma = g \circ \gamma$ in some neighborhood $\widetilde{V}$ of $f(x) = g(x)$.
Say $\widetilde{W} \overset{\text{open}}{\subset} W$ is such that $\exp_x(\widetilde{W}) = \widetilde{U}$.
This gives us

$$f \circ (\exp_x)|_{\widetilde{W}} = g \circ (\exp_x)|_{\widetilde{W}}$$

and thus

$$f|_{\widetilde{U}} = g|_{\widetilde{U}} \quad \text{(right multiply by } \exp_x^{-1}\text{)}.$$

So, $(\phi_1)_{|\widetilde{U}} = (\phi_2)_{|\widetilde{U}}$ and $\widetilde{U} \subset S$ is an open neighborhood of x. Thus, S is clopen and contains x_0, which means $S = M$, as M is connected. □

V is a finite-dimensional vector space and $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form.

Definition 1.5. $\mathrm{O}(V, \langle \cdot, \cdot \rangle) := \{A \in \mathrm{GL}(V) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \ \forall \ \ x, y \in V\}$
If $V = W \oplus W'$ with the standard projection $p : V \to W$, then the **reflection** about W is the linear map

$$\begin{aligned} s : V &\to V \\ v &\mapsto 2p(v) - v \end{aligned}$$

(notice that $s^2 = \mathrm{id}$)

Theorem 1.6. $s \in \mathrm{O}(V, \langle\cdot,\cdot\rangle)$ if and only if

$$W' = \{v \in V \mid \langle v, w\rangle = 0 \;\forall\;\; w \in W\} = W^\perp$$

Proof. We want to see when

$$\langle 2p(x) - x, 2p(y) - y\rangle = \langle x, y\rangle \quad \forall \;\; x, y \in V$$

Let $v, w \in W$, $v', w' \in W'$ such that $x = v + v'$ and $y = w + w'$

$$\mathrm{LHS} = \langle 2v - (v + v'), 2w - (w + w')\rangle = \langle v - v', w - w'\rangle$$

$$\mathrm{RHS} = \langle v + v', w + w'\rangle$$

So LHS = RHS if and only if

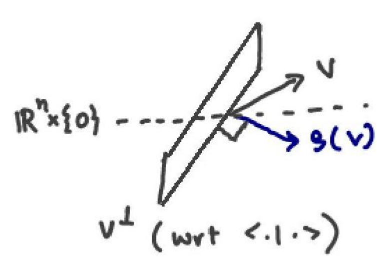
$$2(\langle v', w\rangle + \langle w', v\rangle) = 0 \quad (*)$$

$\forall \;\; v, w \in W \; and \; v', w' \in W'$
Putting $v' = 0$, we want $\langle w', v\rangle = 0 \;\forall\;\; w' \in W',\; v \in V$

$$\Rightarrow W' = W^\perp$$

Conversely, if $W' = W^\perp$ then clearly $*$ is satisfied. □

When $W' = W^\perp$, we call s the reflection with respect to W or the reflection parallel to $W^\perp$. From now on we consider reflections only with respect to a hyperplane (parallel to $\mathbb{R}v$ for some $v \in V$ such that $\langle v, v\rangle \neq 0$).
Note that such a reflection parallel to v in $(\mathbb{R}^{n+1}, \langle \cdot \mid \cdot \rangle_{(n,1)})$ is the same as the standard reflection parallel to $\rho(v)$ with respect to the Euclidean inner product, where ρ is reflection with respect to $\mathbb{R}^n \times \{0\}$.



Let V be an n-dimensional real vector space.
$\langle\cdot,\cdot\rangle \to$ non-degenerate symmetric bilinear form on V
By some basic linear algebra, $\exists\, p \le n$ and a basis $\{v_1, v_2, \dots, v_n\}$ of V such that

$$\langle v_i, v_j\rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{if } i = j \le p \\ -1 & \text{if } i = j > p \end{cases} \qquad (\star)$$

$q := n - p$ and (p, q), called the signature of $\langle\cdot,\cdot\rangle$ is independent of the choice of basis

Theorem 1.7 $\mathrm{O}(V, \langle\cdot,\cdot\rangle)$ is generated by reflections.
Proof. Let $\{v_1, \dots, v_n\}$ be a basis of V satisfying $\star$

- We first show $-I$ is the composition

$$s_{v_n} \circ s_{v_{n-1}} \circ \cdots \circ s_{v_1}$$

where s_{v_i} is the reflection parallel to v_i.

If $v \in V$ then

$$v = \langle v, v_1\rangle v_1 + \cdots + \langle v, v_p\rangle v_p - \langle v, v_{p+1}\rangle v_{p+1} - \cdots - \langle v, v_n\rangle v_n$$

i.e.

$$v = \begin{bmatrix} \langle v, v_1\rangle \\ \vdots \\ \langle v, v_p\rangle \\ -\langle v, v_{p+1}\rangle \\ \vdots \\ -\langle v, v_n\rangle \end{bmatrix} \quad \textit{and} \quad s_{v_1}(v) = \begin{bmatrix} -\langle v, v_1\rangle \\ \langle v, v_2\rangle \\ \vdots \\ \langle v, v_p\rangle \\ -\langle v, v_{p+1}\rangle \\ \vdots \\ -\langle v, v_n\rangle \end{bmatrix}$$

Similarly

$$s_{v_n} \circ s_{v_{n-1}} \circ \cdots \circ s_{v_1}(v) = \begin{bmatrix} -\langle v, v_1 \rangle \\ -\langle v, v_2 \rangle \\ \vdots \\ -\langle v, v_p \rangle \\ \langle v, v_{p+1} \rangle \\ \vdots \\ \langle v, v_n \rangle \end{bmatrix} = -v$$

- We proceed by induction on $n = \dim V$.

  When $n = 1$, $\mathrm{GL}(V) = \{[a] \mid a \in \mathbb{R}\}$ and the only orthogonal elements are $[\pm 1]$
  As $[1] = [-1] \circ [-1]$, the proposition follows.
  Now assume the proposition $\forall \ m \leq n$ and say $\dim V = n + 1$.

  Let $A \in \mathrm{O}(V, \langle \cdot, \cdot \rangle)$.

  Let $v \in V$ such that $\langle v, v \rangle \neq 0$ (we can choose it to be one of the basis elements such that $\langle v, v \rangle = \pm 1$).

- Assume $\langle v - Av, v - Av \rangle \neq 0$

  If it is zero, then $\langle v + Av, v + Av \rangle \neq 0$, otherwise

$$\begin{gathered} \langle v, v \rangle + \langle Av, Av \rangle = 2\langle Av, v \rangle = -2\langle Av, v \rangle \\ \text{But } \langle Av, Av \rangle = \langle v, v \rangle \\ \Rightarrow 2\langle v, v \rangle = 0 \Rightarrow\Leftarrow \end{gathered}$$

  Then A can be replaced with $-A$, but if A is a product of reflections, so is $-A = -I \cdot A$ by the first remark.

- Define ρ as the reflection parallel to $v - Av$

  Notice $\langle v - Av, v + Av \rangle = \langle v, v \rangle - \langle Av, Av \rangle = 0$

  Also $v = \frac{1}{2}(v - Av) + \frac{1}{2}(v + Av)$

  Then $\rho(v) = \frac{1}{2}(v + Av) - \frac{1}{2}(v - Av) = Av$

  So $\rho \circ A(v) = \rho^2(v) = v$

  Note $\rho \circ A \in \mathrm{O}(V, \langle \cdot, \cdot \rangle)$ and then $\rho \circ A|_{v^\perp} \in \mathrm{O}(v^\perp, \langle \cdot, \cdot \rangle)$

Now $v^\perp$ is an n-dimensional hyperplane in V, so by the induction hypothesis,

$$\rho \circ A|_{v^\perp} = \rho_1 \circ \rho_2 \cdots \rho_m \quad \text{for some reflections } \rho_i \text{ in } v^\perp$$

Observe that each ρ_i can be extended to $V = \mathbb{R}v \oplus v^\perp$ by defining $\tilde{\rho}_i(v) = v$, $\tilde{\rho}_{i|v^\perp} = \rho_i$.

If ρ_i were a reflection parallel to some w_i in $v^\perp$ then $\tilde{\rho}_i$ is the reflection parallel to w_i in V

($w_i \in v^\perp$, so the reflection parallel to w_i fixes v).

In fact, as $\rho \circ A(v) = v$, $\rho \circ A = \tilde{\rho}_1 \circ \tilde{\rho}_2 \circ \cdots \circ \tilde{\rho}_m$

$$\Rightarrow \rho^2 \circ A = A = \rho \circ \tilde{\rho}_1 \cdots \tilde{\rho}_m$$

is a product of reflections. □

Definition 1.8 Setting $V = \mathbb{R}^{n+1}$ and $\langle \cdot, \cdot \rangle = \langle \cdot \mid \cdot \rangle_{(n,1)}$ we define $\mathrm{O}(I_n)$ as

$$\mathrm{O}(I_n) := \{A \in \mathrm{O}(\mathbb{R}^{n+1}, \langle \cdot \mid \cdot \rangle_{(n,1)}) \mid A(I_n) = I_n\}$$

$$\mathrm{SO}(I_n) := \mathrm{O}(I_n) \cap \mathrm{SL}_{n+1}(\mathbb{R})$$

Remark 1.8 $\mathrm{O}(\mathbb{R}^{n+1}, \langle \cdot \mid \cdot \rangle_{(n,1)})$ is the same as $\mathrm{O}(n+1)$ because we showed that reflections parallel to non-singular vectors with respect to $\langle \cdot \mid \cdot \rangle_{(n,1)}$ are Euclidean reflections parallel to the reflection of the vectors about $\mathbb{R}^n \times 0$ and both orthogonal groups are generated by such reflections due to Theorem 1.7. We will sometimes omit the subscript while referring to the bilinear form $\langle \cdot \mid \cdot \rangle_{(n,1)}$.

Theorem 1.9 $\mathrm{O}(I_n)$ is generated by the reflections it contains.

Proof.

- Reflections preserve any bilinear form as we have shown before. So,

$$\langle x \mid x \rangle = -1 \Rightarrow \langle \rho(x) \mid \rho(x) \rangle = -1 \quad \forall \quad x \in \mathbb{R}^{n+1},\ \rho \in \mathrm{O}(\mathbb{R}^{n+1}, \langle \cdot \mid \cdot \rangle)$$

Then $\rho(I_n \cup I_n) \subseteq I_n \cup (-I_n)$

Also $I_n \cup (-I_n) = \rho^2(I_n \cup (-I_n)) \subseteq \rho(I_n \cup (-I_n))$

$$\Rightarrow \rho(I_n \cup (-I_n)) = I_n \cup (-I_n)$$

As $\mathrm{O}(\mathbb{R}^{n+1}, \langle \cdot \mid \cdot \rangle)$ is generated by reflections parallel to v such that $\langle v \mid v \rangle \neq 0$, we analyze these reflections.

- We have shown that any such reflection keeps $I_n \cup (-I_n)$ invariant.

  Now we show a reflection parallel to v exchanges the 2 components if and only if $\langle v \mid v \rangle < 0$.

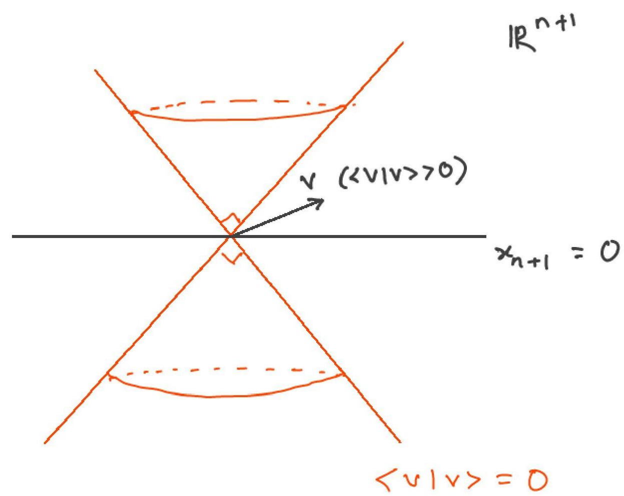
  If $\langle v \mid v \rangle < 0$ then define $w := \frac{T(v)}{\sqrt{|\langle v|v \rangle|}}$ where T is the reflection with respect to $\mathbb{R}^n \times 0$.

  Then $\langle w \mid w \rangle = \frac{1}{|\langle v|v \rangle|} \langle T(v) \mid T(v) \rangle = \frac{1}{|\langle v|v \rangle|} \langle v \mid v \rangle = -1$

  So w is a multiple of $T(v)$ lying on $I_n \cup (-I_n)$

  Then the hyperbolic reflection ρ parallel to v, which is the Euclidean reflection parallel to $T(v)$, is also the same as the Euclidean reflection parallel to w thus $\rho(w) = -w$

  As ρ preserves components, $\rho(I_n) = -I_n$ and vice versa.



Now suppose $\langle v \mid v \rangle > 0$, and β is a reflection parallel to v

Clearly $e_{n+1} \in I_n$

Normalizing v, we assume $\langle v \mid v \rangle = 1$

Note that

$$\beta(e_{n+1}) = 2(e_{n+1} - \langle e_{n+1} \mid v \rangle\, v) - e_{n+1} = e_{n+1} + 2v_{n+1}v \quad \text{where} \quad v_{n+1} = \langle e_{n+1}, v \rangle$$

So, the $(n+1)^{\text{th}}$ coordinate of $\beta(e_{n+1}) = 1 + 2v_{n+1}^2 > 0$

As $\beta(e_{n+1}) \in I_n \cup (-I_n)$, this shows that $\beta(e_{n+1}) \in I_n \Rightarrow \beta(I_n) = I_n$

- Now, let $A \in \mathrm{O}(I_n) \subseteq \mathrm{O}(\mathbb{R}^{n+1}, \langle \cdot \mid \cdot \rangle)$

$$\Rightarrow A = \beta_1 \circ \beta_2 \ldots \beta_k \quad \text{for some reflections } \beta_i \parallel \text{ to } v_i$$

If each $\langle v_i \mid v_i \rangle > 0$, we are good.

Suppose some $\langle v_i \mid v_i \rangle < 0$.

Then v_i can be completed to an orthogonal basis

$$\{v_i, w_1, w_2, \ldots, w_n\} \subset \mathbb{R}^{n+1} \quad \text{where} \quad \langle w_j \mid w_j \rangle > 0 \ \ \forall\, j$$

as the signature of $\langle \cdot \mid \cdot \rangle$ is 1.

Setting $\sigma_j^i :=$ the reflection $\parallel$ to w_j, we have

$$\begin{aligned} \beta_i \circ \sigma_1^i \circ \sigma_2^i \circ \cdots \circ \sigma_n^i &= -I \\ \Rightarrow \quad \sigma_1^i \circ \sigma_2^i \circ \cdots \circ \sigma_n^i &= -\beta_i \end{aligned}$$

If $\langle v_i \mid v_i \rangle < 0$, set $\sigma_1^i \circ \sigma_2^i \circ \cdots \circ \sigma_n^i = -\beta_i$

Then,

$$\beta_1 \circ \beta_2 \ldots \beta_k = \pm \sigma_1^1 \circ \sigma_2^1 \ldots \sigma_n^1 \circ \sigma_1^2 \circ \sigma_2^2 \ldots \sigma_n^n$$

As A does not switch I_n and $-I_n$, the minus sign is absurd, and thus we get A as a composition of reflections in $\mathrm{O}(I_n)$. $\square$

Theorem 1.10
$$\begin{aligned} \mathrm{O}(I_n) &\longrightarrow \mathcal{I}(\mathbb{I}^n) \\ T &\longmapsto T_{|\mathbb{I}^n} \quad \text{is an isomorphism.} \end{aligned}$$

Thus, $\mathcal{I}(\mathbb{I}^n)$ is generated by reflections.
Similarly,

$$\mathrm{SO}(I_n) \simeq \mathcal{I}^+(\mathbb{I}^n)$$

which is the group of orientation-preserving isometries of $\mathbb{I}^n$.

Proof.

- We first show well-definedness.

  Being a linear isomorphism, $T \in \mathrm{O}(I_n)$ is a diffeomorphism and $T_{*,x} = T \ \ \forall \ \ x \in \mathbb{I}^n$

  Note $\forall \ \ v, w \in T_x\mathbb{I}^n$, $\langle T(v) \mid T(w)\rangle = \langle v \mid w\rangle \ \Rightarrow \ T_{|\mathbb{I}^n} \in \mathcal{I}(\mathbb{I}^n)$

- To see $T \mapsto T_{\mathbb{I}^n}$ is injective, suppose $T_{|\mathbb{I}^n} = \mathrm{id}$ for some $T \in \mathrm{O}(I_n)$

  Notice that the $\mathbb{R}$-span of I_n is $\mathbb{R}^{n+1}$, as $e_{n+1} \in I_n$ and then $\forall\ (x_1, \ldots, x_n) \in \mathbb{R}^n$, setting

  $$x_{n+1} = \sqrt{1 + \sum_{i=1}^{n} x_i^2}$$

  we can see that $(x_1, \ldots, x_{n+1}) \in I_n$

- Lastly, for surjectivity, let $f \in \mathcal{I}(\mathbb{I}^n)$

  Then $f_{*,x} : x^\perp \longrightarrow f(x)^\perp$ is an isometry $\ \forall\ x \in \mathbb{I}^n$

  Define a linear map

  $$\begin{aligned} A : \mathbb{R}x \oplus x^\perp &\longrightarrow \mathbb{R}^{n+1} \quad \text{for some } x \in \mathbb{I}^n \\ \lambda x + v &\longmapsto \lambda f(x) + f_{*,x}(v) \end{aligned}$$

  Notice that

  $$\begin{aligned} \langle \lambda x + v \mid \lambda' x + v'\rangle &= -\lambda\lambda' + \langle v \mid v'\rangle, \\ \langle \lambda f(x) + f_{*,x}(v) \mid \lambda' f(x) + f_{*,x}(v')\rangle &= \lambda\lambda'\langle f(x) \mid f(x)\rangle + \langle f_{*,x}(v), f_{*,x}(v')\rangle \\ &= -\lambda\lambda' + \langle v \mid v'\rangle \end{aligned}$$

Thus, A preserves $\langle \cdot \mid \cdot \rangle_{(n,1)}$.

Additionally,

$$\begin{aligned} A_{*,x} : T_x\mathbb{R}^{n+1} &\longrightarrow T_x\mathbb{R}^{n+1} \\ = A : \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^{n+1} \end{aligned}$$

Now $A_{|I^n} = A \circ i$ where $i : I^n \hookrightarrow \mathbb{R}^{n+1}$ (inclusion)

So

$$(A_{|I^n})_{*,x} = A_{*,i(x)} \circ i_{*,x} = A_{*,x} \circ j$$

where $j : T_x I^n \hookrightarrow \mathbb{R}^{n+1}$ is the canonical inclusion (as $I^n \leq \mathbb{R}^{n+1}$ is a regular submanifold.)

Then $(A_{|I^n})_{*,x} = f_{*,x}$, $A(x) = f(x)$, and thus by Theorem 1.4, $A_{|I^n} = f$

Now,

$$\mathrm{O}(I_n) = \{A \in \mathrm{O}(I_n) \mid \det A = 1\} \ \cup \ \{A \in \mathrm{O}(I_n) \mid \det A = -1\}$$

and clearly the ones with $\det = +1$ are the orientation preserving ones so that

$$\mathrm{O}(I_n) \xrightarrow{\sim} \mathcal{I}(\mathbb{I}^n)$$

restricts to an isomorphism:

$$\mathrm{SO}(I_n) \xrightarrow{\sim} \mathcal{I}^+(\mathbb{I}^n)$$

$\square$

We prove an analogous result for $\mathbb{R}^n$ with the standard Euclidean metric and S^n with the induced metric from $\mathbb{R}^{n+1}$.

Theorem 1.11

$$\mathcal{I}(S^n) = \{A|_{S^n} \mid A \in \mathrm{O}(n+1)\}$$

$$\mathcal{I}(\mathbb{R}^n) = \{x \mapsto Ax + b \mid A \in \mathrm{O}(n), b \in \mathbb{R}^n\} = \mathcal{T} \rtimes \mathrm{O}(n)$$

Both groups are generated by reflections.

Proof. Any $A \in \mathrm{O}(n+1)$ sends S^n to S^n as it is norm preserving.
The proof technique is the same as that of Theorem 1.10; we show $A \mapsto A|_{S^n}$ is injective (linear span of $S^n = \mathbb{R}^{n+1}$) and that every element in $\mathcal{I}(S^n)$ and $\mathcal{I}(\mathbb{R}^n)$ agrees up to 1^{st} order with a linear map of the required form at a particular point.

Given any $f \in \mathcal{I}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $T_{f(x)^0} f_{*,x} \circ T_{-x}$ would be the required map agreeing with f up to 1st order. Notice that if $A \in \mathrm{O}(n)$ and T is a translation, then $A \circ T \circ A^{-1}$ is also a translation $\Rightarrow \mathcal{T} \trianglelefteq \mathcal{I}(\mathbb{R}^n)$

We also remark that the translation $T_b : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + b$, can be obtained by the composition $\rho_b \circ \rho_{b/2}$ where ρ_b and $\rho_{b/2}$ are reflections parallel to b and $b/2$, respectively. $\square$

Theorem 1.12 $\mathcal{I}(\mathbb{I}^n)$ acts transitively on the tangent bundle $T\mathbb{I}^n$, i.e.

$$\forall \ (x, v) \in T\mathbb{I}^n \textit{ and } (y, w) \in T\mathbb{I}^n, \ \exists \ \gamma \in \mathcal{I}(\mathbb{I}^n) \text{ such that } \gamma(x) = y \textit{ and } \gamma_{*,x}(v) = w$$

Proof. $\mathcal{I}(\mathbb{I}^n) = \mathrm{O}(I_n)$.

We now show we can send $(e_{n+1}, e_1) \in T_{e_{n+1}}\mathbb{I}^n$ to any $(x, v) \in T\mathbb{I}^n$.

$(x, v) \in T\mathbb{I}^n \ \Rightarrow \ \langle x \mid v \rangle = 0$, so they are orthogonal.

As $\langle \cdot \mid \cdot \rangle$ restricts to a positive-definite inner product on $T_x\mathbb{I}^n$, v can be extended to an orthonomal basis $\{v, v_2, \ldots, v_n\}$ of $T_x\mathbb{I}^n$.

Then

$$A = \begin{bmatrix} | & | & & | & | \\ v & v_2 & \ldots & v_n & x \\ | & | & & | & | \end{bmatrix} \in \mathrm{O}(I_n)$$

by Theorem 1.2 as $x \in \mathbb{I}^n$ and see that $A(e_1) = v$, $A(e_{n+1}) = x$.

Clearly $A^{-1} \in \mathrm{O}(I_n)$ too, and $A^{-1} : (x, v) \mapsto (e_{n+1}, e_1)$. Similarly, there exists an element of $\mathcal{I}(\mathbb{I}^n)$ sending (e_{n+1}, e_1) to any $(y, w) \in T\mathbb{I}^n$, thus proving the statement. $\square$

Chapter 2

Conformal Maps and Hyperbolic Isometries

Euclidean Conformal Geometry

We now study conformal geometry in $\mathbb{R}^n$ and state an important theorem of Liouville. This permits a complete calculation of the isometries of the disc and half-space models of hyperbolic space: we show their isometries are the same as conformal automorphisms with respect to the Euclidean conformal structure endowed by their ambient space.

Definition 2.1 $M, N \to$ Riemannian manifolds
A diffeomorphism $\phi : M \to N$ is said to be **conformal** if $\exists$ a differentiable function $\alpha : M \to \mathbb{R}^+$ such that

$$\langle \phi_{*,x}(v), \phi_{*,x}(w) \rangle_N = \alpha(x) \langle v, w \rangle_M$$

for all $x \in M$, $v, w \in T_x M$.
This means that, when conformal, ϕ preserves angles but not necessarily lengths. Note that this definition agrees with pointwise conformality, i.e., if $\forall\ x \in M$ there exists $\alpha_x > 0$ such that $\forall\ v, w \in T_x M$

$$\langle \phi_{*,x}(v), \phi_{*,x}(w) \rangle = \alpha_x \langle v, w \rangle,$$

then given any C^∞ local section s of $\|s\| = 1$, $\alpha_x = \langle \phi_{*,x}(s(x)), \phi_{*,x}(s(x)) \rangle$ must be a C^∞ function of x.
The set of conformal diffeomorphisms : $M \to N$ is denoted by $\mathrm{Conf}(M, N)$, and if $M = N$, by $\mathrm{Conf}(M)$.

$\mathrm{Conf}^+(M)$ contains the orientation-preserving elements of $\mathrm{Conf}(M)$.

Definition 2.2 Let $x_0 \in \mathbb{R}^n$ and $\alpha > 0$. Define the map $i_{x_0,\alpha} : \mathbb{R}^n \cup \infty \to \mathbb{R}^n \cup \infty$ by

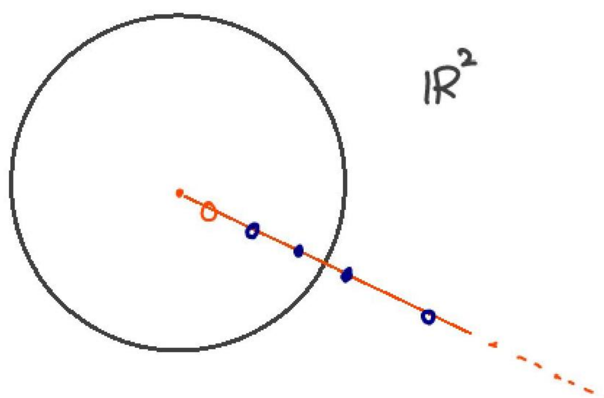
$$i_{x_0,\alpha}(x) := \alpha \frac{x - x_0}{\|x - x_0\|^2} + x_0$$

which is called the **inversion** with respect to the sphere of radius $\sqrt{\alpha}$, centered at x_0. It is clearly smooth.

To see why it is named so, we visualize

$$i_{0,1}(x) = \frac{x}{\|x\|^2}.$$

In $\mathbb{R}^2$, the map simply inverts each ray through the origin so that the part inside the circle is swapped with the part outside.



Overall, it switches the inside and outside of S^1 and this behaviour is also true in $\mathbb{R}^n$, sending points closer to the center further away. Similarly, $i_{x_0,\alpha}$ swaps the inside and outside of $M(x_0, \alpha) :=$ the sphere of radius $\sqrt{\alpha}$ centered at x_0. We sometimes think of $i_{x_0,\alpha}$ as a map from $\mathbb{R}^n \setminus \{x_0\}$ to itself and sometimes as a map from $\mathbb{R}^n \cup \{\infty\}$ to itself.

Note that the definition also makes sense for $\alpha < 0$ and in this case $i_{x_0,\alpha}$ is the composition of the inversion with respect to $M(x_0, -\alpha)$ with the symmetry about x_0 ($x_0 + v \mapsto x_0 - v \ \forall \ v \in \mathbb{R}^n$).

We now enumerate (and prove) a few important properties of inversions. We say 2 (linear) hyperplanes are orthogonal if the lines normal to them are orthogonal and 2 spheres are orthogonal if at each intersection point the tangent planes of the 2 spheres are orthogonal.

Theorem 2.3

1. $i_{x_0,\alpha} \circ i_{x_0,\beta}$ is the dilation centred at x_0 of magnitude α/β.
2. $i_{x_0,\alpha}$ is its own inverse (involution).
3. $i_{x_0,\alpha}(x) = x$ if and only if $x \in M(x_0, \alpha)$.
4. $i_{x_0,\alpha}$ is conformal.
5. Given $\alpha, \beta > 0$ and $x_0 \neq x_1$, the following are equivalent:
   (a) $M(x_1, \beta)$ is $i_{x_0,\alpha}$ invariant
   (b) $M(x_0, \alpha)$ is $i_{x_1,\beta}$ invariant
   (c) $\|x_0 - x_1\|^2 = \alpha + \beta$
   (d) $M(x_0, \alpha)$ and $M(x_1, \beta)$ are orthogonal spheres.
6. Let $i = i_{x_0,\alpha}$
   (a) H is an affine hyperplane containing $x_0 \Rightarrow i(H) = H$
   (b) H is an affine hyperplane missing, $x_0 \Rightarrow i(H)$ is a sphere containing x_0
   (c) M is a sphere containing $x_0 \Rightarrow i(M)$ is a hyperplane missing x_0
   (d) M is a sphere missing $x_0 \Rightarrow i(M)$ is a sphere containing x_0
   (e) i operates bijectively on the set of open balls and open half spaces in $\mathbb{R}^n$

Proof.

1. For any $v \in \mathbb{R}^n$,
$$\begin{aligned} i_{x_0,\alpha} \circ i_{x_0,\beta}(x_0 + v) &= i_{x_0,\alpha}\left(\beta \frac{v}{\|v\|^2} + x_0\right) \\ &= \alpha \frac{\left(\beta \frac{v}{\|v\|^2}\right)}{\left\|\beta \frac{v}{\|v\|^2}\right\|^2} + x_0 \\ &= x_0 + \frac{\alpha}{\beta} v \end{aligned}$$

2. Follows from 1, putting $\alpha = \beta$.

3. $i_{x_0,\alpha}(x) = \alpha \dfrac{x - x_0}{||x - x_0||^2} + x_0$

$$= x \quad \text{iff} \quad x - x_0 = \alpha \frac{x - x_0}{||x - x_0||^2} \quad \text{or} \quad ||x - x_0||^2 = \alpha \quad \text{or} \quad x \in M(x_0, \alpha)$$

4. Let

$$f : x \mapsto \frac{x}{||x||^2} = \frac{(x_1, \ldots, x_n)}{(x_1^2 + \ldots + x_n^2)}$$

$$f_{*,p} = \left[\frac{\partial y^i}{\partial x^j}\Big|_p \right]_{ij} = \left[\frac{\delta_{ij} \cdot ||p||^2 - 2p_j p_i}{||p||^4} \right]_{ij}$$

If ρ_p is the reflection $\|$ to $p \in \mathbb{R}^n \setminus \{0\}$
then note that

$$\begin{aligned}
\rho_p(e_j) &= 2\pi_{p^\perp}(e_j) - e_j \\
&= 2 \left(e_j - \frac{\langle e_j, p \rangle}{||p||^2} p \right) - e_j \\
&= e_j - 2 \frac{\langle e_j, p \rangle}{||p||^2} p \\
&= \sum_i \left(\delta_{ij} - \frac{2p_i p_j}{||p||^2} \right) e_i
\end{aligned}$$

So,

$$f_{*,p} = \frac{1}{||p||^2} \rho_p$$

Since ρ_p is orthogonal $\forall\ p$, $f_{*,p}$ preserves angles. At $p = 0$, $\frac{x}{||x||^2}$ is the standard chart in S^n around ∞ and thus it is, by definition, conformal at 0 too.

$$\frac{x}{||x||^2} : \mathbb{R}^n \cup \{\infty\} \setminus \{0\} \xrightarrow[\text{chart}]{\text{standard}} \mathbb{R}^n \quad : \ \infty \mapsto 0$$

Now

$$i_{x_0,\alpha} = T_{x_0} \circ \alpha \cdot f \circ T_{-x_0}$$

$$\Rightarrow (i_{x_0,\alpha})_{*,x} = (T_{x_0})_{*,\alpha f(x - x_0)} \circ \alpha \cdot f_{*,x-x_0} \circ (T_{-x_0})_{*,x} = \frac{\alpha}{||x - x_0||^2} \rho_{x - x_0}$$

as the differential of a translation map is the identity.

5. (a) $\Rightarrow$ (c)

   Let $x_0 = 0$

   $\mathbb{R}x_1$ intersects $M(x_1, \beta)$ at $\left(1 \pm \sqrt{\frac{\beta}{||x_1||^2}}\right) x_1$ Now both $\mathbb{R}x_1$ and $M(x_1, \beta)$ are invariant under $i_{0,\alpha}$, so

$$\mathbb{R}x_1 \cap M(x_1, \beta) = \left\{ \left(1 \pm \frac{\sqrt{\beta}}{||x_1||}\right) x_1 \right\}$$

   is also invariant.

   By 3, the only points $i_{0,\alpha}$ fixes, are those of $M(0, \alpha)$, thus $i_{0,\alpha}$ exchanges these two points. Then,

$$\begin{aligned} i_{0,\alpha}\left(\left(1 + \frac{\sqrt{\beta}}{||x_1||}\right) x_1\right) &= \left(1 - \frac{\sqrt{\beta}}{||x_1||}\right) x_1 \\ &= \frac{\alpha x_1}{\left(1 + \frac{\sqrt{\beta}}{||x_1||}\right) ||x_1||^2} \end{aligned}$$

   So,

$$\left(1 + \frac{\sqrt{\beta}}{||x_1||}\right)\left(1 - \frac{\sqrt{\beta}}{||x_1||}\right) ||x_1||^2 = \alpha$$

$$\left(1 - \frac{\beta}{||x_1||^2}\right) ||x_1||^2 = \alpha$$

$$||x_1||^2 = \alpha + \beta$$

   Now for a general x_0, when $M(x_1, \beta)$ is $i_{x_0,\alpha}$-invariant,

$$i_{x_0,\alpha} = T_{x_0} \circ i_{0,\alpha} \circ T_{-x_0}$$

   So,

$$T_{x_0} \circ i_{0,\alpha} \circ T_{-x_0}\left(M(x_1, \beta)\right) = M(x_1, \beta)$$

$$i_{0,\alpha} \circ T_{-x_0}\left(M(x_1, \beta)\right) = T_{-x_0}\left(M(x_1, \beta)\right)$$

$$i_{0,\alpha} \circ T_{-x_0}\left(M(x_1, \beta)\right) = T_{-x_0}\left(M(x_1, \beta)\right)$$

   i.e.

$$i_{0,\alpha}\left(M(x_1 - x_0, \beta)\right) = M(x_1 - x_0, \beta)$$

   Then we get

$$||x_1 - x_0||^2 = \alpha + \beta$$

(c) $\Rightarrow$ (a)

Put $x_0 = 0$, $\alpha = 1$

Then $\|x_1\|^2 = \beta + 1$

If $x \in M(x_1, \beta)$, then $\|x - x_1\| = \sqrt{\beta}$

Now,

$$\left\| \frac{x}{\|x\|^2} - x_1 \right\| = \frac{\|x - x_1\|x\|^2\|}{\|x\|^2}$$

But

$$\|x - x_1\|^2 = \|x\|^2 + \|x_1\|^2 - 2\langle x, x_1\rangle = \beta$$

$$\begin{aligned}\|x - x_1\|x\|^2\|^2 &= \|x\|^2 + \|x\|^4\|x_1\|^2 - 2\|x\|^2\langle x, x_1\rangle \\ &= \|x\|^2 \left(1 + \|x\|^2\|x_1\|^2 + \beta - \|x\|^2 - \|x_1\|^2\right) \\ &= \|x\|^2 \left(1 + \|x\|^2(\beta + 1) + \beta - \|x\|^2 - \beta - 1\right) \\ &= \|x\|^4\beta\end{aligned}$$

So,

$$\frac{\|x - x_1\|x\|^2\|}{\|x\|^2} = \sqrt{\beta} \quad \Rightarrow \quad M(x_1, \beta) \text{ is } i_{0,1} \text{ -invariant}$$

Again, use

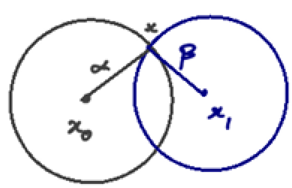
$$i_{x_0,\alpha} = T_{x_0} \circ \alpha \cdot i_{0,1} \circ T_{-x_0}$$

for the general situation.

