

Bell State Measurements in Quantum Information Theory and Quantum Foundations

A Thesis

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BS-MS Dual Degree Programme

by

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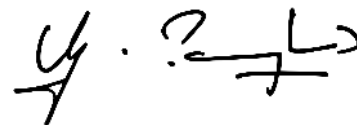
April, 2022

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Certificate

This is to certify that this dissertation entitled Bell State Measurements in Quantum Information Theory and Quantum Foundations towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Abhishek Jamunkar at Defence Institute of Advanced Technology, Pune, under the supervision of Dr. G Raghavan, Professor, School of Quantum Technology, DIAT, Pune, during the academic year 2021-2022.



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This thesis is dedicated to my parents who have always supported me in pursuing what I want.

Declaration

I hereby declare that the matter embodied in the report entitled Bell State Measurements in Quantum Information Theory and Quantum Foundations are the results of the work carried out by me at the School of Quantum Technology, DIAT, Pune, under the supervision of Dr. G Raghavan and the same has not been submitted elsewhere for any other degree.

A handwritten signature in black ink, consisting of a stylized 'A' and 'J' followed by a long horizontal stroke.

Abhishek Jamunkar

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Abstract

A Bell State Measurement is a two-photon measurement that projects the two-photon state represented in a Bell basis into one of the four Bell states. Besides lying at the heart of most protocols in Quantum Information Theory, Bell measurements are also helpful in deepening our understanding of Quantum Theory itself. In the present work, a detailed theoretical and experimental study of Bell State Measurements is undertaken. From a practical side, Bell state measurements are central for the realization of measurement device-independent and device-independent quantum key distribution. Further, One of the earliest foundational questions concerning quantum mechanics is why the quantum state is defined over a complex scalar field and not a real one. Is it possible to rule out 'real quantum mechanics?' experimentally. Recent work suggests that a Bell State Measurement could provide an answer to this question provisional to certain assumptions. In the later part of the thesis, this question is taken up for investigation. Compared to an earlier work which requires the use of two entangled photon sources, we provide an alternate proof that requires only a single entangled photon source but involves a path-polarization hyper-entangled state.

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Chapter 1

Introduction

Quantum Theory is full of counter-intuitive features that are classically impossible to reconcile. Yet, it is an extremely successful theory that has been tested to the highest levels of precision over the widest length scales. Quantum mechanics helps explain superconductors, semiconductors, the origin of magnetism, white dwarfs, and so forth. However, even after more than a century of successful applications, deeply troubling questions about the very foundations of quantum mechanics remain to be addressed satisfactorily. It is possible to create and manipulate single atoms, single photons, and single molecules with rapid technological advancements. This has opened the flood gates for the realization of quantum technologies such as quantum computers, quantum communication, and quantum sensing. These developments are hailed as the second quantum revolution. Coherent superposition, entanglement, and contextuality are distinct features of quantum states and may be viewed as non-classical resources. These resources enable applications that are otherwise impossible to perform using classical resources alone. Before presenting the material of the present dissertation, a brief introduction to quantum mechanics is provided below.

The quantum state

The pure quantum state is a normalized state vector of a complex, normed, inner product space called the Hilbert space. The temporal evolution of the quantum state is unitary in character and it is the solution of a linear partial differential equation called the Schrodinger

equation.

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 |\psi\rangle}{\partial x^2} + V|\psi\rangle$$

Given the linearity of the above equation any arbitrary superposition of the solutions of the Schrodinger equation is also a solution:

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle \quad (1.1)$$

The coefficients are in general complex. An immediate consequence of linearity is that there exist no unitary operator such that for any arbitrary state $|\psi\rangle$:

$$\hat{U}|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle \quad (1.2)$$

Where $|0\rangle$ is an initial state, this implies that perfect copies of arbitrary quantum states cannot be made. This is called the no-cloning theorem[19]. Pure quantum states belonging to an N-dimensional Hilbert space can be expressed in terms of a set of N linearly independent basis vectors. Connection with the world of phenomena is based on the premise that experimentally observable quantities should yield real numbers as measurement outcomes. Based on this belief, operators related to observable quantities are, of necessity, self-adjoint operators (Hermitian operators). When a pure state is expressed in the eigenbasis of an observable, rank one projective measurements are guaranteed to reduce the state to one of the eigenstates in an intrinsically random manner. Classical randomness is, however, the outcome of ignorance and is called epistemic randomness, whereas the former is called ontic randomness. When a single photon is sent through a balanced beam splitter, it is transmitted or reflected in an intrinsically random manner. This random "collapse" of the wave function is the basis on which a quantum random number generator is built. Such randomness of measured outcomes is also the basis upon which quantum key distribution is done. Therefore, coherent superposition is viewed as a quantum resource. When there is uncertainty regarding which pure state we are dealing with or states which are a mixture of pure states weighted from a classical distribution, the general mixed state is defined by a density operator and is represented by a density matrix[9]. The density matrix has a unit trace, and all its eigenvalues are positive.

$$\rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i|$$

Quantum Mechanics is an ensemble theory, meaning it gives only statistically verifiable predictions. So, in theory, it is impossible to know the exact state of a single particle by doing a single measurement. But if we have an ensemble of identically prepared states, then it is possible to reconstruct the density matrix that represents the ensemble by doing projective measurements. The expectation value of an observable is given by the Born formula:

$$\langle \hat{A} \rangle = \text{Tr}(\hat{\rho}\hat{A})$$

The experimental reconstruction of an unknown quantum state through a series of projective measurements is called Quantum State Tomography(QST)[1], which is an important means of authentication of the Quantum State emitted by a source. **In Chapter 2**, we'll look at different methods of such reconstruction. The Python codes that I have written for the reconstruction are supplied in the Appendix. The standard procedure for constructing the state of Composite systems made of two or more subsystems is to enlarge the space through the tensor product structure. If a bipartite state $|\Psi\rangle \in H_a \otimes H_b$ of a composite system can be written as $|\Psi_a\rangle \otimes |\Psi_b\rangle$ then it is called a product state. States that be written as convex combination of product states such as $\sum p_{ij}|\Psi_i\rangle \otimes |\Psi_j\rangle$ are called separable states. States that are not separable are called entangled states. Typical examples of maximally entangled states are the Bell states:

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) & |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}$$

Entangled states exhibit non-local correlations, which can be exploited as a resource. When an arbitrary bipartite state is expressed as a linear combination of the Bell states, we say it is expressed in a Bell basis. The correlations between the subsystems over arbitrary distances in the absence of any signaling between them is the direct consequence of quantum entanglement. This was dubbed by Einstein as 'Spooky action at a distance'. Einstein posited that quantum mechanics was incomplete, and it would be completed by the addition of additional variables often referred to as hidden variables[8]. Einstein was also troubled by the intrinsic randomness of outcomes. This led him to define the notion of what constituted elements of reality. If the value of a physical property of an object could be obtained without, in any way, disturbing it, Einstein called it an element of reality. Classically, it is accepted

that the physical properties of objects have a definite value associated with them quite independent of whether we measure them or not. Such a point of view is called realism. Further, if it is held that local measurements on objects that are in space-like separated regions cannot influence measurement outcomes of each other, this belief is called locality. Theories that are realistic and local are called local realistic theories. Quantum mechanics is not consistent with realism and is definitely non-local. But is there any way through which we can experimentally distinguish between the predictions of any arbitrary local-realistic theory and that of quantum mechanics? In 1964, John Bell came up with inequality, famously called a Bell inequality, that proves that no local hidden-variable theory can reproduce all the correlations predicted by Quantum Mechanics[2]. Subsequently, a family of such inequalities, now called Bell-like inequalities, have been proposed that try to experimentally distinguish between classical theories and Quantum Mechanics[6]. **In Chapter 3**, we'll take a look at the derivation of this inequality, see how Quantum Mechanics predicts a violation, and then perform the experiments on maximally entangled bipartite states to demonstrate the violation of the CHSH inequality, which is a version of Bell's inequalities.

There are many ways to create a Bell state; one method is to project a bipartite state in a Bell state basis. Such measurements are called Bell state measurements[15]. The underlying effect behind the Bell State Measurement is called the Hong-Ou-Mandel effect[11]; it is a two-photon interference effect that has no classical explanation.**In Chapter 4**, we'll first look at the theory of Bell state measurements and then demonstrate the experimental implementation of the same. We shall also analyze the effect of various parameters on the Hong-Ou-Mandel effect. The Hong-Ou-Mandel deep is itself a good indicator of the indistinguishability of the photons. Such a study is important because of the role it plays in the Quantum Information protocols such as Teleportation and Entanglement swapping. The Entanglement swapping, in itself, is important in the present work because it plays a critical role in establishing whether 'Real Quantum Mechanics' can be ruled out. **In Chapter 5** we'll take a look at this foundational question and see how the protocol can be useful in addressing this question[17]. We'll also look at something known as Single Photon Entangled States(SPE)[14] and look at a variant of the protocol based on SPE states and ask ourselves whether any additional insight can be provided by this setup. **In Chapter 6**, We'll see how SPE states can be used to create a Self-Certifying QRNG. In the end, **Chapter 7** gives conclusions about all the different aspects covered in the thesis, along with a scope of improvements and further research.

Chapter 2

Quantum State Tomography

Quantum State Tomography[1] is a process by which an unknown quantum state is reconstructed by performing measurements on an ensemble of identically prepared states. The goal is to get as close as possible to correctly describe an ensemble by constructing a density matrix that close approximates the actual density matrix in the sense of fidelity/trace distance.

2.1 Experimental Setup

To perform QST, the first thing we need is a source. So let's take a brief look at the source used for single and two qubits QST.

2.1.1 Source

The photon source needed for performing QST should be able to, at least theoretically, generate an ensemble of identical quantum states. For single qubit Quantum State Tomography, we used a Single Photon source that creates photons of identical polarization. In order to certify that the source was indeed a single-photon source, we performed the Hanbury Brown and Twiss(HBT)[12] experiment on the source. The experimental setup of HBT is rather trivial, with the Single-photon source connected to a beam splitter whose two outputs

are connected to detectors. If the source is really a single-photon source, we should observe that at a given instant, only one of the detectors clicks. i.e., there are no coincidence counts between the two detectors. The setup is given in the figure below: The figure below shows

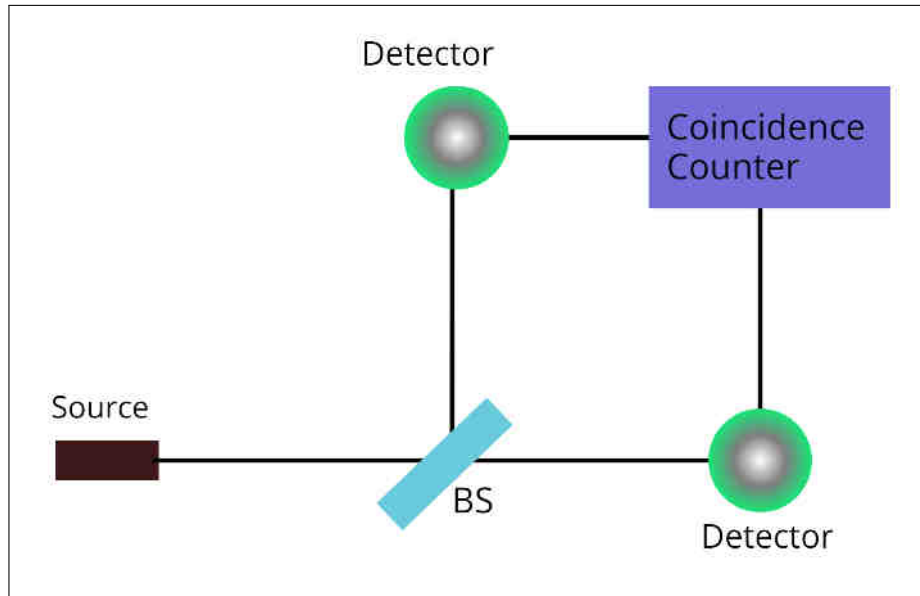


Figure 2.1: Hanbury Brown and Twiss experiment setup

the single counts and coincidence counts observed for the source. It shows that coincidence counts are almost zero compared to single counts. For Two-photon Quantum State Tomog-



Figure 2.2: Results of HBT experiment

raphy, we used a Spontaneous Parametric Down-Conversion[10] (SPDC) source. SPDC is

a process of down-converting a photon of higher frequency into two photons of lower frequencies such that energy and momentum are conserved. In order to achieve that, we used a non-linear type-1 BBO crystal. The BBO crystal was pumped by a 405nm diode laser corrected for beam circularity. The BBO crystal was phase-matched by cutting the face of the crystal at a specific angle with respect to the optic axis. The phase matching was done for degenerate down-conversion such that two down-converted 810 nm photons of polarization orthogonal to that of the pump are emitted at an opening angle of 3° and performed QST on them. The SPDC process is an extremely weak one, and roughly a pair of down-converted photons are produced for about 10^8 pump photons. While I have made a type-I down-converter, the tomography and the analysis of the entangled photons were carried out on a ready-made system. The setup used for these measurements is shown in the figure below:

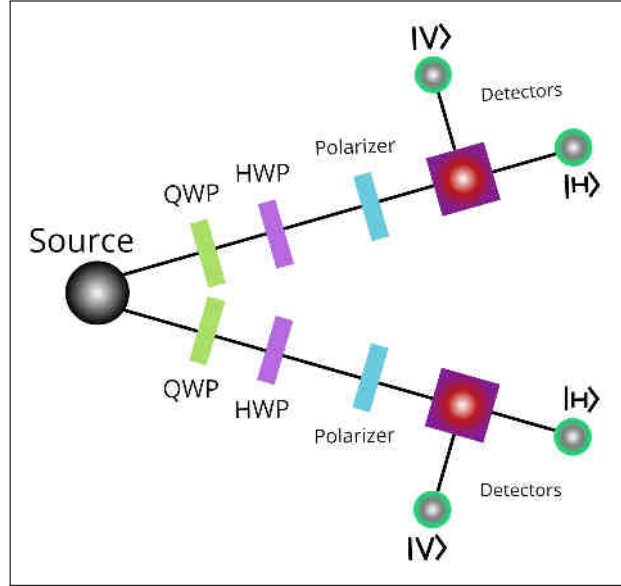


Figure 2.3: Two Photon Quantum State Tomography

Detectors and Coincidence counters

Detectors and coincidence counters are needed to detect the presence of a photon. The detectors used were roughly 30% efficient and had a dark count of around 300 photons per second. Coincidence counters become meaningful only when we consider two or more photon state tomography. For an n -photon state tomography, we need an n -fold coincidence counter

to ensure that they all represent a single quantum state. The coincidence counter used for two photons QST had a coincidence window of 20ns.