(b) $\Longleftrightarrow$ (c)

Follows as c) is symmetric in (x_0, α) *and* (x_1, β)

(d) $\Rightarrow$ (c)



Let $x \in M(x_0, \alpha) \cap M(x_1, \beta)$

$$\begin{aligned}
&\langle x_1 - x \mid x - x_0 \rangle = 0 \\
&\Rightarrow \|x - x_0\|^2 + \|x_1 - x\|^2 = \|x_1 - x_0\|^2 \\
&= \alpha + \beta
\end{aligned}$$

(c) $\Rightarrow$ (d)

$$\|(x - x_0) + (x_1 - x)\|^2 = \|x - x_0\|^2 + \|x - x_1\|^2$$

$$\Rightarrow \langle x_1 - x \mid x - x_0 \rangle = 0$$

6. (a) We first prove thus for $x_0 = 0, \alpha = 1$

If $0 \in H$ then $i_{0,1}(0) = \infty \in H$ (for us ∞ lies in any hyperplane)

Say H is the hyperplane perpendicular to a, i.e. $x \in H$ if and only if $\sum a_i x_i = 0$

Then,

$$\sum a_i x_i = 0 \Rightarrow \sum a_i \frac{x_i}{\|x\|^2} = 0$$

So $i_{0,1}(H) \subseteq H$

But

$$\underbrace{i_{0,1} \circ i_{0,1}(H)}_{\text{id}} \subseteq i_{0,1}(H) \Rightarrow i_{0,1}(H) = H$$

Thus, $i_{0,1}(H) = H \quad \forall$ hyperplanes passing through 0.

So, given any $x_0 \in \mathbb{R}^n \cup \{\infty\}$, $\alpha > 0$ and affine hyperplane (solution set of $\sum a_i x_i = b$ for some real a_i, b) H through x_0,

$$i_{0,1}(T_{-x_0}(H)) = T_{-x_0}(H)$$

$$\alpha \cdot i_{0,1}(T_{-x_0}(H)) = \alpha \cdot (T_{-x_0}(H)) = T_{-x_0}(H)$$

$$T_{x_0} \circ \alpha \cdot i_{0,1} \circ T_{-x_0}(H) = T_{x_0} \circ T_{-x_0}(H)$$

$$i_{x_0,\alpha}(H) = H$$

(b) For $x_0 \notin H$, we examine $0 \notin H$, $i = i_{0,1}$

Then $H = \{x \in \mathbb{R}^n \cup \{\infty\} \mid \sum a_i x_i + b = 0)$ for some $b \neq 0$.

$$\sum a_i x_i + b = 0 \iff \sum \frac{a_i x_i}{\|x\|^2} + \frac{b}{\|x\|^2} = 0$$

$$i(x) = \left(\frac{x_1}{\|x\|^2}, \frac{x_2}{\|x\|^2}, \dots, \frac{x_n}{\|x\|^2} \right) = (y_1, \dots, y_n)$$

$$\|i(x)\| = \frac{1}{\|x\|} = \|y\|$$

Then

$$\sum a_i y_i = -\frac{b}{\|x\|^2} = -b\|y\|^2$$

Let $a' := \frac{1}{b}(a_1, \dots, a_n)$

So $\langle a', y \rangle = -\|y\|^2$

$$\|y + \frac{a'}{2}\| = \left\| \frac{a'}{2} \right\|$$

$$\Rightarrow i_{0,1}(H) = M\left(-\frac{a'}{2}, \frac{(a')^2}{4} \right)$$

Also $i_{0,1}(\infty) = 0 \quad$ and $\quad \infty \in H$

So $i_{0,1}(H)$ is a sphere containing 0.

Thus given a hyperplane $H \not\ni x_0$,

$$i_{x_0,\alpha}(H) = T_{x_0} \circ \alpha \cdot i_{0,1} \circ T_{-x_0}(H)$$

is a sphere containing x_0 (spheres are preserved under dilations and translations).

(c) $0 \in M \Rightarrow M = \{x \mid \|x - a\| = \|a\|\}$

$$i_{0,1}(M) = \left\{ \frac{x}{\|x\|^2} \;\middle|\; x \in M \right\}$$

Let $y = \frac{x}{\|x\|^2} \;\Rightarrow\; \|y\| = \frac{1}{\|x\|}$

Then $\forall\, x \in M$,

$$\begin{aligned}
\frac{\|x-a\|}{\|x\|^2} &= \frac{\|a\|}{\|x\|^2} \\
\left\| \frac{x}{\|x\|^2} - \frac{a}{\|x\|^2} \right\| &= \frac{\|a\|}{\|x\|^2} \\
\|y - a\|y\|^2\| &= \|a\|\|y\|^2 \\
\|y\|^2 - 2\|y\|^2\langle a, y\rangle + \|a\|^2\|y\|^4 &= \|a\|^2\|y\|^4 \quad \text{(squaring)} \\
\Rightarrow y = 0 \quad \text{or} \quad \langle a, y\rangle &= \frac{1}{2} \\
y = 0 \Rightarrow x = \infty, \text{ but } \infty \notin M \quad \Rightarrow\!\Leftarrow
\end{aligned}$$

Thus, $i_{0,1}(M)$ is contained in a hyperplane H missing 0.

But then $M = i_{0,1} \circ i_{0,1}(M) \subseteq i_{0,1}(H)$

By (a), $i_{0,1}(H)$ lies in a sphere containing 0, and this sphere must be M.

Then $i_{0,1}(H) = M$ and $i_{0,1}(M) = H$

So, if $\tilde{M}$ is a sphere containing x_0, $\quad M := T_{-x_0}(\tilde{M})$

$$i_{x_0,\alpha}(\tilde{M}) = T_{x_0} \circ \alpha \ \cdot i_{0,1} \circ T_{-x_0}(\tilde{M}) = \alpha \cdot T_{x_0} \circ i_{0,1}(M)$$

which will be a hyperplane missing x_0.

(d) Now assume $0 \notin M$

So, $M = \{x \mid \|x-a\| = r\}$ for some $a \in \mathbb{R}^n$ and $r > 0$, $\quad r \neq \|a\|$

$i_{0,1}(M) = \left\{ \frac{x}{\|x\|^2} \;\middle|\; \|x-a\| = r \right\}$

$y := \frac{x}{\|x\|^2}, \quad \|y\| = \frac{1}{\|x\|}$

Then,

$$\left\| \frac{y}{\|y\|^2} - a \right\| = r$$

$$\|y - a\|y\|^2\| = r\|y\|^2$$

$$\|y\|^2 - 2\|y\|^2\langle a, y\rangle + \|a\|^2\|y\|^4 = r^2\|y\|^4 \quad \text{(squaring)}$$

$$(\|a\|^2 - r^2)\|y\|^4 + (1 - 2\langle a, y\rangle)\|y\|^2 = 0$$

$$\|y\|^2 - \frac{2}{\|a\|^2 - r^2}\langle a, y\rangle + \frac{1}{\|a\|^2 - r^2} = 0$$

$$\left\| y - \frac{a}{\|a\|^2 - r^2} \right\| = \frac{\|a\|^2}{(\|a\|^2 - r^2)^2} - \frac{1}{\|a\|^2 - r^2}$$
$$= \frac{r^2}{(\|a\|^2 - r^2)^2}$$

So,

$$i_{0,1}(M) \subseteq M\left(\frac{a}{\|a\|^2 - r^2}, \frac{r}{\|a\|^2 - r^2}\right)$$

which is a sphere missing 0 (as $\|a\| \neq r$).

We can show equality by similar arguments as in (c).

Finally, again $i_{x_0,\alpha}(\tilde{M})$ is a sphere missing x_0, when $\tilde{M}$ is a sphere missing x_0.

(e) Obvious from the fact that i preserves connected components and then using (a) to (d) □

We now find the set of conformal diffeomorphisms between 2 domains (connected open subsets) of $\mathbb{R}^n$.
The techniques for doing so are entirely different for $n = 2$ and $n \geq 3$, but perhaps surprisingly the result is the same. For $n = 2$ we assume some results about complex analysis.

$n = 2$

Any connected oriented Riemann surface admits a unique complex structure (assume). Consider S^2 identified with $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ where ∞ corresponds to the north pole. Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

We define 2 classes of maps of $\mathbb{CP}^1$ onto itself.
Definition 2.4 A **homography** is a map : $\mathbb{CP}^1 \to \mathbb{CP}^1$

$$z \mapsto \frac{az + b}{cz + d}$$

Definition 2.5 An **anti-homography** is a map : $\mathbb{CP}^1 \to \mathbb{CP}^1$

$$z \mapsto \frac{a\overline{z} + b}{c\overline{z} + d}$$

Theorem 2.6 $M, N \to$ connected oriented Riemann surfaces
$\mathrm{Conf}(M,N)$ is the same as the set of biholomorphisms (bijective analytic functions) and anti-biholomorphisms (conjugates of biholomorphisms): $M \to N$

Proof. Let $f \in \mathrm{Conf}(M,N)$
Say we prove f is locally a holomorphism or anti-holomorphism.
Now if $U, V \overset{\text{open}}{\subseteq} M$ and f is a holomorphism on U but an anti-holomorphism on V, then on $U \cap V$, f must be both a holomorphism and an anti-holomorphism

$$\Rightarrow f|_{U\cap V} = \text{constant}, \quad \text{but as } f \text{ is bijective}, \quad U \cap V = \varnothing$$

But then, as M is connected, f must be either a holomorphism everywhere or an anti-holomorphism. So it suffices to check holomorphism/anti-holomorphism locally and we assume M, N are domains of $\mathbb{C}$.

$$\begin{aligned} df \ (\text{or } f_*) : TM &\longrightarrow \mathbb{C} \\ (p,v) &\longmapsto f_{*,p}(v) \end{aligned}$$

$$|dz|^2 := dz \otimes \overline{dz}$$

where $\overline{dz} = \overline{d(x+iy)} := dx - idy =: d\overline{z}$
Similarly, $|df|^2 := df \otimes \overline{df}$ i.e. if $df = a\,dz + b\,d\overline{z}$ then

$$\overline{df} := \overline{a}\,d\overline{z} + \overline{b}\,dz$$

We can also think of $|df|^2$ as a 2-form such that

$$|df|^2(v,w) = df(v)\cdot\overline{df}(w) \quad \forall\ (v,w) \in T\mathbb{C}\times T\mathbb{C}$$

Now,

$$\left|\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}\right|^2 = (f_z\,dz + f_{\overline{z}}\,d\overline{z}) \otimes \left(\overline{f_z}\,d\overline{z} + \overline{f_{\overline{z}}}\,dz\right)$$
$$= |f_z|^2|dz|^2 + |f_{\overline{z}}|^2|d\overline{z}|^2 + f_z\overline{f_{\overline{z}}}\,(dz)^2 + f_{\overline{z}}\overline{f_z}\,(d\overline{z})^2 \quad - (*)$$

but

$$\overline{f_z\overline{f}_{\overline{z}}(dz)^2} = \overline{f_z}\,f_{\overline{z}}\,\overline{dz\otimes dz} = \overline{f_z}\,f_{\overline{z}}\,(d\overline{z}\otimes d\overline{z})$$

Then $(*)$ gives

$$|df|^2 = |f_z|^2|dz|^2 + |f_{\overline{z}}|^2|d\overline{z}|^2 + 2\,\mathrm{Re}\left(f_z\overline{f_{\overline{z}}}\,(dz)^2\right) = \alpha\,|dz|^2 \text{ (by conformality)}$$

As $|d\overline{z}|^2 = |dz|^2$,

$$\left(|f_z|^2 + |f_{\overline{z}}|^2\right)|dz|^2 + 2\,\mathrm{Re}\left(f_z\overline{f_{\overline{z}}}\,(dz)^2\right) = \alpha\,|dz|^2$$

which applied to $\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)$ yields

$$\mathrm{Re}\left(f_z\overline{f_{\overline{z}}}\right) = 0$$

and applied to $\left(i\frac{\partial}{\partial z}, i\frac{\partial}{\partial z}\right)$ gives

$$\mathrm{Re}\left(f_z\overline{f_{\overline{z}}}\cdot i\right) = 0$$

so that

$$f_z\,\overline{f_{\overline{z}}} = \frac{\partial f}{\partial z}\cdot\overline{\frac{\partial f}{\partial \overline{z}}} = 0$$

Thus for each $z_0 \in M$,

$$\frac{\partial f}{\partial z}(z_0) = 0 \quad \text{or} \quad \frac{\partial f}{\partial \overline{z}}(z_0) = 0$$

(but not both as $df_{z_0} = f_{*,z_0} \neq 0$)
Then

$$M = \{z_0 : \frac{\partial f}{\partial z}(z_0) = 0\} \sqcup \{z_0 : \frac{\partial f}{\partial \overline{z}}(z_0) = 0\}$$

is a union of 2 closed disjoint sets, so one of them is empty.
We will be done by the following lemma.

Lemma 2.6 If f is a C^∞ function between domains of $\mathbb{C}$

$$f \text{ is holomorphic} \iff \frac{\partial f}{\partial \overline{z}} = 0$$

$$f \text{ is anti-holomorphic} \iff \frac{\partial f}{\partial z} = 0$$

Proof. Let $f = u + iv$
Then

$$2\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = u_x + iv_x + i(u_y + iv_y)$$

$$= u_x - v_y + i(v_x + u_y)$$

$$= 0 \quad \text{iff} \quad u_x = v_y \ \text{ and } \ v_x = -u_y$$

$$2\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = u_x + v_y + i(v_x - u_y) \qquad \square$$

The above calculations can simply be reversed to show holomorphism or antiholomorphism $\Rightarrow$ conformal $\square$

We now identify $\mathbb{R}^2$ with $\mathbb{C}$ such that $\mathbb{R}^2$, D^2 and Π^2 are open subsets of $\mathbb{CP}^1$.
If F is a set of mappings,

$$c(F) := \{\overline{f} : z \mapsto \overline{f(z)} \mid f \in F\}$$

$$-F := \{-f : z \mapsto -f(z) \mid f \in F\}$$

Theorem 2.6 $\mathrm{Conf}^+(\mathbf{S}^2) =$ the set of homographies
$\mathrm{Conf}(\mathbf{S}^2) =$ the set of homographies + anti-homographies
For $M = D^2, \Pi^2, \mathbb{C}$ we have

$$\mathrm{Conf}^+(M) = \{f_{|M} \mid f \in \mathrm{Conf}^+(\mathbf{S}^2) \text{ and } f(M) = M\}$$

$$\mathrm{Conf}(M) = \{f_{|M} \mid f \in \mathrm{Conf}(\mathbf{S}^2) \text{ and } f(M) = M\}$$

In particular

$$\mathrm{Conf}^+(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}$$

$$\mathrm{Conf}(\mathbb{C}) = \mathrm{Conf}^+(\mathbb{C}) \cup c\left(\mathrm{Conf}^+(\mathbb{C})\right)$$

$$\mathrm{Conf}^+(D^2) = \left\{ z \mapsto e^{i\theta}\frac{z - \alpha}{1 - \overline{\alpha}z} \;\middle|\; \theta \in \mathbb{R}, \alpha \in D^2 \right\}$$

$$\mathrm{Conf}(D^2) = \mathrm{Conf}^+(D^2) \cup c\left(\mathrm{Conf}^+(D^2)\right)$$

$$\mathrm{Conf}^+(\Pi^2) = \left\{ z \mapsto \frac{az + b}{cz + d} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

$$\mathrm{Conf}(\Pi^2) = \mathrm{Conf}^+(\Pi^2) \cup \left(-c\left(\mathrm{Conf}(\Pi^2)\right)\right)$$

Proof.

- We first find $\mathrm{Conf}^+(\mathbb{C})$ and then $\mathrm{Conf}^+(\mathbb{CP}^1)$ as a result.

  Let $f : \mathbb{C} \to \mathbb{C}$ be a biholomorphism.

  Then
  $$f(z) = \sum_{i=0}^{\infty} a_i z^i \quad \text{for } a_i \in \mathbb{C}$$
  and the radius of convergence for f, an entire function, is ∞.

  Let $h : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be $h(z) = 1/z$. f is injective $\Rightarrow f \circ h$ is injective $\forall\ z \neq 0$. Now by the Great Picard's Theorem ([Con78]), $f \circ h$ cannot have an essential singularity at 0.

  So $z \mapsto f\left(\frac{1}{z}\right)$ has a pole or removable singularity at 0.

  In either case, $\exists$ holomorphic $g : N_r(0) \to \mathbb{C}$, $m \geq 0$ such that
  $$f\left(\frac{1}{z}\right) = z^{-m} g(z) \quad \forall\ z \in N_r(0) \setminus \{0\}$$
  $$\sum_{i=0}^{\infty} a_i z^{m-i} = g(z)$$
  As $\lim_{z\to 0} g(z)$ must exist, $a_i = 0 \quad \forall\ i > m$.

  $\Rightarrow f$ is a polynomial.

  So, by the fundamental theorem of algebra,
  $$f(z) = (z - a_0)(z - a_1) \cdots (z - a_m)$$
  But by injectivity, $\deg f = 1$.

  So $\exists\ a, b \in \mathbb{C}$ such that $f(z) = az + b \quad \forall\ z \in \mathbb{C}$.

- Let $g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be a biholomorphism.

  If $g(\infty) = \infty$ then $\exists a, b \in \mathbb{C}$ such that $g(z) = az + b \quad \forall\ z \in \mathbb{C}$ by the previous argument.

  If $g(\infty) = c \in \mathbb{C}$ then let $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be
  $$z \mapsto \frac{1}{z - c}$$

Clearly ϕ is a biholomorphism on $\mathbb{C} \cup \{\infty\}$ sending $c \mapsto \infty$.

Then $\phi \circ g : \infty \mapsto \infty$.

$\Rightarrow \exists\, a, b \in \mathbb{C}$ such that $\phi \circ g(z) = az + b \quad \forall\, z \in \mathbb{C}$.

$$\frac{1}{g(z) - c} = az + b \quad \Rightarrow \quad g(z) - c = \frac{1}{az + b}$$

$$g(z) = \frac{c(az + b) + 1}{az + b} = \frac{acz + bc + 1}{az + b}$$

Note $abc - a(bc + 1) = -1 \neq 0$.

So we can write

$$g(z) = \frac{az + b}{cz + d} \quad \text{for } a, b, c, d \in \mathbb{C} \text{ where } ad - bc \neq 0$$

Additionally, each homography is a composition of dilations, translations, inversions which are all holomorphic $\Rightarrow$ conformal

- Given the conformal maps on $\mathbf{S}^2$ are homographies, it can be shown

  those preserving $\mathbb{C}$: $\quad \frac{az+b}{d} \quad (c = 0)$

  those preserving D^2: $\quad e^{i\theta} \frac{z-a}{1-\bar{a}z}$

  those preserving Π^2: $\quad \frac{az+b}{cz+d} \quad$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$

  The methods for anti-biholomorphisms are similar. We omit the explicit calculations.

$\square$

Theorem 2.7 After the identification of $\mathbb{C}P^1$ with $\mathbb{R}^2 \cup \{\infty\}$, $\mathrm{Conf}(\mathbb{CP}^1)$ consists of all and only maps of the form

$$x \mapsto \lambda A\, i(x) + v$$

where $\lambda > 0$, $A \in \mathrm{O}(2)$, $v \in \mathbb{R}^2$ and i is id or an inversion with respect to a sphere.

Proof.

$$z \mapsto \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$= A \circ i_{0,1}$ where A denotes conjugation, i.e. the reflection with respect to the x-axis.

$$z \mapsto re^{i\theta} z = r \circ R_\theta(z) \quad \text{where } R_\theta \in \mathrm{O}(2) \text{ is the rotation by } \theta$$

As each homography is a composition of reflections, translations, inversions and dilations, it is of the form $\lambda A\, i + v$. The situation is very similar for anti-homographies.

Now, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{O}(2)$$

By Theorem 1.7, A is a composition of reflections.
But any reflection can be obtained by a composition $R_\theta \circ (z \mapsto \overline{z}) \circ R_\theta^{-1}$, where the former is a homography and the latter is an anti-homography.
Now

$$i_{0,1} = z \mapsto \frac{1}{\overline{z}}$$

The other inversions can be obtained by conjugating with translations and dilations, which are homographies. □

Corollary 2.7

i) $\mathrm{Conf}(\mathbb{D}^2)$ consists only of maps of the form $x \mapsto A\, i(x)$ where $A \in \mathrm{O}(2)$ and i is id or an inversion with respect to sphere orthogonal to $\partial\mathbb{D}^2$.

ii) $\mathrm{Conf}(\Pi^2)$ consists only of maps of the form

$$x \mapsto \lambda \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} i(x) + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

where $u \in \{\pm 1\}$, $b \in \mathbb{R}$ and i is id or inversion with respect to sphere orthogonal to $\mathbb{R} \times \{0\}$.

Proof. Same as the proof of the more general Theorem 2.10. □

Remark 2.8 As translations, dilations and elements of $\mathrm{SO}(n)$ preserve orientation, while elements of $\mathrm{O}(n) \setminus SO(n)$ and inversions reverse it, we have:

1. $A \circ i \in \mathrm{Conf}^+(D^2)$ if and only if $\{(i = \text{id and } A \in \mathrm{SO}(2))$ or $(i = \text{inv and } A \in \mathrm{O}(2) \setminus \mathrm{SO}(2))\}$

2. $\lambda \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \circ i + \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathrm{Conf}^+(D^2)$ if and only if $\{(i = \text{id},\ u = +1)$ or $(i = \text{inv},\ u = -1)\}$

$n \geq 3$

Theorem 2.9 Liouville's Theorem
Any conformal diffeomorphism between 2 domains of $\mathbb{R}^n$ is of the form $(n \geq 3)$

$$x \mapsto \lambda A i(x) + b$$

where $\lambda > 0$, $A \in \mathrm{O}(n)$, $i \to$ id or an inversion, and $b \in \mathbb{R}^n$.

We skip the the proof as the details are heavily analytic, but it can be found in [BP92].
Remark 2.10

i) The assumption $n \geq 3$ cannot be dropped as in 2-dimensions, for example: on D^2, every holomorphic function is conformal onto its range and $\exists$ many injective holomorphic functions on the disc which are not of the form $\lambda A i + b$.

ii) By the Riemann Mapping Theorem ([Con78]), every simply connected proper domain in $\mathbb{C}$ is conformally equivalent to D^2, but by Liouville's theorem only open balls and open half-spaces are conformally equivalent to D^n for $n \geq 3$. This is an early example of "rigidity" for $n \geq 3$.

Despite the differences in the cases $n = 2$ and $n \geq 3$, the results about the conformal groups for S^2, D^2, Π^2 can be generalized word to word in higher dimensions.

Corollary 2.9 $\mathrm{Conf}(S^n)$ consists of all and only mappings of the form

$$x \longmapsto \lambda A i(x) + b$$

where $A \in \mathrm{O}(n)$, $\lambda > 0$, $b \in \mathbb{R}^n$ and $i =$ id or an inversion with respect to a sphere.
If $M, N \subseteq S^n$ are domains then

$$\mathrm{Conf}(M, N) = \left\{ f_{|M} \;\middle|\; f \in \mathrm{Conf}(S^n),\ f(M) = N \right\}$$

Proof. We identify S^n with $\mathbb{R}^n \cup \{\infty\}$ and analyse any $f \in \mathrm{Conf}(S^n)$.

- If $f : \infty \mapsto \infty$ then $f_{|\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is conformal and by Theorem 2.9, $f_{|\mathbb{R}^n} = \lambda A + b$ with $\lambda > 0$, $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$ (note that no inversion can be defined on $\mathbb{R}^n$).

- If $f(\infty) \in \mathbb{R}^n$ then let i be an inversion on S^n with respect to a sphere centred at $f^{-1}(\infty) \ \Rightarrow f(i(\infty)) = \infty$.

  Then $f \circ i : \infty \mapsto \infty$

  $$\Rightarrow (f \circ i)_{|\mathbb{R}^n} = \lambda A + b$$

  So $f_{|S^n \setminus \{f^{-1}(\infty)\}} = \lambda A \circ i + b$

  But $f(f^{-1}(\infty)) = \infty = \lambda A \circ i(f^{-1}(\infty)) + b$

  So $f = \lambda A \circ i + b$

- Now we remark that the translations on the sphere, more precisely

  $$\phi(x) := \begin{cases} p^{-1} \circ T \circ p(x) & \forall \ x \in S^n \setminus \{\mathrm{N}\} \\ \infty & \text{if } x = \mathrm{N} \text{ (north pole)} \end{cases}$$

  $$\text{for some translation} \quad \begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto x + b \end{aligned}$$

  are conformal at all $x \in S^n \setminus \{\mathrm{N}\}$ being a composition of conformal maps. Thus $\exists \ \alpha : S^n \setminus \{\mathrm{N}\} \to \mathbb{R}_+$ such that

  $$\forall \ x \in S^n \setminus \{N\} \quad \text{and} \quad v, w \in T_x S^n,$$

  $$\langle \phi_{*,x}(v), \ \phi_{*,x}(w) \rangle = \alpha(x) \langle v, w \rangle$$

  $$\phi_{*,x} = (p^{-1})_{*,T \circ p(x)} \circ T_{*,p(x)} \circ p_{*,x}$$

  $$= (p^{-1})_{*,T \circ p}(x) \circ p_{*,x}$$

  $$\phi_{*,\mathrm{N}} = \lim_{x \to \mathrm{N}} \left((p^{-1})_{*,T \circ p(x)} \circ p_{*,x} \right) \quad \left(\text{ by the continuity of } x \mapsto \phi_{*,x} \ \right)$$

  As $\lim_{x \to \mathrm{N}} T \circ p(x) = p(x), \quad \phi_{*,N} = \mathrm{id}$

  Thus ϕ is conformal at all $x \in S^n \ \Rightarrow \ \phi \in \mathrm{Conf}(S^n)$

- Now if $A \in \mathrm{O}(n)$ then A is a composition of reflections parallel to vectors i.e. with respect to hyperplanes through $\vec{0}$ in $\mathbb{R}^n$. But the reflection in $\mathbb{R}^n$ with respect to a hyperplane H corresponds to a reflection in S^n coming from the reflection in $\mathbb{R}^{n+1}$ with respect to $\mathrm{span}\{H, e_{n+1}\} \ \Rightarrow \ A$ is conformal on S^n.

- We already know conformality of inversions, thus we have shown all maps of the form

$$x \mapsto \lambda A i(x) + b$$

are conformal diffeomorphisms on S^n.

As for domains of S^n, if they miss a point on S^n then the claim reduces to Liouville's Theorem as they directly correspond to domains in $\mathbb{R}^n$.

□

Theorem 2.11

i) $\mathrm{Conf}(D^n)$ is all mappings of the form

$$x \mapsto A \circ i(x)$$

where $A \in \mathrm{O}(n)$, $i = \mathrm{id}$ or inversion with respect to sphere orthogonal to ∂D^n

ii) $\mathrm{Conf}(\Pi^n)$ is all mappings of the form

$$x \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \circ i(x) + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

where $A \in \mathrm{O}(n-1)$, $b \in \mathbb{R}^{n-1}$ and $i = \mathrm{id}$ or inversion with respect to a sphere orthogonal to $\mathbb{R}^{n-1} \times \{0\}$.

Proof.

i) Clearly all $A \in \mathrm{O}(n)$ and inversions with respect to spheres orthogonal to ∂D^n preserve ∂D^n and thus D^n too. So all mappings : $x \mapsto A \circ i(x) \in \mathrm{Conf}(D^n)$.

Now let $f \in \mathrm{Conf}(D^n)$. By Theorem 2.9,

$$f(x) = \lambda A i(x) + b$$

where $A \in \mathrm{O}(n)$, $i = \mathrm{id}$ or inversion with respect to a sphere, $b \in \mathbb{R}^n$, $\lambda > 0$.

If $i = \mathrm{id}$ then $b = 0$ and $\lambda = 1$ as $A(\partial D^n) = \partial D^n$.

If $i = i_{x_0,\alpha}$ then $x_0 \notin \bar{D}^n$, else $f(x_0) = \infty$ but we need $f(\bar{D}^n) = \bar{D}^n$ (as $f(D^n) = D^n$).

Then set $j =$ inversion with respect to $M(x_0, \sqrt{\|x_0\|^2 - 1})$

$j(\partial D^n) = \partial D^n$ as ∂D^n is orthogonal to $M(x_0, \sqrt{\|x_0\|^2 - 1})$

Additionally $j(x_0) = \infty$, and as both x_0 and ∞ are outside D^n,

$$j(D^n) = D^n \text{ and } j \in \mathrm{Conf}(D^n)$$

Now $f \circ j = \lambda A \circ i \circ j + b$
As $i \circ j$ is a dilation (Theorem 2.3), $f \circ j = \lambda' A + b$ for some $\lambda' > 0$
$f \circ j \in \mathrm{Conf}(D^n) \Rightarrow b = 0$ and $\lambda' = 1$
$\Rightarrow f \circ j = A$ and $f = A \circ j$

ii) Let $f \in \mathrm{Conf}(\Pi^n)$ be of the form $f = \lambda A + b$
Note that if we want Π^n to be invariant under f, then $\mathbb{R}^{n-1} \times 0$ must also be invariant under f $\big($because for any conformal map f, $f(\overline{\Pi}_n) \subseteq \overline{f(\Pi_n)} = \overline{\Pi}_n$ and same for f^{-1} $\big)$
Then $f(0) = b \in \mathbb{R}^{n-1} \times 0$ i.e. $b_n = 0$
So A also preserves $\mathbb{R}^n \times 0 = e_n^\perp$ and, in turn, $A(\mathbb{R}e_n) = \mathbb{R}e_n$
Then $A(e_n) = \pm e_n$, but as A must preserve Π^n, it will fix e_n
Thus

$$A = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$$

where $B \in \mathrm{O}(n-1)$

Now if $f = \lambda A i_{x_0,\alpha} + b$ then $x_0 \in e_n^\perp$, else $i_{x_0,\alpha}(e_n^\perp)$ will be a sphere and $f(e_n^\perp)$ will be bounded (we want $f(e_n^\perp) = e_n^\perp$). Then $i_{x_0,\alpha} \in \mathrm{Conf}(\Pi^n)$.
So $f \circ i_{x_0,\alpha} = \lambda A + b \in \mathrm{Conf}(\Pi^n)$
Finally,

$$f = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \circ i_{x_0,\alpha} + \begin{pmatrix} c \\ 0 \end{pmatrix}$$

where $B \in \mathrm{O}(n-1)$, $c \in \mathbb{R}^{n-1}$ and $x_0 \in \mathbb{R}^{n-1} \times 0$ □

We now prove a technical result.
Lemma 2.12 $n \geq 2$, $\phi \in \mathrm{Conf}(S^n) \setminus \{\mathrm{id}\}$
If $\exists$ a codimension 1 submanifold N of S^n such that $\phi|_N = \mathrm{id}$, then exactly one of the following holds:
a) $N \subseteq$ (a sphere) and ϕ is the inversion with respect to that sphere.
b) $N \subseteq$ (a hyperplane) and ϕ is the reflection with respect to that hyperplane.

Proof. By Corollary 2.9, we know ϕ can be of 2 forms:
i) $\phi = \lambda A + b$
ii) $\phi = \lambda A i + b$ ($i =$ inversion)

We analyse them separately.
i) Let $x \in N \setminus \{\infty\}$
Then $\phi|_N = \mathrm{id} \Rightarrow \phi_{*,x}|_{T_xN} = \mathrm{id}$
But $\phi_{*,x} = \lambda A$, so $\lambda = 1$
Choosing an ONB $\{v_1, \dots, v_n\}$ of T_xS^n such that $v_1, \dots, v_{n-1} \in T_xN$ we see $Av_i = v_i$ $\forall\ i \leq n-1$.
As A is orthogonal, $Av_n \pm v_n$, and as $A \neq$id, $Av_n = -v_n$.

$\phi = A + b$ where A is the reflection with respect to the hyperplane $T_xN = \mathbb{R}\langle v_1, \dots, v_{n-1}\rangle$

- If $b \perp T_xN$, then $A+b$ is the reflection with respect to the hyperplane $H := T_xN + b/2$
- If $b \not\perp T_xN$ then A doesn't fix any point $\Rightarrow\Leftarrow$
- If $b = 0$, $\phi = A$

In each case ϕ is a reflection and N lies in the hyperplane fixed under ϕ.
ii) If $i = i_{x_0,\alpha}$, $x \in N$ and $v \in T_xN$,

$$\|v\| = \|\phi_{*,x}(v)\| = \frac{\lambda\alpha}{\|x - x_0\|^2}\|v\|$$

So $\beta := \lambda\alpha = \|x - x_0\|^2 \ \Rightarrow\ N \subseteq M(x_0, \beta)$
Now

$$\phi \circ i_{x_0,\beta} = \lambda A \circ i_{x_0,\alpha} \circ i_{x_0,\beta} + b$$
$$= \lambda A \left(\frac{\alpha}{\beta}\right) + b = A + b \quad \text{(type i))}$$

is identity on $N \Rightarrow b = 0$ (this time we can't have $b \perp T_xN$ $\forall\ x \in N$ as N is in a sphere)
So $\phi \circ i_{x_0,\beta} = A = \begin{bmatrix} I_n & 0 \\ 0 & \pm 1 \end{bmatrix}$ (by i)) but -1 isn't possible as then $\phi = A \circ i_{x_0,\beta}$ won't be id on N. Thus $\phi = i_{x_0,\beta}$ $\square$

Hyperbolic Isometries

Theorem 2.13 $\mathcal{I}(\mathbb{D}^n) = \mathrm{Conf}(D^n)$ (with Euclidean conformal structure on D^n)
$\mathcal{I}^+(\mathbb{D}^n) = \mathrm{Conf}^+(D^n)$

Proof. First, we show that $\quad p : \mathbb{I}^n \longrightarrow D^n$ (Euclidean)

$$(x_1, \ldots, x_{n+1}) \longmapsto \frac{1}{1+x_{n+1}}(x_1, x_2, \ldots, x_n, 0)$$

is a conformal diffeomorphism.

$$y_i := \frac{x_i}{1+x_{n+1}} \quad \forall\ i \leq n \quad \textit{and} \quad y_{n+1} = 0$$

$$\begin{aligned} dy_i &= \frac{(1+x_{n+1})\,dx_i - x_i\,dx_{n+1}}{(1+x_{n+1})^2} \quad \forall \quad i \leq n \\ &= \frac{1}{1+x_{n+1}}\left(dx_i - \frac{x_i}{1+x_{n+1}}\,dx_{n+1}\right) \end{aligned}$$

$$\sum_{i=1}^{n+1} dy_i^2 = \frac{1}{(1+x_{n+1})^2}\left(\sum_{i=1}^{n} dx_i^2 - \sum_{i=1}^{n} \frac{2x_i}{1+x_{n+1}}\,dx_i\,dx_{n+1} + \frac{dx_{n+1}^2}{(1+x_{n+1})^2}\sum_{i=1}^{n} x_i^2\right)$$

Note that $\sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1 \;\Rightarrow\; \sum_{i=1}^n x_i\,dx_i = x_{n+1}\,dx_{n+1}$
So,

$$\begin{aligned} \sum_{i=1}^{n+1} dy_i^2 &= \frac{1}{(1+x_{n+1})^2}\left(\sum dx_i^2 - \frac{2x_{n+1}\,dx_{n+1}^2}{1+x_{n+1}} + \frac{x_{n+1}^2 - 1}{(1+x_{n+1})^2}\,dx_{n+1}^2\right) \\ &= \frac{1}{(1+x_{n+1})^2}\left(\sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2\right) \end{aligned}$$

Thus, p is conformal.