Projective measurements and Change of Basis

The general methodology behind tomography is to project the incoming unknown state into different basis states, measure the counts on detectors for each basis state, calculate the probability of collapse associated with each state, and then reconstruct the state. To do that, we need to find such basis states that would reveal maximum information about the system at a minimum number of experiments. Moreover, we need to be able to project our unknown quantum states into those basis states experimentally. In order to get maximum information, Our basis states should be such that each choice of basis would reveal a different aspect of the unknown state. One possible choice is;

- The horizontal and vertical basis.

$$|H\rangle, |V\rangle$$

- The diagonal and anti-diagonal basis.

$$|+\rangle, |-\rangle$$

- The left and right circular basis.

$$|L\rangle, |R\rangle$$

We will use these three basis states to reconstruct our unknown quantum state. Each of these states is an eigenvector of the Pauli matrices, and each eigenvector in one basis is an equal superposition of the two states in the other basis. Such basis are called Mutually Unbiased Basis(MUB). See the conversion relations below;

$$|+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle) \quad |R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$$

$$|H\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle) \quad |V\rangle = \frac{1}{\sqrt{2}}(i|L\rangle - i|R\rangle)$$

Consider a combination of a quarter-wave plate, half-wave plate, and a polarizer kept between a source of an unknown quantum state and the detector. Let us assume that the polarizer is kept in the horizontal direction, and initially, the fast axes of both the waveplates are at 0° with the horizontal axis. Given that we do not know the polarization of the incoming photon state, instead of acting the wave plate on the unknown polarization, we will act the inverse of it on the horizontal polarizer. This should be equivalent to acting the wave plate on the unknown state as one is allowed to either act the operator on a state or act the inverse of it on the basis it is being projected onto. See the table below to see how the initial horizontal basis can be changed to the rest of the five basis states by simply changing the angle between the fast axes of wave plates and the horizontal. Notice that if we keep the fast axes at 0 or 90 degrees to the horizontal, there will be no change in the basis because there will be no relative phase shift in the components of the incoming photon.

QWP	HWP	Initial basis	Final basis
$0^\circ/90^\circ$	$0^\circ/90^\circ$	$ H\rangle\langle H $	$ H\rangle\langle H $
0°	45°	$ H\rangle\langle H $	$ V\rangle\langle V $
45°	0°	$ H\rangle\langle H $	$ L\rangle\langle L $
45°	45°	$ H\rangle\langle H $	$ R\rangle\langle R $
0°	22.5°	$ H\rangle\langle H $	$ +\rangle\langle + $
0°	67.5°	$ H\rangle\langle H $	$ -\rangle\langle - $

Reconstructing the Quantum State of a Polarization Qubit

When a photon is projected in a basis, the photon wave function collapses into one of the basis states depending on the probability associated with each basis state. After arranging the wave plates to set the desired basis, we will note the counts of photons detected by the detector. For a single polarization qubit tomography, we will look at the counts at the detector; but for multiple photon state tomography, we will look at the coincidence count of detectors to ensure that all the photons belong to the same wave function. There are two ways to perform the experiment. The first way is by using a quarter-wave plate, Half wave

plate, a linear polarizer, and a detector. The other is by using a polarizing beam splitter and two detectors. Refer to the diagrams below;

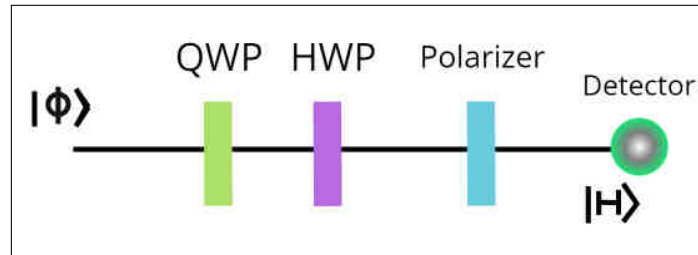


Figure 2.4: One detector per photon

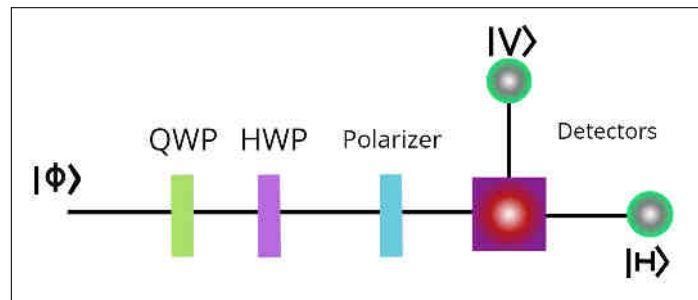


Figure 2.5: Two detectors per photon

2.2 Methods of Reconstruction

There are several methods to reconstruct the density matrix from the data collected. Two methods are mentioned here, along with the Python code in the Appendix.

Linear Inversion

Linear inversion[1] follows a relatively straightforward logic. For the observed counts of photons in a given basis, we calculate the probability of the wavefunction $|\Psi\rangle$ to collapse into the basis states and then follow rigorous maths to express each element of the density matrix into the probabilities that we obtained experimentally.

Stokes parameters The density matrix of a single polarization qubit can be uniquely identified on a Poincare sphere by providing its 3 stokes parameters[1]. The stokes parameters are the coefficients associated with the pauli matrices when the density matrix is represented as their sum. Mathematically, it looks something like this;

$$\hat{\rho} = \frac{1}{2} \sum_{i=0}^3 S_i \hat{\sigma}_i \quad S_i = Tr\{\hat{\rho} \hat{\sigma}_i\}$$

For a multiple qubit system, the density matrix and the stokes parameters will be given by;

$$\hat{\rho} = \frac{1}{2^n} \sum_{i_1, i_2, \dots, i_n=0}^3 S_{i_1, i_2, \dots, i_n} \hat{\sigma}_{i_1} \otimes \hat{\sigma}_{i_2} \otimes \dots \otimes \hat{\sigma}_{i_n} \quad S_i = Tr\{\hat{\rho} \hat{\sigma}_{i_1} \otimes \hat{\sigma}_{i_2} \otimes \dots \otimes \hat{\sigma}_{i_n}\}$$

In terms of probabilities, the stokes parameters will be given as;

$$S_{i_1, i_2, \dots, i_n} = (P_{\Psi_1} - P_{\Psi_1^\perp}) \otimes (P_{\Psi_2} - P_{\Psi_2^\perp}) \otimes \dots \otimes (P_{\Psi_n} - P_{\Psi_n^\perp})$$

After performing the measurements, we can calculate all the Stokes parameters from the above relations and then construct the density matrix. However, there is a catch here, the density matrix constructed out of this method may not be the correct one. In fact, it may not even be real. It turns out the restriction on stokes parameters are not enough to always construct the density matrix because of experimental errors in measuring the Stokes parameters. Also, the data that we have obtained contains some errors because of inefficiencies of detectors, improper alignment of the waveplates, and other experimental errors. So it is pretty likely that the stokes parameters will also not be the correct ones. Before seeing how to tackle this issue, let us take a moment to look at the basic properties of a physically real density matrix. These properties are;

1. The trace of the density matrix should be unity. This is the normalization condition.
i.e. $Tr\{\rho\} = 1$
2. The matrix should be hermitian. i.e $\rho = \rho^\dagger$
3. The projection of a density matrix on any state ψ should be greater than or equal to 0. i.e. $\langle \psi | \rho | \psi \rangle \geq 0$. It also means that all the eigenvalues of ρ should be positive.

Maximum Likelihood Estimation

We have already seen why we cannot rely on linear inversion to reconstruct a correct density matrix if we do not have error-free data, which is almost always the case. So we need a better method, a practical method that does not guarantee to give the exact density matrix but only maximizes the likelihood of getting one[1]. We have already seen the basic properties all the physically real density matrices should satisfy. Now we need to use them to exclude all the choices of incorrect density matrices. The general flow of the MLE method is as follows;

1. Using the properties, construct a density matrix with n^2 parameters (where n is the number of qubits).
2. Calculate the likelihood that this matrix will generate the data that we have collected.
3. Maximize this likelihood function and refer to the parameters at which it attains the maximum.

The density matrix corresponding to these parameters will be the matrix with the highest probability of correctly representing the unknown quantum state.

Consider a matrix T with parameters $t_1, t_2 \dots t_{n^2}$. Let $T^\dagger$ be its conjugate transpose matrix. Now, we know that a matrix of the form $T^\dagger T$ will be a hermitian matrix. Now let's divide this matrix by its $\text{Trace}\{T^\dagger T\}$. The final matrix will have the form as;

$$\rho = \frac{1}{\text{Tr}\{T^\dagger T\}} T^\dagger T$$

This matrix is hermitian and has a unit trace. So, two of the three properties are satisfied. The third property is that its projection on any state should be greater than or equal to 0. This property is also satisfied, which becomes evident on acting the matrix on a state ψ .

$$\langle \psi | \rho | \psi \rangle = \langle \psi | \frac{1}{\text{Tr}\{T^\dagger T\}} T^\dagger T | \psi \rangle = \frac{\langle \psi' | \psi' \rangle}{\text{Tr}\{T^\dagger T\}} \geq 0$$

Now that we have a parameterized matrix that satisfies all the properties of a physical density matrix let us construct a likelihood function. Let us assume that the errors in the

data follow a Gaussian distribution. Let n_μ be the data of the μ^{th} measurement out of all the κ measurements. Here, we will approximate the standard deviation $\sigma = \sqrt{\bar{n}}$. The counting statistics for each of them will be given as follows;

$$P(n) = \frac{1}{\sqrt{2\pi\bar{n}}} e^{-\frac{(n-\bar{n})^2}{2\bar{n}}}$$

Where $\bar{n}$ is the average number of counts given by;

$$\bar{n}_\mu = N \langle \psi_\mu | \rho | \psi_\mu \rangle$$

Here, N is the total number of photons measured. In theory, N is unknown to us. However, we can estimate it by summing the counts of all the orthogonal states of a basis. The probability that our parametrized density matrix will generate the data $n_1, n_2 \dots n_\kappa$ will be given by;

$$P(n_1, n_2 \dots n_\kappa | \rho) = \prod_{\mu=0}^{\kappa} \frac{1}{\sqrt{2\pi\bar{n}_\mu}} e^{-\frac{(n_\mu - \bar{n}_\mu)^2}{2\bar{n}_\mu}}$$

Now, this looks like a complicated function to be maximized. So instead of dealing with this, we will minimize the negative log of this function, which is equivalent to finding the maximum of the first function. Let that function be denoted by $F(\rho)$.

$$\begin{aligned} F(\rho(t_1, t_2 \dots t_n)) &= \sum_{\mu=0}^{\kappa} \left(\frac{(n_\mu - \bar{n}_\mu)^2}{2\bar{n}_\mu} + \frac{\log \bar{n}}{2} \right) \\ &= \sum_{\mu=0}^{\kappa} \left(\frac{(n_\mu - N \langle \psi_\mu | \rho(t_1, t_2 \dots t_n) | \psi_\mu \rangle)^2}{2N \langle \psi_\mu | \rho(t_1, t_2 \dots t_n) | \psi_\mu \rangle} + \frac{\log (N \langle \psi_\mu | \rho(t_1, t_2 \dots t_n) | \psi_\mu \rangle)}{2} \right) \end{aligned}$$

The last step is to numerically minimize this function and calculate the values of all the $t_{i's}$. The density matrix constructed out of these values will be the matrix with the highest probability of generating the data obtained. This was the Maximum Likelihood Estimation

method. The biggest possible source of error in MLE is the uncertainty in the exact value of N . Fortunately, with more data, this error could be significantly decreased. Another possible source of error is that depending on the method and function we use to find the minimum of the above function, sometimes we could find a local minimum instead of a global one, which we desire. If that happens, then we will end up with an incorrect density matrix. However, if we could prove that the above function has only a global minimum, then we could be more certain about the unknown state.

2.3 Experimental Reconstruction

There are two ways to set up the experiment. The first is by using a single detector, and the second is by using two detectors per qubit[1]. This creates a difference in the minimum number of measurement setups and the possible number of measurement data obtained from each setup. We need to perform at least 4^n projection measurement for a n qubit system for a single detector setup. For a double detector setup, we need to perform 3^n measurements, and we obtain two types of data from each measurement setup, so we need to collect 6^n different types of data. Python codes for linear inversion and MLE for single and double photon state reconstruction out of measurement counts are mentioned in the Appendix.

2.3.1 Single Photon State

Here, reconstructed density matrices for different reconstruction methods are given below.