Now if $\phi \in \mathcal{I}(\mathbb{D}^n)$ then $p^{-1} \circ \phi \circ p \in \mathcal{I}(\mathbb{I}^n) \subseteq \mathrm{Conf}(\mathbb{I}^n)$ and by the conformality of p and p^{-1}, $\phi \in \mathrm{Conf}(D^n)$. So $\mathcal{I}(\mathbb{D}^n) \subseteq \mathrm{Conf}(D^n)$.

For the reverse inclusion, we show that all $A \in \mathrm{O}(n)$ and inversions with spheres $\perp \partial D^n$ are isometries of $\mathbb{D}^n$.

$$p^{-1} : x \mapsto \frac{1}{1-\|x\|^2}\begin{bmatrix} 2x \\ 1+\|x\|^2 \end{bmatrix}$$

One can check

$$p \circ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \circ p^{-1} = A$$

But $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{I}(\mathbb{I}^n)$, so

$$A = p \circ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \circ p^{-1} \in \mathcal{I}(\mathbb{D}^n)$$

Now we consider the reflection ρ parallel to $(y,t) \in \mathbb{R}^{n+1}$ with respect to $\langle \cdot \mid \cdot \rangle$, assuming $\langle (y,t) | (y,t) \rangle = 1$, $t > 0$.
Let

$$\begin{aligned} N &:= \{x \in \mathbb{D}^n \mid \langle p^{-1}(x) | (y,t) \rangle = 0\} \\ &= \{x \in \mathbb{D}^n \mid 2\langle x|y \rangle - (1 + ||x||^2)t = 0\} \\ &= \{x \in \mathbb{D}^n \mid 2\langle x|y \rangle = (1 + ||x||^2)\sqrt{||y||^2 - 1}\} \\ &= \{x \in \mathbb{D}^n \mid 2\langle x|w \rangle = 1 + ||x||^2\} \quad (\text{where } w := \frac{y}{\sqrt{||y||^2 - 1}}) \\ &= \{x \in \mathbb{D}^n \mid ||x - w||^2 = ||w||^2 - 1\} \\ &\subseteq M(w, ||w||^2 - 1) \end{aligned}$$

$$p \circ \rho \circ p^{-1}(x) = x \quad \forall\, x \in N$$

Being a conformal map on D^n, $p \circ \rho \circ p^{-1}$ extends to a conformal map on S^n (Corollary 2.9). As $p \circ \rho \circ p^{-1}|_N = id_N$ and $N \subseteq M(w, ||w||^2 - 1)$, by Lemma 2.12,

$$p \circ \rho \circ p^{-1} = i_{w,||w||^2-1}$$

As $\rho \in \mathcal{I}(\mathbb{I}^n)$, $i_{w,||w||^2-1} \in \mathcal{I}(\mathbb{D}^n)$. Note that $w = \frac{y}{|t|}$ can be any vector in $\mathbb{R}^{n+1}$ with norm $>$ 1, so this includes all inversions with respect to spheres $\perp \partial D^n$ $\square$

Corollary 2.13 $\mathcal{I}(\mathbf{\Pi}^n) = \mathrm{Conf}(\Pi^n)$ and $\mathcal{I}^+(\mathbf{\Pi}^n) = \mathrm{Conf}^+(\Pi^n)$

Proof. Let $f \in \mathcal{I}(\mathbf{\Pi}^n)$ and $i : \mathbb{D}^n \to \mathbf{\Pi}^n$ be the isometric inversion used to define $\mathbf{\Pi}^n$. $i \circ f \circ i^{-1} \in \mathcal{I}(\mathbb{D}^n) \subseteq \mathrm{Conf}(D^n)$. We know i, i^{-1} are conformal. So $f \in \mathrm{Conf}(\Pi^n)$.

Now, if $g \in \mathrm{Conf}(\Pi^n)$, then $i \circ g \circ i^{-1} \in \mathrm{Conf}(D^n) = \mathcal{I}(\mathbb{D}^n)$. Also, i, i^{-1} are isometries between $\mathbb{D}^n$ and $\mathbf{\Pi}^n$. So $g \in \mathcal{I}(\mathbf{\Pi}^n)$. $\square$

Metrics on $\mathbb{D}^n$ and $\mathbf{\Pi}^n$

The isometry groups of $\mathbb{D}^n$ and $\mathbf{\Pi}^n$ allow us to easily compute their Riemannian metrics.

Theorem 2.14 For $x \in \mathbb{D}^n, (y,t) \in \mathbb{I}^n$ and $v \in \mathbb{R}^n,$ the metrics are

$$ds_x^2(v) = \left(\frac{2}{1-||x||^2}\right)^2 ||v||^2$$

$$ds_{(y,t)}^2(v) = \frac{||v||^2}{t^2}$$

Proof.

$$\text{Let} \quad \widetilde{p} : \mathbb{R}^{n+1} - \{x_{n+1} = -1\} \longrightarrow \mathbb{R}^n$$

$$(x_1, x_2, \ldots, x_{n+1}) \longmapsto \frac{1}{1+x_{n+1}}(x_1, x_2, \ldots, x_n)$$

Then $\widetilde{p}|_{\mathbb{I}^n} = p =$ the stereographic projection from $\mathbb{I}^n$ to $\mathbb{D}^\ltimes$

$$\widetilde{p}_{*,x} = \frac{1}{1+x_{n+1}} \begin{bmatrix} 1 & 0 & -\frac{x_1}{1+x_{n+1}} \\ 0 & \ddots & \vdots \\ 0 & 1 & -\frac{x_n}{1+x_{n+1}} \end{bmatrix}_{n\times(n+1)}$$

$$p_{*,e_{n+1}} = \widetilde{p}_{*,e_{n+1}}|_{T_{e_{n+1}}\mathbb{I}^n} = \widetilde{p}_{*,e_{n+1}}|_{\mathbb{R}^n\times\{0\}} = \frac{1}{2}I_n$$

$$p^{-1} : \mathbb{D}^n \longrightarrow \mathbb{I}^n$$

$$(p^{-1})_{*,0} = (p_{*,(0,\ldots,0,1)})^{-1} = 2\,\mathrm{Id}$$

$$(p^{-1})^*\langle v|v\rangle_{(0,\ldots,0,1)} = \langle 2\,\mathrm{Id}(v)|2\,\mathrm{Id}(v)\rangle$$

$$ds_0^2(v) = 4||v||^2 \quad (\text{as } \langle\cdot \mid \cdot\rangle \text{ reduces to the standard inner product on } \mathbb{R}^n \times \{0\})$$

Now for any $x \in \mathbb{D}^n$, we know

$$i := i_{\frac{x}{||x||^2}, \frac{1}{||x||^2}-1} \in \mathcal{I}(\mathbb{D}^n)$$

so that

$$p^{-1} \circ i : \mathbb{D}^n \longrightarrow \mathbb{I}^n$$

Note that

$$i(0) = x \quad \Rightarrow \ i(x) = 0 \quad \text{too}$$

Now, let $v \in T_x\mathbb{D}^n$

$$i_{*,x}(v) \in T_0\mathbb{D}^n$$

$$\begin{aligned}(p^{-1})^*\langle v, v\rangle &= (p^{-1})^*\langle i_{*,x}(v), i_{*,x}(v)\rangle_0 \quad (i \in \mathcal{I}(\mathbb{D}^n))\\ &= 4\|i_{*,x}(v)\|^2\\ &= \frac{4}{(\|x\|^2-1)^2}\|v\|^2\end{aligned}$$

Similarly, if

$$j : \mathbb{D}^n \longrightarrow \Pi^n$$

$$x \longmapsto 2\frac{x+e_n}{\|x+e_n\|^2} - e_n$$

then

$$j_{*,0} = \frac{2}{\|e_n\|^2}P_{e_n} \qquad (P_{e_n} \text{ is the Euclidean reflection parallel to } e_n)$$

$$\Rightarrow \langle j_{*,0}(v), j_{*,0}(v)\rangle = 4\|v\|^2$$

$$\langle j_{*,e_n}(w), j_{*,e_n}(w)\rangle = \frac{1}{4}\|w\|^2$$

So,

$$ds^2_{e_n}(w) = 4 \times \frac{1}{4}\|w\|^2 = \|w\|^2$$

Since horizontal translations $\left(x \mapsto x + \begin{pmatrix} b \\ 0 \end{pmatrix}\right)$ are isometries in $\mathbf{\Pi^n}$,

$$ds^2_{e_n} = ds^2_{e_n + \binom{b}{0}}$$

Now

$$(t \cdot \mathrm{Id})_{*,e_n} = t \cdot \mathrm{Id} \quad \forall \ t > 0$$

so that

$$ds^2_{t\,e_n}(v) = \frac{\|v\|^2}{t^2} \quad \forall \ v \in T_{t\,e_n}\mathbf{\Pi}^n$$

As a corollary, at any point of $\mathbb{D}^n$ and $\mathbf{\Pi}^n$, the hyperbolic metric is a positive multiple of the Euclidean one. $\square$

Remark 2.14 This tells us that the notion of conformal diffeomorphisms on $\mathbb{D}^n$ is the same for the Euclidean and hyperbolic metrics. By Theorem 2.13, all conformal diffeomorphisms with respect to the hyperbolic metric on $\mathbb{D}^n$ are then isometries.

$$\mathcal{I}(\mathbb{D}^n) = \mathrm{Conf}(\mathbb{D}^n) = \mathrm{Conf}(D^n)$$

This means that a sufficient condition for a diffeomorphism of $\mathbb{H}^n$ to preserve lengths is that it preserves angles. Heuristically, this tells us a unit of measure is intrinsically defined on $\mathbb{H}^n$ and can't be changed.

Chapter 3

Geodesics and the Boundary of $\mathbb{H}^n$

Hyperbolic Geodesics and Subspaces

We now talk about geodesics on $\mathbb{H}^n$, starting with the hyperboloid model $\mathbb{I}^n$.

Theorem 3.1 Given any $x \in \mathbb{I}^n$ and $v \in T_x\mathbb{I}^n$, such that $\langle v \mid v \rangle = 1$, the geodesic starting at x with velocity v is

$$t \longmapsto \cosh(t) \cdot x + \sinh(t) \cdot v$$

In particular, its image is the intersection of $\mathbb{I}^n$ with the plane spanned by x and v in $\mathbb{R}^{n+1}$.

Proof. Let W be the plane spanned by x and v.
Extend x, v to an orthogonal basis and define $\phi \in \mathrm{O}(\mathbb{R}^{n+1}, \langle \cdot \mid \cdot \rangle)$
by $\phi(x) = x$, $\phi(v) = v$ and $\phi|_{W^\perp} = -\mathrm{id}$.
Clearly $\phi(x) = x \Rightarrow \phi \in \mathrm{O}(I_n)$.
Now let ω be the maximal unit-speed geodesic starting at x with velocity v; by this we mean start with the geodesic $\omega' : (a, b) \to M$ with the same initial conditions and define

$$b_0 := \sup\{t \in \mathbb{R} \text{ such that } \omega' \text{ can be extended to } (a, t)\}$$

$$a_0 := \inf\{y \in \mathbb{R} \text{ such that } \omega' \text{ can be extended to } (y, b_0)\}$$

and ω is the geodesic defined on (a_0, b_0).
Note $\phi_{*,x} = [\phi]_{|T_x\mathbb{I}^n}$ (ϕ is a linear map).
So $\phi_{*,x}(v) = \phi(v) = v$ and as $\phi(x) = x$, ω is invariant under $\phi \Rightarrow \mathrm{Im}\ \omega \subseteq W \cap \mathbb{I}^n$.
Let $\gamma : \mathbb{R} \to \mathbb{I}^n$ be the map

$$\gamma : t \mapsto \frac{e^t + e^{-t}}{2}x + \frac{e^t - e^{-t}}{2}v,$$

$$\begin{aligned}\langle \gamma(t) \mid \gamma(t) \rangle &= -\left(\frac{e^t + e^{-t}}{2}\right)^2 + \left(\frac{e^t - e^{-t}}{2}\right)^2 \\ &= \frac{-1}{2} - \frac{1}{2} = -1 \quad \text{(well-defined)}\end{aligned}$$

$$\gamma' : t \mapsto \frac{e^t - e^{-t}}{2}x + \frac{e^t + e^{-t}}{2}v$$

$$\langle \gamma'(t) \mid \gamma'(t) \rangle = 1$$

Also if $y \in W \cap \mathbb{I}^n$ then $y = ax + bv$ for $a, b \in \mathbb{R}$ such that

$$\begin{aligned}\langle y|y \rangle &= a^2\langle x|x \rangle + b^2\langle v|v \rangle = -a^2 + b^2 \\ &= -1\end{aligned}$$

i.e. $a^2 - b^2 = 1$ so $\exists\ t \in \mathbb{R}$ such that $a = \frac{e^t+e^{-t}}{2}$, $b = \frac{e^t-e^{-t}}{2}$ or $y \in \operatorname{Im}\gamma$
γ gives a unit-speed parametrization of $W \cap \mathbb{I}^n$ and has the same initial conditions as ω, so it must agree with ω. □

Corollary 3.1.1 $\mathbb{H}^n$ is a complete Riemannian manifold
Proof. By Theorem 3.1 the domain of each geodesic can be extended to $\mathbb{R}$, so that $\mathbb{H}^n$ is geodesically complete and by the Hopf-Rinow Theorem ([Lee18])$\mathbb{H}^n$ is complete. □

Corollary 3.1.2 $\exists!$ geodesic between any 2 points of $\mathbb{H}^n$.
(Again by Hopf-Rinow Theorem)

In exactly the same way as Theorem 3.1, we can prove
Theorem 3.2 If $x \in S^n$ and $v \in T_xS^n$, the unique geodesic starting at x with velocity v is given by

$$t \mapsto (\cos t) \cdot x + (\sin t) \cdot v$$

In particular, its image is the intersection of S^n with the plane spanned by x and v in $\mathbb{R}^{n+1}$.

Definition 3.3 $N \subseteq \mathbb{H}^n$ is said to be a **hyperbolic subspace** if it contains the entire geodesic passing through any 2 of its points.

Theorem 3.4 $N \subseteq \mathbb{I}^n$ is a hyperbolic subspace if and only if it is the intersection of a linear subspace of $\mathbb{R}^{n+1}$ with $\mathbb{I}^n$. Thus $N \overset{\text{submanifold}}{\subseteq} \mathbb{I}^n$

Proof. By Theorem 3.1, N is the intersection of $\mathbb{I}^n$ with a collection of planes through $\vec{0}$.
For linear closure, say $x, y \in N$
Then $\mathbb{R}\langle x, y\rangle$ intersects $\mathbb{I}^n$ in a geodesic $\subseteq N$ □

Using the geodesic completeness of $\mathbb{H}^n$, we can prove the following useful result.

Theorem 3.5 $\mathcal{I}(\mathbb{H}^n)$ acts transitively on pairs of points in $\mathbb{H}^n$, i.e. if $x, x' \in \mathbb{H}^n$ and $y, y' \in \mathbb{H}^n$ are such that $d(x, x') = d(y, y')$ then $\exists\ \phi \in \mathcal{I}(\mathbb{I}^n)$ such that $\phi(x) = y$ and $\phi(x') = y'$

Proof. Let γ be the maximal geodesic between x and x' (starting at x)
Let $\widetilde{\gamma}$ be the maximal geodesic between y and y' (starting at y)
By Theorem 1.12, $\exists\ \phi \in \mathcal{I}(\mathbb{I}^n)$ such that

$$\phi(x) = y \text{ and } \phi_{*,x}(\gamma'(0)) = \widetilde{\gamma}'(0)$$

Then $\phi(\gamma) = \widetilde{\gamma}$ because $\gamma, \widetilde{\gamma}$ are determined by their value and velocity at a point.
Now $\phi(x') \in \mathrm{tr}(\widetilde{\gamma})$ but $d(\phi(x), \phi(x')) = d(x, x') = d(y, y')$
So $d(\phi(x'), y) = d(y', y) \Rightarrow \phi(x') = y'$ □

We now consider the other models of $\mathbb{H}^n$.

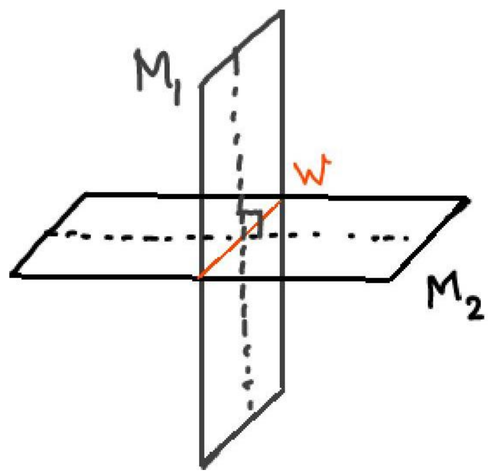
Definition 3.6 An affine subspace Y of $\mathbb{R}^n$ is said to be **vertical** if

$$Y = Y' + \mathbb{R}e_n$$

where Y' is an affine subspace of $\mathbb{R}^{n-1} \times \{0\}$ and

$$e_n = (0, \ldots, 0, 1)$$

We now allow a **sphere** in $\mathbb{R}^n$ to have dimension less than $n-1$ too. However while talking of inversion with a sphere we always assume the sphere has maximal dimension $(n-1)$.
Let $M_1, M_2 \to$ spheres/affine subspaces (or parts of them) in $\mathbb{R}^n$.
m_1, m_2 are their respective dimensions.
We say M_1 and M_2 are **orthogonal** or $M_1 \perp M_2$ if they intersect and $\forall\ x \in M_1 \cap M_2$, $W = T_xM_1 \cap T_xM_2$ is a linear subspace of dimension $\max\{0, m_1 + m_2 - n\}$ and the orthogonal complements of W in T_xM_1 and T_xM_2 are orthogonal to each other.

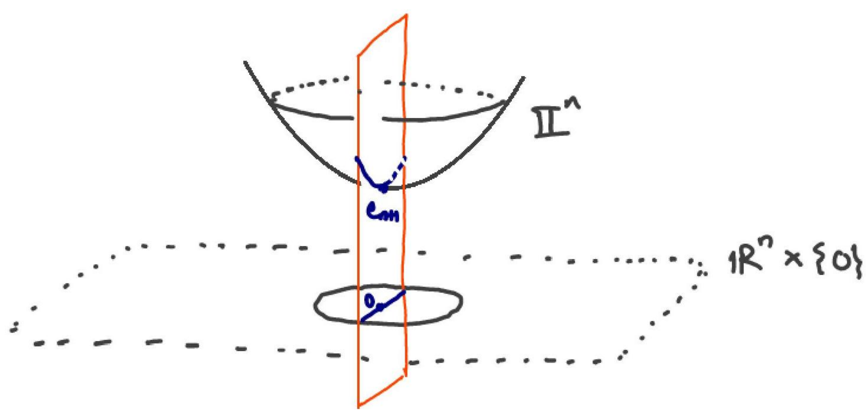


Theorem 3.7

i) $N \subseteq \mathbb{D}^n$ is a hyperbolic subspace if and only if it is the intersection of $\mathbb{D}^n$ with a linear subspace of $\mathbb{R}^n$ or a sphere orthogonal to ∂D^n. Thus, geodesics are obtained by parametrizing the diameters of D^n and the circles orthogonal to ∂D^n.

ii) $N \subseteq \mathbf{\Pi}^n$ is a hyperbolic subspace if and only if it is the intersection of $\mathbf{\Pi}^n$ with an affine vertical subspace or a sphere $\perp \partial\mathbf{\Pi}^n$. Thus, geodesics are obtained by the parameterization of vertical lines and semicircles $\perp \partial\mathbf{\Pi}^n$.

Proof. i) We first find the geodesics through $\vec{0}$ in $\mathbb{D}^n$. If $p : \mathbb{I}^n \to \mathbb{D}^n$ is the stereographic projection, clearly p sends any vertical linear subspace of $\mathbb{R}^{n+1}$ to itself and thus geodesics in $\mathbb{I}^n$ through e_{n+1} go to their projections onto $\mathbb{R}^n \times \{\vec{0}\}$, which are diameters of D^n.



Now, if γ is a geodesic through $x \in D^n$ then

$$i_{\frac{x}{||x||^2}, \frac{1}{||x||^2}-1}$$

being an isometry on $\mathbb{D}^n$, sends γ to a geodesic γ' through $\vec{0}$ ($i(x) = 0$).
So $i \circ \gamma = \gamma' =$, a diameter of D^n

$$\Rightarrow \gamma = i \circ \gamma' = \text{an arc} \perp \partial D^n$$

Moreover, every such arc $\perp \partial D^n$ can be sent to a diameter via an inversion.

In fact, any hyperbolic subspace H through $\vec{0}$ (which we now know to be the intersection of a linear subspace of $\mathbb{R}^{n+1}$ with D^n) will go to a hyperbolic subspace through x via $i_{\frac{x}{||x||^2}, \frac{1}{||x||^2}-1}$ which by Theorem 2.3 is a sphere $\perp \partial D^n$ if $x \notin H$ (angles are preserved).

ii) Directly follows from i) as the map $i : D^n \to \mathbf{\Pi}^n$ used in defining $\mathbf{\Pi}^n$ is an inversion and thus it maps spheres and affine subspaces to spheres and affine subspaces. □

Corollary 3.7 A p-dimensional hyperbolic subspace in $\mathbb{H}^n$ is isometric to $\mathbb{H}^p$.

Proof Consider the disc model and assume the hyperbolic subspace contains $\vec{0}$. We showed in 3.7 that it must be the intersection of $\mathbb{D}^n$ with a linear subspace of $\mathbb{R}^n \times \{0\}$, which is clearly a p-dimensional disc with the inherited hyperbolic metric, which is the same as the metric on $\mathbb{D}^p$ (or we can think of $\mathbb{D}^p/\mathbf{\Pi}^p$ naturally embedded in $\mathbb{D}^n/\mathbf{\Pi}^n$ by setting the last $n-p$ coordinates as zero and the metric defining map: $\Pi^p \to \mathbb{D}^n$ sends Π^p to $\mathbb{D}^p$). If the hyperbolic subspace doesn't contain $\vec{0}$ but passes through some $x \in \mathbb{D}^n$ then $i_{\frac{x}{||x||^2}, \frac{1}{||x||^2}-1}$ isometrically takes it to $\mathbb{D}^p$.

Theorem 3.7 allows us to explicitly compute the hyperbolic distance in D^n and $\mathbf{\Pi}^n$.
th = hyperbolic tan function, ath = th^{-1}.

Theorem 3.8 i) If $x, y \in \mathbb{D}^n$, we have

$$d(x,y) = 2\,\mathrm{ath}\left(\frac{||x-y||}{(1 - 2\langle x \mid y\rangle + ||x||^2 \cdot ||y||^2)^{1/2}}\right)$$

ii) If $(x,t), (y,s) \in \mathbf{\Pi}^n$ then

$$d((x,t),(y,s)) = 2\,\mathrm{ath}\left(\frac{||x-y||^2 + (t-s)^2}{||x-y||^2 + (t+s)^2}\right)$$

Proof. i) Notice that the function $h : \mathbb{R} \to (-1, 1)$

$$t \mapsto \operatorname{th}(t/2) = \frac{e^{t/2} - e^{-t/2}}{e^{t/2} + e^{-t/2}}$$

is bijective and monotonically increasing. Let $v \in \partial D^n$
Now we claim $\gamma : \mathbb{R} \to \mathbb{D}^n$

$$t \mapsto \operatorname{th}(t/2) \cdot v$$

is a unit speed parametrisation of the diameter of $\mathbb{D}^n$ towards v.

$$\dot\gamma(t) : t \mapsto \frac{\frac{1}{2}\left(e^{t/2} + e^{-t/2}\right)^{-2} - \frac{1}{2}\left(e^{t/2} - e^{-t/2}\right)^{-2}}{(e^{t/2} + e^{-t/2})^2} \cdot v$$

$$= \frac{1}{2\cosh^2(t/2)} \cdot v$$

$$ds^2_{\gamma(t)}(\dot\gamma(t)) = \left(\frac{2}{1 - \|\gamma(t)\|^2}\right)^2 \|\dot\gamma(t)\|^2$$

$$= \left(\frac{2}{1 - \operatorname{th}^2(t/2)}\right)^2 \left(\frac{1}{2\cosh^2(t/2)}\right)^2$$

$$= \left(\frac{2}{\frac{1}{\cosh^2(t/2)}}\right)^2 \left(\frac{2}{\cosh^2(t/2)}\right)^2 = 1$$

Thus $\forall\ x \in \mathbb{D}^n, \quad d(x, 0) = 2\operatorname{ath}(\|x\|)$
We can similarly find $d(x, y) \quad \forall\ x, y \in \mathbb{D}^n$, but we omit the calculation.

ii) If $i : \mathbb{D}^n \to \mathbf{\Pi}^n$ is the metric defining inversion, to find $d(x, y) \ \ \forall\ x, y \in \mathbf{\Pi}^n$ we simply need to compute $d(i(x), i(y))$ as $i(x), i(y) \in \mathbb{D}^n$.
We avoid the explicit calculation. □

The Boundary of $\mathbb{H}^n$

We can already think of $\partial\mathbb{D}^n$ as the S^{n-1} bounding $\mathbb{D}^n$ and $\partial\mathbf{\Pi}^n$ as $\mathbb{R}^n \times \{0\} \cup \{\infty\}$ but now we give an intrinsic definition of this boundary (independent of embedding in $\mathbb{R}^n$ and the chosen model).

Definition 3.9 $S :=$ the set of half geodesics in $\mathbb{H}^n$, parametrized by arc-length on $[0, \infty)$
If $\gamma_1, \gamma_2 \in S$ we say $\gamma_1 \sim \gamma_2$ if and only if

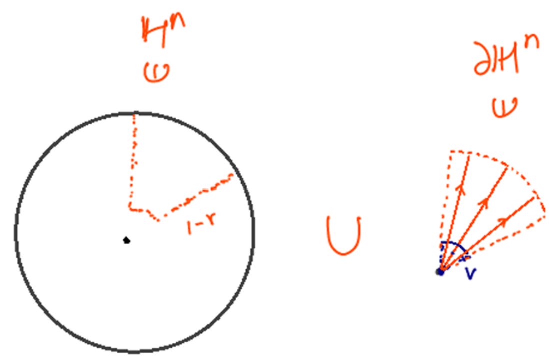
$$\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$$

As a set we define $\partial\mathbb{H}^n := S/\sim$ and $\overline{\mathbb{H}^n} := \mathbb{H}^n \cup \partial\mathbb{H}^n$.
We give $\overline{\mathbb{H}^n}$ a topology such that $\mathbb{H}^n \overset{\text{open}}{\subseteq} \overline{\mathbb{H}^n}$ and it inherits its own topology as a subspace.
Let $p \in \partial\mathbb{H}^n$, choose $\gamma \in [p]$ such that $x := \gamma(0)$ and V is a neighbourhood of $\dot{\gamma}(0)$ in the unit sphere in $T_x\mathbb{H}^n$.
Let $r > 0$. Then,

$$U(\gamma, V, r) := \{\gamma_1(t) \mid \gamma_1 \in S, \gamma_1(0) = x, \dot{\gamma}_1(0) \in V, t > r\} \cup \{[\gamma_1] \mid \gamma_1 \in S, \gamma_1(0) = x, \dot{\gamma}_1(0) \in V\}$$

is defined to be a neighbourhood of p.

$U(\gamma, V, r) =$



We claim that these neighbourhoods along with the open sets of $\mathbb{H}^n$ form the basis for a topology on $\overline{\mathbb{H}^n}$. Clearly, they cover all of $\partial\mathbb{H}^n$ on varying γ.
If $\gamma_1(0) = \gamma_2(0)$:

$$U(\gamma_1, V_1, r_1) \cap U(\gamma_2, V_2, r_2) = U(\gamma_1, V_1 \cap V_2, \max(r_1, r_2))$$

In general, with some imagination, one can convince oneself that every point of the intersection of two basic elements will have a neighbourhood of this form contained in the intersection.

Theorem 3.10 $\partial\mathbb{H}^n$ is homeomorphic to S^{n-1} and $\overline{\mathbb{H}^n}$ is homeomorphic to $\overline{D^n}$. Thus, considering the disc model, $\overline{\mathbb{D}^n}$ is canonically identified with the closure of $\mathbb{D}^n$ as a subset of $\mathbb{R}^n$.

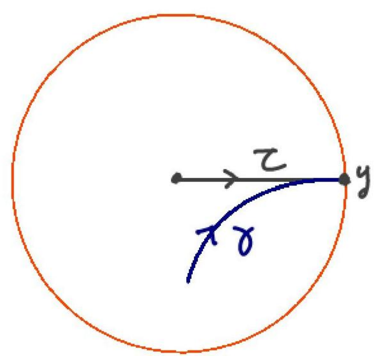
Proof.

$$f : \overline{D^n} \longrightarrow \overline{\mathbb{H}^n} \quad \text{(disc model)}$$

$$x \mapsto x \quad \text{if } x \in D^n$$

$$y \mapsto \text{(radial half geodesic from 0 to } y) \quad \text{if } y \in \partial D^n$$

As $\overline{D^n}$ is compact, it suffices to show that f is a continuous bijection.
We have covered geodesics ending at each point on the boundary via f.
Claim: f is surjective.
We show any half geodesic γ ending at y is equivalent under $\sim$ to the radial half geodesic from 0 to y.



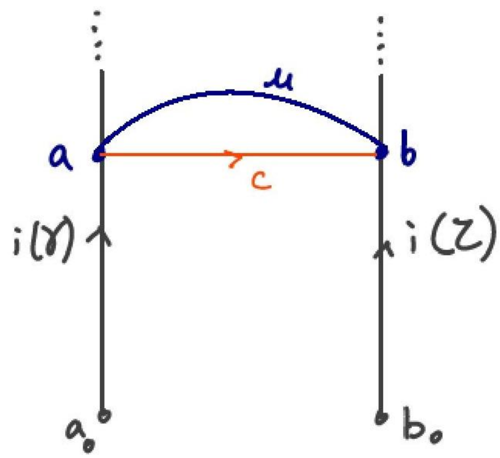
Recall $i : \mathbb{D}^n \to \mathbf{\Pi}^n$

$$x \mapsto 2\frac{x + e_n}{\|x + e_n\|^2} - e_n$$

Perform a rotation (isometry) on $\mathbb{D}^n$ sending y to $-e_n$ so that without loss of generality, $i(y) = \infty$.
Note that here we have extended i to a continuous map: $\overline{\mathbb{D}}^n \longrightarrow \overline{\mathbf{\Pi}}^n$.
Thus, $i(\gamma)$ and $i(\tau)$ become vertical rays in $\mathbf{\Pi}^n$.
First, assume that they start at the same height h_0.
Let $t \in \mathbb{R}$ and say

$$i(\gamma)(t) = a$$

$$i(\mathcal{Z})(t) = b$$



Now $d(a, b) = l(\mu) \leq l(c)$.
Let $c : [0, 1] \longrightarrow \mathbf{\Pi}^n$

$$s \mapsto (1 - s)a + s \cdot b$$

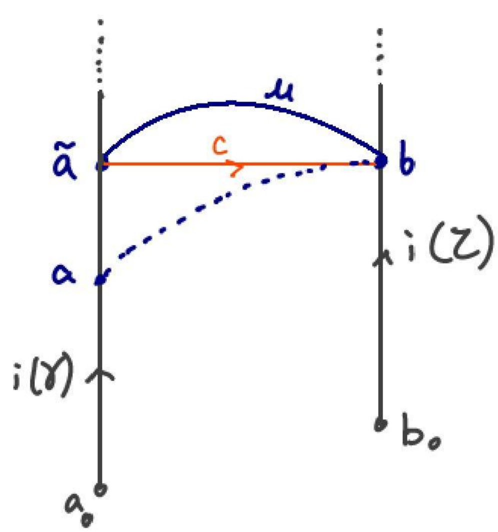
$$c'(s) = b - a \quad \forall\ s$$

Let $h = a_n = b_n$ (same height as horizontal translations are isometries).

$$l(c) = \int_0^1 \sqrt{\langle c'(s), c'(s)\rangle}\, ds = \int_0^1 \frac{\|b - a\|}{h}\, ds = \frac{\|b - a\|}{h} \quad \text{(Euclidean norm)}$$

Thus $d(a, b) \leq l(c) \leq \frac{\|b-a\|}{h_0}$

Now say they start at different heights h_1 and h_2.