Linear Inversion

1. Consider a known state $\rho = |H\rangle\langle H|$. The density matrix we be;

$$\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The experimentally obtained data was:

$$\begin{aligned}
H &= 9983 & V &= 24 \\
D &= 4924 & R &= 5323
\end{aligned}$$

Using the code for linear inversion, the matrix that we obtain is:

$$\rho = \begin{bmatrix} 0.99760 & -0.00794 - i0.03192 \\ -0.0079 + i0.03192 & 0.00239 \end{bmatrix}$$

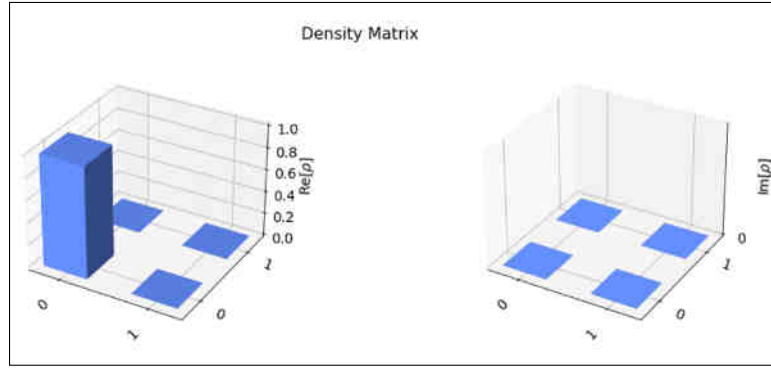


Figure 2.6: Reconstructed Density Matrix of a known Single Photon state

The eigenvalues of the matrix are;

$$\lambda_1 = 0.99868 + i1.60071 \times 10^{-20} \quad \lambda_2 = 0.00131 - i1.60071 \times 10^{-20}$$

2. Consider a state ψ which is unknown to us. The experimental data we have collected is;

$$\begin{aligned}
H &= 13867 & V &= 7299 \\
D &= 10093 & R &= 10449
\end{aligned}$$

The matrix reconstructed from the data is:

$$\rho = \begin{bmatrix} 0.65515 & -0.02315 + i0.00631 \\ -0.02315 - i0.00633 & 0.34484 \end{bmatrix}$$

The eigenvalues of the matrix are;

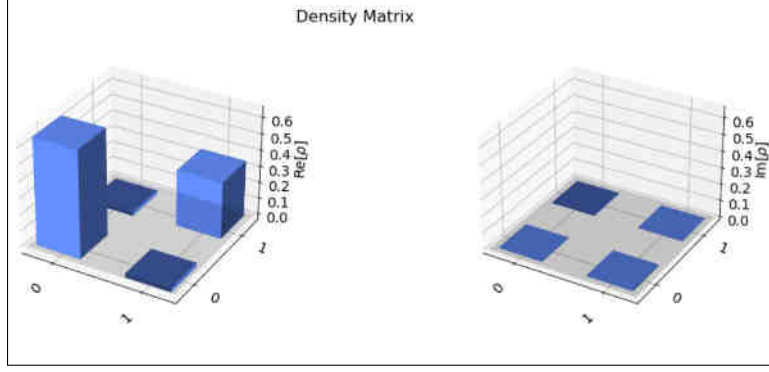


Figure 2.7: Reconstructed Density Matrix of an unknown state

$$\lambda_1 = 0.65699 + i3.84046 \times 10^{-20}$$

$$\lambda_2 = 0.34300 - i3.84046 \times 10^{-20}$$

The eigenvalues of the reconstructed matrices have a small imaginary part, which is small enough to ignore safely. The real part is greater than 0, which is what we desire. For a single qubit state tomography, the number of measurements taken is less, and hence the chance of getting an error in the final density matrix is also low. So, linear inversion proves effective for a single qubit tomography. Let us take a look at MLE.

Maximum Likelihood Estimation

1. The theoretical state is $\rho = |+\rangle\langle+|$. The density matrix would be;

$$\rho = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

The data obtained was:

$$H = 4894 \quad V = 5102$$

$$D = 9987 \quad R = 5323$$

The matrix reconstructed from the data is;

$$\rho = \begin{bmatrix} 0.48959 & 0.49883 - i0.03245 \\ 0.49883 + i0.03245 & 0.51040 + 0.j \end{bmatrix}$$

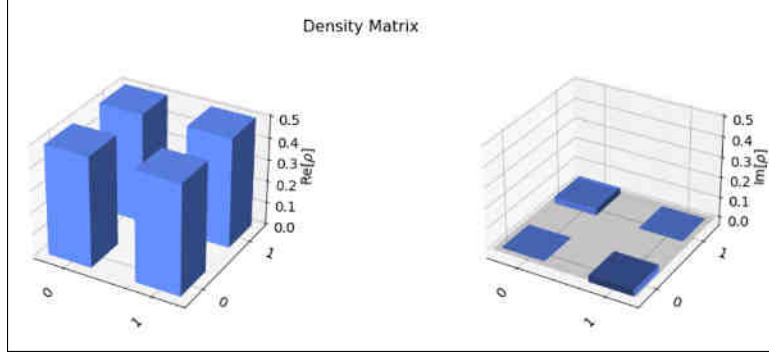


Figure 2.8: Reconstructed Density Matrix using Maximum Likelihood Estimation

The eigenvalues of the matrix are;

$$\lambda_1 = 2.94827 \times 10^{-10} - 1.7580210^{-18} \quad \lambda_2 = 1.00000 - i5.18086 \times 10^{-18}$$

2. Let us try MLE for our previous unknown state ψ . The experimental data is the same as before;

$$H = 13867 \quad V = 7299$$

$$D = 10093 \quad R = 10449$$

Again, using the code for MLE, the matrix that we obtain is;

$$\rho = \begin{bmatrix} 0.65516 & -0.02317 + i0.00635 \\ -0.02315 - i0.00635 & 0.34483 \end{bmatrix}$$

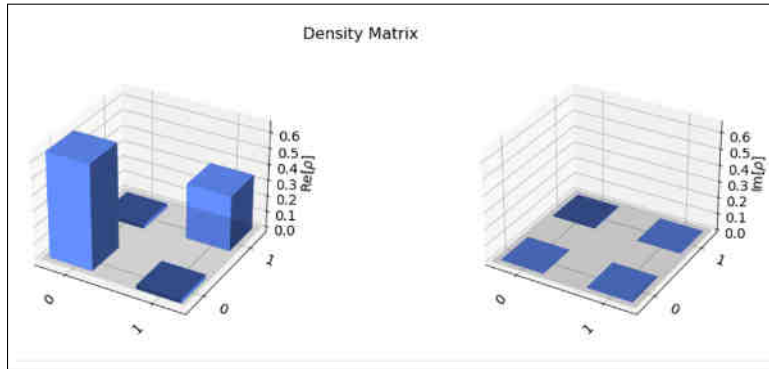


Figure 2.9: Reconstructed Density Matrix using MLE for an unknown state

The eigenvalues of the matrix are;

$$\lambda_1 = 0.65701 + i3.7904 \times 10^{-20} \quad \lambda_2 = 0.34298 - i3.79047 \times 10^{-20}$$

The matrix obtained by MLE and linear inversion differs at 5th decimal place. This is because the error in linear inversion is low due to the lesser number of measurements. When we apply these methods to a multiple qubit state, we see how linear inversion outputs illegal density matrices.

2.3.2 Two Photon State Tomography

Suppose we generate two photons by Spontaneous Parametric Down-conversion. We will use two detectors per photon. So, we will need to perform the measurement in at least nine different configurations. Moreover, we will get four different types of data belonging to each basis state in each configuration. In total, we will have 36 different types of data. Arranging it in a 9×4 matrix will look something like this:

$$Counts = \begin{bmatrix} 40230 & 150 & 220 & 39094 \\ 19970 & 20533 & 18542 & 20050 \\ 21988 & 18910 & 18003 & 21054 \\ 20556 & 19706 & 20010 & 18629 \\ 333 & 39918 & 38011 & 440 \\ 20475 & 19975 & 18931 & 19544 \\ 19900 & 19649 & 20761 & 19330 \\ 17423 & 21755 & 21207 & 18804 \\ 38717 & 401 & 483 & 39382 \end{bmatrix}$$

Linear Inversion

The experimental setup would look similar to the setup in Figure 2.3

The matrix reconstructed from the data is:

$$\rho = \begin{bmatrix} 0.50480 & -0.00341 - i0.01731 & 0.00403 + i0.0060 & -0.48949 + i0.01773 \\ -0.00341 + i0.01731 & 0.0018 & -0.00068 + i0.02478 & 0.00740 - i0.00133 \\ 0.00403 - i0.0060 & -0.00068 - i0.02478 & 0.00276 & -0.00938 + i0.02100 \\ -0.48949 - i0.01773 & 0.00740 + i0.00133 & -0.00938 - i0.02100 & 0.49055 \end{bmatrix}$$

The eigenvalues of the matrix are;

$$\lambda_1 = 0.98823867 + i4.22000232 \times 10^{-18}$$

$$\lambda_2 = -0.02884852 - i7.07573863 \times 10^{-18}$$

$$\lambda_3 = 0.01094133 + i5.60884514 \times 10^{-19}$$

$$\lambda_4 = 0.02966852 + i1.21064962 \times 10^{-18}$$

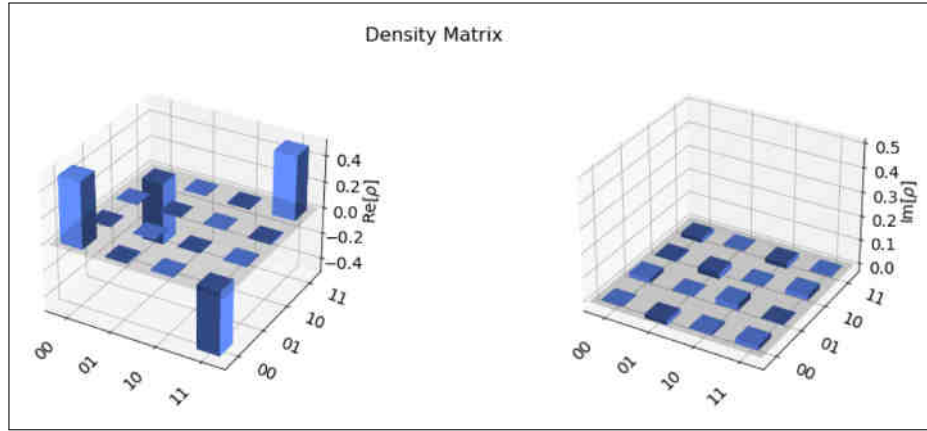


Figure 2.10: Two Photon QST using Linear Inversion

Clearly, we can see that one of the eigenvalues is not positive. This means our density matrix is not physical. This could have happened because of the error in the data we used to calculate the stokes parameters. To construct a legal matrix with the highest probability of generating the above data, we will MLE for the same data and see what happens.

Maximum Likelihood Estimation

The code for MLE for a two-qubit state is;

The matrix reconstructed from the data is;

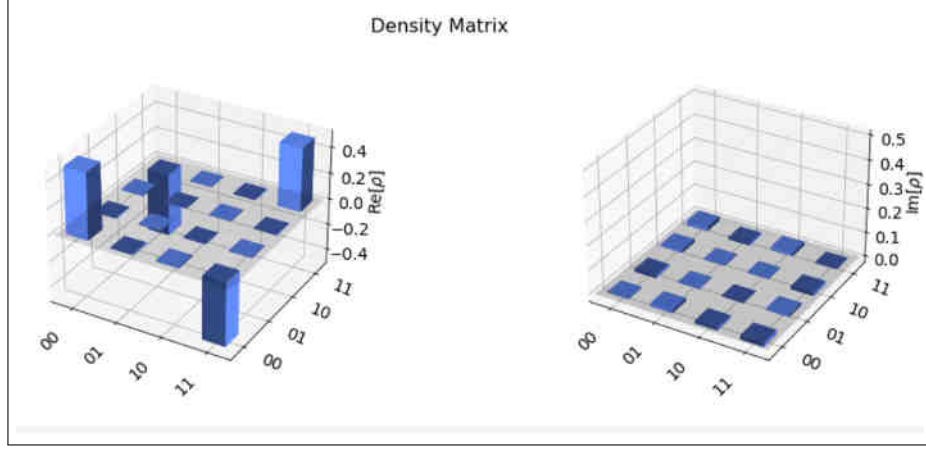


Figure 2.11: Two Photon QST using Maximum Likelihood Estimation

$$\rho = \begin{bmatrix} 0.50856 & -0.00605 + i0.01163 & 0.00752 - i0.01133 & -0.48941 - i0.01751 \\ -0.00605 - i0.01163 & 0.00219 & -0.00027 - i0.00167 & 0.00386 + i0.00806 \\ 0.00752 + i0.01133 & -0.00027 + i0.00167 & 0.00292 + 0.j & -0.00649 - i0.01405 \\ -0.48941 + i0.01751 & 0.00386 - i0.00806 & -0.00649 + i0.01405 & 0.48631 \end{bmatrix}$$

The eigenvalues of the matrix are;

$$\lambda_1 = 0.98797 + i1.91936 \times 10^{-18}$$

$$\lambda_2 = 0.00946 - i2.36709 \times 10^{-17}$$

$$\lambda_3 = 0.00255 + i3.00823 \times 10^{-18}$$

$$\lambda_4 = 1.18885 \times 10^{-13} + i2.06949 \times 10^{-17}$$

All the eigenvalues have a positive real part and negligibly small complex part, which can be ignored. So the density matrix obtained by MLE is physical and has the highest probability of generating the data we have obtained.

Chapter 3

Falsification of Local Hidden-Variable Theories

When the phenomenon of entanglement was newly discovered, physicists all around the world had trouble accepting its implications. The phenomenon was so counter-intuitive that people did not think it existed at all. They came up with alternate explanations of the phenomena, which did not involve any need for non-local correlations. This class of theories is called Local Hidden-Variable theories, which try to reproduce the correlation possible between entangled particles[8]. The name Local hidden-variable comes from the fact that we assume Quantum Mechanics is not a complete description of reality, and hence there exists a local variable, which is hidden from us, that governs the behavior of the two entangled particles. If we had complete information about this 'hidden variable,' we could explain the correlations shown by the entangled particles. But in a paper published by John Bell in 1964, inequality was proposed, which gave an upper bound on the maximum correlations possible between two spatially separated particles based on any local hidden-variable theory[2]. It was also shown that Quantum Mechanics violates this inequality, meaning, Quantum correlations exceed the maximum correlations possible by any classical theory. A few years later, a CHSH inequality was proposed, which was more robust with fewer assumptions and an experimentally friendly margin between the two theories[6]. First, we'll look at the derivation of the inequality, and then we'll see how Quantum Mechanics predicts its violation. After that, we'll perform an experimental verification using an entangled photon pair generated using the Spontaneous Parametric Down-Conversion Process.

3.1 Derivation

Adopting the derivation from[3]. Consider two spatially separated observers, Alice and Bob. They are provided with a pair of particles that are classically correlated. Let the correlations depend on the value of λ , which is locally present with both of the particles but is hidden from us. Let the value of λ present in each pair vary as per the probability distribution $\rho(\lambda)$. Let Alice's and Bob's outcome be denoted by $x, y \in \{1, -1\}$, which follows a probability distribution of $A(a, \lambda)$ & $B(b, \lambda)$ where a & b denote the experimental settings of their measurement devices which they can change as per our choice.

The expectation value of xy for choices of a & b will be given as:

$$E(a, b) = \int_{\lambda} A(a, \lambda) B(b, \lambda) \rho(\lambda) d\lambda$$

$$\begin{aligned} E(a_1, b_1) - E(a_1, b_2) &= \int_{\lambda} A(a_1, \lambda) B(b_1, \lambda) \rho(\lambda) d\lambda - \int_{\lambda} A(a_1, \lambda) B(b_2, \lambda) \rho(\lambda) d\lambda \\ &= \int_{\lambda} \left[A(a_1, \lambda) B(b_1, \lambda) (1 \pm A(a_2, \lambda) B(b_2, \lambda)) \right. \\ &\quad \left. - A(a_1, \lambda) B(b_2, \lambda) (1 \pm A(a_2, \lambda) B(b_1, \lambda)) \right] \rho(\lambda) d\lambda \end{aligned} \quad (3.1)$$

Taking the absolute values of both side;

$$\begin{aligned} |E(a_1, b_1) - E(a_1, b_2)| &= \int_{\lambda} \left| A(a_1, \lambda) B(b_1, \lambda) (1 \pm A(a_2, \lambda) B(b_2, \lambda)) \right. \\ &\quad \left. - A(a_1, \lambda) B(b_2, \lambda) (1 \pm A(a_2, \lambda) B(b_1, \lambda)) \right| \rho(\lambda) d\lambda \end{aligned} \quad (3.2)$$

Using the inequality:

$$|a - b| \leq |a| + |b|$$

$$\begin{aligned} |E(a_1, b_1) - E(a_1, b_2)| &\leq \int_{\lambda} \left\{ \left| A(a_1, \lambda) B(b_1, \lambda) \right| \left| (1 \pm A(a_2, \lambda) B(b_2, \lambda)) \right| \right. \\ &\quad \left. + \left| A(a_1, \lambda) B(b_2, \lambda) \right| \left| (1 \pm A(a_2, \lambda) B(b_1, \lambda)) \right| \right\} \rho(\lambda) d\lambda \end{aligned} \quad (3.3)$$

Since $|A(a, \lambda)B(b, \lambda)| \leq 1$, we can rewrite equation 6.3 as:

$$|E(a_1, b_1) - E(a_1, b_2)| \leq \int \left\{ (1 \pm A(a_2, \lambda)B(b_2, \lambda)) + (1 \pm A(a_2, \lambda)B(b_1, \lambda)) \right\} \rho(\lambda) d\lambda \quad (3.4)$$

Rewriting this in terms of expectation values:

$$|E(a_1, b_1) - E(a_1, b_2)| \leq 2 \pm (E(a_2, b_2) - E(a_2, b_1))$$

The terms on RHS is less than or equal to both $2 + (E(a_2, b_2) + E(a_2, b_1))$ & $2 - (E(a_2, b_2) - E(a_2, b_1))$. Taking this into consideration and using inequality $|a + b| \leq |a| + |b|$:

$$S = |E(a_1, b_1) + E(a_2, b_1) + E(a_2, b_2) - E(a_1, b_2)| \leq 2 \quad (3.5)$$

Hence, we have obtained an upper bound on the maximum possible correlation between the observables of Alice's and Bob's measurements, given that there exists a local hidden variable that dictates their correlations. Notice that in the derivation, we did not use Quantum Theory at all. So let us use Quantum Mechanics and see what we can say about the maximum correlations possible between two quantum particles.

3.2 Violation predicted by QM

In the above scenario, the two observables were taken to be any general dichotomic observables. We can take a special case of an entangled photon pair and measure its polarization, which will be our dichotomic observable. Let us calculate the joint expectation value of Alice's and Bob's observables for their ϕ_a & ϕ_b orientations of their polarizers. The figure below depicts the experimental setup.

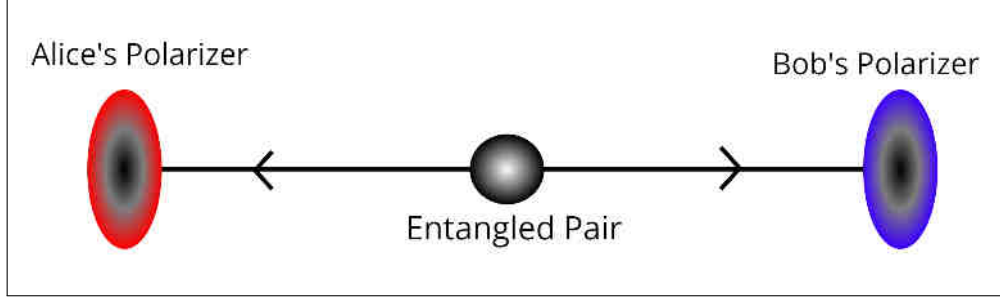


Figure 3.1: Experimental Setup for CHSH violation

Let the source in the middle emit an entangled pair described by:

$$|\psi\rangle = \frac{|HH\rangle + |VV\rangle}{\sqrt{2}}$$

Alice and Bob are allowed to project the polarization of their half of the pair along any axis on the Bloch sphere. Suppose they decide to keep the projection only on the x-z plane. They can change the orientation of their polarizer, which is now dependent on only one angle. Let the angle of orientation for them be α & β . The polarization projection operators of Alice and Bob will be:

$$\Pi_a = |n_a^+\rangle\langle n_a^+| - |n_a^-\rangle\langle n_a^-| \quad \& \quad \Pi_b = |n_b^+\rangle\langle n_b^+| - |n_b^-\rangle\langle n_b^-|$$

Where n^+ & n^- are the two eigenvectors of the polarization observables of Alice and Bob. Rewriting the operators in terms of pauli vector σ and orientation vector n_a & n_b :

$$\Pi_a = \vec{\sigma} \cdot \vec{a} \qquad \Pi_b = \vec{\sigma} \cdot \vec{b}$$

$$\vec{a} = \cos(\alpha)\hat{x} + \sin(\alpha)\hat{z}$$

$$\vec{b} = \cos(\beta)\hat{x} + \sin(\beta)\hat{z}$$

The joint operator acting on the entire state $|\Psi\rangle$ will be given by:

$$\Pi = \Pi_a \otimes \Pi_b$$

The expectation value of the two observables will be given by:

$$\begin{aligned}
E(a, b) &= \langle \Psi | \Pi_a \otimes \Pi_b | \Psi \rangle \\
&= \frac{\langle HH | \Pi_a \otimes \Pi_b | HH \rangle}{2} + \frac{\langle HH | \Pi_a \otimes \Pi_b | VV \rangle}{2} \\
&\quad + \frac{\langle VV | \Pi_a \otimes \Pi_b | HH \rangle}{2} + \frac{\langle VV | \Pi_a \otimes \Pi_b | VV \rangle}{2}
\end{aligned} \tag{3.6}$$

Substituting the values of $\vec{a}$ & $\vec{b}$:

$$\begin{aligned}
E(a, b) &= \frac{1}{2} \left[\langle HH | (\cos(\alpha)\vec{\sigma}_x + \sin(\alpha)\vec{\sigma}_z) \otimes (\cos(\beta)\vec{\sigma}_x + \sin(\beta)\vec{\sigma}_z) | HH \rangle \right. \\
&\quad + \langle HH | (\cos(\alpha)\vec{\sigma}_x + \sin(\alpha)\vec{\sigma}_z) \otimes (\cos(\beta)\vec{\sigma}_x + \sin(\beta)\vec{\sigma}_z) | VV \rangle \\
&\quad + \langle VV | (\cos(\alpha)\vec{\sigma}_x + \sin(\alpha)\vec{\sigma}_z) \otimes (\cos(\beta)\vec{\sigma}_x + \sin(\beta)\vec{\sigma}_z) | HH \rangle \\
&\quad \left. + \langle VV | (\cos(\alpha)\vec{\sigma}_x + \sin(\alpha)\vec{\sigma}_z) \otimes (\cos(\beta)\vec{\sigma}_x + \sin(\beta)\vec{\sigma}_z) | VV \rangle \right]
\end{aligned} \tag{3.7}$$

On simplifying the equation further, we get:

$$\begin{aligned}
E(a, b) &= \frac{1}{2} \left[\cos(\alpha)\cos(\beta)\langle HH | VV \rangle + \cos(\alpha)\sin(\beta)\langle HH | VH \rangle \right. \\
&\quad + \sin(\alpha)\cos(\beta)\langle HH | HV \rangle + \sin(\alpha)\sin(\beta)\langle HH | HH \rangle \\
&\quad + \cos(\alpha)\cos(\beta)\langle VV | HH \rangle - \cos(\alpha)\sin(\beta)\langle VV | HV \rangle \\
&\quad - \sin(\alpha)\cos(\beta)\langle VV | VH \rangle + \sin(\alpha)\sin(\beta)\langle VV | VV \rangle \\
&\quad + \cos(\alpha)\cos(\beta)\langle HH | HH \rangle - \cos(\alpha)\sin(\beta)\langle HH | HV \rangle \\
&\quad - \sin(\alpha)\cos(\beta)\langle HH | VH \rangle + \sin(\alpha)\sin(\beta)\langle HH | VV \rangle \\
&\quad + \cos(\alpha)\cos(\beta)\langle VV | VV \rangle + \cos(\alpha)\sin(\beta)\langle VV | VH \rangle \\
&\quad \left. + \sin(\alpha)\cos(\beta)\langle VV | HV \rangle + \sin(\alpha)\sin(\beta)\langle VV | HH \rangle \right]
\end{aligned} \tag{3.8}$$

This simplifies to:

$$\begin{aligned}
E(a, b) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
&= \cos(\alpha - \beta)
\end{aligned} \tag{3.9}$$

Rewriting the inequality derived in equation 5.5 in terms of expectation values presented

above:

$$\begin{aligned}
S &= |E(a_1, b_1) + E(a_2, b_1) + E(a_2, b_2) - E(a_1, b_2)| \\
&= |\cos(\alpha_1 - \beta_1) + \cos(\alpha_2 - \beta_1) + \cos(\alpha_2 - \beta_2) - \cos(\alpha_1 - \beta_2)|
\end{aligned} \tag{3.10}$$

Now, for the choices of $\alpha_1 = 0^\circ$, $\alpha_2 = 90^\circ$ and $\beta_1 = 45^\circ$, $\beta_2 = 135^\circ$:

$$\begin{aligned}
S &= |\cos(-45) + \cos(45) + \cos(-45) - \cos(-135)| \\
&= 2\sqrt{2} \\
&= 2.82
\end{aligned} \tag{3.11}$$

Clearly, this value exceeds the maximum correlation possible by local hidden-variable theories. Hence, we have successfully proved that predictions of Quantum Mechanics cannot be fully reproduced by any local hidden variable theory. The next thing to do is to perform the experiment and see what we observe.

3.3 Experimental verification

In order to create polarization-entangled photon pairs, we used a type-1 BBO crystal. A polarized laser beam of 405nm was incident on a sandwich of two BBO crystals with their optic axes oriented at 90° to each other. The two crystals will down-convert the incoming



Figure 3.2: Experimental Setup for measuring the correlations

beam into two beams of 810nm whose polarizations will be entangled. This happens because we take the down-converted beam along paths in which the down-converted photons are not known to have been down-converted by the first crystal or the second one. The polarization state of these photons is described as:

$$|\psi\rangle = \frac{|HH\rangle + |VV\rangle}{\sqrt{2}}$$

The photons are then projected along different polarizations, and the coincidence counts are measured to check correlations between them. The picture below shows the experimental setup. On calculating the expectation value by obtaining the coincidence counts for different

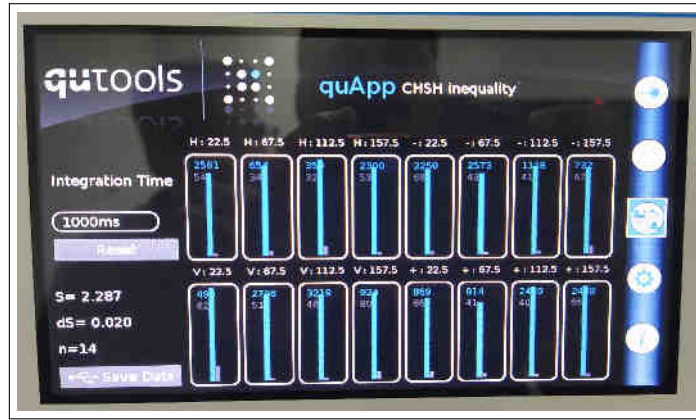


Figure 3.3: Coincidence obtained for different polarization orientations

orientations of polarizers of Alice and Bob, we found that the CHSH inequality was violated up to 2.28 with a standard deviation of 0.02.



Figure 3.4: Visibility in H/V and D/A basis

The visibility of the state in the horizontal/vertical and diagonal/anti-diagonal basis was plotted to check the validity of the result. The figure below shows the uncorrected and corrected visibility for both the basis.

Chapter 4

Bell-State Measurement

A Bell-State measurement projects a bipartite system into a bell basis[15]. It is at the heart of most of the protocols in Quantum Information Theory. In this chapter, we'll take a look at the Bell State Measurement, Hong-Ou-Mandel effect[11], which is the underlying effect behind the Bell state measurement and some protocols where BSM is used.