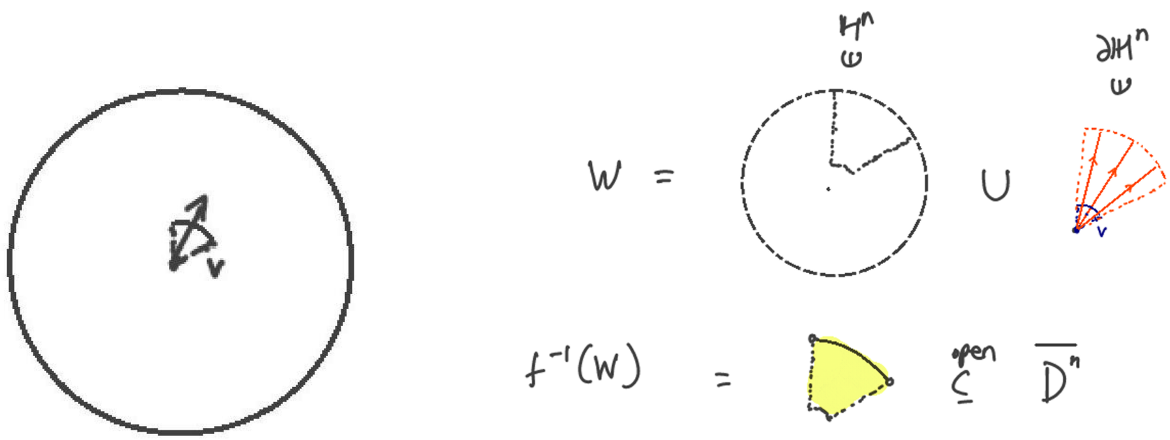
$$d(a,b) \le d(a,\tilde{a}) + d(\tilde{a},b)$$

Now by a horizontal translation we can overlay $i(\tau)$ onto $i(\gamma)$ so that $\tilde{a} = b$ and $i(\gamma)$ remains unit speed.
Then if $d(a,b_0) = t_0$, we get

$$i(\gamma)(t) = i(\tau)(t + t_0) \quad \forall\ t$$

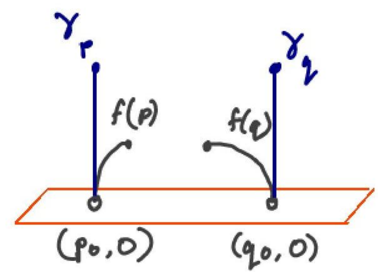
Thus $d(a,\tilde{a}) = t_0$ (independent of a and b).
So $d(a,b) \le t_0 + \frac{\|b-a\|}{h_0} \quad \forall\ a,b$ on the vertical lines. This shows that $i(\gamma)$ and $i(\tau)$, and thus γ and τ always have bounded distance; so that $\gamma \sim \tau$ and f is surjective.

Now let $p \in \partial H^n$ and consider a neighbourhood $U(\gamma, V, r) =: W$ of p, such that γ is the radial half geodesic representative of p.



So f is continuous.

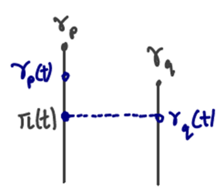
For injectivity on the boundary, let $p, q \in \partial D^n$. We show the distance between $f(p)$ and $f(q)$ is unbounded.
In the half-space model $\mathbf{\Pi}^n$, assume $\infty \notin \{p,q\}$.
Now we can replace $f(p)$ and $f(q)$ with vertical half lines with endpoints p and q respectively, as we have shown that the distance between half geodesics ending at the same point is bounded.



Assume first that γ_p and γ_q start at the same height.
Then at any t, γ_p and γ_q are at the same height h_t.

$$d(\gamma_p(t), \gamma_q(t)) = 2 \text{ ath} \left(\frac{\|p_0 - q_0\|}{\|p_0 - q_0\| + 2h_t} \right)$$

As $t \to \infty$, $2h_t \to 0$, so $\frac{\|p_0 - q_0\|}{\|p_0 - q_0\| + 2h_t} \to 1$
and thus $d(\gamma_p(t), \gamma_q(t)) \to \infty$
If they started at different heights, such that $\gamma_p(t_0)$ is at the same height as $\gamma_q(0)$,



$$\begin{aligned} d(\gamma_p(t), \gamma_q(t)) + d(\gamma_p(t), \pi(t)) &> d(\pi(t), \gamma_q(t)) \\ \| \ (\gamma_p, \gamma_q &\to \text{unit speed}) \\ t_0 & \\ \Rightarrow d(\gamma_p(t), \gamma_q(t)) &> d(\pi(t), \gamma_q(t)) \end{aligned}$$

and the RHS blows up as $t \to \infty$. So we have shown f is a continuous bijection. $\square$

Remark 3.11

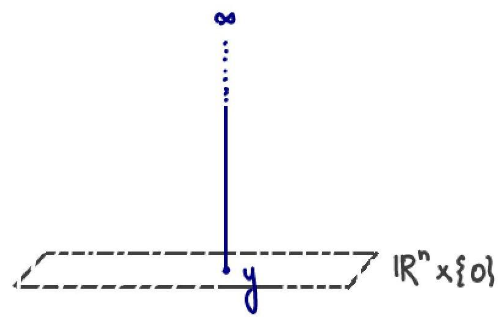
- $i : \mathbb{D}^n \to \mathbf{\Pi}^n$ maps $\partial \mathbb{D}^n$ onto $R^n \times \{0\} \cup \{\infty\}$ which is the natural boundary of $\mathbf{\Pi}^n$.
- Since $\partial \mathbb{D}^n \simeq S^n$ and $\mathbb{D}^n$ already has the conformal structure of D^n, we can give $\partial \mathbb{D}^n$ the conformal structure of the S^{n-1} bounding D^n.
- We refer to points in $\partial \mathbb{H}^n$ as the points at ∞ of $\mathbb{H}^n$.
- We say p is an end point of a geodesic γ if

$$f(\gamma_{|[0,\infty)}) = p \quad \text{or} \quad f(\gamma_{|(-\infty,0]}) = p.$$

- Each geodesic has 2 end points.

Theorem 3.12 $\forall\, x, y \in \mathbb{H}^n$, $\exists!$ geodesic having x and y as its endpoints.

Proof. We know $\mathcal{I}(\mathbb{H}^n)$ acts transitively on geodesics, so of course it also acts transitively on equivalence classes of half geodesics. Thus, $\mathcal{I}(\mathbb{H}^n)$ acts transitively on points in $\partial\mathbb{H}^n$.
Then assume $x = \infty$ in the half-space model $\mathbf{\Pi}^n$.
Then $y \in \partial\mathbf{\Pi}^n \setminus \{\infty\}$ and it is clear that the vertical line arising from y is the unique geodesic with endpoints x and y.



□

Definition 3.13 We say that two geodesics in $\mathbb{H}^n$ are

1. **incident** if they have a common point in $\mathbb{H}^n$
2. **asymptotically parallel** if they have a common endpoint
3. **ultra-parallel** if they have no intersection in $\overline{\mathbb{H}^n}$

Theorem 3.14 Let $N, M \subseteq \mathbb{H}^n$ be hyperbolic subspaces

1. If N, M meet in $\partial\mathbb{H}^n$ and not in $\mathbb{H}^n$, then they have exactly 1 common point at infinity and no geodesic orthogonal to both N and M.
2. If N, M don't meet in all of $\overline{\mathbb{H}^n}$ then $\exists$ geodesic γ orthogonal to both N and M, and the distance between N and M is the length of the arc on γ between N and M.

Proof.

1. Let x be a common point at infinity of N and M. We assume $x = \infty$. Then the images of N and M are vertical. If they don't intersect in $\mathbb{H}^n$, clearly they can meet in $\partial\mathbb{H}^n$ only at ∞. Any geodesic intersecting them must be a semicircle and is orthogonal to N if and only if it intersects N only at its topmost point, in which case it cannot be orthogonal to M.

2. $d(N, M) = \inf\{d(a, b) \mid a \in N, b \in M\}$ so $\exists$ sequence $a_n \in N, b_n \in M$ such that $d(a_n, b_n) \to d(N, M) =: \delta \geq 0$.

   Now if x is a limit point of $\{a_n\}$ in ∂N then $\exists$ subsequence $a_{n_k} \to x$.

   But $d(a_{n_k}, b_{n_k}) \to \delta \Rightarrow x$ is a limit point of $\{b_n\}$ (as distance blows up near $\partial\mathbb{H}^n$) and thus $x \in \partial M$ too $\Rightarrow\!\Leftarrow$

   So any limit points of $\{a_n\}$ and $\{b_n\}$ must be in $\mathbb{H}^n$.

   As $\overline{N}$ is compact, $\{a_n\}$ has a limit point $a \in \overline{N} \cap \mathbb{H}^n = N$.

   So $\exists$ subsequence $\{a_{n_k}\} \to a$, without loss of generality, $\{a_n\} = \{a_{n_k}\}$.

   Now $\{b_n\}$ has a limit point in M.

   So $\exists$ subsequence $\{b_{n_k}\} \to b$.

   So $\{a_{n_k}\} \to a$, $\{b_{n_k}\} \to b$ and $d(a_{n_k}, b_{n_k}) \to \delta \Rightarrow d(a, b) = \delta$.

   Note that as N and M are closed in $\mathbb{H}^n$, $\delta = 0 \Rightarrow a = b$ and thus $a \in M \cap N \Rightarrow\!\Leftarrow$

   So $\delta > 0$

   Consider the disc model $\mathbb{D}^n$ and by isometry, assume $a = \vec{0}$.

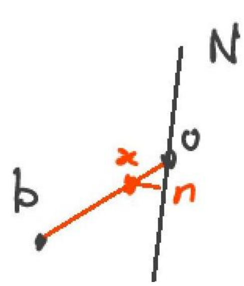
   Consider a small enough neighbourhood $N_\epsilon(0)$.

   Note $\forall\ x \in N_\epsilon(0)$, $d(0, x) = 2\ \mathrm{ath}(||x||)$ where $||x|| < \epsilon$.

   For small enough t, $th(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} \approx \frac{1+t-(1-t)}{1+t+1-t} = t$.

   Then $\mathrm{ath}(t) \approx t$ and we can say that in $N_\epsilon(0)$, $d(0, x) \approx 2||x||$.

   Now if the geodesic γ between 0 and b isn't $\perp N$, we can see $\exists\ x \in N_\epsilon(0)$, $n \in N$ such that $d(x, 0) > d(x, n)$.



   But then $d(M, N) \leq d(b, n) \leq d(b, x) + d(x, n) < d(b, x) + d(x, 0) = \delta \ \Rightarrow\!\Leftarrow$

   So $\gamma \perp N$ and repeating this for M, we see $\gamma \perp M$. $\square$

Theorem 3.15
1) All isometries of $\mathbb{H}^n$ extend to homeomorphisms of $\overline{\mathbb{H}^n}$ with a fixed point in $\overline{\mathbb{H}^n}$.
2) Any $\phi \in \mathcal{I}(\mathbb{H}^n)$ is uniquely determined by its action on $\partial\mathbb{H}^n$.
3) If $M = \mathbb{D}^n$ or $\mathbf{\Pi}^n$ then the restriction to the boundary $r : \mathcal{I}(M) \xrightarrow{\sim} \mathrm{Conf}(\partial M)$ is an isomorphism.
Proof.
1) Any conformal map on the open disc $\mathbb{D}^n$ extends to $\mathbb{R}^n$, so of course it also extends to the boundary. It has a fixed point by Brouwer's fixed point theorem.
2) Again in $\mathbb{D}^n$, we only analyse $A \in \mathrm{O}(n)$ and inversion with spheres $\perp \partial\mathbb{D}^n$. Clearly A is fully determined by $A_{|\partial D^n}$.
Any inversion $\perp \partial\mathbb{D}^n$ is of the form $i_{\frac{x}{\|x\|^2}, \frac{1}{\|x\|^2}-1}$ for some $x \in \mathbb{D}^n$.
So knowing its value at any point of $\partial\mathbb{D}^n$ gives us x and thus determines the inversion.
3) Give the conformal structure of S^{n-1} to ∂M.
Now again in the disc model each isometry is a composition of elements of $\mathrm{O}(n)$ and inversions $\perp \partial\mathbb{D}^n$. These maps are conformal on $\mathbb{R}^n$ and thus their restriction to $\partial\mathbb{D}^n$ is conformal.
Thus, r is well-defined.
By 2), r is injective.
For surjectivity, observe that when $M = \mathbf{\Pi}^n$, the restriction to the boundary of

$$\lambda \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \circ i + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

is $\lambda A\bar{i} + b$.
$\forall\ \lambda > 0, A \in \mathrm{O}(n-1), b \in \mathbb{R}^{n-1}$, and inversion i with respect to $M\left(\begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \alpha\right)$
$\bar{i} \rightarrow$ inversion with respect to $M(x_0, \alpha)$. $\square$

Fixed Point Classification of Isometries

Theorem 3.16 If $\phi \in \mathcal{I}(\mathbb{H}^n)$ then exactly 1 of the following holds:
i) ϕ has some fixed point in $\mathbb{H}^n$
ii) ϕ has no fixed point in $\mathbb{H}^n$ and exactly 1 fixed point in $\partial\mathbb{H}^n$
iii) ϕ has no fixed point in $\mathbb{H}^n$ and exactly 2 fixed points in $\partial\mathbb{H}^n$

Proof. It suffices to show that if $\phi \in \mathcal{I}(\mathbb{H}^n)$ fixes no points in $\mathbb{H}^n$ then it cannot fix > 2 points in $\partial\mathbb{H}^n$.
In $\mathbb{H}^n$, assume ϕ fixes $a, b \in \partial\mathbb{H}^n$.
Then, by the transitivity of $\mathcal{I}(\mathbb{H}^n)$ on geodesics and thus on pairs of points in $\partial\mathbb{H}^n$, assume $a = 0$, $b = \infty$ in the half-space model.
So $\phi = \lambda \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ (fixing $\infty \Rightarrow$ no inversion, fixing $0 \Rightarrow$ no translation)
As $\phi\left(\begin{pmatrix} \vec{0} \\ 1 \end{pmatrix}\right) \neq \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix}$, $\lambda \neq 1$ and thus $\forall\ y \in \mathbb{R}^{n-1} \times \{0\}$

$$\phi\left(\begin{pmatrix} \vec{y} \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \lambda A\vec{y} \\ 0 \end{pmatrix}$$

such that $\|\lambda Ay\| = |\lambda|\|y\| \neq \|y\|$ □

Definition 3.17 We say $\phi \in \mathcal{I}(\mathbb{H}^n)$ is:
- **elliptic** if it is of type i)
- **parabolic** if it is of type ii)
- **hyperbolic** if it is of type iii)

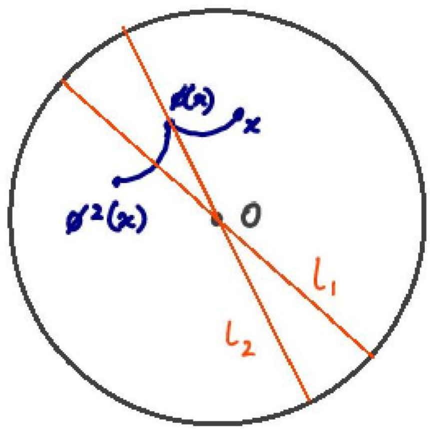
We now specialize this classification to dimensions 2 and 3 for orientation preserving isometries. For $n = 2$ we give a geometric and an algebraic classification of the isometries. If $a, b, c \in IH^2$, $[a, b]$ will denote the geodesic segment between a and b, and (a, b, c) denotes the angle between $[a, b]$ and $[b, c]$ at b.

Theorem 3.18 Let $\phi \in \mathcal{I}^+(\mathbb{H}^2) \setminus \{\mathrm{id}\}$ and $x \in \mathbb{H}^n$ such that $\phi(x) \neq x$.
Let $l_1 :=\perp$ bisector geodesic of $[\phi(x), \phi^2(x)]$
$l_2 := \angle$ bisector geodesic of $(x, \phi(x), \phi^2(x))$
i) l_1 and l_2 are incident if and only if ϕ is elliptic
ii) l_1 and l_2 are asymptotically parallel if and only if ϕ is parabolic
iii) l_1 and l_2 are ultra-parallel if and only if ϕ is hyperbolic.

Proof. First we remark that if ϕ has 2 fixed points in $\mathbb{H}^2$, then it fixes the geodesic through them. Looking at $\mathbb{D}^2$, let $\vec{0}$ be one of the fixed points.
$\Rightarrow \phi = A$ (reflection with respect to a diameter) $\rightarrow$ orientation reversing
Thus ϕ can fix at most 1 point in $\mathbb{H}^2$. Suppose ϕ is:

i) *elliptic*

By isometry, assume $0 \in \mathbb{D}^2$ is fixed.



$\phi \in \mathrm{SO}(2)$, so it is simply a rotation

As $\phi(x)$ and $\phi^2(x)$ are equidistant from $\vec{0}$, l_1 contains $\vec{0}$ and the situation is shown above.

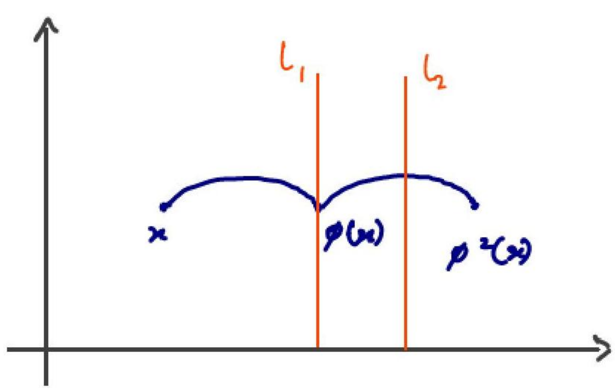
ii) *parabolic*

Assume $\infty \in \partial\mathbf{\Pi}^2$ is fixed

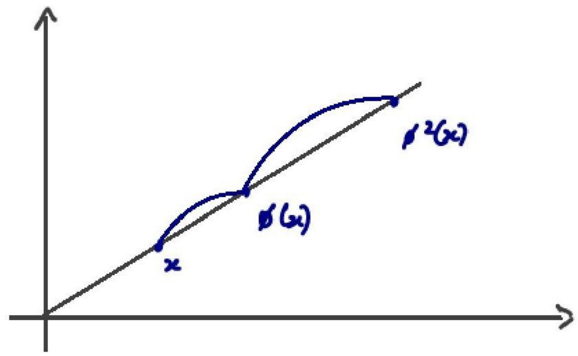
$$\phi = \lambda \begin{bmatrix} +1 & 0 \\ 0 & 1 \end{bmatrix} + b \quad \text{(no inversions because } \infty \text{ is fixed)}$$

Note $b \neq 0$ else 0 will also be fixed.

By dilation (isometry), assume $\phi(x) = x + b \quad \forall\, x \in \Pi^2$



iii) *hyperbolic*
Assume $0, \infty \in \partial\mathbf{\Pi}^2$ are fixed
Then $\phi(x) = \lambda x$ for some $\lambda > 0$, $\forall\ x \in \Pi^2$



$\square$

Theorem 3.19 Let $\phi \in \mathcal{I}^+(\mathbb{H}^2) \setminus \{\mathrm{id}\}$ be represented by a matrix A in $\mathrm{SL}_2(\mathbb{R})$. Then
i) ϕ is elliptic if and only if $|tr(A)| < 2$
ii) ϕ is parabolic if and only if $|tr(A)| = 2$
iii) ϕ is hyperbolic if and only if $|tr(A)| > 2$
Proof. Suppose that ϕ fixes a point at infinity. without loss of generality, we assume ϕ fixes ∞ as the trace of a matrix is conjugation invariant.
Then in $z \mapsto \frac{az+b}{cz+d}$, $c = 0$ and $a \neq 0$
Note that $c = 0 \Rightarrow \phi$ has no fixed point in $\mathbf{\Pi}^2$ (unless $\phi =$id).

$$ad - bc = 1 \Rightarrow d = 1/a$$

So $\phi : z \mapsto a^2 z + ab$
If $a = \pm 1$ ($\iff$ $\mathrm{tr}(A) = \pm 2$) then ϕ is a translation $\Rightarrow$ parabolic
However, if $a \neq \pm 1$ then ϕ fixes $\left(\frac{ab}{1-a^2}, 0\right) \in \partial\mathbf{\Pi}^2$
$\Rightarrow \phi$ is hyperbolic
Conversely, ϕ is parabolic or hyperbolic $\Rightarrow |tr(A)| = a + \frac{1}{a} \geq 2$
So lastly ϕ is elliptic if and only if $|tr(A)| < 2$ $\square$

We now move on to $\mathbb{H}^3$

Notice $\mathcal{I}^+(\mathbb{H}^3) \simeq \mathrm{Conf}^+(\partial\mathbb{H}^3) \simeq \mathrm{SL}_2(\mathbb{C})/\{I, -I\} =: \mathrm{PSL}_2(\mathbb{C})$

Theorem 3.20 If $\phi \in \mathcal{I}^1(\mathbb{H}^3) \setminus \{\mathrm{id}\}$ is represented by a matrix A in $\mathrm{SL}_2(\mathbb{C})$. Then

i) ϕ is elliptic if and only if $\mathrm{tr}(A) \in \mathbb{R}$ and $|\,\mathrm{tr}(A)| < 2$

ii) ϕ is parabolic if and only if $\mathrm{tr}(A) = \pm 2$

iii) ϕ is hyperbolic if and only if $|\,\mathrm{tr}(A)| > 2$ or $\mathrm{tr}(A) \neq \mathbb{R}$

Proof. $\phi_{|\partial\Pi^3}$ is a map: $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} : z \mapsto \frac{az+b}{cz+d}$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{C})$

Note by the algebraic closure of $\mathbb{C}$, $z = \frac{az+b}{cz+d}$ has a complex root.

Thus, $\phi_{|\partial\Pi^3}$ always fixes a point in $\mathbb{C} \cup \{\infty\}$.

As trace is conjugation invariant, let ∞ be a fixed point.

Then again $c = 0,\ a \neq 0$ and $\phi_{|\partial\Pi^2}(z) = a^2 z + ab$

$\Rightarrow d = 1/a$

Now $\mathrm{tr}(A) = a + \frac{1}{a}$

Recall $\lambda M + b$ extends to $\lambda \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} \quad \forall\ M \in \mathrm{O}(2), \quad \lambda > 0,\ b \in \mathbb{R}^2$

Here $M(z) = \frac{a^2}{|a|^2} z$, so $\lambda = |a|^2 \Rightarrow \phi(z,t) = (a^2 z + ab, |a|^2 t)$

i) If $|a| = 1$, $a \neq \pm 1$ (iff $|\mathrm{tr}(A)| < 2$ and $\mathrm{tr}(A) \in \mathbb{R}$), then $a^2 z + ab = z$ has the solution, $z_0 = \frac{ab}{1-a^2}$, and $|(z_0, t)|$ is fixed under $\phi\ \forall\ t \Rightarrow \phi$ is elliptic.

ii) If $a = \pm 1$ ($\mathrm{tr}(A) = \pm 2$), then $z \pm b = z$, has no solution except ∞, which is the only fixed point $\Rightarrow \phi$ is parabolic.

iii) If $|a| \neq 1$ ($|\mathrm{tr}(A)| > 2$ or $\mathrm{tr}(A) \notin \mathbb{R}$), then $(z_0, 0)$ and ∞ are the only points fixed under $\phi \Rightarrow \phi$ is hyperbolic. $\square$

Horospheres and Horoballs

Definition 3.21 If $p \in \partial\mathbb{H}^n$, we say a closed hypersurface $N \subseteq \mathbb{H}^n$ is a **horosphere** centred at p if it is $\perp$ to all geodasic lines through p.

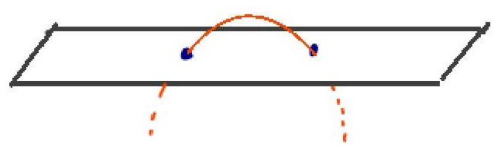
Theorem 3.22 Given $p \in \partial\mathbb{H}^n$, $\mathbb{H}^n$ is the disjoint union of all horospheres centred at p. As a subspace of $\mathbb{H}^n$, each horosphere inherits the Euclidean Riemmannian metric on $\mathbb{R}^{n-1}$.

Proof. By the transitivity on points in $\partial\mathbb{H}^n$, we assume $p = \infty \in \partial\mathbf{\Pi}^n$. Then it is clear that each horosphere centred at ∞ is a horizontal hypersurface and the first part of the theorem follows. On a horizontal hypersurface, the Riemmannian metric

$$ds^2_{(y,t)}(v) = \frac{\|v\|^2}{t^2}$$

is a scalar multiple of the Euclidean metric as $t =$ fixed. $\square$

Remark 3.22 Although horospheres in $\mathbb{H}^n$ inherit the Riemannian structure of $\mathbb{R}^{n-1}$, they don't inherit the metric space structure i.e. the distance function of $\mathbb{H}^n$ restricted to them is not Euclidean. This is because in defining the distance between 2 points on a horizontal hyperplane, we use geodesics that go beyond it.



Thus any horosphere centred at $p \in \partial\mathbb{H}^n$ divides $\mathbb{H}^n$ into 2 connected regions, and we call the one meeting $\partial\mathbb{H}^n$ in p a **horoball** centred at p.

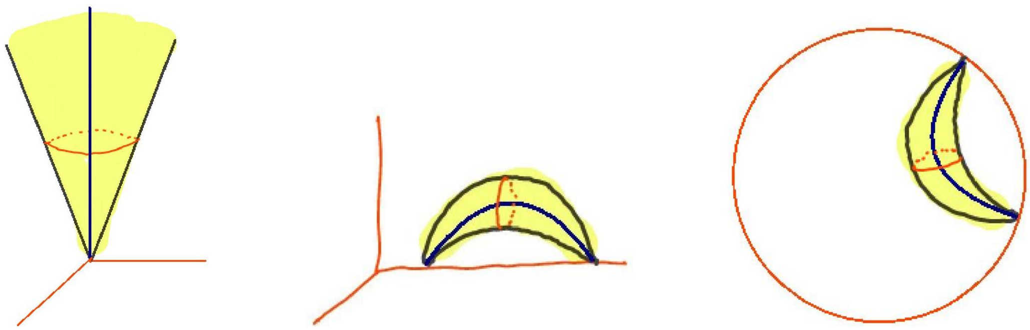
Theorem 3.23 A hyperbolic ball in $\mathbb{D}^n$ is a Euclidean ball, with a possibly different centre and radius.

Proof. We first note that the hyperbolic ball centred at $\vec{0} \in \mathbb{D}^n$ with radius r is simply the Euclidean ball centred at $\vec{0}$ with radius $\operatorname{th}(r/2)$ $(d(0, y) = 2 \operatorname{ath} \|y\|)$.
Now, the conformal maps on the disc are compositions of inversions and orthogonal linear maps, both of which send balls to balls, and thus the hyperbolic ball centred at any other $x \in \mathbb{D}^n$ is also a Euclidean ball. $\square$

We now describe the general shape of the geodesic neighbourhoods in $\mathbb{H}^n$. Given a geodesic Y in $\mathbb{H}^n$ and $\varepsilon > 0$, we consider

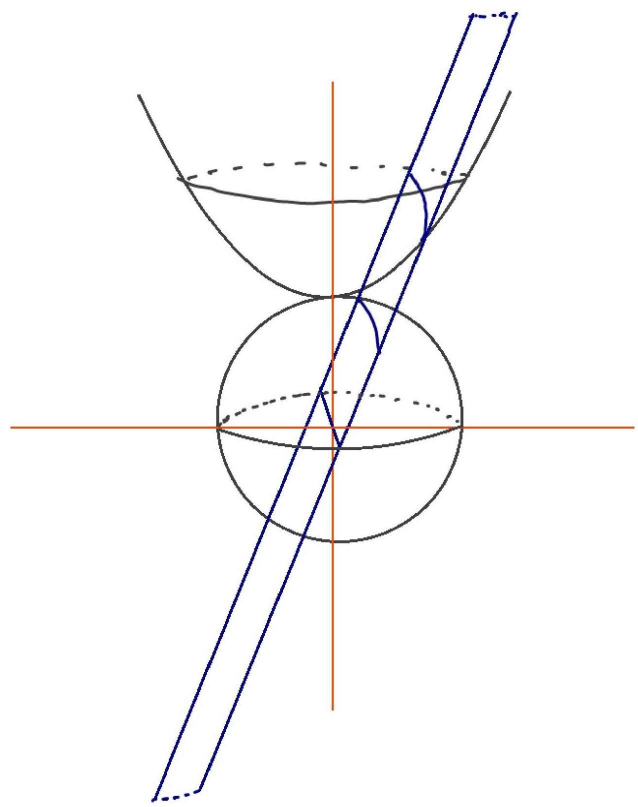
$$N_\varepsilon(Y) := \{x \in \mathbb{H}^n \mid d(x, Y) \leq \varepsilon\}$$

In $\mathbb{H}^n$, if $Y = \{0\} \times \mathbb{R}_+$ then $N_\varepsilon(Y)$ is invariant under dilations (as Y is) and thus $N_\varepsilon(Y)$ is the same one shown below. Now due to the properties of inversions, the general shape of $N_\varepsilon(Y)$ in $\mathbb{H}^n$ is as follows.



Remark 3.24 By restricting the canonical projection: $\mathbb{R}^{n+1} \to \mathbb{RP}^n$ to $\mathbb{I}^n$, we can obtain another model of $\mathbb{H}^n$, called the **Klein** (or projective) model. Its main peculiarity is that if we canonically identify it with the unit disc D^n then its inherited conformal structure differs from the usual $\mathrm{Conf}(D^n)$ i.e. angles are misrepresented (the same way lengths are misrepresented in $\mathbf{\Pi}^n$ or $\mathbb{D}^n$).

However, it has a nice property, the fact that all its geodesics are straight line segments! So a geodesic polygon is in this model a geodesic Euclidean polygon. This is sometimes useful in generalizing arguments applying to Euclidean polyhedra.



The Curvature of $\mathbb{H}^n$

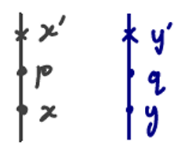
We first state a result on the strict convexity of the distance function, which qualitatively expresses the negative curvature of hyperbolic space.

Theorem 3.25

Let γ_1, γ_2 be closed geodesic arcs in $\mathbb{H}^n$, both part of different maximal geodesics. Assume their interiors don't intersect and $\exists$ at most 1 common endpoint. Let $x, x' \in \gamma_1$ and $y, y' \in \gamma_2$ with $x \neq x'$, $y \neq y'$ and $p := \frac{x+x'}{2}$, $q := \frac{y+y'}{2}$ (mid-point of y and y' on γ_2).

$$d(p,q) < \frac{d(x,y) + d(x',y')}{2}$$

Note that in $\mathbb{R}^n$, the LHS can be equal to the RHS; an example is



The proof proceeds by assuming $p = \vec{0} \in \mathbb{D}^n$ and analysing the symmetries about $\vec{0}$. The details are presented in [BP92].

Definition 3.26 We define the sectional curvature at a point x of a Riemannian manifold M with respect to a 2-subspace $V \subseteq T_x M$ as the curvature of the oriented Riemiannian surface obtained as the image of a suitably small neighbourhood of $\vec{0} \in V$ under the exponential mapping at x.

Theorem 3.27 The sectional curvature of $\mathbb{H}^n$ at x with respect to $V \subseteq T_x\mathbb{H}^n$ is independent of x, V and n!

Proof. We first show that $\mathcal{I}(\mathbb{H}^n)$ acts transitively on pairs $(x, V) \in M \times \{$2-subspaces of $T_x\mathbb{H}^n\}$. Pick an ONB $\{v_1, v_2\}$ of V (with respect to $\langle \cdot \mid \cdot \rangle_{(n,1)}$) and extend to ONB $\{v_1, \dots, v_n\}$ of $T_x\mathbb{H}^n$. Then

$$A := \begin{bmatrix} | & | & & | & | \\ v_1 & v_2 & \dots & v_{n-1} & x \\ | & | & & | & | \end{bmatrix} \in \mathrm{O}(I_n)$$

is such that its inverse sends $(x, V) \mapsto (e_{n+1}, \mathbb{R}[e_1, e_2])$.

Since this can be done for any (y, W) as well, the action is transitive.

For independence with respect to n, note that a hyperbolic subspace of dimension $n-1$ in $\mathbb{D}^n$ obtained by intersecting an $(n-1)$-dimensional subspace with $\mathbb{D}^n$ is a regular submanifold of $\mathbb{D}^n$ and we showed that it is isometric to $\mathbb{H}^{n-1}$. Now $\forall\ x \in \mathbb{D}^{n-1}$, $\exp_x\colon T_x\mathbb{D}^n \to \mathbb{D}^n$ restricts to $\overline{\exp}_x\colon T_x\mathbb{D}^{n-1} \to \mathbb{D}^{n-1}$ (since the geodesics are the same), and thus $\exp_x(V)$ for some 2-dimensional subspace $V \leq T_x\mathbb{H}^{n-1} \leq T_x\mathbb{H}^n$ is equal to $\overline{\exp}_x(V)$. □

Corollary 3.27 All sectional curvatures of $\mathbb{H}^n$ are -1.
Proof. Follows from the well known fact that the Gaussian curvature of $\mathbb{H}^2$ is -1. □

We can also thus conclude that the sectional curvatures of $\mathbb{R}^n$ are 0 and S^n are 1.

Theorem 3.28 Let T be a geodesic triangle in $\mathbb{H}^2$ with inner angles α, β, γ.

$$Area(T) = \pi - (\alpha + \beta + \gamma)$$

Proof. By the Gauss-Bonnet Formula ([Tu2]), $\int \mathcal{K}_g\, ds = 2\pi - \sum \varepsilon_i - \int \mathcal{K} \cdot \mathrm{vol}$
For the geodesic triangle, $\varepsilon_1 = \pi - \alpha,\ \varepsilon_2 = \pi - \beta,\ \varepsilon_3 = \pi - \gamma$ and $\mathcal{K}_g = 0$. Thus,

$$0 = 2\pi - (3\pi - \alpha - \beta - \gamma) - \int (-1) \cdot \mathrm{vol}$$

$\Rightarrow \pi - (\alpha + \beta + \gamma) = \mathrm{vol}(T)$ □

This theorem tells us that the area of any geodesic triangle only depends on its angles, i.e any similar triangles are congruent! This observation aligns with our earlier remark about the existence of an intrinsic measure on $\mathbb{H}^n$.

Chapter 4

Geometric Structures on Manifolds

Defining (X, G)-manifolds

We will now introduce the notion of a hyperbolic manifold via the definition of a more general class of manifolds

Definition 4.1 $X \to$ simply connected + oriented n-dimensional manifold ($n \geq 2$)

$G \leq \mathrm{Diff}(X) :=$ group of self-diffeomorphisms of X

We say that a C^∞ n-dimensional manifold M has an (X, G)-structure if $\exists$ an open cover $\{U_i\}$ of M and a set of C^∞ open mappings $\phi_i \colon U_i \to X$ such that:

i) $\phi_i : U_i \to \phi_i(U_i)$ is a diffeomorphism

ii) The restriction of $\phi_i \circ \phi_j^{-1}$ to each connected component of $\phi_j(U_i \cap U_j)$ extends to an element of G whenever $U_i \cap U_j \neq \emptyset$

$\{(U_i, \phi_i)\}$ will be called an *atlas* defining the (X, G)-structure. It lies in a unique maximal atlas having the same properties.