4.0.1 Bell-State Basis

A Bell State basis consists of four maximally entangled basis states of a bipartite system. These states can be used to represent any bipartite state of a system. Let us first write the basis states of a bipartite system in a computational basis.

$$(|00\rangle, |11\rangle, |10\rangle, |01\rangle)$$

Any bipartite state can be expressed as a linear combination these four states. Now, Bell states are defined as:

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) & |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}$$

These are the four maximally entangled Bell States. Note that they can be used to express the computational basis states of the bipartite system. So, any bipartite state can also be expressed in a Bell State basis. Now, let us look at the underlying principle behind BSM.

4.1 Hong-Ou-Mandel effect

Hong-Ou-Mandel effect[11] is a two-photon interference effect that occurs when we spatially overlap two photons and make them indistinguishable. It is the underlying effect behind a Bell state measurement that projects the bipartite state onto a bell state. First, we'll briefly discuss the theory and then talk about the experimental results obtained in the lab.

Assumptions

There are some conditions that are needed for the phenomena to occur. In order to simplify things, we will assume that these conditions are already met. These conditions are;

1. The two photons should be indistinguishable. i.e., they should have the same frequency, polarization, and phase.
2. The physical wave-packet of the photons should spatially overlap in the beam-splitter. We will examine what will happen if they do not overlap physically.
3. The overlap should happen in the coherence time of the photons. Coherence time is an interval in which the phase of a photon wave packet is assumed to be constant.

4.1.1 Theory

Consider two photons arriving at a beam-splitter from two different sides. Each photon can either reflect or transmit through from the surface of the beam-splitter. So there are four possible outcomes. They are as follows;

- The first photon will reflect, and the second will transmit.

- Both the photons will transmit.
- Both the photons will reflect.
- The first photon will transmit, and the second will reflect.

Refer the figure below for the four possibilities.

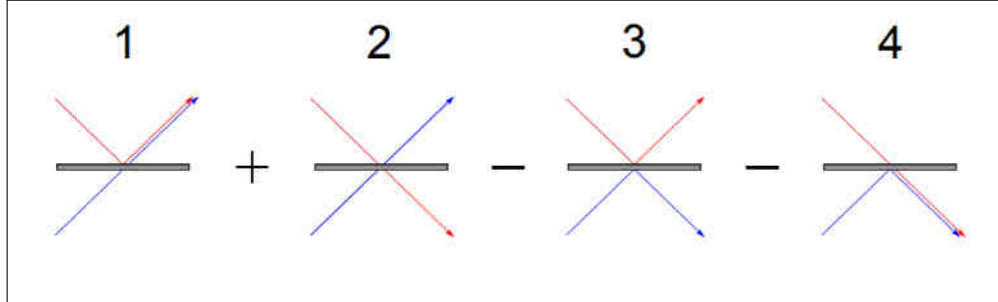


Figure 4.1: Possibilities on incidenting photons on a BS

The negative sign in front of the third and fourth possibility is because of a relative phase change that happened due to reflection. When a photon present in a rarer medium is reflected off from a denser medium surface, its phase is shifted by π . It should be noted that the reflection of both the photons happens at the same surface. Suppose that surface is the bottom surface of the beam-splitter, then a photon coming from below will experience a phase shift on reflection, but a photon coming from above will not because it will be inside the beam-splitter, and it will reflect off a rarer medium surface.

Recall that we had assumed initially that the two photons are indistinguishable. That means we have no way of knowing which photon was reflected and which one was transmitted. So, the second and third possibilities are identical to each other, and hence they will cancel out.

So the outcome is that we have only two possibilities left. Either both the photons will exit the beam-splitter from the bottom surface or from the top surface. This means the

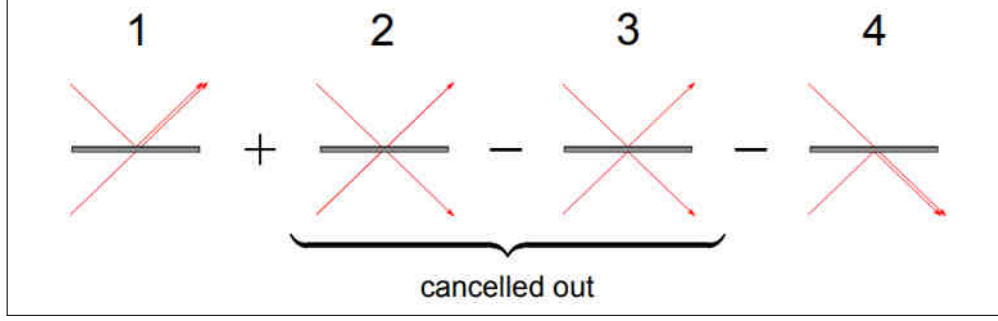


Figure 4.2: Possibilities after spatially overlapping the photons

two photons are entangled. If the initial state before the photons enter the beam-splitter is denoted by Ψ .

$$\Psi = |1\rangle|2\rangle$$

after passing through the beam-splitter, the state will be;

$$\Psi = \frac{1}{\sqrt{2}}(|3\rangle|3\rangle + |4\rangle|4\rangle)$$

Where the index 1,2,3,4 denote the input and output paths as shown below:

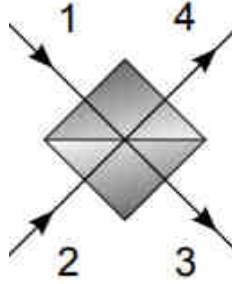


Figure 4.3: Input and output ports of the Beam Splitter

The final state Ψ is a maximally entangled bell-state. After passing through the beam-splitter, there will be a detector along each paths 3 and 4. These detectors will force the wave function to collapse into one of the states. And depending on the amount of spatial overlap of the two-photon wave packets, we will observe a decrease in the coincidence count of the two detectors. For a perfect overlap, there should be no coincidence count, theoretically.

4.2 Experimental Result

The experiment was performed with two photons generated by the Spontaneous Parametric Down-Conversion process. A laser beam of 405nm, horizontally polarized, was incident on a type -1 BBO crystal. The crystal down-converted the photons to 810nm vertically polarized photons. These photons, which are identical in all degrees of freedom, were incident on a fiber-based beam splitter from the two ports. The output ports were linked to Single Photon detectors with a coincidence counter. By changing the path length of one of the ports of the beam splitter, the two photons were made to spatially overlap, and coincidence counts were noted as we changed the path length. It was found that while changing the path length, at one point, the coincidence counts suddenly drop. This point is supposedly the point where the wave function of the two photons overlap and the HOM effect is observed. Shown below is the picture of the coincidence counter's screen: To analyze the effect in more

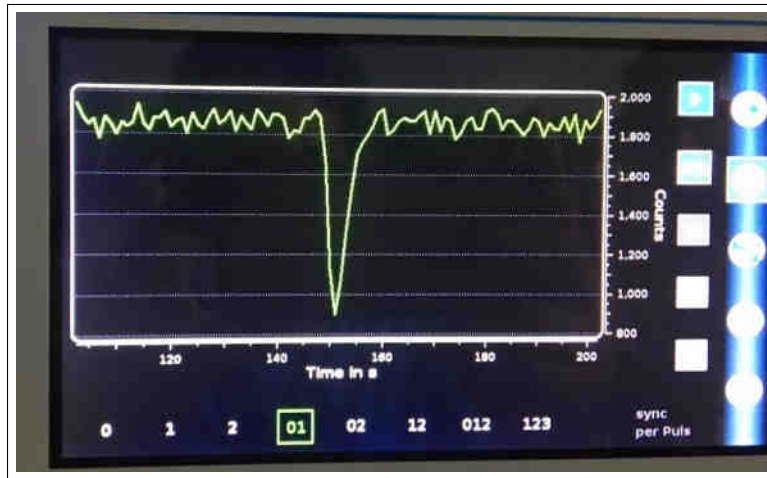


Figure 4.4: Deep in the coincidence counts on spatially overlapping the photons

detail, the integration time of the device was varied. Integration time is the time for which the device will take the signal from the detectors before taking an average of the signal and showing it on the screen. The picture shown below captures the HOM deep for two different integration times. On the left side of the above figure, we can see that the average value of the coincidence counts fluctuates a lot. This is because the integration time is set to 0.25 seconds. The shorter the integration time, the more fluctuation will be reflected in the data. As we increase the integration time, as seen on the right side, we see that the average value doesn't fluctuate much. Another parameter that comes into play is the coincidence window. A coincidence window is the time window in which if the two detectors click, then a coinci-

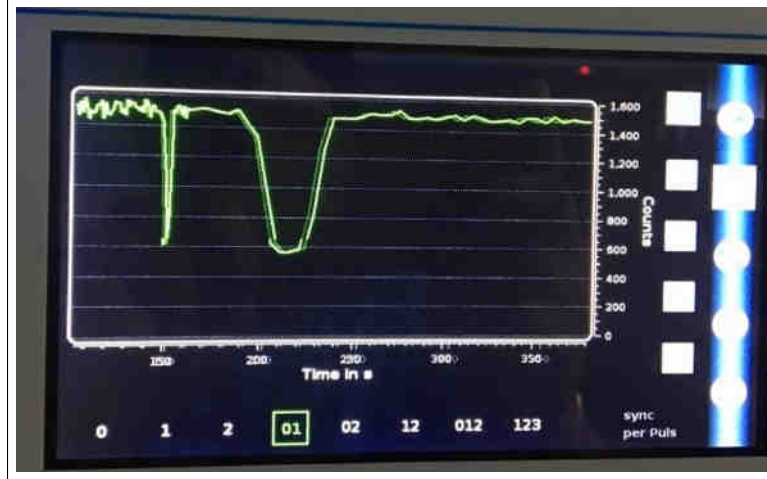


Figure 4.5: Deep in the coincidence counts for different integration times

dence count is registered. One can guess that as we increase the coincidence window, we will register an increase in the coincidence counts. The graphs below are plotted for two different coincidence windows of 20ns and 40ns: For the 20ns, the average coincidence counts were

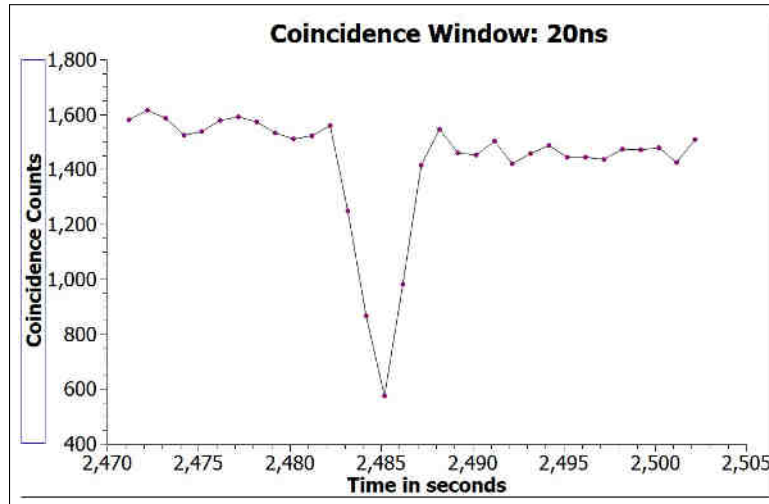


Figure 4.6: Coincidence window of 20ns

around 1500, and the deep was at around 500, a decrease of around 1000 counts. As we increased the coincidence window to 40ns, it was observed that the average coincidence counts went up to 1800, but the deep was observed at around 900, a gap of around 900. On further increasing the coincidence window, it was confirmed that the coincidence counts go up, but the gap between the deep and the average count decreases. This suggests that as we increase the coincidence window, we also increase the possibility of registering a coincidence count

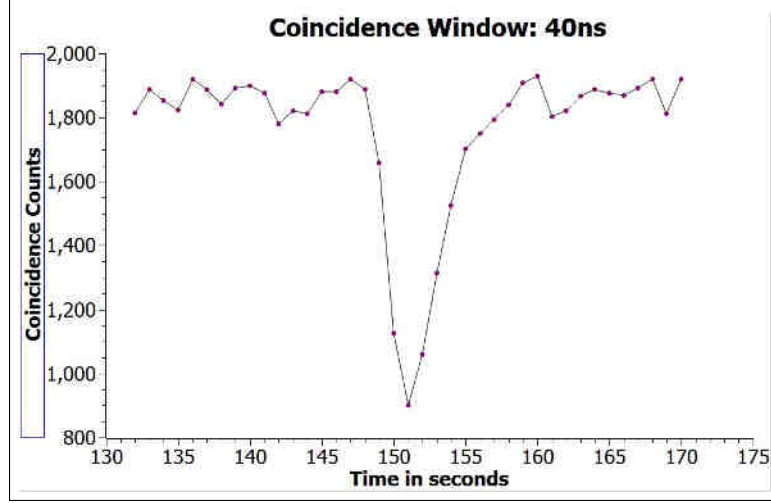


Figure 4.7: Coincidence window of 40ns

between two photons whose wave functions are not spatially overlapping. Such coincidence counts would be unaffected as we cross the point of the spatial overlap.

4.3 Partial indeterminacy of BSM

A Bell State Measurement projects the bipartite system in a Bell State Basis, but just projection is not enough; we also need the information regarding the basis the state has collapsed in. A Bell State Analyzer is a setup that can aid in obtaining this information. The setup of a Bell State Analyzer is depicted in the figure below: Two photons enter the Beam Splitter from the two sides. Based on which two of the four detectors click, we say that the bipartite system has been projected in a particular Bell state. Let us now see which detector clicks represent which Bell state. The total wave function of the two photons should be symmetric under the exchange of the first and second photons. For the three symmetric polarization mode wave functions of Φ^+ , Φ^- & Ψ^+ , the spatial mode wave functions will be symmetric. And for the singlet state Ψ^- , the spatial mode wave function will be anti-symmetric. So, the wave functions of the four bell states will be as follows:

$$|\Phi^+\rangle = \frac{(|H_1H_2\rangle + |V_1V_2\rangle)}{\sqrt{2}} \otimes \frac{(|a_1b_2\rangle + |b_1a_2\rangle)}{\sqrt{2}}$$

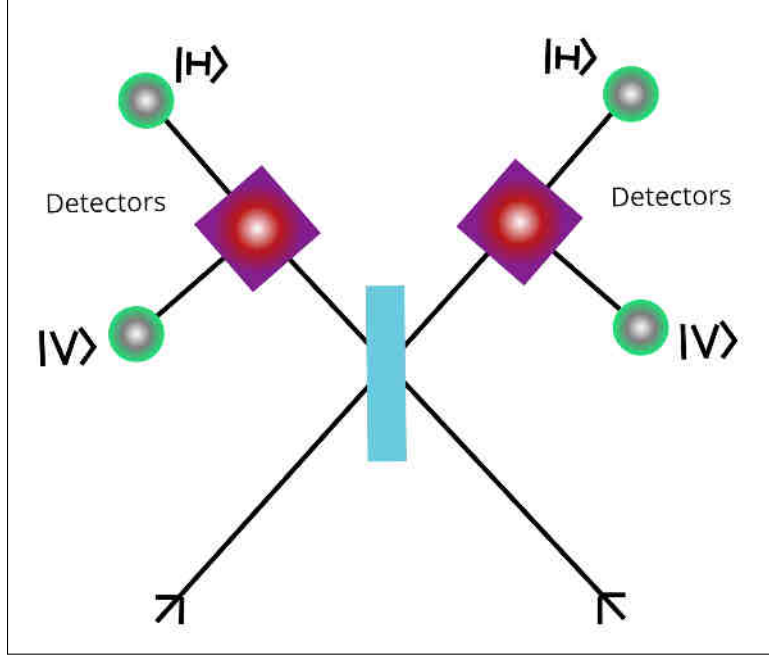


Figure 4.8: Bell State Analyzer

$$\begin{aligned}
 |\Phi^-\rangle &= \frac{(|H_1H_2\rangle - |V_1V_2\rangle)}{\sqrt{2}} \otimes \frac{(|a_1b_2\rangle + |b_1a_2\rangle)}{\sqrt{2}} \\
 |\Psi^+\rangle &= \frac{(|H_1V_2\rangle + |V_1H_2\rangle)}{\sqrt{2}} \otimes \frac{(|a_1b_2\rangle + |b_1a_2\rangle)}{\sqrt{2}} \\
 |\Psi^-\rangle &= \frac{(|H_1V_2\rangle - |V_1H_2\rangle)}{\sqrt{2}} \otimes \frac{(|a_1b_2\rangle - |b_1a_2\rangle)}{\sqrt{2}}
 \end{aligned}$$

If the incoming photons exit the beam splitter from the two different ports and the detectors click for two different polarizations, we can be certain that the state has been projected in Ψ^- . If the photons exit from the same port but the polarizations are found to be opposite, we know that Ψ^+ state was observed. But if the photons exit from the same port and are found to have the same polarization, we cannot be certain whether it was Ψ^+ or Ψ^- . So, we can be certain about the outcome only 50% of the time. This indeterminacy is said to be a fundamental constraint. There have been No-go theorems that prove that [18][13]. Now, let's take a look at two important protocols in QIT that require BSM for their implementation.

4.4 Protocols using BSM

4.4.1 Teleportation

Quantum Information is fundamentally different than classical information of 0 & 1. A qubit, which is the smallest amount of information in QI, can exist in a superposition of both 0 & 1. This counter-intuitive nature of Quantum Information enables us to implement protocols that are otherwise impossible with Classical information. One of such protocols is the protocol of Teleportation[4]. It enables one to teleport a qubit to a different party using an entangled pair and classical communication. The sender party may not be aware of the state of the qubit to be teleported. And the two parties should share an entangled pair with them. Let us look at the mathematical formalism of the Teleportation protocol. Let Alice and Bob share an entangled pair denoted by $|\Phi^+\rangle$.

$$|\Phi^+\rangle = \frac{|H_a H_b\rangle + |V_a V_b\rangle}{\sqrt{2}}$$

Here, the subscripts a & b denote the qubits possessed by Alice and Bob. Let the qubit to be teleported be denoted by $|\Psi_T\rangle$.

$$|\Psi_T\rangle = \alpha|H_a\rangle + \beta|V_a\rangle$$

The qubit to be teleported is assumed to be unknown. The subscript a denote that the qubit is possessed by Alice. Now, let us write down the complete state of the system.

$$\begin{aligned} |\Psi_{complete}\rangle &= |\Psi_T\rangle \otimes |\Phi^+\rangle \\ &= (\alpha|H_a\rangle + \beta|V_a\rangle) \otimes \frac{(|H_a H_b\rangle + |V_a V_b\rangle)}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} (\alpha|H_a H_a H_b\rangle + \alpha|H_a V_a V_b\rangle + \beta|V_a H_a H_b\rangle + \beta|V_a V_a V_b\rangle) \end{aligned} \tag{4.1}$$

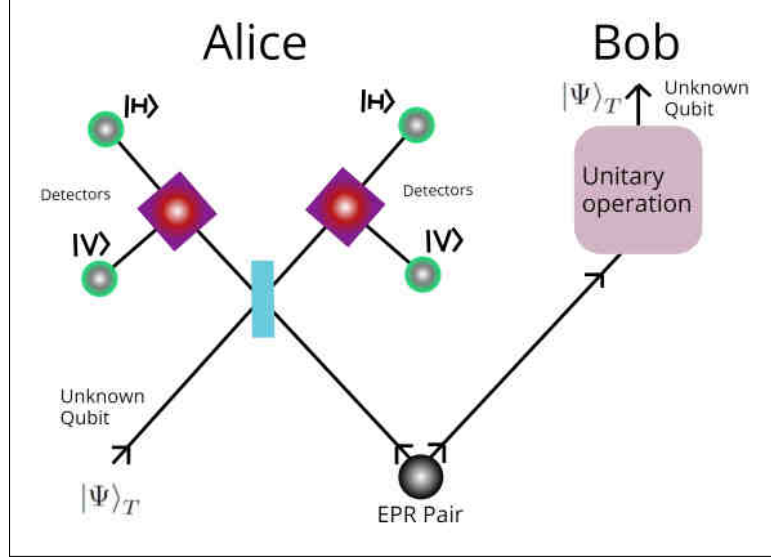


Figure 4.9: Setup for Teleportation

Now, Alice will perform a Bell State Measurement on the qubits in her possession. So let's write down Alice's qubits in Bell State Basis.

$$|\Psi_{complete}\rangle = \frac{1}{\sqrt{2}} \left(\alpha \frac{(|\Phi^+\rangle + |\Phi^-\rangle)}{\sqrt{2}} |H_b\rangle + \alpha \frac{(|\Psi^+\rangle + |\Psi^-\rangle)}{\sqrt{2}} |V_b\rangle + \beta \frac{(|\Psi^+\rangle - |\Psi^-\rangle)}{\sqrt{2}} |H_b\rangle + \beta \frac{(|\Phi^+\rangle - |\Phi^-\rangle)}{\sqrt{2}} |V_b\rangle \right) \quad (4.2)$$

On rearranging the terms:

$$|\Psi_{complete}\rangle = \frac{1}{2} \left(|\Phi^+\rangle (\alpha |H_b\rangle + \beta |V_b\rangle) + |\Phi^-\rangle (\alpha |H_b\rangle - \beta |V_b\rangle) + |\Psi^+\rangle (\alpha |V_b\rangle + \beta |H_b\rangle) + |\Psi^-\rangle (\alpha |V_b\rangle - \beta |H_b\rangle) \right) \quad (4.3)$$

Based on the outcome of the measurement, Bob's qubit will end in the respective state. To recover the state $|\Psi_T\rangle$, he must perform one of the four unitary operations on his qubit after he receives the information of the outcome from Alice. Since Alice needs to communicate her result through a classical channel, the speed of the Teleportation protocol cannot exceed the speed of light. So, though teleportation happens instantaneously, it cannot be used for Quantum Superluminal Communication.

4.4.2 Entanglement Swapping^[7]

A Bell State Measurement can entangle two qubits that were not entangled before. But what if the two qubits were entangled with some other qubits? Let us do the math and see for ourselves. Consider two entangled pairs, $|\Psi_{12}\rangle$ & $|\Psi_{34}\rangle$. The subscripts represents the qubits.

$$|\Psi_{12}\rangle = \frac{|H_1H_2\rangle + |V_1V_2\rangle}{\sqrt{2}}$$

$$|\Psi_{34}\rangle = \frac{|H_3H_4\rangle + |V_3V_4\rangle}{\sqrt{2}}$$

The wave function of the total system will be tensor products of the two.

$$\begin{aligned} |\Psi_{Total}\rangle &= |\Psi_{12}\rangle \otimes |\Psi_{34}\rangle \\ &= \frac{1}{2}(|H_1H_2H_3H_4\rangle + |H_1H_2V_3V_4\rangle + |V_1V_2H_3H_4\rangle + |V_1V_2V_3V_4\rangle) \end{aligned} \quad (4.4)$$

Now, suppose we perform a Bell State measurement between the Qubits 2 and 3:

$$\begin{aligned} |\Psi_{Total}\rangle &= \frac{1}{2} \left(|H_1\rangle \frac{(|\Phi^+\rangle + |\Phi^-\rangle)}{\sqrt{2}} |H_4\rangle + |H_1\rangle \frac{(|\Psi^+\rangle + |\Psi^-\rangle)}{\sqrt{2}} |V_4\rangle \right. \\ &\quad \left. + |V_1\rangle \frac{(|\Psi^+\rangle - |\Psi^-\rangle)}{\sqrt{2}} |H_4\rangle + |V_1\rangle \frac{(|\Phi^+\rangle - |\Phi^-\rangle)}{\sqrt{2}} |V_4\rangle \right) \end{aligned} \quad (4.5)$$

On rearranging the terms:

$$\begin{aligned} |\Psi_{Total}\rangle &= \frac{1}{2} \left(|\Phi^+\rangle \frac{(|H_1H_4\rangle + |V_1V_4\rangle)}{\sqrt{2}} \right. \\ &\quad + |\Phi^-\rangle \frac{(|H_1H_4\rangle - |V_1V_4\rangle)}{\sqrt{2}} \\ &\quad + |\Psi^+\rangle \frac{(|H_1V_4\rangle + |V_1H_4\rangle)}{\sqrt{2}} \\ &\quad \left. + |\Psi^-\rangle \frac{(|H_1V_4\rangle - |V_1H_4\rangle)}{\sqrt{2}} \right) \end{aligned} \quad (4.6)$$

Based on the outcome of the Bell State Measurement between 2nd & 3rd qubits, 1st & 4th

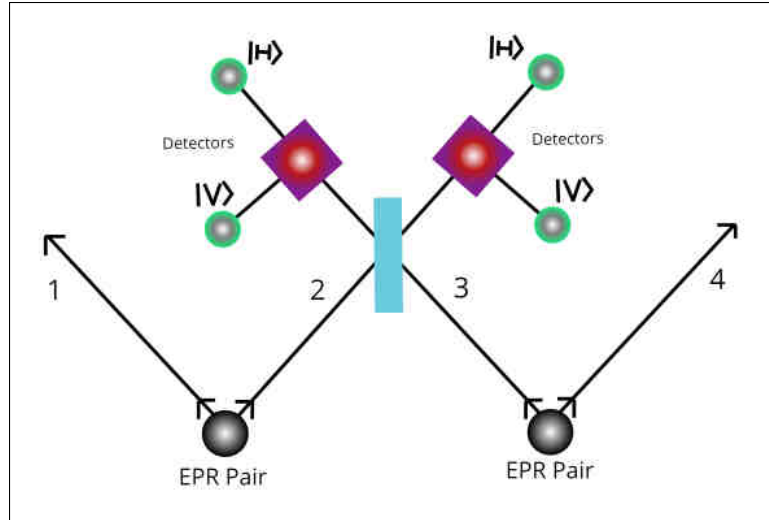


Figure 4.10: Entanglement Swapping Setup

qubits end up being entangled in one of the four bell states. So the entanglement between 1,2 & 3,4 is swapped. This protocol is called the Entanglement swapping protocol. The most peculiar thing about this protocol is that 1 & 4, which end up being entangled, have never interacted with each other. In fact, they may be separated by a great distance. Still, they will end up being entangled. Another important use of the Entanglement swapping protocol is that it holds the key to addressing a Foundational question in Quantum Mechanics. In the next chapter, we will take a look at this Foundational question and see how Bell State Measurement plays a role in addressing it.

Chapter 5

Status of Complex Numbers in QM

Even though complex numbers appear in a number of physical theories like Electromagnetic wave theory, they are used for computational convenience. Real numbers are enough to describe most physical phenomena. But is that also true in Quantum Mechanics? We are well aware that the standard formalism of QM requires states to be members of a Hilbert space which of necessity is a vector space defined over a complex scalar field. This has been deeply puzzling even to the founders of quantum mechanics. Alternate formalisms of QM suggested that do not involve complex numbers. There has also been mathematical proof that both Real and Complex Hilbert Spaces are equivalent in reproducing any bipartite quantum correlations[16]. It seemed that complex numbers, like in other physical theories, are just a computational trick. It has been argued that quantum mechanical states requiring complex numbers could always be replaced by states in higher dimension Vector spaces described over a real field so long as expectation values computed using the Born formula give identical results. Bell's inequalities such as CHSH like inequality have been proposed that give different predictions for the Real and Complex Hilbert Spaces[17]. The experimental results were in favor of Complex Hilbert Spaces. Here, we'll look at the proposed inequality and the experiment. Then we'll look at a new variant of the experiment that I propose and see what more can be understood about the status of complex numbers in Quantum Mechanics.