Remark 4.1 If M is an (X, G)-manifold and and is oriented, then M is also an (X, G^+)-manifold where $G^+ :=$ orientation preserving diffeomorphisms in G

Proof. We prove this by showing that given an (X, G) atlas for M, we can modify the charts slightly so that the transition maps are all orientation-preserving on X.
Suppose G contains an orientation-reversing map ψ (if not then $G = G^+$ and we are done)
If $\phi_i : U_i \to X$ is orientation-preserving, then let $\phi_i' = \phi_i$
If ϕ_i is orientation-reversing, then $\phi_i' := \psi \circ \phi_i$.

Now each ϕ_i' preserves orientation and the transition maps, being compositions of orientation-preserving maps, must preserve orientation, i.e. $\{(U_i, \phi_i')\}$ define an (X, G^+) structure on M. $\square$

Remark 4.2 If $(X, g) \to$ Riemannian manifold and $G = \mathcal{I}(X)$, then M has a natural Riemannian structure acquired by requiring the ϕ_i's to be isometries.
For well-definedness, if U_i intersects U_j, is $\phi_i^* g = \phi_j^* g$?

$$\begin{aligned} \phi_i \circ \phi_j^{-1} \in \mathcal{I}(X) &\Rightarrow (\phi_i \circ \phi_j^{-1})^* g = g \\ (\phi_j^{-1})^* \circ (\phi_i^* g) &= g \\ \phi_i^* g &= \phi_j^* g \end{aligned}$$

From now on we assume that X is a Riemannian manifold and $G = \mathcal{I}(X)$ (or $\mathcal{I}^+(X)$ in case of an oriented (X, G)-manifold).

Definition 4.3 An n-dimensional (X, G)-manifold is:

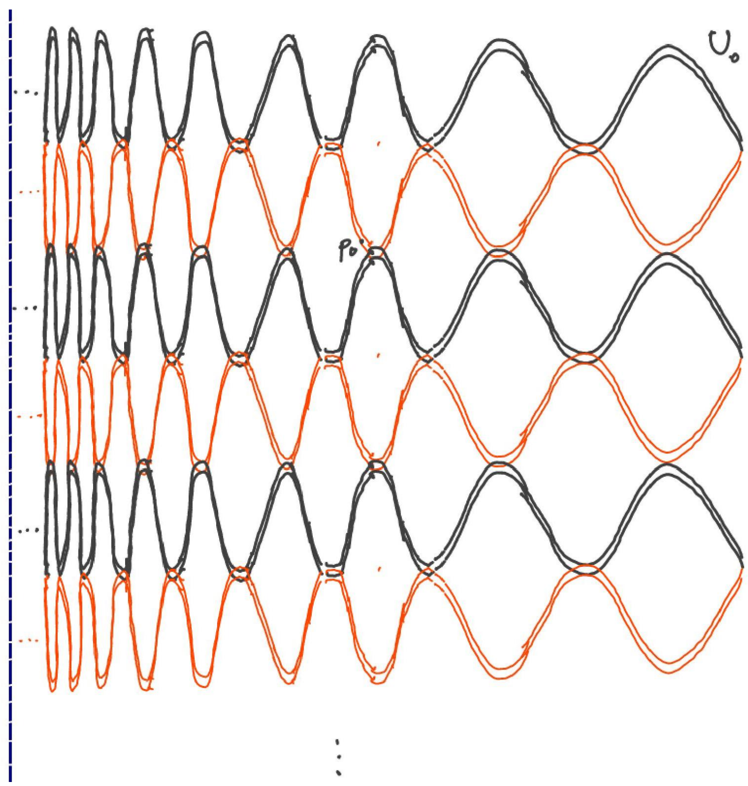
- *flat* if $X = \mathbb{R}^n$
- *elliptic* if $X = S^n$
- *hyperbolic* if $X = \mathbb{H}^n$

Developing Maps and Completeness

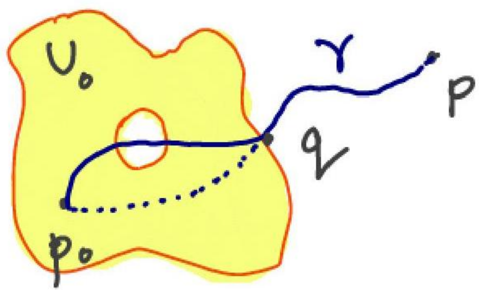
Theorem 4.4 Let $M \to$ simply-connected (X, G)-manifold
(U_0, ϕ_0) is a chart in the (X, G)-atlas of M and U_0 is connected.
Then $\exists!$ local isometry $D : M \to X$ extending ϕ_0.

Proof. Uniqueness follows immediately from Theorem 1.4.
For existence, we fix a point $p_0 \in U_0$.
We wish to define D at any $p \in M$.
There exists a piecewise smooth path from p_0 to p.
Smoothen out the finitely many problematic points by choosing a Euclidean neighbourhood around each of them, then choosing 2 points on either side and drawing a Euclidean smooth curve with the required velocities at those points.

So we have a C^∞ path γ from p_0 to p (assume locally injective).
In order to define D we want this path to only exit U_0 once.
However, it may not be possible to have such a path. For example, if U_0 looks like:



But notice that we can consider a smaller open neighbourhood V_0 (choose ϵ small enough and $V_0 := N_\epsilon(p_0)$ such that $\overline{V_0}$ lies in a normal neighbourhood of p_0) of p_0 in U_0 whose closure is path connected and then by Theorem 1.4, any extension of $\phi_{0|V_0}$ will also be an extension of ϕ_0. Then let V_0 be our U_0.

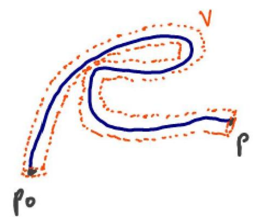


$$t_0 := \sup\left(\gamma^{-1}(U_0)\right)$$
$$q := \gamma(t_0)$$

Then $\gamma_{|\{t|t>t_0\}}$ is a C^∞ path from q to p

Now, as $t_0 \in \overline{\gamma^{-1}(U)},\ q \in \overline{U}_0$

As $\overline{U}_0 = \overline{N_\varepsilon(p_0)}$, $\exists$ geodesic τ from p_0 to q, such that $\tau([0,1)) \subseteq U_0$ and $\tau(1) = q$
Concatenating τ with $\gamma_{|\{t>t_0\}}$, we get a piecewise C^∞ path from p to q, which can be smoothened out to give a path; without loss of generality, call it γ, and then $\gamma^{-1}(U_0)$ is an interval.
Now, let V be a tubular neighbourhood of γ i.e. at any point $v \in V$, $|\gamma^{-1}(v)| \leq 1$.



Let $\{(U_\alpha, \phi_\alpha)\}$ be an (X, G)-atlas of M.
As each U_α is a union of balls (with respect to the Riemannian distance on M, assume that each open set in the atlas is a ball.
Then $S := \{U_\alpha \cap V \mid U_\alpha$ is a ball centred at a point in $\gamma \setminus U_0\}$ is an open cover of $\gamma \setminus U_0$ and by compactness it has a finite open subcover $\{U_i\}_{i=1}^n$. Choose it to be minimal. The following hold:
i) $\gamma^{-1}(U_i) =: I_i$ is an interval $\forall\ 0 \leq i \leq n$
ii) $I_i \cap I_j = \emptyset$ if $|i - j| \geq 2$
iii) $U_i \cap U_j$ is connected $\forall\ i, j$
We now define D on γ as follows, where g_i is the isometry of X extending $\phi_{i-1} \circ \phi_i^{-1}$
$\forall\ i \in \{1, 2, \ldots, n\}$:

$$D_{|U_0 \cap \gamma} = \phi_{0|U_0 \cap \gamma}$$

$$D_{|U_1 \cap \gamma} = g_1 \circ \phi_{1|U_1 \cap \gamma}$$

$$\vdots$$

$$D_{|U_n\cap\gamma} = g_1 \circ g_2 \cdot \dots \cdot g_n \circ \phi_{n|U_n\cap\gamma}$$

In particular, we have defined $D(p)$.

Next, we show:

A) $D(p)$ does not depend on $U_1, \dots, U_n$

B) $D(p)$ does not depend on γ.

Then we will have a well-defined D on M (in fact, it clearly extends to a map on $\cup_{i=1}^n U_i$) and this will be a local isometry as it is of the form $g_1 \circ g_2 \cdots \circ g_k \circ \phi_k$ on each U_k. It is also obvious that D extends ϕ_0.

A) First assume we have 2 covers $\{(U_0, U_1)\}$ and $\{(U_0, \widetilde{U}_1)\}$

Now $\gamma^{-1}(U_0) = (a, b) \subseteq \mathbb{R}$ (suppose)

If U_0 and U_1 cover γ then $\gamma^{-1}(U_1)$ intersects $\gamma^{-1}(U_0)$ and for some $\varepsilon > 0$, $\gamma^{-1}(U_1)$ contains $(b - \varepsilon, b)$.

Similarly, for some $\widetilde{\varepsilon} > 0$, $\gamma^{-1}(\widetilde{U}_1)$ contains $(b - \widetilde{\varepsilon}, b)$.

So $\gamma^{-1}(U_0) \cap \gamma^{-1}(U_1) \cap \gamma^{-1}(\widetilde{U}_1) \neq \emptyset$

$\Rightarrow U_0 \cap U_1 \cap \widetilde{U}_1 \neq \emptyset$

So, if $g \in \mathcal{I}(X)$ is the isometry extending $\phi_1\circ, \widetilde{\phi_1}^{-1}$, then

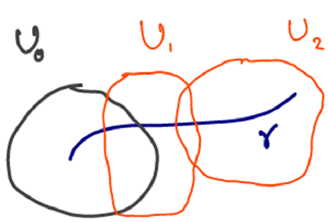
$$g = g_1^{-1} \circ \widetilde{g}_1$$

Now notice that U_0 and $U_1 \cap \widetilde{U}_1$ cover γ as they both contain $\gamma(\{t > b\})$. Then

$$\begin{aligned}
D_{|U_1\cap\widetilde{U}_1\cap\gamma} &= g_1 \circ \phi_{1|U_1\cap\widetilde{U}_1\cap\gamma} \\
\widetilde{D}_{|U_1\cap\widetilde{U}_1\cap\gamma} &= \widetilde{g}_1 \circ \widetilde{\phi}_{1|U_1\cap\widetilde{U}_1\cap\gamma} \\
&= g_1 \circ g \circ \widetilde{\phi}_{1|U_1\cap\widetilde{U}_1\cap\gamma} \\
&= g_1 \circ \phi_{1|U_1\cap\widetilde{U}_1\cap\gamma}
\end{aligned}$$

and of course $D_{|U_0\cap\gamma} = \widetilde{D}_{|U_0\cap\gamma} = \emptyset$.

Now suppose we have



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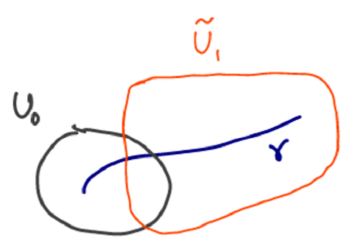
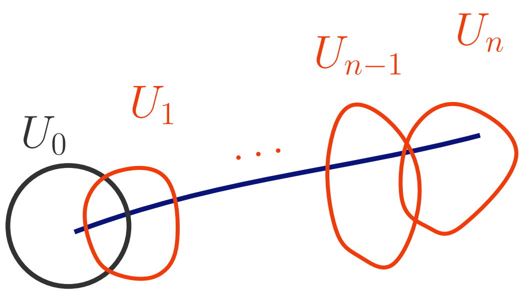
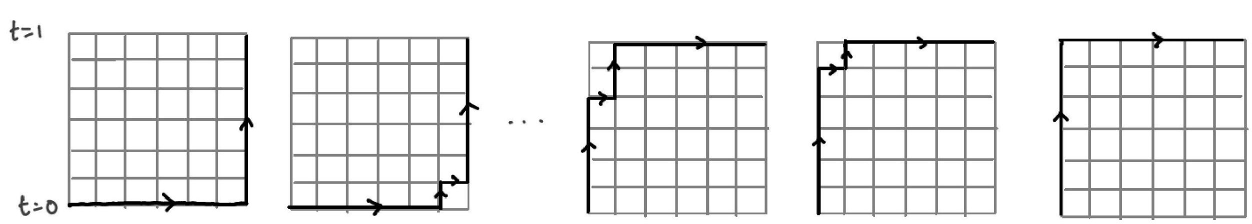


Let $z \in \gamma^{-1}(U_1 \cap U_2)$
Then on $\gamma((a,z))$, D agrees with $\widetilde{D}$ by the previous argument. So we only need to check $D_{|U_2\cap\gamma} = \widetilde{D}_{|U_2\cap\gamma}$ (observe that $U_2 \cap \gamma = U_2 \cap \widetilde{U}_1 \cap \gamma$). This can also be done using similar arguments.
Now, if we have D with respect to $\{U_0, U_1, \ldots, U_n\}$ and $\widetilde{D}$ with respect to $\{U_0, \widetilde{U}_1, \ldots, \widetilde{U}_n\}$ (assume sizes are the same by adding empty sets if required), we perform induction on n.



Without loss of generality, $\gamma^{-1}(U_n) \supseteq \gamma^{-1}(\widetilde{U}_n)$
Then if $z \in U_{n-1} \cap U_n$, by the induction hypothesis, $D_{|\gamma((a,z))} = \widetilde{D}_{|\gamma((a,z))}$
and by the last part $D_{|\gamma(\{t>z\})} = \widetilde{D}_{|\gamma(\{t>z\})}$ as this part is covered by U_n and $\widetilde{U}_{n-1} \cup \widetilde{U}_n$, respectively.

B) Let γ_0, γ_1 be two C^∞ paths joining p_0 and p.
$\exists$ a homotopy, and thus a C^∞ homotopy H from γ_0 to γ_1 as M is a simply-connected smooth manifold (see the Whitney approximation theorems [Lee12]).
Now by the Lebesgue Number Lemma we can subdivide $I \times I$ into a grid such that each small square is mapped into an open set of the (X,G)-atlas.



Then, the above sequence of steps takes γ_0 to γ_1 and it is clear that at each step the modification is within a chart element, so that D remains unchanged. $\square$

The map D defined above is called the **developing function** of M with respect to ϕ_0

Remark 4.4 If D and D' are two developing functions on M, then $\exists\, g \in G$ such that

$$D' = g \circ D$$

To see this, let D extend (U_0, ϕ_0). If $x_0 \in U_0$ then D' was defined on a neighbourhood U (assume $U \subseteq U_0$) of x_0 by the composition of an element g of G with a chart ϕ_n. Then, if g' extends $\phi_n \circ \phi_0^{-1}$ on U, we get $D_{|U} = g \circ g' \circ D_{|U}$, which gives $D = g \circ g' \circ D$ by Theorem 1.4.

When M is simply-connected and complete, this local isometry ends up being an isometry!

Theorem 4.5 Let M be a complete simply-connected (X, G)-manifold.
Then M is isometric to X. In particular, X is complete.

Proof. We show the local isometry $D : M \to X$ is a bijection.

- *Claim:* D satisfies the path-lifting property, i.e., for any path α in X starting at $\alpha \in D(M)$ and given $\widetilde{\alpha} \in D^{-1}(x)$, $\exists!$ path $\widetilde{\alpha}$ in M such that $D \circ \widetilde{\alpha} = \alpha$.

- As the path is arbitrary, for any $y \in X$ we can take a path α from x to y and then $D \circ \widetilde{\alpha} = \alpha$ would imply $y \in D(M)$. Thus the claim implies surjectivity.

  Given surjectivity and local homeomorphism, we will show D is injective by taking a path between $a, b \in D^{-1}(x)$, pushing it to a singular loop in X, and then lifting it to a singular loop in M via D, so that $a = b$. For this we will show homotopy lifting.

- To prove the claim, fix $x \in D(M)$, $\widetilde{x} \in D^{-1}(x)$ and a path α starting at x. We define $\alpha^t := \alpha_{[0,t]}$. Note that any path lift of α^t is unique (use the Lebesgue Number Lemma and the local injectivity of D).

  In particular, if $t_1 < t_2 \in [0, 1]$ then $\widetilde{\alpha}^{t_1} = \widetilde{\alpha}^{t_2}_{|[0,t_1]}$, if the lifts are defined.

  $t_0 := \sup\{t \in [0, 1] \mid$ a path lift $\widetilde{\alpha}^t$ of α^t with respect to D exists$\}$

  Note that $t_0 > 0$ as D is a local isometry. We show $t_0 = 1$.

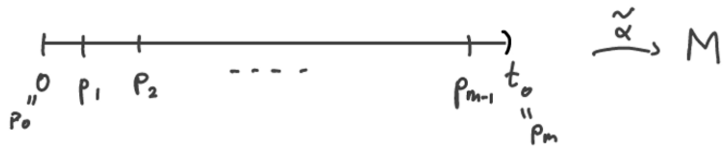
  We do this by showing that this supremum is a maximum, i.e., $\widetilde{\alpha}^{t_0}$ exists. After this, again the lift will exist in a neighbourhood of t_0 since D is a local isometry, so $t_0 = 1$.

  Suppose $\{t_n\}$ is an increasing sequence converging to t_0. Then we show $\widetilde{\alpha}(t_n)$ is a Cauchy sequence in M.

  If not, then $\exists\ \varepsilon > 0$ such that $\forall\ n \in \mathbb{N},\ \exists\ a_n, b_n > n$ such that $d(\widetilde{\alpha}(t_{a_n}), \widetilde{\alpha}(t_{b_n})) \geq \varepsilon$

  Choose $a_1 < b_1 < a_2 < b_2 \cdots$

  But then $l(\widetilde{\alpha}_{|[0,t_0]}) \geq \sum_{n=1}^{\infty} d(\widetilde{\alpha}(t_{a_n}), \widetilde{\alpha}(t_{b_n})) \geq \sum_{n=1}^{\infty} \varepsilon = \infty$



Subdividing $[0, t_0]$ into intervals that map into charts of the (X, G)-atlas of M under $\widetilde{\alpha}$,

$$l(\widetilde{\alpha}) = \sum_{i=0}^{m-1} l(\widetilde{\alpha}([p_i, p_{i+1}]))$$

But as D is an isometry on each $\widetilde{\alpha}([p_i, p_{i+1}])$,

$$l(\widetilde{\alpha}([p_i, p_{i+1}])) = l(\alpha([p_i, p_{i+1}])) < \infty \quad \forall\ i \ \Rightarrow\ l(\widetilde{\alpha}) < \infty \Rightarrow\!\Leftarrow$$

So $\widetilde{\alpha}(t_n)$ is Cauchy and converges to some p. Define $\widetilde{\alpha}^{t_0} : [0, t_0] \to M$ as

$$\widetilde{\alpha}^{t_0}(t) := \begin{cases} \widetilde{\alpha}(t) & \forall\ t < t_0 \\ p & \text{if } t = t_0 \end{cases}$$

$D(\widetilde{\alpha}(t_0)) = D(\lim \widetilde{\alpha}(t_n)) = \lim \alpha(t_n) = \alpha(t_0)$ so that $\widetilde{\alpha}^{t_0}$ is a lift of α^{t_0}

$\Rightarrow t_0$ is a maximum

- We now show each path homotopy in X lifts to a path homotopy in M. Let H be a path homotopy in X from a path f to a path g.

  Say $f,\ g$ start at x, choose $\tilde{x} \in D^{-1}(x)$, and supppose that $\tilde{f}, \tilde{g}$ are lifts at $\tilde{x}$.

  Each H_t is a path in $X \Rightarrow \tilde{H}_t$ starting at $\tilde{x}$ exists. Define

$$\begin{aligned} \tilde{H} : I \times I &\to M \\ (s, t) &\mapsto \tilde{H}_t(s) \\ D \circ \tilde{H}(s, t) &= D \circ \tilde{H}_t(s) \\ &= H_t(s) = H(s, t) \\ &\Rightarrow D \circ \tilde{H} = H \end{aligned}$$

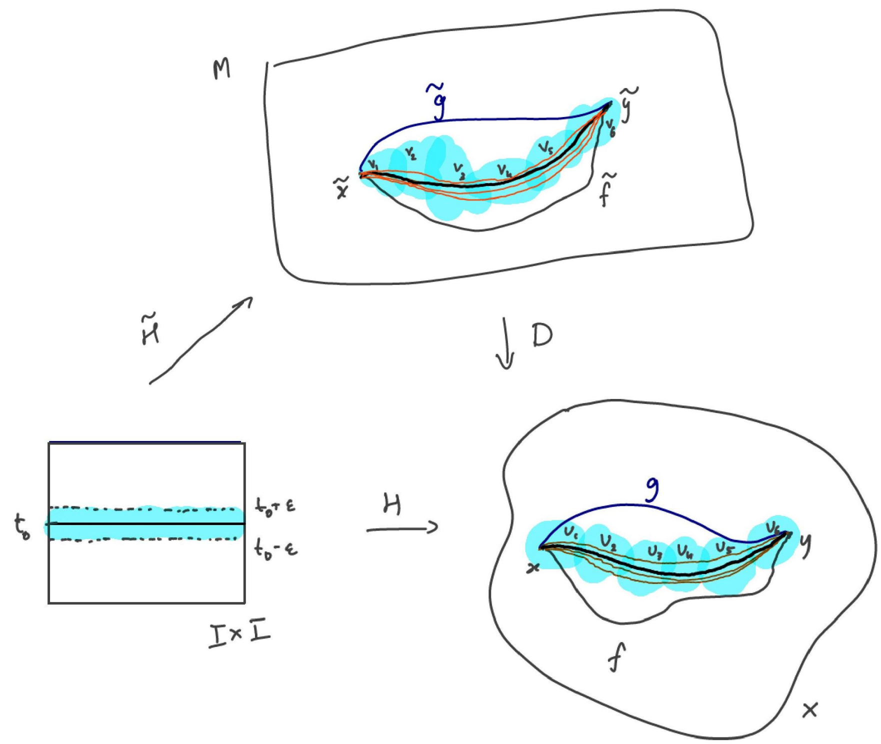
Note $\tilde{H}_0 = \tilde{f}$ and $\tilde{H}_1 = \tilde{g}$, so $\tilde{H}$ is a promising candidate for the path homotopy from $\tilde{f}$ to $\tilde{g}$.

We only need to show the continuity of $\tilde{H}$.

It suffices to consider open neighbourhoods of each $I \times \{t_0\} \subseteq I \times I$ such that the restriction of $\tilde{H}$ to the open neighbourhood is continuous.

$t_0 \in I$ and let $\|V_i\|_{i=1}^{k}$ be a finite open cover of $\tilde{H}_{t_0}(I)$ in M and $D_{|V_i}$ is an isometry $\forall\ i$.

Let $U_i := D(V_i) \overset{\text{open}}{\subseteq} X$.



We choose small enough $\varepsilon > 0$ such that $H(I \times N_\varepsilon(t_0)) \subseteq \bigcup_{i=1}^{k} U_i$ (tube lemma [Mun00])

Now define $F : I \times N_\varepsilon(t_0) \longrightarrow M$ such that

$$F_{|H^{-1}(U_i)} = (D_{|V_i})^{-1} \circ H$$

F is well-defined on intersections $H^{-1}(U_i) \cap H^{-1}(U_j) = H^{-1}(U_i \cup U_j)$, as

$$(D_{|V_i})^{-1}_{|U_i \cap U_j} = (D_{|V_i \cap V_j})^{-1} = (D_{|V_j})^{-1}_{|U_i \cap U_j}$$

Clearly F is continuous, and as $F_t := F_{|I \times \{t\}}$ is a lift of H_t and $t \in N_\varepsilon(t_0)$,

$$\tilde{H}_{|I \times N_\varepsilon(t_0)} = F \Rightarrow \tilde{H} \text{ is continuous}$$

- Finally, for the injectivity of D, suppose $x, y \in M$ and $D(x) = D(y) = x_0$. Then choose an embedded path α from x to y in M. $D \circ \alpha$ is a loop at x_0 in X.

  $\exists$ a path homotopy H from $D \circ \alpha$ to the constant loop at x_0. This lifts to a path homotopy from α to the constant loop at $x = y$. So D is injective. $\square$

From now on, we also assume that X is complete.

X as a Universal Cover

We now prove certain facts about groups acting on topological spaces. Let T be a locally compact + Hausdorff topological space. $\Gamma \leq \mathrm{Aut}(T)$ (self homeomorphisms of T).

Definition 4.6.

- Γ is said to act freely on T if $\forall\, \gamma \in \Gamma,\ x \in T,\ \gamma(x) = x \ \Rightarrow\ \gamma = \mathrm{id}$
- Γ acts properly discontinuously on T if $\forall$ compact $H, K \subseteq T$, the set $\Gamma(K, H) := \{\gamma \in \Gamma \mid \gamma(H) \cap K \neq \emptyset\}$ is finite. Equivalently $\forall$ compact K, $\Gamma(K, K) < \infty$ (check)

Recall that we have

$$\begin{aligned} &\mathcal{I}(\mathbb{I}^n) \simeq O(I_n) \\ &\mathcal{I}(\mathbf{S}^n) \simeq O(n+1) \\ &\mathcal{I}(\mathbb{R}^n) \simeq \mathbb{R}^n \rtimes O(n) \end{aligned}$$

$\forall\, n \geq 2$ and the sets on the right can have natural topologies.

Theorem 4.7 $T \to$ Hausdorff + locally compact topological space.
$\Gamma \subseteq \mathrm{Aut}(T)$. The following are equivalent

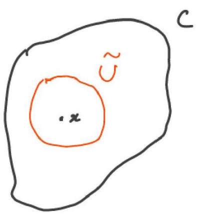
1. Γ acts freely and properly discontinuously on T.
2. T/Γ is Hausdorff and $\forall\, x \in T$, $\exists$ neighbourhood U of x, such that
$$\gamma(U) \cap U = \emptyset \quad \forall\, \gamma \in \Gamma \setminus \{\mathrm{id}\}$$
3. T/Γ is Hausdorff and the quotient map: $T \to T/\Gamma$ is a covering map.

   Moreover, if $T \in \{\mathbb{I}^n, S^n, \mathbb{R}^n\}$ and $\Gamma \leq \mathcal{I}(T)$ then

4. Γ operates freely on T and is a discrete subset of $\mathcal{I}(T)$.

Proof. 1 $\Rightarrow$ 2

We start with the second part of 2.



Fix $x \in T$

Let C be a compact neighbourhood of x

$S := \{\gamma \in \Gamma \mid \gamma(C) \cap C \neq \emptyset\}$ is finite

Let $P := \{\gamma^{-1}(x) \mid \gamma \in S \setminus \{\mathrm{id}\}\}$ (finite)

By Hausdorffness, we can separate x and P by disjoint open neighbourhood. So $\exists$ open neighbourhood $W \ni x$ such that $\overline{W} \cap P = \emptyset \quad \forall\, \gamma \in \Gamma$

$W \cap \widetilde{U} \subseteq C$, so without loss of generality, we assume $W \subseteq C$

Now, define $U := W \setminus \bigcup_{\gamma \in S} \gamma(\overline{W})$

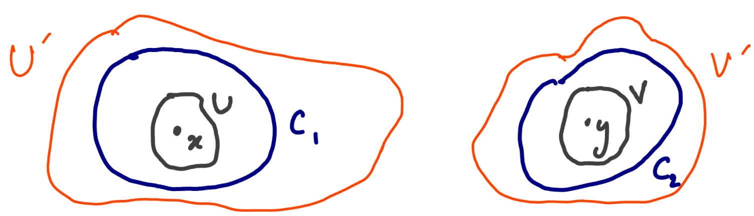
As $W \cap P = \emptyset$, $\bigcup_{\gamma \in S} \gamma(\overline{W})$ does not contain x, so $x \in U$

Clearly $U \cap \gamma(W) = \emptyset \Rightarrow U \cap \gamma(U) = \emptyset \quad \forall\, \gamma \in \Gamma \setminus \{\mathrm{id}\}$

For the Hausdorffness of T/Γ, we need something similar.

Let U and V be disjoint open neighbourhoods of x and y in T respectively.

For $[U] \cap [V] = \emptyset$, we want $U \cap \gamma(V) = \emptyset \; \forall\, \gamma \in \Gamma$



Taking U and V to lie in disjoint compact sets, we get $\{\gamma \in \Gamma \mid C_1 \cap \gamma(C_2) \neq \emptyset\}$ and $S := \{\gamma \in \Gamma \mid U \cap \gamma(V) \neq \emptyset\}$ are finite.

Again $P := \{\gamma^{-1}(x) \mid \gamma \in S\}$ choose open $W \subseteq C_1$ containing x, such that $\overline{W} \cap P = \emptyset$

$\widetilde{U} := W \setminus \bigcup\limits_{\gamma \in S} \gamma(\overline{W})$ and keep V as it is.

2 $\iff$ 3

The quotient projection $p\colon T \to T/\Gamma$ is continuous and surjective.

$$\forall\, U \overset{\text{open}}{\subseteq} T, \quad p^{-1}(U) = \bigcup_{\gamma \in \Gamma} \gamma(U)$$

So, for p to be a covering map, we must have an open neighbourhood of x such that this union is a disjoint union, i.e. $\gamma_1(U) \cap \gamma_2(U) = \emptyset \; \forall\, \gamma_1, \gamma_2 \in \Gamma$ or $U \cap \gamma(U) = \emptyset \; \forall\, \gamma \in \Gamma \setminus \{\mathrm{id}\}$

2 $\Rightarrow$ 1

$$\gamma(U) \cap U = \emptyset \quad \Rightarrow \quad \gamma(x) \neq x \quad \forall\, \gamma \in \Gamma \setminus \{\mathrm{id}\}$$

So Γ acts freely on T

For proper discontinuity, we first show that $\Gamma(K,H)$ is finite for a specific class $\mathcal{S}$ of compact sets $K,H\subseteq T$, such that $\exists$ open neighbourhoods

$$U\supseteq K \text{ and } V\supseteq H \text{ satisfying } U\cap\gamma(U)=V\cap\gamma(V)=\emptyset \quad \forall\,\gamma\in\Gamma\setminus\{id\}$$

Once this is done, for arbitrary compact subsets $H,K\subseteq T$, we can consider a cover of each point in $H\cup K$ by open neighbourhoods $U_p\ni p$, such that $U_p\cap\gamma(U_p)=\emptyset$ if $\gamma\in\Gamma\setminus\{\mathrm{id}\}$. Then take compact neighbourhoods $C_p\subseteq U_p$ containing p, so that $\exists$ open $V_p\subseteq C_p$. $\{V_p\}_{p\in H\cup K}$ is an open cover for $H\cup K$ and thus $\exists$ finite subovers $\{V_{p_i}\}_{i=1}^n$ and $\{V_{q_i}\}_{i=1}^m$ of H and K respectively. Consequently, $\{C_{p_i}\}_{i=1}^n$ and $\{C_{q_i}\}_{i=1}^m$ are finite compact covers of H and K respectively.

As $C_{p_i},C_{q_i}\in\mathcal{S}\quad\forall\,i$, we have assumed that the finiteness of $\Gamma(C_{p_i},C_{q_j})$ holds $\forall\,i,j$. But

$$\begin{aligned}\gamma(H)\cap K&=\gamma\Big(\bigcup_{i=1}^n C_{p_i}\Big)\cap\Big(\bigcup_{j=1}^n C_{q_j}\Big)\\&=\bigcup_{j=1}^m\bigcup_{i=1}^n\big(\gamma(C_{p_i})\cap C_{q_j}\big)\\&=\emptyset\quad\text{for all but finitely many }\gamma\in\Gamma\setminus\{\mathrm{id}\}\end{aligned}$$

Thus, $\Gamma(K,H)$ is finite.

Hence, we have reduced to the case $K,H\in\mathcal{S}$

Let $\mathcal{C}:=\{\gamma(U)\cap K\mid\gamma\in\Gamma(K,H)\}$

Then $\mathcal{C}$ is a collection of disjoint non-empty open subsets of K. If $\pi:T\to T/\Gamma$ is the quotient map, check that

$$\pi(\Gamma(H)\cap K)=\pi(H)\cap\pi(K)$$

so that $\pi(\Gamma(H)\cap K)$ is compact, therefore closed as T/Γ is Hausdorff

Then $\pi^{-1}\pi(\Gamma(H)\cap K)\cap K=\Gamma(H)\cap K\overset{\text{closed}}{\subseteq}T$

Thus, $\Gamma(H)\cap K$ is compact and $\mathcal{C}$ is a disjoint open cover of it

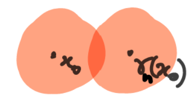
$\Rightarrow\mathcal{C}$ is finite and thus $\Gamma(K,H)$ is finite.

4 $\Rightarrow$ 2

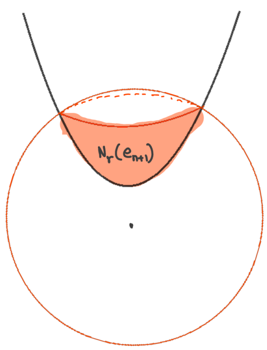
Assume $T=\mathbb{I}^n$ and $\Gamma\leq\mathcal{I}(\mathbb{I}^n)$

We first prove that $\forall\,x_0\in T$, $\exists$ open neighbourhood $U\ni x_0$ such that $U\cap\gamma(U)=\emptyset$ $\forall\,\gamma\in\Gamma\backslash\{\mathrm{id}\}$

If not, then each $N_{1/n}(x_0)$ intersects $\gamma_n(N_{1/n}(x_0)) = N_{1/n}(\gamma_n(x_0))$ for some $\gamma_n \in \Gamma$



$\Rightarrow d(x_0, \gamma_n(x_0)) < 2/n$ and thus $\gamma_n(x_0) \longrightarrow x_0$
$\forall\ r > 0,\ \exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N,\ \gamma_n(x_0) \in \overline{N_r(x_0)}$
Let $x \in \overline{N_r(x_0)}$
Then $d(x_0, \gamma_n(x)) \leq d(x_0, \gamma_n(x_0)) + d(\gamma_n(x_0), \gamma_n(x)) = d(x_0, \gamma_n(x_0)) + d(x_0, x)$
$\leq \quad r \quad + \quad r$



So $\forall\ n \geq N,\ \gamma_n(\overline{N_r(x_0)}) \subseteq \overline{N_{2r}(x_0)}$
Let $\mathcal{S} := \{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid A(\overline{N_r(e_{n+1})}) \subseteq \overline{N_{2r}(e_{n+1})}\}$
As the boundary of $\overline{N_r(e_{n+1})}$ is circular, $\exists$ Euclidean ball $\overline{B_l(0)} \subseteq \mathbb{R}^{n+1}$ such that $\overline{B_l(0)} \cap \mathbb{I}^n = \overline{N_r(e_{n+1})}$
Then it can be shown that $\mathcal{S}$ is a compact subset of $\mathbb{R}^{(n+1)^2}$.
$\mathcal{C} := \mathcal{S} \cap \det^{-1}\{-1, +1\}$ is then also a compact set.
Replace e_{n+1} with x_0.
$\exists\ B \in \mathrm{O}(I_n) \subseteq \mathrm{O}(n+1)$ such that $B(x_0) = e_{n+1}$.
$\det B = \pm 1$

Then $\{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid A(N_r(x_0)) \subseteq N_{2r}(x_0),\ \det A = \pm 1\}$
$= \{A \in \mathrm{GL}_{n+1}(\mathrm{R}) \mid A \cdot B^{-1}(N_r(e_{n+1})) \subseteq B^{-1}(N_{2r}(e_{n+1})), \det A = \pm 1\}$
$= \{A' \cdot B \mid A' \in \mathcal{C}\} = \mathcal{C} \cdot B$
Right multiplication by B is a continuous function : $\mathbb{R}^{(n+1)^2} \longrightarrow \mathbb{R}^{(n+1)^2}$
Thus, $\mathcal{C} \cdot B$ is also compact.
But $\forall\ n \geq N,\ \gamma_n \in \mathcal{C} \cdot B$
Now $\Gamma = \mathrm{O}(I_n) = \mathrm{O}(n+1) \cap \{A \in \mathrm{M}_{n+1}(\mathbb{R}) \mid A_{n+1,\ n+1} \geq 0\}$ (as the hyperboloid does not intersect $\mathbb{R}^n \times 0$) $\overset{\text{closed}}{\subseteq} \mathrm{GL}_{n+1}(\mathbb{R})$
Thus, every subset of Γ (discrete in $\mathrm{O}(I_n)$) is also closed in $\mathrm{GL}_{n+1}(\mathbb{R})$, so Γ is also a discrete subspace of $\mathrm{GL}_{n+1}(\mathbb{R})$.
But a discrete subset of a compact space is finite, so then $\{\gamma_n \mid n \geq N\}$ is finite, i.e. $\{\gamma_n\}_{n \in \mathbb{N}}$ is finite and then $\gamma_n(x_0) \to x_0 \Rightarrow \exists\ m \in \mathbb{N}$ such that $\gamma_m(x_0) = x_0 \Rightarrow\!\Leftarrow$

Now, as each $x \in T$ has a neighbourhood $U \ni x$ such that $U \cap \gamma(U) = \emptyset \quad \forall\ \gamma \in \Gamma \setminus \{\mathrm{id}\}$, in particular, $\gamma(x) \notin U \Rightarrow \Gamma(x)$ is discrete
Given $y \in T \setminus \{x\}$, choose $\varepsilon > 0$ small enough such that $d(y, \Gamma(x)) > 2\varepsilon$

Then if $y' \in N_\varepsilon(y)$, $\gamma \in \Gamma \setminus \{\mathrm{id}\}$,

$$d(x, \gamma(y')) + d(\gamma(y), \gamma(y')) \geq d(x, \gamma(y)) = d(\gamma^{-1}(x), y)$$

$$d(x, \gamma(y')) > 2\varepsilon - \varepsilon$$

Thus, $\gamma(y') \notin N_\varepsilon(x) \Rightarrow N_\varepsilon(x) \cap \gamma(N_\varepsilon(y)) = \emptyset$
So $[N_\varepsilon(x)]$ and $[N_\varepsilon(y)]$ are disjoint neighbourhoods of $[x]$ and $[y]$ in T/Γ, and thus T/Γ is Hausdorff.

1 ⇒ 4

Assume $T = \mathbb{I}^n$ and $\Gamma \leq \mathcal{I}(\mathbb{I}^n)$
We first show id $\in \Gamma$ has a finite open neighbourhood in Γ.
Let $K := N_\varepsilon(e_{n+1})$ for some $\varepsilon > 0$
Recall that $\exists$ Euclidean ball $B_\delta(0) \subseteq \mathbb{R}^{n+1}$ such that $B_\delta(0) \cap \mathbb{I}^n := N_\varepsilon(e_{n+1}) = K$
Let $r := \varepsilon\delta \cdot (n+1)$ and $H := N_r(e_{n+1})$ (note that $\delta > 1$, so $K \subseteq H$)

Claim: $V := \{\gamma \in \Gamma \mid \gamma(K) \subseteq H\}$ contains $\Gamma \cap N_\varepsilon(\mathrm{id})$, where $N_\varepsilon(\mathrm{id}) \subseteq O(I_n)$
If $\gamma \in N_\varepsilon(\mathrm{id})$ then $\gamma(e_i) \in B_\varepsilon(e_i) \subseteq \mathbb{R}^{n+1}\ \forall\ i$
Now $v = \sum a_i e_i \in K$, then $\|v\| = \sqrt{\sum a_i^2} < \delta$
In particular, each $|a_i| < \delta$
Now $\gamma(v) = \sum a_i \gamma(e_i)$ and $\|\gamma(e_i)\| < \varepsilon\ \forall i$
$\Rightarrow \|\gamma(v)\| \leq \sum |a_i| \cdot ||\gamma(e_i)|| < \varepsilon\delta \cdot (n+1)$ or $\gamma(v) \in H$
Then $\gamma \in V$, proving the claim

But $V \subseteq \Gamma(K, H)$ and is thus finite.
So $\exists$ a finite open neighbourhood of id in $\Gamma \Rightarrow \{\mathrm{id}\} \overset{\text{open}}{\subseteq} \Gamma$
For any other element of Γ, by translation via isometries we can find a similar finite neighbourhood and thus Γ is discrete. □

Theorem 4.8 $M \to$ complete + connected (X, G)-manifold

$$\pi_1(M) \simeq \text{ some } \Gamma \leq G = \mathcal{I}(X)$$

Γ acts freely and properly discontinuously on X and M is isometric to X/Γ or $X/\pi_1(M)$

Remark 4.8 The converse can also easily be deduced, i.e. if $\Gamma \leq G$ acts freely and properly discontinuously on X, then the quotient manifold X/Γ is complete and connected, and $\pi_1(X/\Gamma) \simeq \Gamma$

This is because, by 4.7 (**1** $\Rightarrow$ **3**),

$$\begin{array}{c} X \\ \downarrow \pi \\ X/\Gamma \end{array} \quad \text{is a covering map, and then } \pi_1(X/\Gamma) \simeq \mathrm{Aut}\,(X/\,(X/\Gamma)) \simeq \Gamma$$

X/Γ acquires an (X,G)-structure with charts $\{\pi^{-1}_{|V} \mid \pi \text{ is a homeomorphism on } V\}$ as the transition maps end up being restrictions of elements of Γ, which are isometries of X. This makes π a locally isometric covering map, and thus the completeness of $X \Rightarrow$ completeness of X/Γ.

Proof. Let $\widetilde{M}$ be the universal cover of M.
Note that $\widetilde{M}$ acquires the C^∞ manifold structure of M.

- We claim that $\widetilde{M}$ acquires an (X,G)-structure from M.
  Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be an atlas defining the (X,G)-structure on M.
  Let $\{V_i\}_{i\in I}$ be an open cover of M by evenly covered sets.
  Then $\{U_\alpha \cap V_i \mid \alpha \in A, i \in I\}$ is a refined open cover of M.
  Without loss of generality, $\{(U_\alpha, \phi_\alpha)\}$ is an atlas consisting only of evenly covered open sets. $\forall\ \alpha \in A$,
  $$p^{-1}(U_\alpha) = \coprod_{j\in J_\alpha} \widetilde{U}_{\alpha j}$$
  Set $\quad \widetilde{\phi}_{\alpha j} : \widetilde{U}_{\alpha j} \longrightarrow X \quad = \quad \phi_\alpha \circ p_{|\widetilde{U}_{\alpha j}} \quad \forall\ j \in J_\alpha$
  which is clearly a diffeomorphism onto $\phi_\alpha(U_\alpha)$ $\forall\ \alpha,\ j$
  $$\widetilde{\phi}_{\alpha j} \circ \widetilde{\phi}^{-1}_{\beta j} = (\phi_\alpha \circ p) \circ (p^{-1} \circ \phi_\beta^{-1}) = \phi_\alpha \circ \phi_\beta^{-1} \in G$$

- Thus, $\widetilde{M}$ has acquired an (X,G)-structure and we can now endow it with a Riemann metric such that the covering map $p : \widetilde{M} \longrightarrow M$ becomes a local isometry.
  Note that $\pi_1(M) \simeq \mathrm{Aut}(\widetilde{M}/M)$ from the Galois correspondence of covering spaces [Hat02]. As M is complete, by the Hopf-Rinow Thoerem ([Lee18]) it is geodesically complete, and each geodesic defined on $\mathbb{R}$ can be lifted via the covering map to a geodesic defined on $\mathbb{R}$ in $\widetilde{M}$. Then $\widetilde{M}$ is also complete, and by Theorem 4.5, $\widetilde{M}$ is isometric to X, so we can identify the two.

- We show that $\Gamma := \mathrm{Aut}(X/M) \subseteq \mathcal{I}(X)$. $\forall\ \phi \in \mathrm{Aut}(X/M)$, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ & p \searrow \swarrow p & \\ & M & \end{array}$$

  i.e. $\phi \in \mathrm{Aut}(\widetilde{M}/M) \Rightarrow p \circ \phi = p$
  If $p_{|U}$ and $p_{|V}$ are isometries, then $p \circ \phi_{|\phi^{-1}(U)\cap V} = p_{|\phi^{-1}(U)\cap V}$
  $\phi_{|\phi^{-1}(U)\cap V} = (p_{|U\cap\phi(V)})^{-1} \circ p_{|\phi^{-1}(U)\cap V}$, an isometry.
  So ϕ is a local isometry + homeomorphism $\Rightarrow$ isometry.
  From some basic covering space theory [Hat02],

$$\pi_1(M) \simeq \mathrm{Aut(X/M)} = \Gamma \ \text{ and } \ X/\mathrm{Aut}(X/M) \simeq M$$

  so that $M \simeq X/\Gamma$.

  Then, by Theorem 4.7 (**3** $\Rightarrow$**1**), we get that Γ operates freely and properly discontinuously on X. □

This theorem implies, in particular, that any complete hyperbolic, elliptic, or flat manifold M is the quotient of $\mathbb{H}^n$, S^n, or $\mathbb{R}^n$ respectively, under the action of $\pi_1(M)$ which is identified with a discrete group of isometries acting freely.

Corollary 4.8 Suppose M is an (X, G)-manifold and $\widetilde{M}$ is its universal cover. Then M is complete if and only if the developing map $D : \widetilde{M} \to X$ is a covering map.

Proof. If M is complete, we showed above that $\widetilde{M}$ is also complete, and then, from the proof of Theorem 4.5, the developing map D satisfies the unique path-lifting property. Then by [Kap], D is a covering map. Conversely, suppose that $D : \widetilde{M} \to X$ is a covering map. Then it clearly satisfies the unique path-lifting and homotopy-lifting property, which is all we needed in the proof of Theorem 4.5. Thus, D is an isometry, and the completeness of $\widetilde{M}$ implies the completeness of $\widetilde{M}$. This, in turn, will guarantee the completeness of M. □

Incomplete Geometric Structures

When X is not complete, we will show that, under a certain hypothesis on X (satisfied by $\mathbb{H}^n$, S^n, and $\mathbb{R}^n$), all (X, G)-structures on a manifold M determine (up to conjugation) a group homomorphism of $\pi_1(M)$ into G.

Definition 4.9 We say that a Riemannian manifold X has the *isometries-extension property* (**IEP**) if $\exists\ r_0 > 0$, such that $\forall\ r < r_0$ and $x, y \in X$ any isometry : $N_r(x) \to N_r(y)$ extends to an isometry of X.

Lemma 4.10 $\mathbb{H}^n$, S^n, and $\mathbb{R}^n$ have the IEP.

Proof. This fact holds for all three with $r_0 = \infty$ by the explicit determination of their isometries. We prove the fact for $\mathbb{H}^n$, and the others can be deduced similarly. Let $x, y \in \mathbb{H}^n$ and $\phi : N_r(x) \to N_r(y)$ be an isometry. We must have $\phi(x) = y$. By composition on the left with an isometry sending y to x, we can assume $x = y$. Now, let $x = 0 \in \mathbb{D}^n$. $N_r(0)$ is then a Euclidean ball $B_\delta(0)$.Then, $\phi \in \mathrm{Conf}(B_\delta(0))$ and as $\phi(0) = 0$, it follows by Theorem 2.11 that $\phi \in \mathrm{O}(n)$, which obviously extends to $\mathbb{D}^n$. □

Now, assume that X is a simply connected Riemannian manifold and M is a connected $(X, \mathcal{I}(X))$-manifold.

Lemma 4.11 If (U, ϕ) and (V, ψ) are charts in the (X, G)-structure on M, and $\gamma \in \mathcal{I}(M)$ such that $\gamma(U) \subseteq V$, then $\psi \circ \gamma \circ \phi^{-1}$ extends to an element of $\mathcal{I}(X)$.

Proof. Let $N_r(x)$ be a ball in $\phi(U)$. Being a composition of isometries, $\psi \circ \gamma \circ \phi^{-1}$ maps $N_r(x)$ to some ball $N_r(y) \subseteq \psi(V)$. Then $\psi \circ \gamma \circ \phi^{-1}_{|N_r(x)}$ extends to some $\alpha \in \mathcal{I}(X)$. By Theorem 1.4, any extension of $\psi \circ \gamma \circ \phi^{-1}_{|N_r(x)}$ is also an extension of $\psi \circ \gamma \circ \phi^{-1}$. □

Theorem 4.12 If X has the IEP, $\exists!$ group homomorphism

$$D_* : \mathcal{I}(\widetilde{M}) \to \mathcal{I}(X)$$

such that $D_*(\phi) \circ D = D \circ \phi \quad \forall\ \phi \in \mathcal{I}(\widetilde{M})$.

Proof. $\forall\ \phi \in \mathcal{I}(\widetilde{M})$, define $\phi_* : D(\widetilde{M}) \to X$ such that $\forall\ x \in D(\widetilde{M})$,

$$\phi_*(x) := D(\phi(m)) \quad \text{for some } m \in D^{-1}(x)$$

We first show well-definedness. Let $m, m' \in \widetilde{M}$, such that $D(m) = D(m') = x$. Then let γ be a C^∞ path in $\widetilde{M}$ such that $\gamma(0) = m$ and $\gamma(1) = m'$. From Lemma 4.11, for each $t \in [0, 1]$, we can find φ_t such that if U_t is a small neighbourhood of $\gamma(t)$, then

$$D \circ \phi \circ (D_{|U_t})^{-1} = \varphi_t$$

The continuous function $t \mapsto \varphi_t$ is locally constant and as $[0,1]$ is connected, it must be globally constant. Then $\varphi_0 = \varphi_1$, and

$$D(\phi(m)) = \varphi_0 \circ D(m) = \varphi_1 \circ D(m') = D(\phi(m'))$$

As X has the IEP, for a ball $B \subseteq D(\widetilde{M})$, $\phi_{*|B}$ extends uniquely to X, and by Theorem 1.4, so does ϕ_*, so we can assume $\phi_* \in \mathcal{I}(X)$. Then we can define $D_*(\phi) := \phi_*$. The group homomorphism property can be verified by using uniqueness. $\square$

Theorem 4.13 X has the IEP. Each (X, G)-structure on M determines the conjugacy class of a group homomorphism

$$\Psi : \pi_1(M) \to \mathcal{I}(X)$$

Proof. We again use $\pi_1(M) \simeq \mathrm{Aut}(\widetilde{M}/M) \subseteq \mathcal{I}(\widetilde{M})$ (see the proof of Theorem 4.8).
Then there is a homomorphism $i : \pi_1(M) \hookrightarrow \mathcal{I}(\widetilde{M})$ and $\Psi := D_* \circ i$ is the required homomorphism. Note that this depends on the choice of the developing function D. For every $\phi \in \mathcal{I}(\widetilde{M})$, $D_*(\phi)$ is an extension of $D \circ \phi \circ (D_{|U})^{-1}$, and as any other developing function $D' = g \circ D$ for some $g \in \mathcal{I}(X)$,

$$D'_* = g \circ D_* \circ g^{-1}$$

and the same holds for Ψ and Ψ'. $\square$

Definition 4.14 The conjugacy class $[\Psi]$ as above is called the **holonomy** of M, and for a fixed D and an element $\gamma \in \pi_1(M)$, $\Psi(\gamma)$ is called the *holonomy of* γ with respect to D.

Mostow-Prasad Rigidity

Theorem 4.15 Mostow's Rigidity Theorem
$n \geq 3$ and $M_1,\ M_2 \to$ compact, connected, oriented n-dimensional hyperbolic manifolds
If $f : M_1 \to M_2$ is a homotopy equivalence, then f is homotopic to an isometry : $M_1 \to M_2$
In particular, homotopy equivalent manifolds satisfying the above condition are isometric.

We present an outline of the proof presented in [BP92].

Proof Outline.

Let $\widetilde{f}$ be the lift of f such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\widetilde{f}} & \mathbb{H}^n \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

The starting point of $\widetilde{f}$ is chosen such that

$$\widetilde{f} \circ \gamma = f_*(\gamma) \circ f \quad \forall\, \gamma \in \Gamma_1 \simeq \pi_1(M_1)$$

where f_* denotes the map : $\pi_1(M_1) \to \pi_2(M_2)$ induced by f.

Step 1

For metric spaces X and Y, we say a map $h : X \to Y$ is a *pseudo-isometry* if $\exists\ c_1,\ c_2 > 0$ such that

$$\frac{1}{c_1} \cdot\ d(x_1, x_2) - c_2 \leq d(h(x_1), h(x_2)) \leq c_1 \cdot d(x_1, x_2)$$

$\widetilde{f}$ is homotopic to a pseudo-isometry. It can then be shown that $\widetilde{f}$ can be extended to a continuous map : $\overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$ such that its restriction to $\partial \mathbb{H}^n$ is injective and the relation

$$\widetilde{f} \circ \gamma = f_*(\gamma) \circ \widetilde{f} \quad \forall\, \gamma \in \Gamma_1$$

holds in all of $\overline{\mathbb{H}^n}$.

Step 2

A regular simplex is one for which every permutation of its vertices can be obtained by an isometry of the ambient space.

$\mathcal{S}_n :=$ the set of n-simplices in $\mathbb{H}^n$ having hyperbolic subspaces as faces

The volume function (defined by integration using the Riemannian metric [Lee18]) has a maximum v_n on $\mathcal{S}^n$. Additionally, v_n is the volume of all regular ideal simplices.

Step 3

Let $\widetilde{f}$ be the map constructed in Step 1.

$\widetilde{f}$ maps the vertices of a maximal volume simplex to the vertices of a maximal volume simplex. By Step 3, we can replace "maximal volume" with "regular".

Step 4
$n \geq 3$ and $P : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is continuous and injective. Now for an ideal hyperbolic simplex with vertices $u_0, ...u_n$ and volume v_n, if $P(u_0), ...P(u_n)$ also has volume v_n, then P extends to an element of $\mathcal{I}(\mathbb{H}^n)$.

Step 5 From the above steps, we have the following.

- $\widetilde{f}$ continuously extends to $\partial\mathbb{H}^n$
- $\widetilde{f} \circ \gamma = f_*(\gamma) \circ \widetilde{f} \quad \forall\, \gamma \in \Gamma_1$ is true in all of $\overline{\mathbb{H}^n}$
- $\exists\, q \in \mathcal{I}(\mathbb{H}^n)$ such that $\widetilde{f}_{|\partial\mathbb{H}^n} = q_{|\partial\mathbb{H}^n}$

From these facts,
$$q \circ \gamma = f_*(\gamma) \circ q \quad \forall\, \gamma \in \Gamma_1 \quad \text{on } \partial\mathbb{H}^n$$
$\forall\, x \in \mathbb{H}^n$, set $F(p_1(x)) = p_2(q(x))$. The conclusion of the proof can be deduced from:

i $F : M_1 \to M_2$ is well-defined and injective, i.e, $p_1(x) = p_1(x') \Leftrightarrow p_2(q(x)) = p_2(q(x'))$.

ii F is surjective. In fact, $p_2(y) = F(p_1(q^{-1}(y))$.

iii F is a local isometry as $p_1,\ p_2$ and q are local isometries, and thus F is an isometry.

iv F is homotopic to f via
$$H(t, p_1(x)) := p_2(t \cdot \widetilde{f}(x) + (1-t) \cdot q(x))$$

□

Prasad ([Pra73]) generalized this as follows
Theorem 4.16 $M_1, M_2 \to$ finite-volume, complete, connected hyperbolic n-manifolds such that $n \geq 3$. If $\phi : \pi_1(M_1) \to \pi_2(M_2)$ is a group isomorphism, then $\exists$ an isometry $f : M_1 \to M_2$ such that $f_* = \phi$.

The proof is essentially based on the same methods used for the compact case. Note that this theorem applies to the complements of all hyperbolic knots in S^3.

Chapter 5

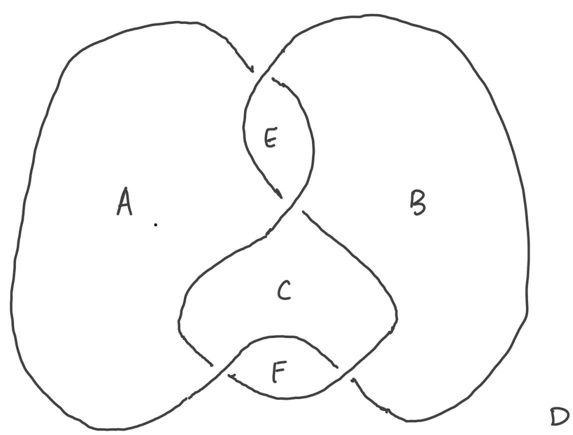
Two-polyhedron Decomposition of Knots

Polyhedral Decomposition of the Figure-8 Knot

We now describe a combinatorial procedure to obtain an ideal triangulation of the figure-8 knot complement. This process easily generalizes to any alternating knot

Definition 5.1 A **polyhedron** is a closed 3-ball in S^3 along with a finite graph on its boundary such that each component of its complement in the boundary is simply connected.

Step 1. The Knot Diagram as a Graph

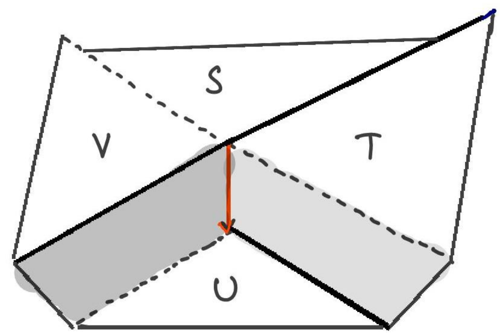


Fix a diagram of the knot. Ignoring the crossings and looking at it as a graph, we can label the faces as above. We imagine 2 balloons expanding on either side of the knot and meeting at the faces marked above. These balloons will become the polyhedra decomposing the knot complement in S^3.

Step 2. Top Polyhedron

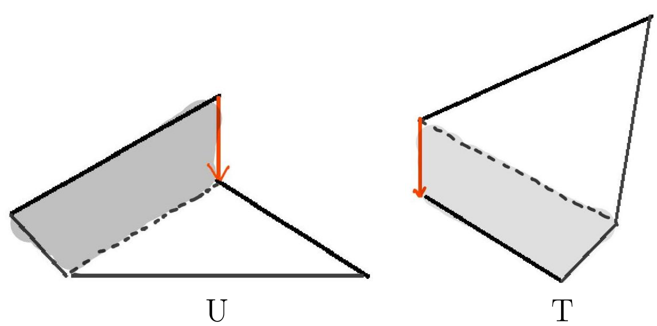
Introducing edges

At each crossing, we add a vertical edge from the overstrand to the understrand.

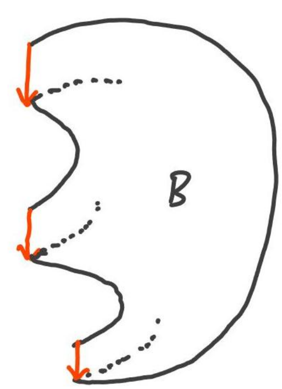


Local picture at a crossing (Figure 1.3 [Pur20])

The bold diagonals are parts of the knot. This edge cuts out the shown part of the knot complement into 4 faces, two of which are shown below.

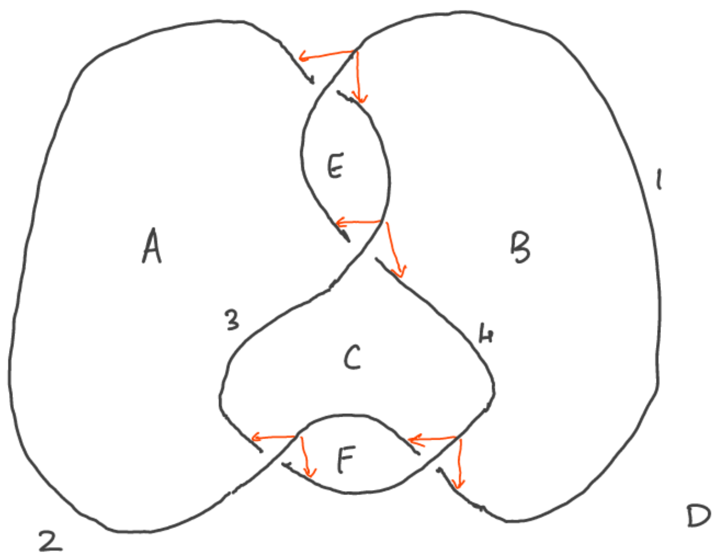


An image of B with the added edges looks like

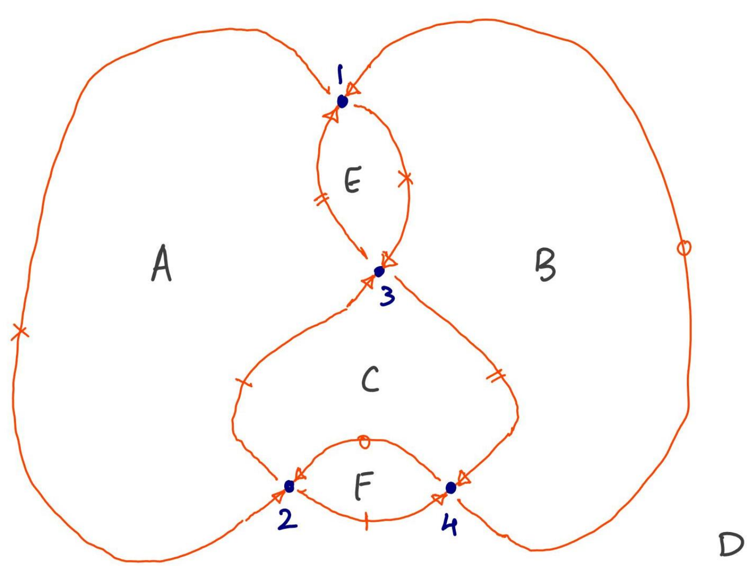


Adding edges to the diagram

We label each arc (part of the knot between undercrossings) and draw edges from each overstrand to understrand. We are looking at the top polyhedron from inside it.



Collapsing arcs to a point

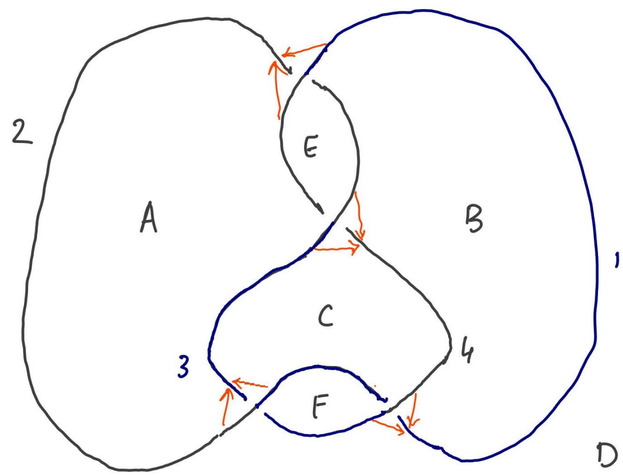


One by one, every arc of the knot between undercrossings is collapsed to a distinct point. There are 4 such arcs. This can be done as the complement of the arc in S^3 is homeomorphic to the complement of a point, and for now we are only concerned with the topology of the knot complement. The orange edges are stretched out during the collapse.

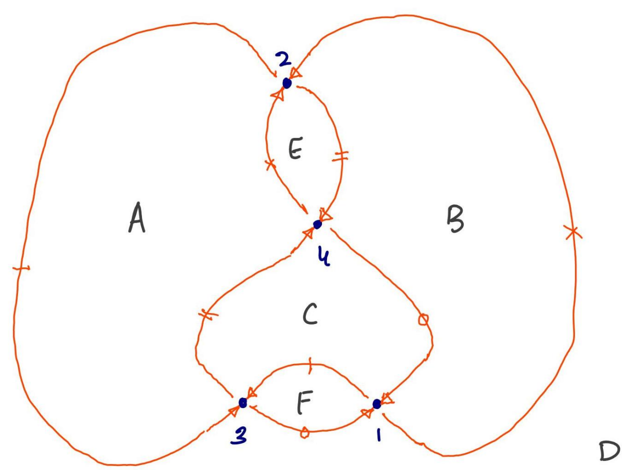
Note that both edges coming from a vertex are in fact the same, and thus are identified. All vertices correspond to the original knot.

Step 3. Bottom Polyhedron

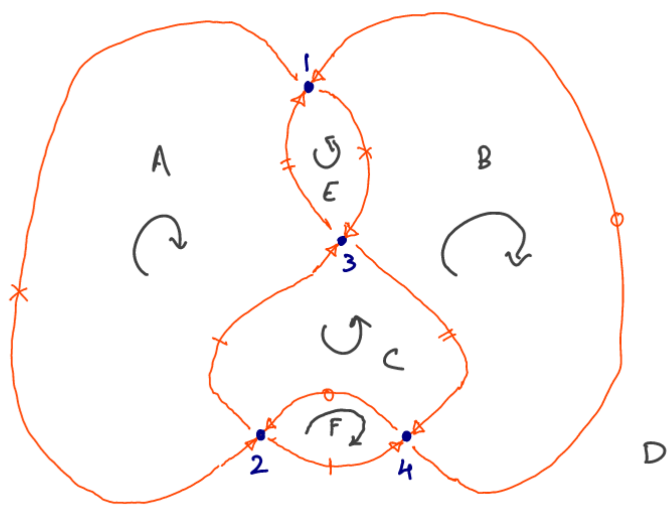
For the bottom polyhedron, understrands are overstrands, so the edges are drawn accordingly. We are looking down at the bottom polyhedron from outside it.



Note that the direction of the orange arrows should have been reversed, but we have kept them this way to glue correctly with the corresponding edges of the top polyhedron.

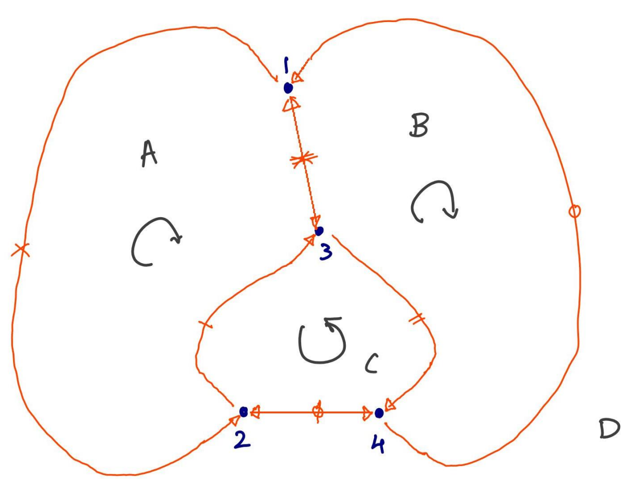


Now, the edges on each crossing must be identified with the corresponding edges in the top polyhedron diagram. This completes the decomposition of the figure-8 knot complement into two polyhedra. The way these polyhedra are glued together is explained below. Gluing the top to the bottom polyhedron is via a rotation of each face, called a "gear rotation".



Step 4. Removing Bigons

A **bigon** is a region of a graph bounded by two edges and two vertices. Notice that this diagram has some bigons, which can be adapted to single edges by deforming the bigon.

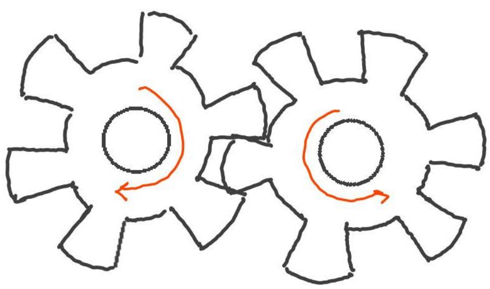


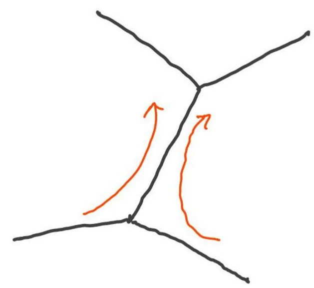
This diagram illustrates how an ideal tetrahedron from the top of the graph can be stuck to one on the bottom, since this graph is the 1-skeleton of a tetrahedron. Thus, we have a two-tetrahedron decomposition of the figure-8 knot complement.

Gear Rotation for Alternating Knots

Definition 5.2 We say that a gluing of two polyhedra is by **gear rotation** if the following hold:

- Each face glues to a corresponding face by a rotation of its edges
- The rotation is in the opposite direction for any two faces separated by an edge.





Observe that this condition is required for the continuity of the gluing.
With these conditions, the gear rotation is a well-defined quotient map, giving rise to a polyhedral complex

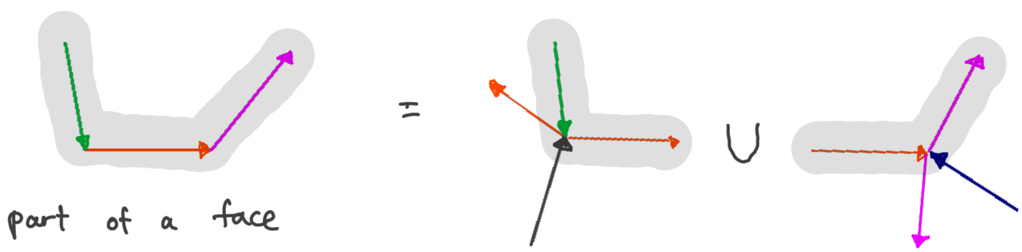
Definition 5.3 An **alternating knot** is a knot with a diagram such that while traveling along the knot, the crossings alternate between overcrossing and undercrossing.
If a knot is alternating, then each arc will contain exactly one overcrossing. Extending the edges at both endpoints of this arc to the overcrossing will collapse the arc to a point and replace it with two edges of the polyhedron. The resultant graph is thus the same as the diagram, and each vertex is of degree 4. so, the method used for the figure-8 knot will give us a polygonal decomposition for any alternating diagram of a knot. The gluing also, will be through a gear rotation, as we shall prove next. The only thing that may change is the fact that the polyhedra need not be tetrahedra.