5.1 CHSH₃ inequality

Any Real Hilbert Space formalism has to be higher dimensional than the Complex Hilbert Space (CHS) formalism to make up for the imaginary numbers. Since each complex number requires two real numbers to be described, the dimensions of Real Hilbert space (RHS) should be at least twice as the dimensions of CHS. It is perfectly fine to do so since the dimensions of Hilbert space need not be restricted; in fact, the dimensions of RHS could be twice, thrice, or any fold times the dimensions of CHS. For the two formalisms to be equivalent, the two formalisms should give the same experimental predictions. As long as bipartite states are considered, it has been proved that any bipartite correlations can be fully reproduced by using only Real vectors and Real Hilbert Spaces[16]. One of the postulates of Quantum Mechanics is that the joint state of two independently prepared subsystems is the tensor product of the individual states of the two subsystems. It was speculated that this postulate could hold the key to differentiating between the CHS and RHS formalisms of QM[17]. This is because any RHS formalism is dimensionally at least twice as large as the CHS formalism. So a 4 partite system that requires 8 dimensions in CHS will require 16 dimensions in RHS formalism. But for certain 4-partite systems, we can perform a protocol called Entanglement swapping. The protocol reduces the system to an entangled bipartite state. Now, one can ask whether this protocol can be explained by using only real vectors. Intuitively, one can expect that an RHS formalism could run into trouble because a 4-partite system in RHS will require at least 16 dimensions. And the final bipartite state will require 8 dimensions. So the outcome of entanglement swapping has to trace out 8 dimensions. But we are aware that entanglement swapping requires a Bell state measurement that has only 4 possible outcomes. So, how shall we trace out the other 4 dimensions in an RHS formalism? But this is not mathematically rigorous proof. So, let's have a look at this scenario in more detail.

The entanglement swapping protocol involves two entangled pairs. One qubit from each pair is then projected on a bell-state basis. Depending on the outcome, the other two qubits end up being entangled in that respective bell state. So the entanglement is swapped between the two pairs. An inequality named CHSH₃ can be defined that is dependent on the outcome of the Bell state measurement and looks for correlations along all the three x-y, x-z, and y-z planes.

$$CHSH_3 = |CHSH_{(1,2;1,2)} + CHSH_{(1,3;3,4)} + CHSH_{(2,3;5,6)}| \leq 6 \quad (5.1)$$

Where on the RHS, $CHSH_{(i,j;m,n)}$ represents i and j choices of Alice and m and n choices of Charlie. The three inequalities correspond to measurements along the three z-x, z-y, and x-y planes. The $CHSH_3$ inequality also depends on Bob's Bell state measurement outcome. If we denote the expectation value S_{xz} as the expectation value of the joint projection operator of Alice and Charlie for their choices x and z, then;

$$S_{xz} = \langle \Psi | A_x \otimes C_z | \Psi \rangle$$

Rewriting the individual CHSH inequalities on the RHS in terms of the expectation values and the measurement outcome of Bob, we get;

$$\begin{aligned} CHSH_3 = & \left| (-1)^{b_2}(S_{11} + S_{12}) + (-1)^{b_1}(S_{21} - S_{22}) \right. \\ & + (-1)^{b_2}(S_{13} + S_{14}) - (-1)^{b_1+b_2}(S_{33} - S_{34}) \\ & \left. + (-1)^{b_1}(S_{25} + S_{26}) - (-1)^{b_1+b_2}(S_{35} - S_{36}) \right| \end{aligned} \quad (5.2)$$

Where b_1 and b_2 are binary numbers obtained from the outcome of Bob's measurement $b = b_1b_2 = 00, 01, 10, 11$. The setup of the experiment looks something like this; Here, three

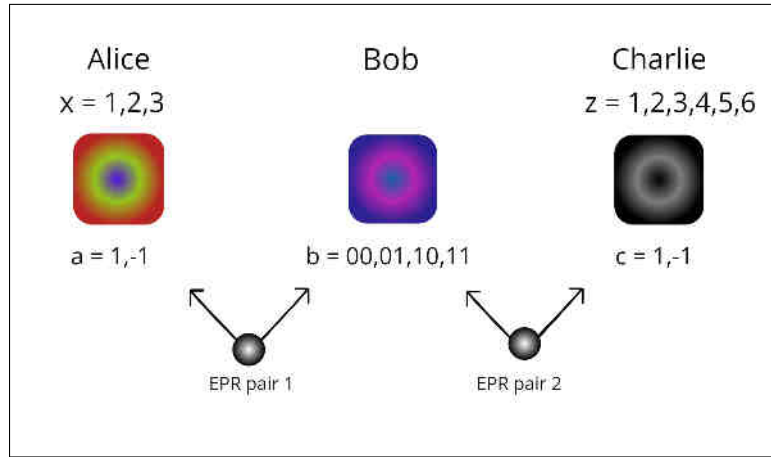


Figure 5.1: CHSH₃ implementation

choices of Alice and six choices of Charlie correspond to measurements along the following axes:

Alice	Basis	Charlie	Basis
x = 1	σ_z	z = 1	$\frac{\sigma_z + \sigma_x}{\sqrt{2}}$
x = 2	σ_x	z = 2	$\frac{\sigma_z - \sigma_x}{\sqrt{2}}$
x = 3	σ_y	z = 3	$\frac{\sigma_z + \sigma_y}{\sqrt{2}}$
		z = 4	$\frac{\sigma_z - \sigma_y}{\sqrt{2}}$
		z = 5	$\frac{\sigma_x + \sigma_y}{\sqrt{2}}$
		z = 6	$\frac{\sigma_x - \sigma_y}{\sqrt{2}}$

Since each inequality on the RHS in equation (5.6) can reach a maximum violation of $2\sqrt{2}$, the maximum violation of $CHSH_3$ goes up to $6\sqrt{2} = 8.48$. However, this bound is for the Complex Hilbert Space formalism. Let us look at what we can say about the maximum violation if we use only real vectors.

Let $P(a, b, c|x, z)$ be the joint probability of getting the outcomes a,b, and c for Alice's and Charlie's choices.

$$P(a, b, c|x, z) = \sum_{\lambda} P(\lambda) Tr((\rho_{AB_1} \otimes \rho_{B_2C})(A_{a|x} \otimes B_b \otimes C_{c|z})) \quad (5.3)$$

Where ρ_{AB_1} and ρ_{B_2C} denote the density matrices of the two EPR pairs. $P(\lambda)$ denote probability distribution of a local hidden variable as in general Bell inequalities. $A_{a|x}$ and $C_{c|z}$ denote Alice and Charlies' projection operators for a and c outcomes. B_b is a projection operator that projects in Bell-state Basis. Mathematically, $A_{a|x}$ & $C_{c|z}$ are as described in the table above and B_b is written as:

$$B_b = \{|\Phi^+\rangle\langle\Phi^+|, |\Phi^-\rangle\langle\Phi^-|, |\Psi^+\rangle\langle\Psi^+|, |\Psi^-\rangle\langle\Psi^-|\}$$

The group that came up with the $CHSH_3$ inequality has also given two separate proofs[17] that if we restrict ourselves to Real Vectors and Real Hilbert spaces, then the joint probability distribution for the above scenario cannot be expressed in the form presented in the equation (5.8). But this proof assumes that the RHS formalism follows the following four postulates of Quantum Mechanics. Any other formalism that does not follow these postulates and does

not involve complex numbers is not ruled out by this proof. So, the proof only rules out a certain class of mathematical formalisms of QM; this is also mentioned in the paper provided by the group[17]. The four postulates are:

1. The state of a physical system is represented by a normalized state vector that resides in a Hilbert space.
2. A measurement on the physical system is described by a set of projection operators (Π_r) that act on the Hilbert space of the system.
3. Born rule: the probability that the measurement on the system will yield a result r is given by $P(r) = \langle \Psi | \Pi_r | \Psi \rangle$.
4. the joint state of two independently prepared subsystems is the tensor product of the individual states of the two subsystems $\sigma_{Total} = \sigma_A \otimes \sigma_B$.

Another approach adopted by the same group was to use the Semidefinite program. A Semidefinite program can be used to optimize an inequality of semidefinite matrices over some given constraints. It basically converts the matrix inequality into a polynomial inequality with constraints like the polynomials do not commute. A numerical optimum can be obtained for this inequality using a set of hierarchical conditions. To calculate the maximum violation possible by the Real Hilbert Space formulation, One can convert the CHSH₃ inequality into a Semidefinite program and obtain all the required constraints, including the one that all the vectors should be real.

On running the Semidefinite program on MATLAB, the maximum possible violation by the inequality involving only real vectors and Hilbert spaces turned out to be ≤ 7.66 [17]. Clearly, this contrasts with the maximum bound predicted by the CHS formalism with enough margin to test experimentally. Recently, a group performed the experiment and reported that they measured violation beyond 7.66[5], proving that Real vectors are not enough to reproduce all the quantum correlations predicted by Complex Hilbert Space formalism.

If we decide to use photons for the experiment, the implementation of the protocol is quite non-trivial. Firstly, the generation of an entangled pair, which is mostly done by Spontaneous Parametric Down Conversion, occurs randomly. And to successfully swap the entanglement, the two photons coming from two pairs should spatially superimpose for the Bell-State

Measurement. Also, Bell-State Measurements are deterministic only 50% of the time. The experimental implementation in [5] did favor the Complex Hilbert Space formalism, but the Bell-inequality violation was not loophole-free. The measurement apparatus of Alice and Charlie were not distant enough to close the locality loophole. And since we're discarding half of the states when we're not sure about the outcome of the Bell-State measurement, the detection loophole is also present. So, technically, there still exists a local hidden variable model that could explain the correlations.

All that being said, what if we decide to use a Single Photon Entangled state instead of an EPR pair? Here, we propose a variant of the 5.1 experiment using SPE states and ask what more can be understood about the status of Complex numbers in Quantum Mechanics. But first, let's have a quick look at what SPE states are and how they are prepared.

5.2 Single Photon Entangled State

Single Photon entangled states[14] are superposition states of single photons where their one degree of freedom is entangled with another degree of freedom. The state vector is non-separable into state vectors of individual degrees of freedom. This means that the observables of the two commuting operators are entangled. It has been shown[14] that SPE states can be used to violate a variant of CHSH inequality that rules out non-contextual hidden variable models, which are a broader class of local hidden-variable models. Here, we will be using path-polarization entangled SPE states created by using the setup below.

A horizontally polarized photon passes through a half-wave plate at 22.5° and is converted to a diagonally polarized photon. It then enters the PBS from the left, which transmits the horizontal component and reflects the vertical component. We introduce a phase modifier Φ_a along with one of the paths. The two paths are then combined with a non-polarizing beam splitter. Before performing the measurement, we introduce two birefringent phase modifiers Φ_b along the two paths. Here, the two observables are path amplitude and polarization amplitude. The maths of the transformation of the photon's state is as follows;

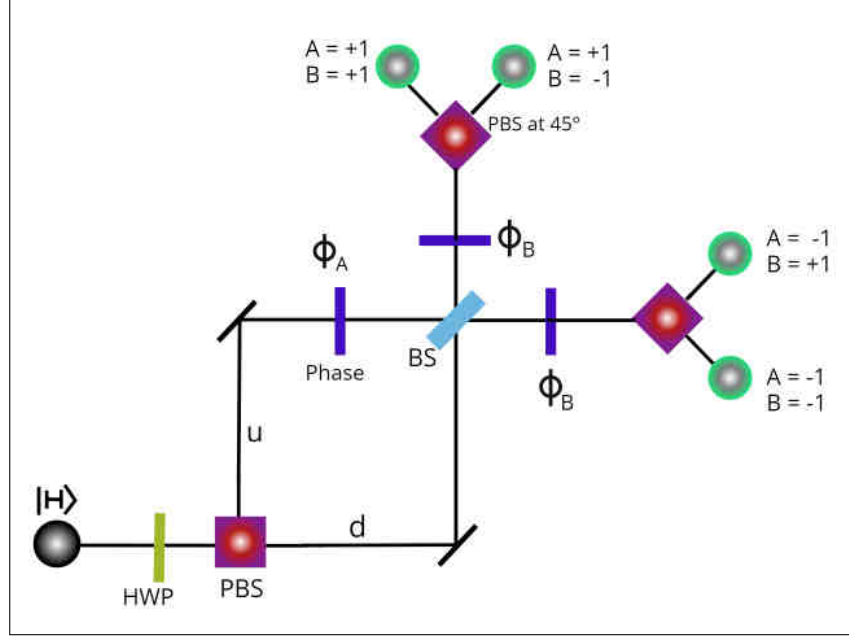


Figure 5.2: Single Photon Entangled State Setup

$$\begin{aligned}
 |\Psi\rangle &= \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \\
 &= \frac{1}{\sqrt{2}}(|H\rangle|d\rangle + e^{i\Phi_a}|V\rangle|u\rangle) \\
 &= \frac{1}{2} (i|H\rangle|u'\rangle + |H\rangle|d'\rangle + e^{i\Phi_a}|V\rangle|u'\rangle + ie^{i\Phi_a}|V\rangle|d'\rangle) \\
 &= \frac{1}{2} [(i|H\rangle + e^{i(\Phi_a+\Phi_b)}|V\rangle)|u'\rangle + (|H\rangle + ie^{i(\Phi_a+\Phi_b)}|V\rangle)|d'\rangle]
 \end{aligned} \tag{5.4}$$

This is the final state that we'll be projecting on the below two projectors;

$$\begin{aligned}
 P_a &= |+\rangle\langle+| - |-\rangle\langle-| \\
 P_b &= |d'\rangle\langle d'| - |u'\rangle\langle u'|
 \end{aligned} \tag{5.5}$$

Here, the first projector measures the photon's polarization, and the second one measures its path. These two observables of the same photon are entangled. They can be thought of as entangled vectors on a Bloch sphere for polarization and the Bloch-sphere equivalent of a Mach-Zender interferometer for the path. Refer to the pictures below.