Theorem 5.4 For any alternating knot, applying the illustrated procedure yields a polygonal decomposition of its complement in S^3, such that the face gluings are via a gear rotation.

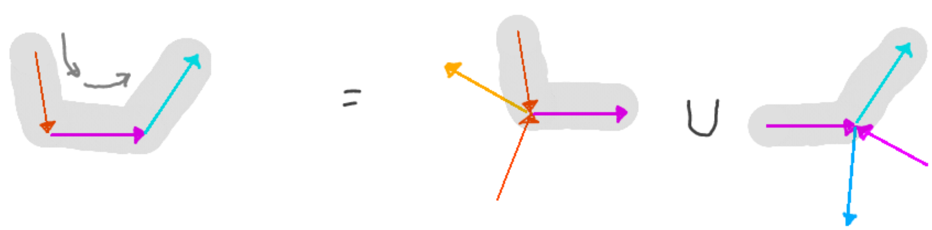
Proof.

- Apply the same procedure to obtain a 4-valent directed graph agreeing with the knot diagram for both the top and the bottom polyhedron.
- First, we show the gluing on any face is via a rotation on its edges. If we show that the gluing direction is the same about two successive edges in a face, then we will be done by induction. We look at the colours of edges at each vertex.

Top



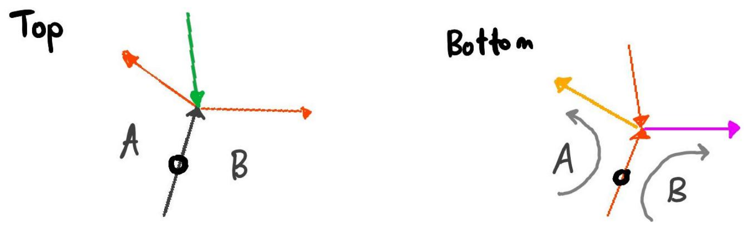
Bottom



Starting with the above part of a face, the colours of edges at each vertex determnined the colours of edges at the corresponding vertices of the bottom polyhedron. Notice that the two identified edges leaving a vertex in the top polyhedron determine the colour of the edges entering the corresponding vertex in the bottom polyhedron.

This is how we obtained the colours of two edges in the bottom polyhedron, and we can see that the direction of gluing at these edges is in the same direction.

We could have started with a part of the face with edges in the opposite direction, but one can convince oneself that a similar result would be obtained.

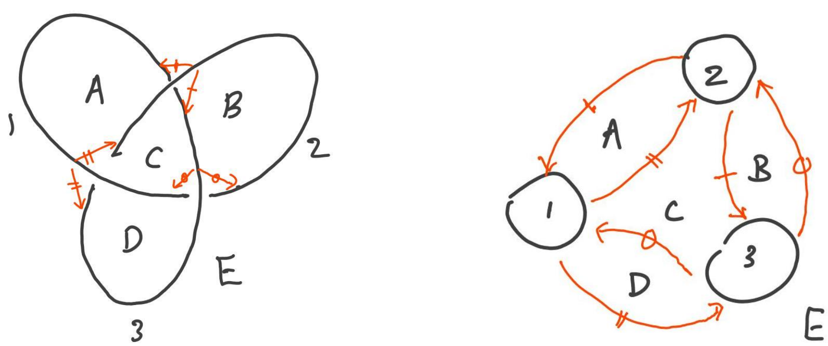


- We now show that the direction of rotation is opposite across any edge. Each vertex is surrounded by 4 faces. We restrict our attention to the marked edge and faces A, B.

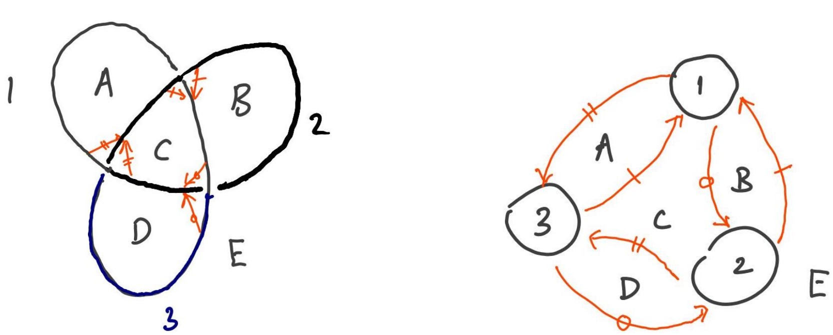
  As the orange edges are identified, the claim clearly holds. □

Example 5.5 Polyhedral Decomposition of the Trefoil

Using the same method, we can find an even simpler polygonal decomposition for the trefoil complement.

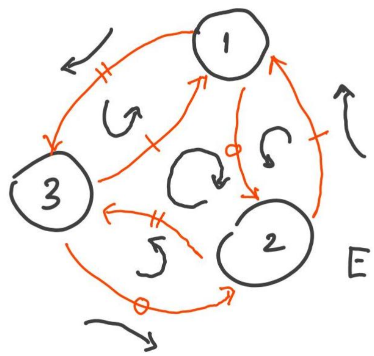


Top Polyhedron



Bottom Polyhedron

Again, we can see that the bottom polyhedron can be stuck to the top one by a gear rotation.



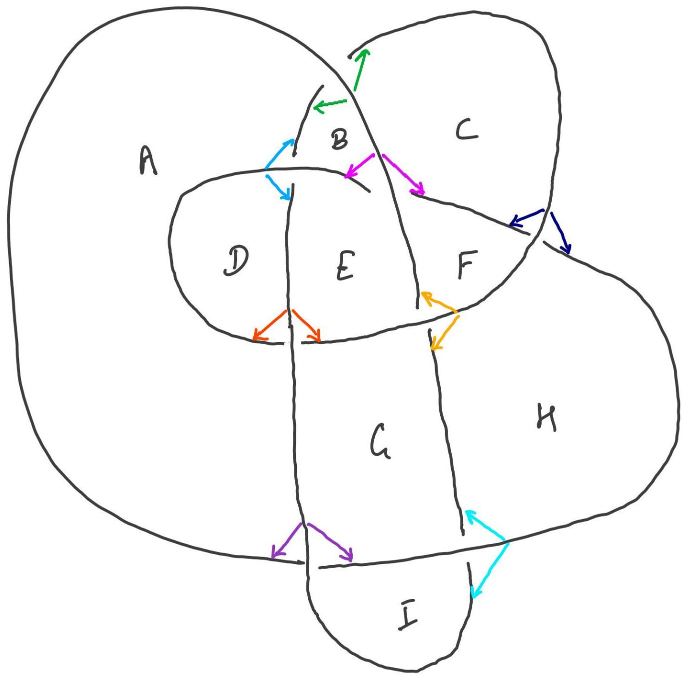
Polyhedral Decomposition of Non-alternating Knots

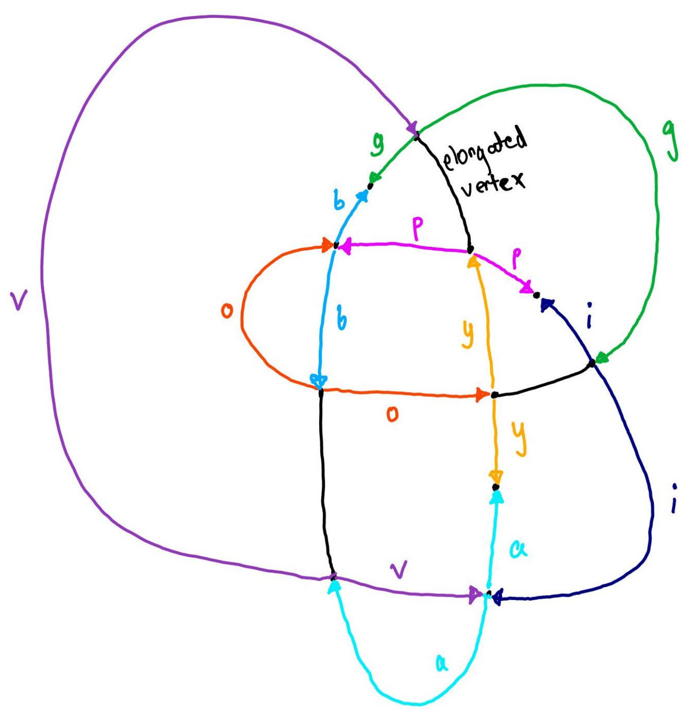
The first example of a non-alternating knot in the knot table is 8_{19}. Next, we give a polyhedral decomposition of this knot following the procedure in the previous section.

Example 5.6 Polyhedral Decomposition of 8_{19}

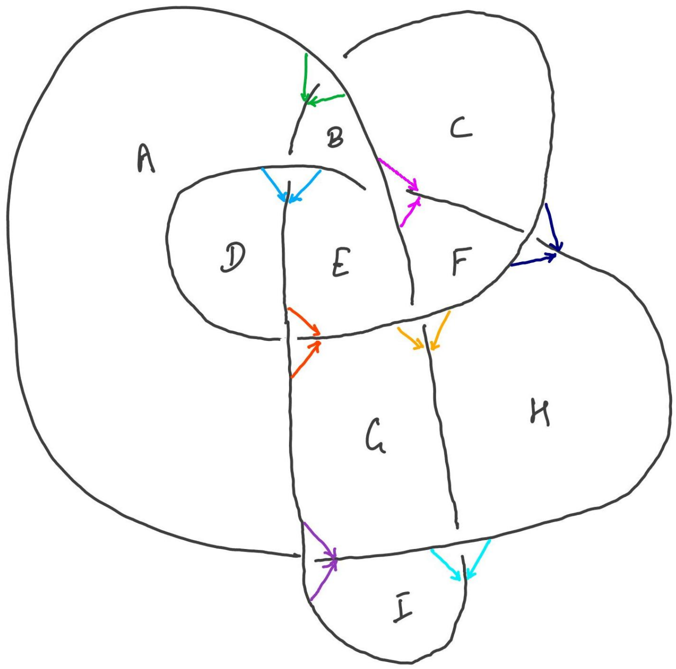
As in the case for alternating knots, we collapse each arc to a point. In order to obtain a graph that looks visually Like the knot diagram, if an arc contains several overcrossings, we represent the subarc from the first to the last overcrossing by a single "elongated" vertex. If an arc contains no overcrossings, then we introduce a new degree two vertex along the arc to which it collapses. Similar expression with the role of over and undercrossings exchanged is made for the bottom polyhedron.

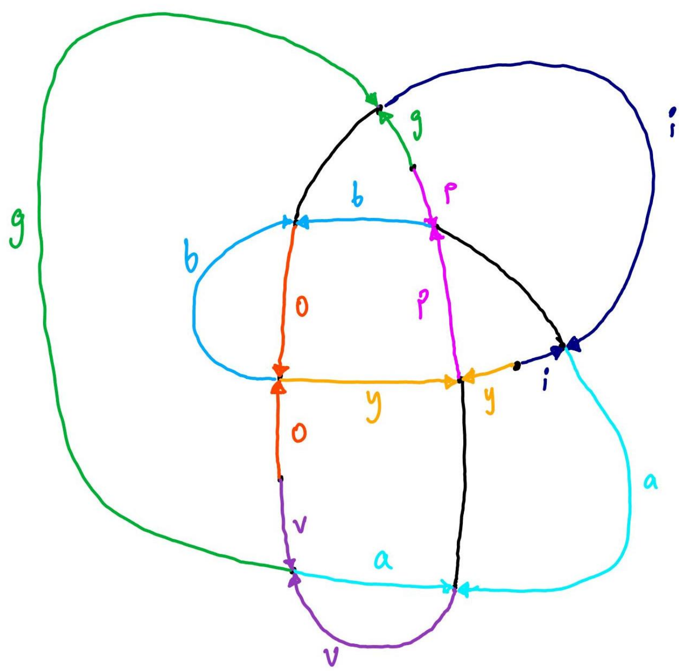
Top Polyhedron



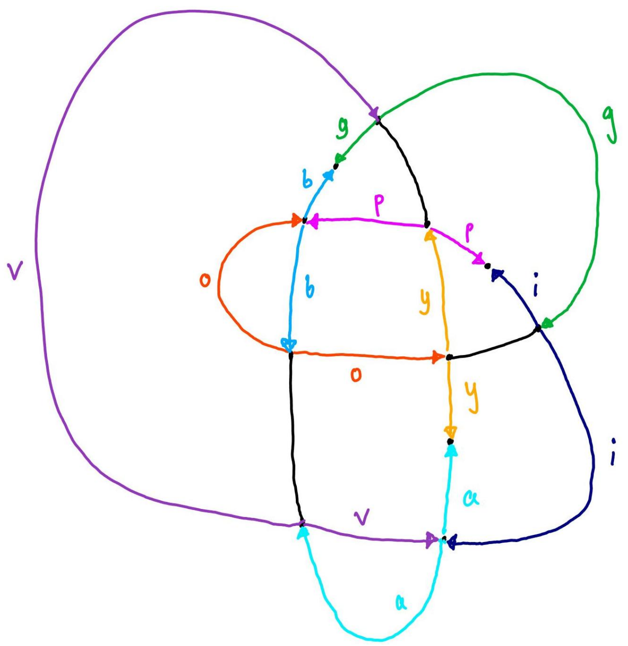


Bottom Polyhedron

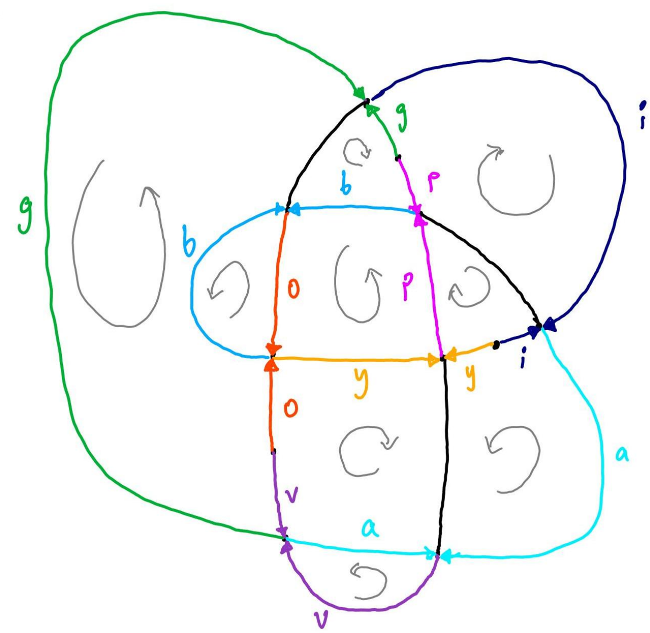




Top



Bottom



Notice that the gear rotation still holds!

Thus, we have a polyhedral decomposition of 8_{19}.

Chapter 6

The Geometry of Polygonal Decompositions

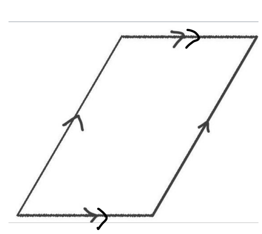
Metrics on Polygonal Decompositions

Definition 6.1

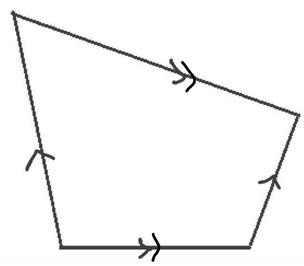
i) A **topological polygonal decomposition** of a 2-manifold M is any topological gluing of polygons (2-dimensional polyhedra) at their edges, so that the resultant quotient space is homeomorphic to M. Edges glue like in a delta complex ([Hat02]).

ii) A **geometric polygonal decomposition** of M is a topological polygonal decomposition along with a metric on each polygon such that the edge gluing maps are isometries, and the resultant manifold is a Riemannian manifold isometric to M. We shall allow vertices to be ideal, i.e. absent from the decomposition. Why do we need the gluing maps to be isometries?

Example 6.2



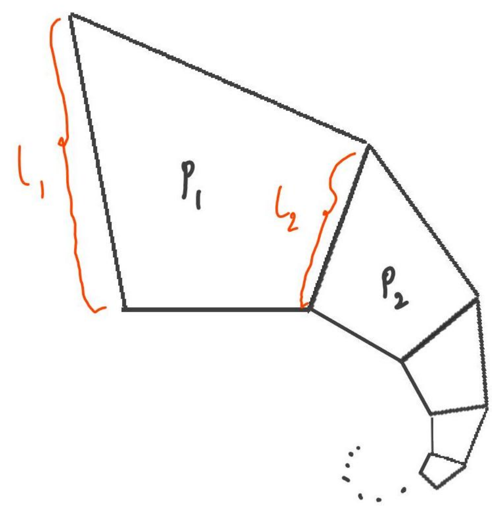
We know that a parallelogram with the above identifications is a torus, and by gluing together multiple parallelograms in $\mathbb{R}^2$, we get a tiling of the plane.



P

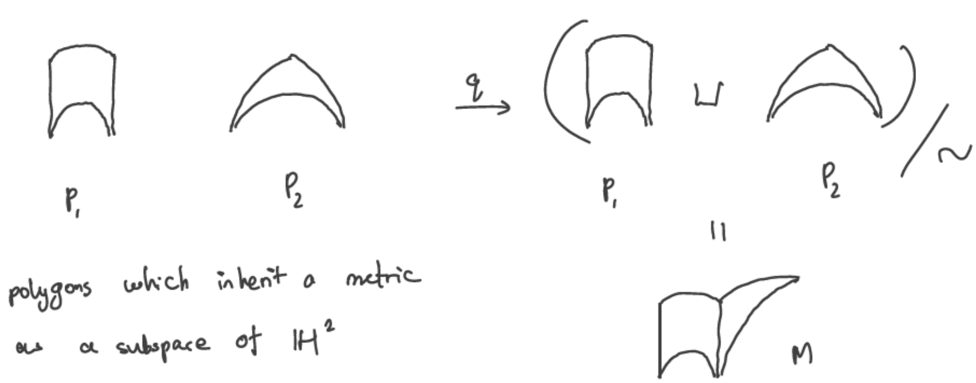
However, identifying edges of a quadrilateral as in the left figure also gives the torus. Let us try gluing copies of this together.

$l_1 > l_2$



We had to scale the down the polygon P to P_2 in order to stick its larger edge to the smaller edge of P_1, i.e., the gluing is not via an isometry. Thus the metrics on P_1 and P_2 do not match the metric on P, and we don't get a tiling of the plane, but an infinite spiral of polygons.

How should we think of a polygonal decompositions?



The condition that the edge gluing maps are isometries alone does not give us a hyperbolic structure on a gluing of hyperbolic polygons. The following lemma provides a necessary and sufficient condition for this. Assume that the faces are hyperbolic and edges are geodesic. Note that we want the hyperbolic structure to respect the existing geometric structure on the interior of each polygon.

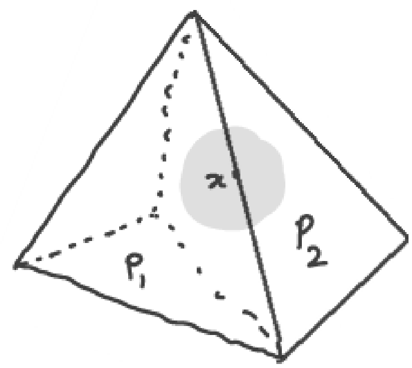
Definition 6.3 We say that a hyperbolic structure on a gluing of polyhedra *agrees with the structure in the interior of the polyhedra* if for all polyhedra P and $x \in \mathring{P}$, a ball around P along with the inclusion map provides a chart in the hyperbolic structure.

Theorem 6.4 A gluing of hyperbolic polyhedra of dimension n results in a hyperbolic n-manifold M with structure agreeing with that on the interior of the polyhedra if and only if each point in the codimension-2 skeleton of the gluing has a neighbourhood U and a homeomorphism $\phi : U \longrightarrow B$ (ball) $\subseteq \mathbb{H}^n$ such that $\phi_{|\mathring{P}}$ is an isometry with respect to the metric on the interior of any polyhedron P that intersects U.

Proof. If M has a hyperbolic structure agreeing with the structure in the interior of the polyhedra, then we give it the metric induced by the structure, and then, by definition, each point has a neighbourhood U and a chart $\phi : U \to \mathbb{H}^n$ which is an isometry onto its image. By shrinking U sufficiently, we can assume that $\phi(U)$ is a ball in $\mathbb{H}^n$.
For the converse, we check the neighbourhood of a point x based on its location in the gluing.

- *Interior point:* If $x \in \mathring{P}$, take a neighbourhood $U \ni x$ such that $U \subseteq P$ (polyhedron) and define the chart $\phi : U \hookrightarrow P \longrightarrow \mathbb{H}^n$.

- *Interior of codimension-1 face:* Suppose x lies in the interior of $P_1 \cap P_2$ Take a small neighbourhood U of x only intersecting the face, and two n-dimensional polyhedra P_1 and P_2 (see the picture on the next page). Define the chart on $U \cap P_1$ as the restriction of the inclusion of P_1 into $\mathbb{H}^n$ and on $U \cap P_2$ as the composition of inclusion $P_2 \hookrightarrow \mathbb{H}^n$ with the face gluing isometry.



- *Codimension-2 skeleton:* Use the given homeomorphism from a neighbourhood of x onto a ball in $\mathbb{H}^n$. We claim that the transition maps are isometries onto open subsets of $\mathbb{H}^n$. As the restriction of such a homeomorphism to the interior of a polyhedron is an isometry, any transition map, being a composition of such homeomorphisms, is an isometry on the image of the interiors of polyhedra. But the union of the interiors of polyhedra is a dense subset of the gluing, so the transition map is a homeomorphism whose restriction to a dense subset is an isometry and thus, the entire map is an isometry. □

Corollary 6.4 A gluing of hyperbolic polygons admits a hyperbolic structure agreeing with the structure on the interior of the polygons if and only if the angle sum around each finite (non-ideal) vertex of the polygons is 2π.

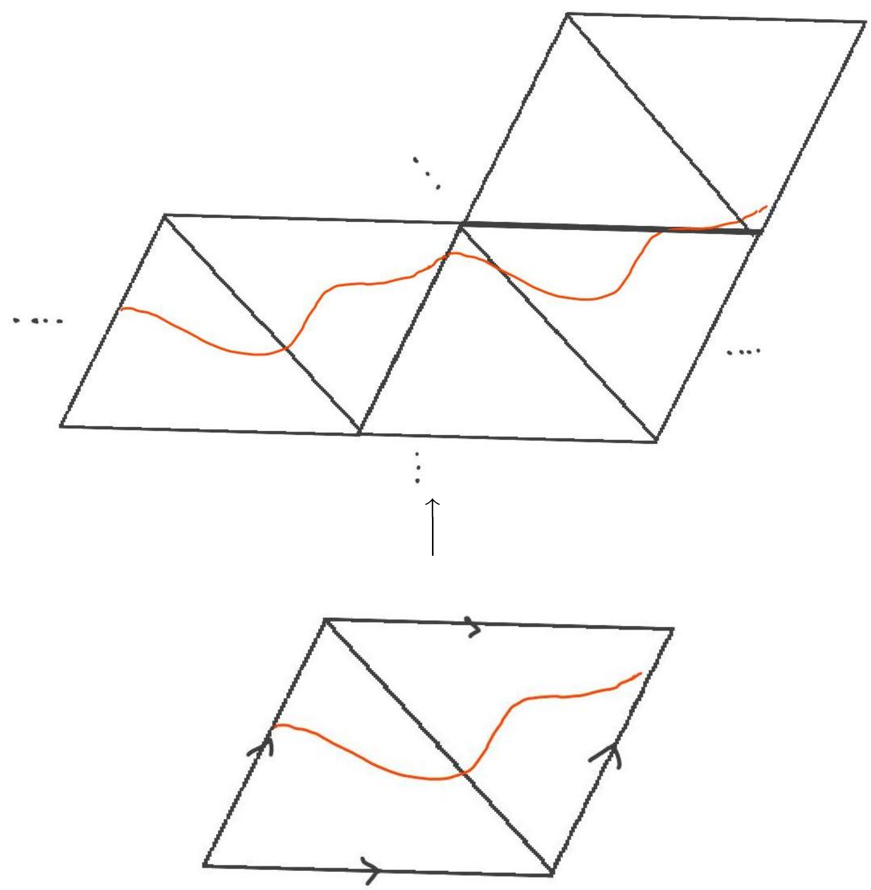
Proof. For a gluing of polygons, the only codimension-2 faces are vertices. By Theorem 6.4, the problem reduces to when each vertex in the gluing has a neighbourhood isometric to a disc in $\mathbb{H}^n$. This is indeed true only when the sum of the interior angles of polygons having the point as a vertex is 2π. □

Completeness of Geometric Polygonal Decompositions

Having a hyperbolic structure alone is often not enough. We are interested in knowing when this hyperbolic structure is complete, so that the gluing has nicer properties like having $\mathbb{H}^2$ as its universal cover and the developing map as a covering map (Corollary 4.8). Now, we assume that the gluing gives a hyperbolic structure on the surface, i.e., angle sum $= 2\pi$

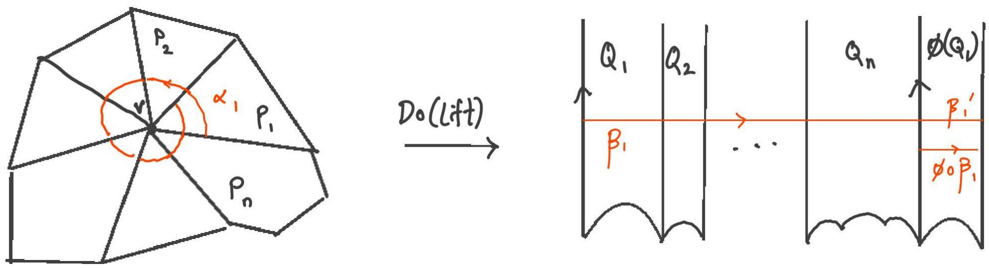
around each finite vertex, and explore conditions for completeness. We also assume that there are finitely many polygons in the decomposition. Note that ideal polygons in a gluing of polygons correspond to hyperbolic polygons in $\mathbb{H}^2$ whose vertices lie in $\partial\mathbb{H}^2$.

If P is a polygon in a hyperbolic polygonal decomposition of a 2-manifold M, then as P is simply-connected, by the General Lifting Lemma ([Mun00]), we can lift the inclusion map (not necessarily injective on the boundary): $P \longrightarrow M$ to a map: $P \longrightarrow \widetilde{M}$, and then we can compose this with the developing map $D : \widetilde{M} \to \mathbb{H}^2$ to get a map: $P \longrightarrow \mathbb{H}^2$. This is how we can use the developing map D to develop the image of a series of polygons belonging to M, in $\mathbb{H}^2$. Specifically, if α is a curve in M, we can develop its image in $\mathbb{H}^2$ by lifting it to $\widetilde{M}$ and then applying D.



Definition 6.5 A horosphere in a hyperbolic polygon will be called a **horocycle**.

Start with an ideal vertex v of M. Then, v is a vertex of some ideal polygon P_1. Draw a horocycle α_1 in P_1 about v. This meets an edge of P_1 shared by another polygon P_2, and starting at that point, there is a unique horocycle α_2 about v in P_2. Now extend α_2, and repeat this process until we reach P_n and come back to P_1 (there are finitely many polygons). Extend the concatenation of $\alpha_1, \ldots, \alpha_n$ to get another horocycle α_1' in P_1 starting at the endpoint of α_n. α_1' does not necessarily agree with α_1, as shown below.

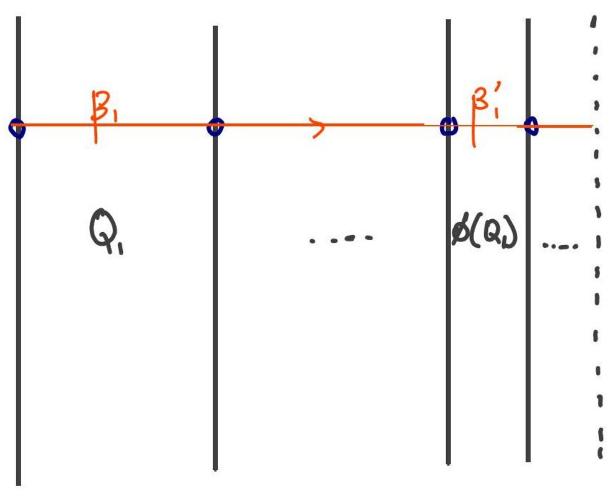


$\phi \in \mathcal{I}(\mathbb{H}^2)$ is not a horizontal translation

We can send v to $\infty \in \partial\mathbb{H}^2$, and then the horocycles become horizontal lines. Now assume that the gluing on the left edge of Q_1 to the right edge of Q_n is by some scaling by an amount < 1, so that points in Q_1 are at a decreased height in $\phi(Q_1)$ in the half-space model, and thus to preserve lengths $(l(\beta_1) = l(\phi \circ \beta_1))$, $\phi(Q_1)$ is narrower than Q_1. $\phi \circ \beta_1$ is below β_1'. Note also that ϕ is the holonomy of the loop in M going around v once.

Note that ϕ sends the left edge of Q_2 to the right edge of $\phi(Q_1)$, and thus ϕ also gives an isometric copy of P_2 to the right of $\phi(Q_1)$. $\phi(Q_2)$ is also narrower than $\phi(Q_1)$ in order to satisfy $l(\phi \circ p_2) = l(p_2)$.

Continuing this process, we get an infinite sequence of narrower and narrower polygons going right, and the extension of the horocycle line $\beta_1, \ldots, \beta_n$ is a horizontal line through them.



With knowledge of the isometries and lengths in $\mathbb{H}^2$, one can verify that

$$l(\beta_1) + l(\beta_1') + l(\beta_1'') \cdots < \infty$$

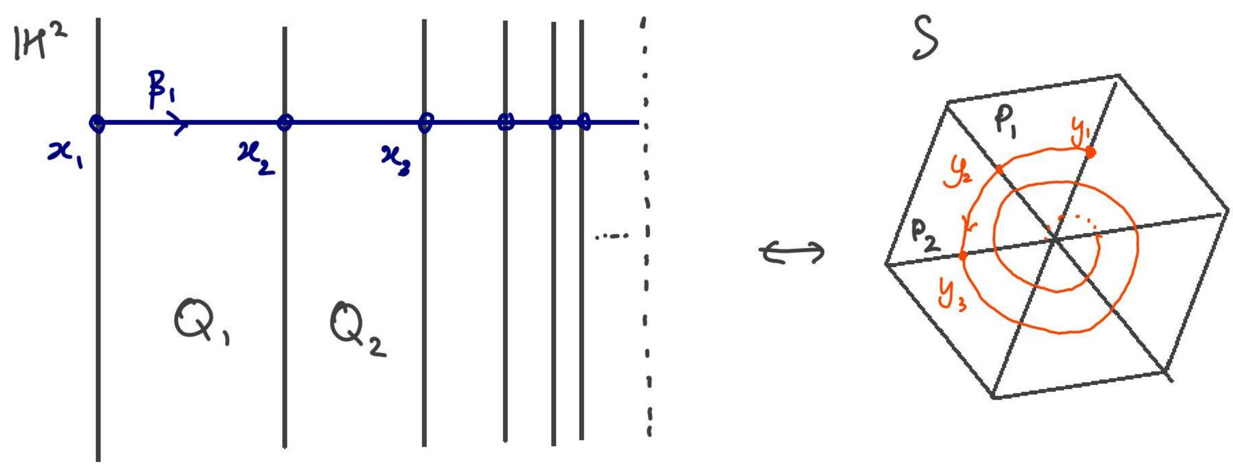
and, in turn, the length of the infinite extension of $\beta_1, \ldots, \beta_n$ is finite.

Definition 6.6 $d(v)$ is defined to be the signed hyperbolic distance between α_1 and α_1' in P_1.

It can be shown that $d(v)$ is independent of α_1.

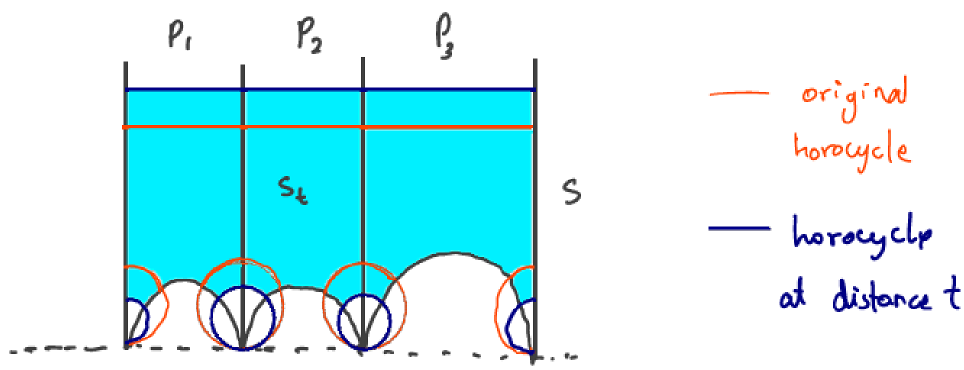
Theorem 6.7 Let S be a surface with hyperbolic structure from a gluing of polygons The metric on S is complete if and only if $d(v) = 0$ for each ideal vertex v in the gluing.

Proof. When $d(v) \neq 0$, there exists an ideal vertex v of a polygon P_1 and a horocycle α_1, such that α_1' does not intersect α_1, just as in the previous example. If like the previous example, the polygons in the developing picture in $\mathbb{H}^2$ get narrower towards the right, we have the following infinite set of polygons.



The sequence $\{x_n\}$ is clearly Cauchy but it does not converge in the developing image, which does not contain the dotted line. Then, the corresponding sequence $\{y_n\} \subseteq S$ is also Cauchy and does not converge as the distance along an edge between consecutive points on the edge is at least $d(v)$. Thus, S is not complete.
Instead, if the gluing isometry was such that the polygons get wider as we go right, then they would get narrower towards the left, and considering the extension of a horizontal horocycle travelling leftwards would do the trick.

If $d(v) = 0$ for all ideal vertices v, then all horocycles close up around each finite vertex. Then, we can remove the interior of a horoball around each ideal vertex from M, and as we started with finitely many polygons, the resultant manifold is compact with boundary. $\forall\, t > 0$, let S_t be the compact manifold obtained by removing the interiors of horoballs, such that their boundary is at a distance t from the boundary of the initially chosen horoball. Notice that $\bigcup_{t>0} S_t = S$.



Any Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded in S.

$$\Rightarrow \exists\, r > 0 \;\text{ such that } \{x_n\} \subseteq N_r(x_1)$$

Choosing t large enough, $N_r(x_1) \subseteq S_t$. Then, $\{x_n\}_{n\in\mathbb{N}} \subseteq S_t$. As S_t is compact, $\{x_n\}_{n\in\mathbb{N}}$ has a limit point, and the sequence x_n converges. $\square$

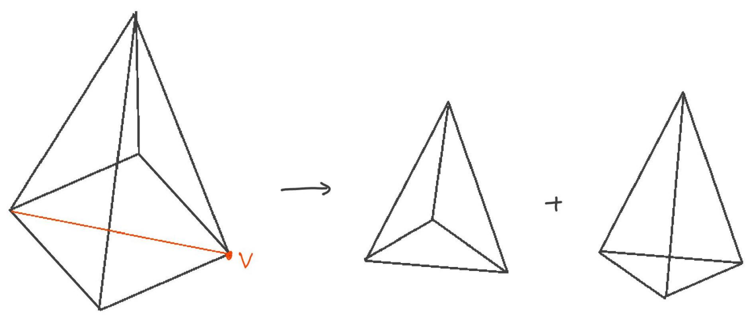
For the last part of this thesis, we find explicit conditions for the existence of a hyperbolic structure on a triangulation of a 3-manifold, with knot complements being our primary examples.

Geometric Ideal Triangulations

Definition 6.8 A **hyperbolic tetrahedron** is the convex hull of four non-coplanar points (vertices) in $\mathbb{H}^3$, so that the edges are geodesics and faces are 2-dimensional hyperbolic subspaces. An **ideal hyperbolic tetrahedron** has all its vertices in $\partial\mathbb{H}^3$.

Definition 6.9 A **topological ideal triangulation** of a 3-manifold M is a gluing of ideal tetrahedra such that the faces glue to entire faces, edges glue to entire edges, and the result is homeomorphic to M. If the tetrahedra are geometric, their gluing should be via isometries.

We constructed a topological ideal triangulation of the figure-8 knot complement in chapter 5. Given an ideal topological polyhedral decomposition of a 3-manifold, one can easily make it a topological ideal triangulation. One way to do this is by choosing an ideal vertex v in each polyhedron and coning to that vertex; which means that we add edges from v to all other edges, 2-simplices between v and all pairs of edges meeting v, and tetrahedra between v and any three 2-simplices meeting v. We now remove the resultant tetrahedra, in particular we remove v; so that the resultant polyhedral complex has at least one lesser ideal vertex. We repeat this process to obtain a collection of tetrahedra.



One can refer to [Pur20] for a carefully worked out example decomposing the 6_1 knot complement into five tetrahedra.

Definition 6.10 A **geometric ideal triangulation** is a topological ideal triangulation with a complete hyperbolic structure. We sometimes simply call this a geometric triangulation.

It is not yet known whether every 3-manifold admitting a complete hyperbolic structure admits a geometric ideal triangulation. Therefore, it might be interesting to explore whether two-polyhedron decompositions of knot complements can be made geometric instead, by simply looking at the knot diagram. In the rest of this chapter, however, we only work with triangulations. We usually assume that the number of tetrahedra is finite.

Edge Gluing Equations

In this chapter, we have seen that a gluing of hyperbolic polygons admits a hyperbolic structure if and only if the angle sum around each finite vertex is 2π. We have similar conditions in the 3-dimensional case, but this time around edges.

Let T be an ideal tetrahedron in $\mathbb{H}^3$. Suppose e is an edge of T. Choose an isometry of $\mathbb{H}^3$ sending the endpoints of e to 0 and ∞, and a third vertex to $1 \in \mathbb{C} \subseteq \partial\mathbb{H}^3$.

If z_1, z_2 are the endpoints of e and z_3 is the third vertex, the isometry is

$$z \mapsto \left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right)$$

Then, $z(e)$ is defined as the image of the fourth vertex under this isometry, which is

$$\left(\frac{z_4 - z_1}{z_4 - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right)$$

Note that exchanging z_1 and z_2 simply inverts $z(e)$; and so does exchanging z_3 and z_4.

$$\frac{1}{z(e)} = \frac{\overline{z(e)}}{\|z(e)\|^2}$$

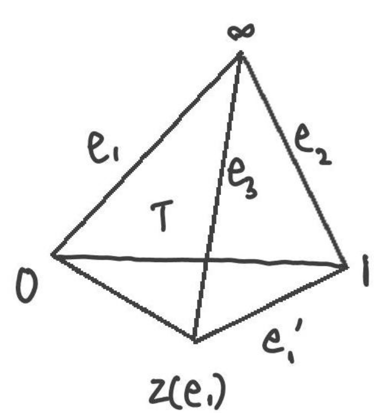
So, in order to have a well-defined $z(e)$, we require it to have non-negative imaginary part.

Definition 6.11 For an edge e of an ideal tetrahedron in $\mathbb{H}^3$, the complex number $z(e)$ defined above is called the edge invariant of e.

Remark 6.12 If $z(e) \in \mathbb{R}$, then the tetrahedron that we started with is planar and is said to be flat. An ideal triangulation with flat tetrahedra is not a geometric ideal triangulation, so we must rule out these tetrahedra and require the imaginary part of $z(e)$ to be positive.

When we glue together tetrahedra, since they often share edges, it is sometimes impossible to choose all edge invariants to have positive imaginary parts. A tetrahedron having an edge invariant with negative imaginary part is said to be **negatively-oriented**.

Any edge invariant of a tetrahedron determines all others as follows.



Lemma 6.13 *T is an ideal tetrahedron with edges e_1, e_2 and e_3 as above, and vertices mapped to $0, 1, \infty$ and $z(e_1)$.*

i) The edge e_1' opposite e_1 has edge invariant

$$z(e_1') = z(e_1)$$

ii) The edge e_2 has edge invariant

$$z(e_2) = \frac{1}{1 - z(e_1)}$$

iii) The edge e_3 has edge invariant

$$z(e_3) = \frac{z(e_1) - 1}{z(e_1)}$$

Then, the edge invariants satisfy

$$z(e_1)z(e_2)z(e_3) = -1 \quad \text{and} \quad 1 - z(e_1) + z(e_1)z(e_3) = 0$$

Proof. i) We need an isometry sending: $1 \mapsto 0$

$$z(e_i) \mapsto \infty$$

$$0 \mapsto 1$$

which is

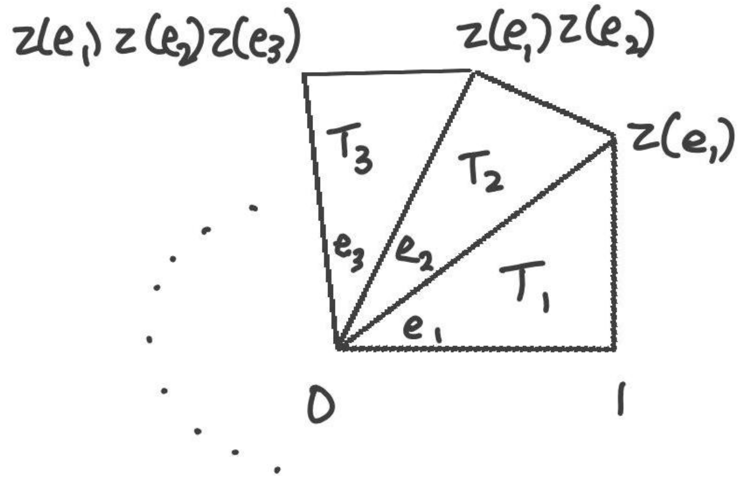
$$z \mapsto \left(\frac{z-1}{z - z(e_1)}\right) \frac{1}{z(e_1)}$$

Note that ∞ is mapped to $\frac{1}{z(e_1)}$, which has negative imaginary part so we invert it, and get $z(e_1') = z(e_1)$.

ii), iii) Same method; we omit the calculation. □

Now, say we have a gluing of ideal tetrahedra. Fix an edge e in the gluing. Suppose T_1 is a tetrahedron with an edge e_1 that glues to e. T_1 can be mapped to $\mathbb{H}^3$ with the vertices of e_1 mapped to 0 and ∞. The third vertex is mapped to 1 and then the final vertex to $z(e)$. F_1 denotes the face of T_1 with vertices corresponding to $0, z(e_1)$ and ∞. F_1 is then glued to a face F_1' of another tetrahedron T_2. Instead of sending its vertices to $0, 1$ and ∞

again, in order to match with the image of T_1, we map vertices of e_2 to 0 and ∞, the third vertex to $z(e_1)$, and the fourth then goes to $z(e_1)z(e_2)$. We continue attaching tetrahedra anticlockwise around e until we reach a tetrahedron T_n which glues back to T_1. The fourth vertex of T_n will be mapped to $z(e_i)\ldots z(e_n) \in \mathbb{H}^3$.



Theorem 6.14 Suppose that M admits a topological ideal triangulation by hyperbolic tetrahedra. M admits a hyperbolic structure agreeing with that on the tetrahedra if and only if for each edge e in the gluing,

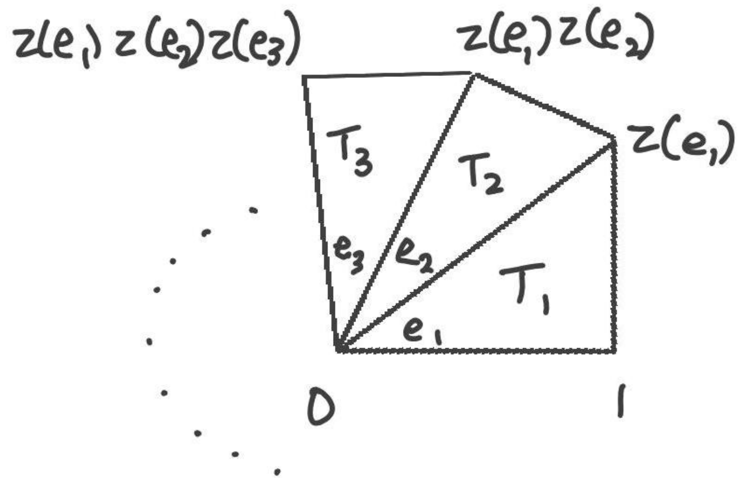
$$\prod z(e_i) = 1 \quad \text{and} \quad \sum \arg(z(e_i)) = 2\pi$$

where the sum and product are over all the edges of tetrahedra that glue to e.

Proof. From Theorem 6.4, admitting the required hyperbolic structure is equivalent to having neighbourhoods of every point isometric to balls in $\mathbb{H}^3$. In chapter 6, we showed how we can lift gluings of polygons to $\mathbb{H}^2$ using the developing map, and we can use the same procedure to lift gluings of tetrahedra, even ideal tetrahedra. This lifting is a local isometry and thus conformal. Now, we focus on a point on the edge e in M.
For it to have a neighbourhood isometric to a ball in $\mathbb{H}^3$, we must have the sum of dihedral angles to be 2π, and thus its lift in $\mathbb{H}^3$ should also satisfy the same condition.
The dihedral angle of T_i at e is

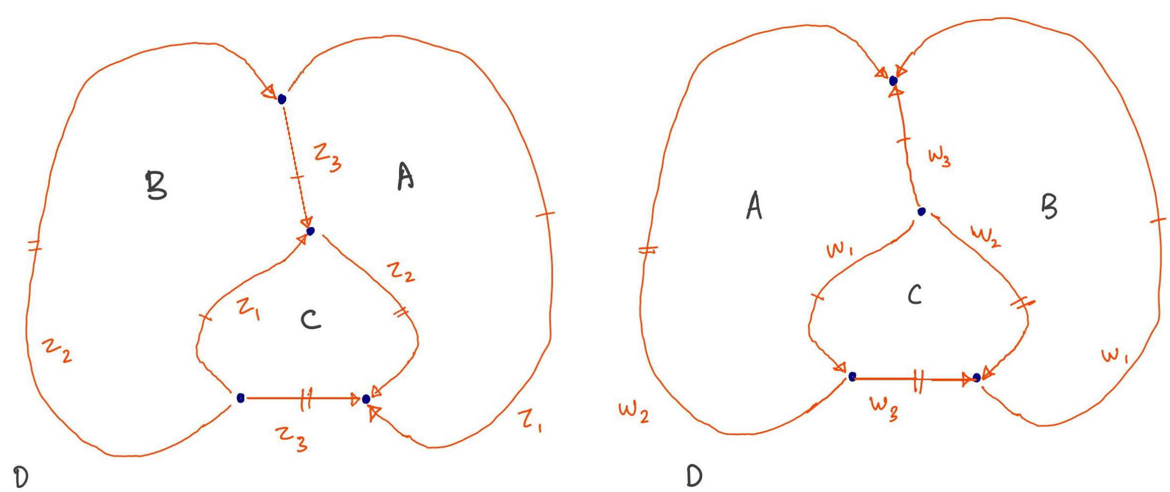
$$\arg\left(z\left(e_i\right)\ldots z\left(e_i\right)\right) - \arg\left(z\left(e_i\right)\ldots z\left(e_{i-1}\right)\right) = \arg\left(z\left(e_i\right)\right)$$

Then, the condition becomes $\sum_{i=1}^{n} \arg(z(e_i)) = 2\pi$.



Lastly, for T_n to glue to T_1, we must have $\prod_{i=1}^{n} z(e_i) = 1$.
Conversely, these two conditions guarantee the existence of a neighbourhood isometric to a ball in $\mathbb{H}^3$.
The two equations above are called the *edge-gluing equations*.
Each edge gives a pair of equations. Since the edge invariants of all edges of a single tetrahedron are related as in lemma 6.13, every tetrahedron contributes no more than one unknown to the equations.

Example 6.15 The figure-8 complement decomposition is into two ideal tetrahedra, as in chapter 5. We now use the labellings presented in [Pur20] (Example 4.8).



We choose both tetrahedra to be regular, and thus all the dihedral angles are $\pi/3$. We have labelled each edge on both tetrahedra with its edge invariant, and these are related by the equations

$$z_1^2 z_3 w_1^2 w_3 = 1$$

for the edge with one tick mark and

$$z_2^2 z_3 w_2^2 w_3 = 1$$

for the edge with two tick marks.

Setting $z_1 = z$, $w_1 = w$ and using Lemma 6.13, the first equation becomes

$$\begin{aligned} & z^2 \left(\tfrac{z-1}{z}\right) w^2 \left(\tfrac{w-1}{w}\right) = 1 \\ \Rightarrow\ & z(z-1)w(w-1) = 1 \end{aligned}$$

The second equation becomes

$$\left(\frac{1}{1-z}\right)^2 \left(\frac{z-1}{z}\right) \left(\frac{1}{1-w}\right)^2 \left(\frac{w-1}{w}\right) = 1$$

$$\frac{1}{(1-z)z} \cdot \frac{1}{(1-w)(w)} = 1$$

which is the same equation.

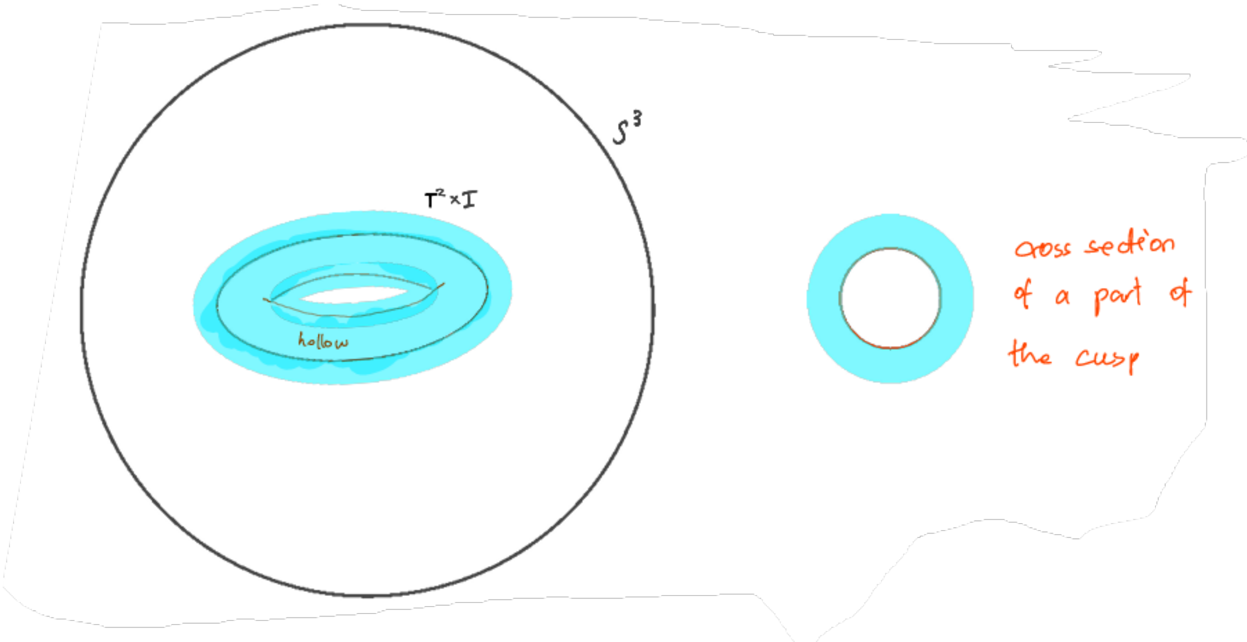
Note that $z = w = (-1)^{1/3}$ is a solution to the equations.

We will show that this gives the complement of the figure-8 knot a complete hyperbolic structure.

Completeness Equations

Suppose that M is a 3-manifold with torus boundary components. For most of the section, we will assume that $\mathring{M}$ has a topological ideal triangulation which has a solution to the edge gluing equations. Hence, $\mathring{M}$ admits a hyperbolic structure. We must examine the cusps of the manifold to determine whether this structure is complete.

Definition 6.16 A **cusp** or a cusp neighbourhood of M is a closed neighbourhood of a component of ∂M which is homeomorphic to $T^2 \times I$. A **cusp torus** is a torus component of the cusp boundary not in ∂M.



Definition 6.17 We say that the hyperbolic structure on $\mathring{M}$ *induces a Euclidean structure on a cusp* if the holonomies of all loops on the cusp torus are Euclidean isometries.

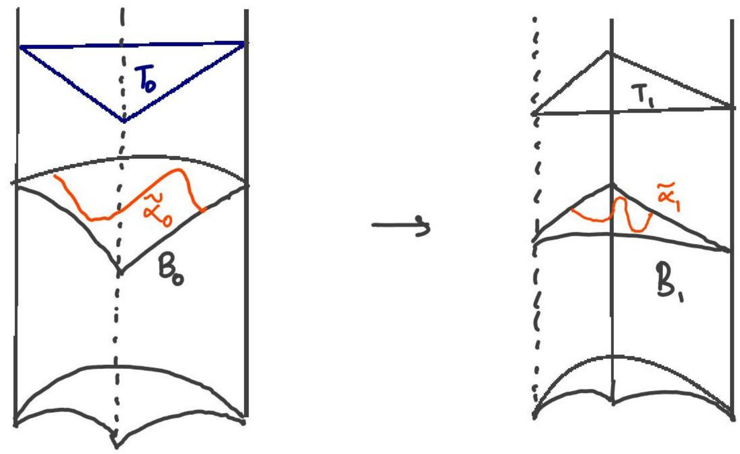
Note that since cusps are homeomorphic to $T^2 \times I$, each cusp torus retracts to a torus boundary component of ∂M. Thus, we sometimes also refer to this component as a cusp torus.

Theorem 6.18 The hyperbolic structure on $\mathbb{M}$ is complete iff the induced structure on each cusp of M is Euclidean.

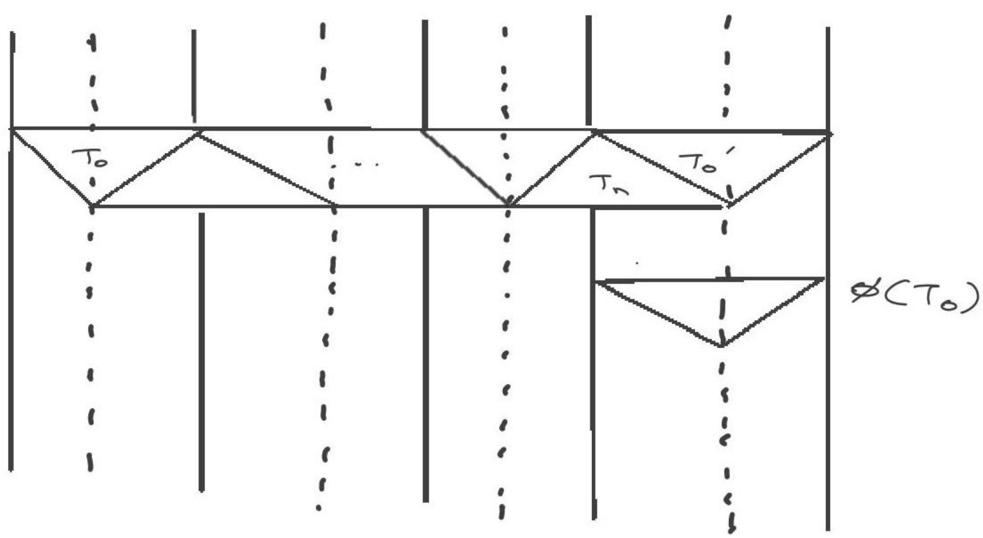
Proof. ($\Rightarrow$)
Assume the contrapositive, i.e., there is some cusp $C \subseteq M$ which does not have a Euclidean induced structure. Then there exists a loop α on the cusp torus $\widetilde{C}$ such that the holonomy of α is not a Euclidean isometry. We know $\mathcal{I}^+(\mathbb{H}^3) \simeq \mathrm{PSL}_2(\mathbb{C})$, which is made up of translations, dilations, and inversions. We omit pure rotations as their action is not free. The only remaining Euclidean isometries are translations, so the holonomy of α must have some non-trivial scaling factor. We assume that the cusp torus inherits a triangulation from the ideal triangulation of $\mathring{M}$.

C is the quotient of a collection of ideal vertices, so we pick one of them, say v, and let A_0 be a triangle in the triangulation of $\widetilde{C}$ around v, containing the starting point of α. Send v to ∞ in $\mathbb{H}^3$, and suppose that A_0 lifts to a triangle B_0.

A segment of α also lifts to $\widetilde{\alpha}_0$ within B_0. Let T_0 be a horospherical triangle about ∞ in the same tetrahedron as B_0.



Continuing $\widetilde{\alpha}$ along cusp triangles $B_0, B_1, \ldots, B_n$, we can develop a triangulation $T_0, T_1, \ldots, T_n$ of a horizontal planar region in $\mathbb{H}^3$. Since the number of tetrahedra around v is finite, we can assume an edge of B_n is glued to B_0, so that an edge of T_n is glued back to T_0. For this gluing, the isometry ϕ applied to the tetrahedron containing T_0 is the holonomy of α. Since this isometry has a scaling factor, the image of T_0 will be above or below the plane of the triangles $T_0, \ldots, T_n$ in $\mathbb{H}^3$.



Then, using the same idea as in the proof of Theorem 6.7, we can construct a Cauchy sequence in the extended image of the triangles, which does not converge.

($\Leftarrow$)
If each cusp acquires a Euclidean structure, then the holonomies of each loop in every cusp torus are pure translations. In this case, the developing horospherical image above closes up to give us a cylinder in $\mathbb{H}^3$, and if we continue this triangulation in the z-direction, we get the triangulation of a horospherical torus about ∞.

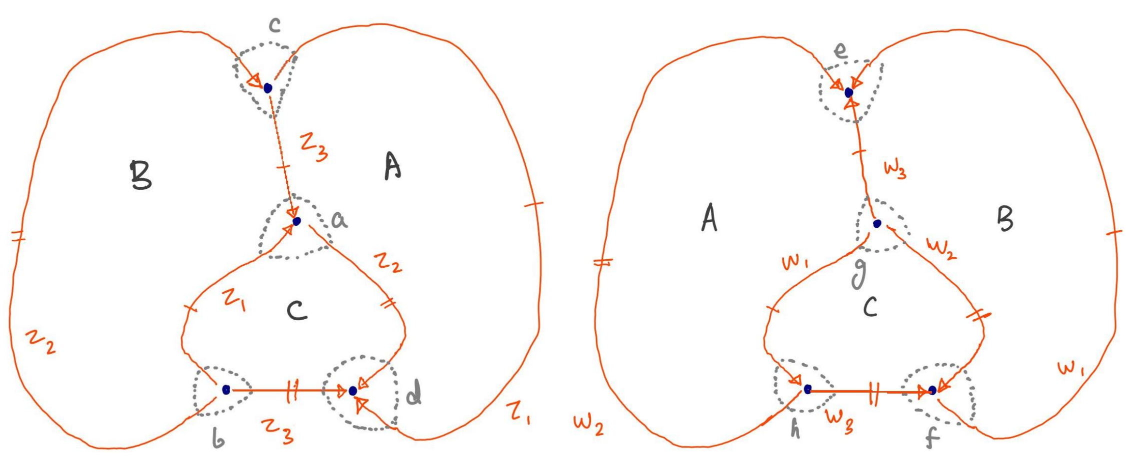
Choosing such a horospherical torus about each cusp, we can remove their interiors from $\mathring{M}$ to obtain a new compact manifold M'. Removing the interiors of horospherical tori at distance $t > 0$ from the initial choice of horospherical tori, we obtain a collection of compact manifolds $\{M_t\}_{t>0}$ just as in the proof of Theorem 6.7, so that

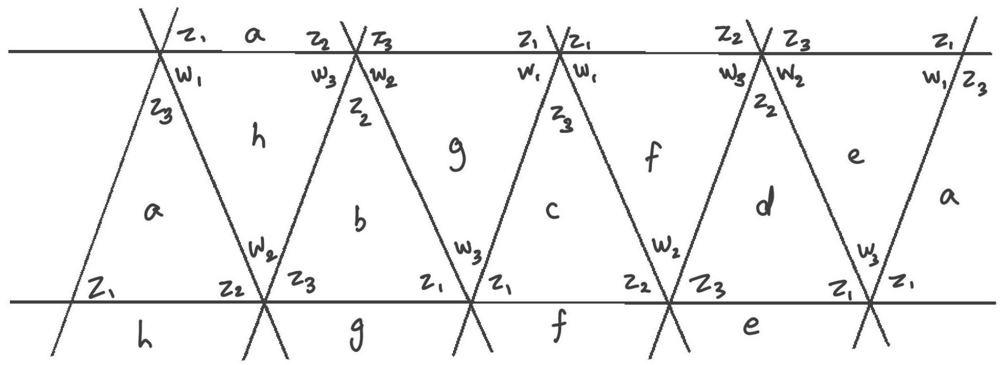
$$\mathring{M} = \bigcup_{t>0} M_t$$

Again, any Cauchy sequence being bounded in $\mathring{M}$, must lie in some M_ε, and then must converge. $\square$

Definition 6.19 $M \to$ 3-manifold with topological ideal triangulation
The triangulation of a cusp torus obtained by truncating the vertices of each ideal tetrahedron is called a **cusp triangulation**.

We illustrate an example for the figure-8 knot.

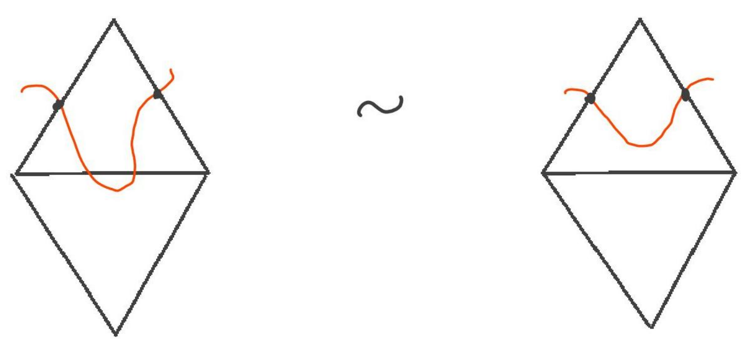




The corner of each triangle is labelled with the edge invariant of the edge meeting that corner of the triangle in the diagram. By Theorem 6.18, the structure on the knot complement will be complete if and only if all cusp tori are Euclidean, or the holonomy of every loop in a cusp torus is a translation. This will happen only when the holonomy of each loop in the cusp torus sends any triangle in its cusp triangulation to another triangle of the same size, pointing in the same direction. We describe an efficient method to check this.

Definition 6.20
$\mathring{M}$ has a topological ideal triangulation and T is a cusp torus. $[\alpha] \in \pi_1(T)$. We define a complex number $H(\alpha)$ in the following way.

Give α an orientation and T a cusp triangulation. By a homotopic modification, α can be made to travel *monotonically*, i.e., it cuts out exactly one vertex in each triangle it traverses.

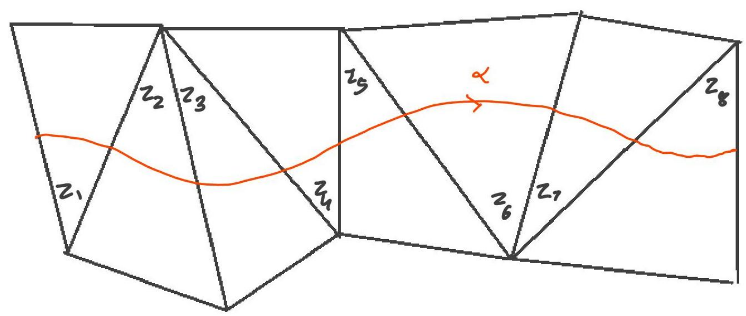


This is because each triangle in a cusp triangulation of T is simply-connected, so that any two paths that share endpoints on the edges of a triangle are homotopic.

Now, we let the edge invariants of the corners cut off by α be $z_1, z_2, \ldots, z_n$. To each corner, we associate a number $\varepsilon_i = \pm 1$ as follows. If the i^{th} corner is to the left of α, then $\varepsilon_i = +1$. Else, if the i^{th} corner lies to the right of α then $\varepsilon_i = -1$.

Finally, we set

$$H(\alpha) = \prod_{i=1}^{n} z_i^{\varepsilon_i}$$



In the above example,

$$H(\alpha) = z_1^{-1} z_2 z_3 z_4^{-1} z_5 z_6^{-1} z_7^{-1} z_8$$

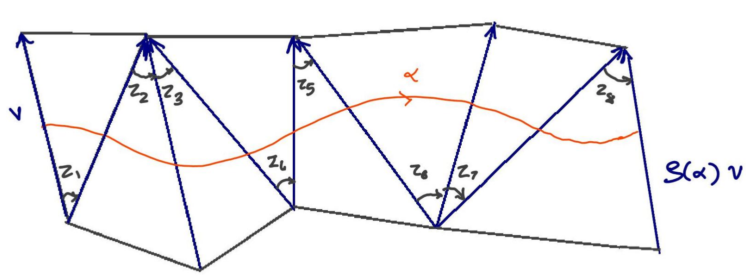
Clearly, $H(\alpha)$ is independent of the homotopy class of α.

Theorem 6.21 $T \to$ cusp torus in M. Say α and β are the loops generating $\pi_1(T)$. The structure on $\mathring{M}$ is complete if and only if

$$H(\alpha) = H(\beta) = 1 \quad \text{(completeness equations)}$$

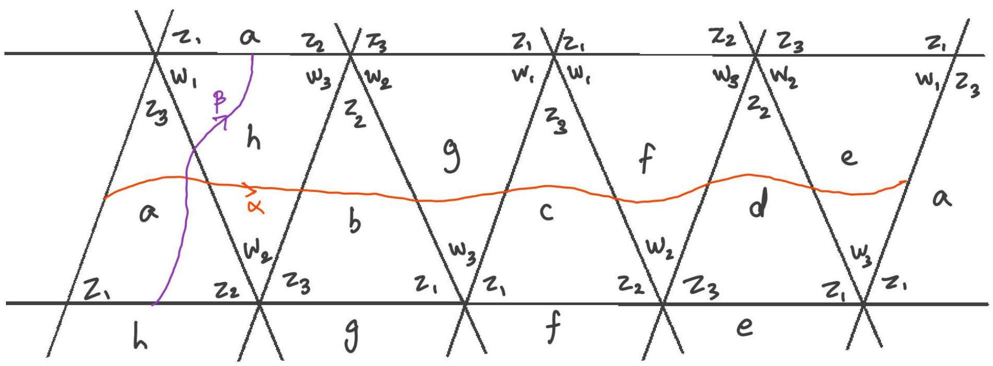
for all such cusp tori.

Proof. We must only check that the induced structure on each such T is Euclidean (Theorem 6.18). For this, it suffices to show that the holonomies $S(\alpha)$ and $S(\beta)$ are Euclidean isometries. This will hold only if $S(\alpha)$ and $S(\beta)$ do not have a scaling factor. Let T_1 be a triangle meeting α, and suppose α meets a side e_1 of T_1. Define $\mathbf{v}$ to be the vector with the same length and direction of e_1, oriented such that α and $\mathbf{v}$ obey the right-hand rule. Orient all the other edges that α meets in a similar fashion, as shown in the example below.



We rotate $\mathbf{v}$ about a vertex of T_1 and scale it such that its length is equal to the next edge e_2 meeting α. Just as in Theorem 6.14, rotating $\mathbf{v}$ anticlockwise is equivalent to multiplying $\mathbf{v}$ by z_1, and rotating clockwise is equivalent to dividing by z_1. The image of $\mathbf{v}$ after performing these rotation and coming back to T_1 will be parallel to $\mathbf{v}$, and its length will be $H(\alpha)$. Thus, for $S(\alpha)$ to not scale v, $H(\alpha) = 1$. The same holds for β □

Example 6.22 The figure-8 knot



Notice that

$$H(\alpha) = z_3 w_2^{-1} z_2 w_3^{-1} z_3 w_2^{-1} z_2 w_3^{-1} = \left(\frac{z_2 z_3}{w_2 w_3}\right)^2$$

$$H(\beta) = z_2^{-1} w_1 = \frac{w_1}{z_2}$$

From Lemma 6.13, these can be written in terms of w and z as

$$H(\alpha) = \left(\frac{1}{1-z} \cdot \frac{z-1}{z} \cdot \frac{1-w}{1} \cdot \frac{w}{w-1}\right)^2 = \left(\frac{w}{z}\right)^2$$

$$H(\beta) = w(1-z)$$

For completeness, $H(\alpha) = H(\beta) = 1$.
Then, $z = w = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\pi/3}$
Note that $z^3 = e^{i\pi} = -1$, which is consistent with the equations in Example 6.15. Thus, the figure-8 knot complement admits a complete hyperbolic structure made up of two ideal tetrahedra.

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