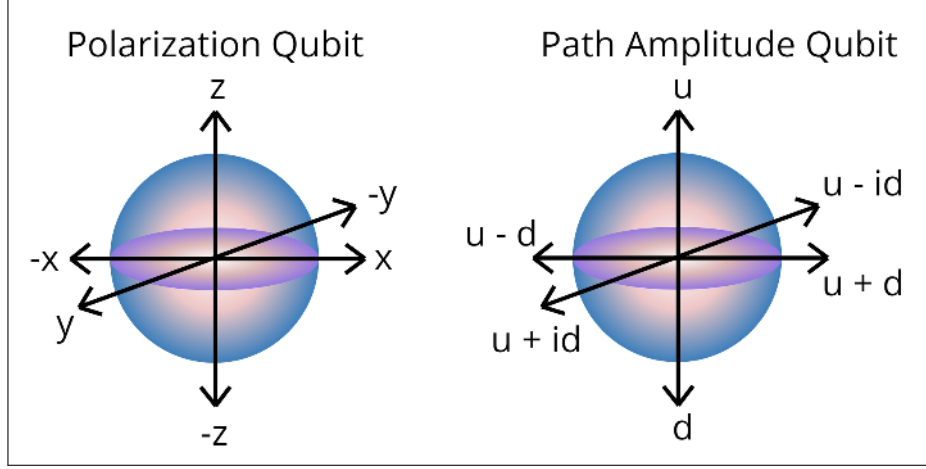


Figure 5.3: Polarization and Path Amplitude Qubits

If we calculate the expectation value of the joint operator $P_a \otimes P_b$, we'll get;

$$\begin{aligned}
E &= \langle \Psi | P_a \otimes P_b | \Psi \rangle \\
&= \frac{1}{4} \left(-i \langle H | + e^{-i(\Phi_a + \Phi_b)} \langle V | \right) P_a \left(i | H \rangle + e^{i(\Phi_a + \Phi_b)} | V \rangle \right) \langle u' | P_b | u' \rangle \\
&\quad + \frac{1}{4} \left(\langle H | - i e^{-i(\Phi_a + \Phi_b)} \langle V | \right) P_a \left(| H \rangle + i e^{i(\Phi_a + \Phi_b)} | V \rangle \right) \langle d' | P_b | d' \rangle \\
&= \sin(\Phi_a + \Phi_b)
\end{aligned} \tag{5.6}$$

The CHSH inequality with four expectation values for 2 distinct choices of $\Phi_a = 0, \frac{\pi}{2}$ and $\Phi_b = \frac{\pi}{4}, \frac{3\pi}{4}$ will be;

$$CHSH^{SPE} = |E_{1,1} + E_{1,2} + E_{2,1} - E_{2,2}| \leq 2 \tag{5.7}$$

For the above choices of Φ_a and Φ_b , we get;

$$\begin{aligned}
CHSH^{SPE} &= \left| \sin\left(0 + \frac{\pi}{4}\right) + \sin\left(0 + \frac{3\pi}{4}\right) + \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) \right| \\
&= 2\sqrt{2}
\end{aligned} \tag{5.8}$$

This shows that the Single-photon entangled states can violate the CHSH inequality. It has to be noted that the CHSH inequality here is formulated by assuming that observables of two commuting operators have an independent existence. Based on this, the correlation

value possible between the two observables is restricted to up to 2. Any non-contextual hidden variable theory has to respect this bound. So, the violation of the above inequality rules out the class of non-contextual hidden variables models that try to reproduce quantum predictions.

Another important aspect to notice here is that the Polarization qubit can be measured along any axis on the Bloch sphere, but how are we supposed to measure the Path amplitude qubit along any general axis? Since we have only two paths, it seems impossible that we can project the path amplitude qubit along any general axis. But the answer to this question can be found with the below setup:

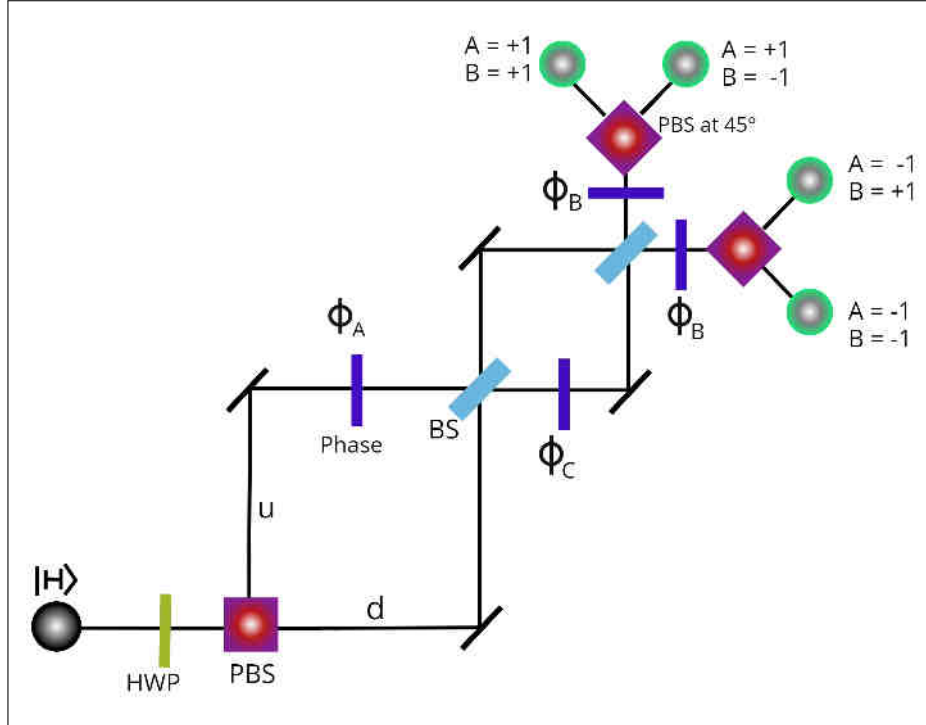


Figure 5.4: Modified Single Photon Entangled Setup

In the earlier setup Figure 5.2, we could project the path amplitude state along $(|u'\rangle \text{ \& } |d'\rangle)$. But now, we can project the path amplitude along $(\frac{|u'\rangle+|d'\rangle}{\sqrt{2}} \text{ \& } \frac{|u'\rangle-|d'\rangle}{\sqrt{2}})$ or $(\frac{|u'\rangle+i|d'\rangle}{\sqrt{2}} \text{ \& } \frac{|u'\rangle-i|d'\rangle}{\sqrt{2}})$ by changing the value of the phase modifier Φ_c along one of the paths. A more detailed math is shown as below:

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \\
&= \frac{1}{\sqrt{2}}(|H\rangle|d\rangle + e^{i\Phi_a}|V\rangle|u\rangle) \\
&= \frac{1}{2} (i|H\rangle|u'\rangle + |H\rangle|d'\rangle + e^{i\Phi_a}|V\rangle|u'\rangle + ie^{i\Phi_a}|V\rangle|d'\rangle) \\
&= \frac{1}{2} [e^{i\Phi_c} (i|H\rangle + e^{i\Phi_a}|V\rangle) |u'\rangle + (|H\rangle + ie^{i\Phi_a}|V\rangle) |d'\rangle] \\
&= \frac{1}{2} \left(\left[\frac{i(1 + e^{i\Phi_c})|H\rangle + e^{i(\Phi_a+\Phi_b)}(1 - e^{i\Phi_c})|V\rangle}{\sqrt{2}} \right] \frac{(|u'\rangle - ie^{-i\Phi_c}|d'\rangle)}{\sqrt{2}} \right. \\
&\quad \left. + \left[\frac{(-1 + e^{i\Phi_c})|H\rangle + ie^{i(\Phi_a+\Phi_b)}(1 + e^{i\Phi_c})|V\rangle}{\sqrt{2}} \right] \frac{(-i|u'\rangle + e^{-i\Phi_c}|d'\rangle)}{\sqrt{2}} \right)
\end{aligned} \tag{5.9}$$

For $\Phi_c = 0$ the path amplitude is projected along $(\frac{|u'\rangle+i|d'\rangle}{\sqrt{2}} \ \& \ \frac{|u'\rangle-i|d'\rangle}{\sqrt{2}})$ and for $\Phi_c = \frac{\pi}{2}$ it is projected along $(\frac{|u'\rangle+|d'\rangle}{\sqrt{2}} \ \& \ \frac{|u'\rangle-|d'\rangle}{\sqrt{2}})$. So, this makes the Polarization and Path amplitude qubit equivalent. Now, let's see how we can use a Single Photon Entangled state instead of an EPR pair in the scenario described in previous section.

5.3 Equivalent of CHSH₃ for SPE states

A Single Photon Entangled state cannot show any non-local correlations. So, we have to modify the setup such that Alice and Charlie are present at the same place. They just measure different observables of the same state. Let Alice measure the path observable and Charlie measure the polarization observable.

Let us define creation operators for path and polarization modes. Let $a_a^\dagger, a_b^\dagger, a_c^\dagger, a_d^\dagger, a_m^\dagger, a_n^\dagger, a_p^\dagger, a_q^\dagger, a_k^\dagger, a_s^\dagger$ denote creation operators for the respective paths as labeled in the subscript. Let $a_H^\dagger, a_V^\dagger, a_+^\dagger, a_-^\dagger, a_L^\dagger, a_R^\dagger$ denote the creation operators for polarization mode as labeled in the subscript. The figure given below can be referred along with the math.

Initially, two down-converted Horizontally polarized photons are released from a type-1 BBO crystal which is being pumped by a laser beam. The two photons are not entangled, but they are released at the same time. These photons are converted to diagonally polarized

photons by Half wave-plates placed along the paths.

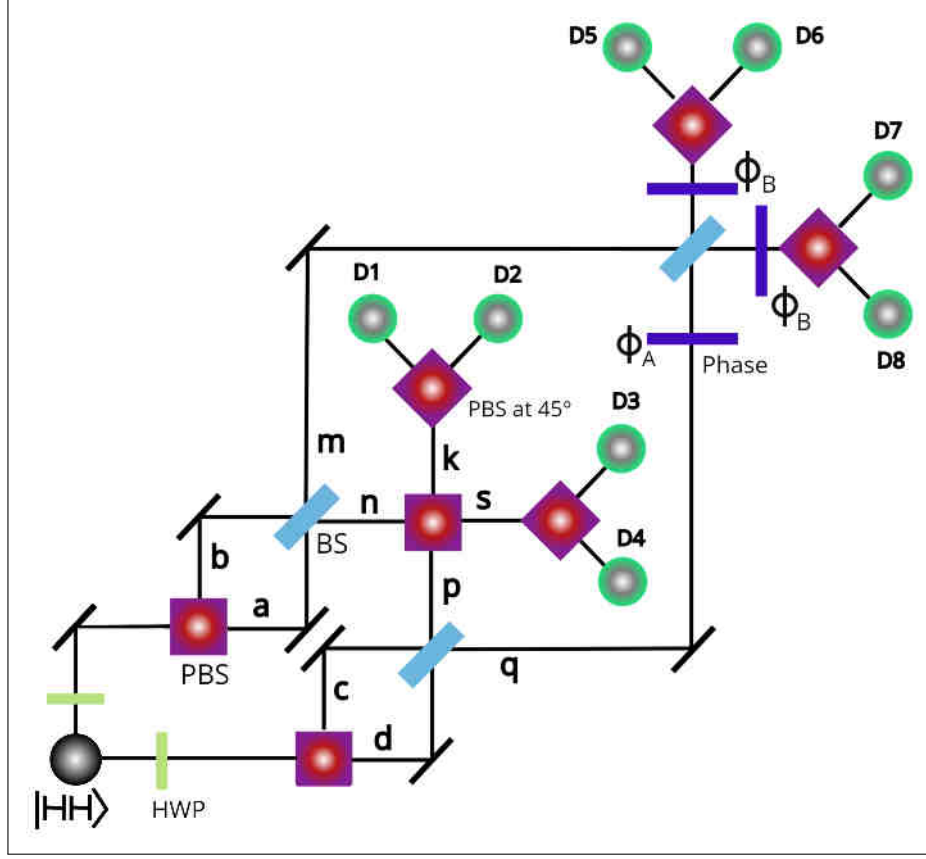


Figure 5.5: Equivalent of Entanglement Swapping for SPE states

$$|\Psi\rangle = \frac{(a_{1H}^\dagger + a_{1V}^\dagger)}{\sqrt{2}}|0\rangle \otimes \frac{(a_{2H}^\dagger + a_{2V}^\dagger)}{\sqrt{2}}|0\rangle$$

After passing through the polarizing beam splitter, the path and polarization modes become entangled, and we have a 4-partite system.

$$|\Psi\rangle = \frac{(a_{1H}^\dagger a_{1a}^\dagger + a_{1V}^\dagger a_{1b}^\dagger)}{\sqrt{2}}|00\rangle \otimes \frac{(a_{2H}^\dagger a_{2c}^\dagger + a_{2V}^\dagger a_{2d}^\dagger)}{\sqrt{2}}|00\rangle$$

$$|\Psi\rangle = \frac{1}{2} \left(a_{1H}^\dagger a_{1a}^\dagger a_{2H}^\dagger a_{2c}^\dagger + a_{1H}^\dagger a_{1a}^\dagger a_{2V}^\dagger a_{2d}^\dagger + a_{1V}^\dagger a_{1b}^\dagger a_{2H}^\dagger a_{2c}^\dagger + a_{1V}^\dagger a_{1b}^\dagger a_{2V}^\dagger a_{2d}^\dagger \right) |0000\rangle$$

As this state passes through the non-polarizing beam splitters, it is transformed into;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{2} & \left[a_{1H}^\dagger a_{2H}^\dagger \frac{(a_{1m}^\dagger + ia_{1n}^\dagger)}{\sqrt{2}} \frac{(a_{2p}^\dagger + ia_{2q}^\dagger)}{\sqrt{2}} \right. \\
& + a_{1H}^\dagger a_{2V}^\dagger \frac{(a_{1m}^\dagger + ia_{1n}^\dagger)}{\sqrt{2}} \frac{(ia_{2p}^\dagger + a_{2q}^\dagger)}{\sqrt{2}} \\
& + a_{1V}^\dagger a_{2H}^\dagger \frac{(ia_{1m}^\dagger + a_{1n}^\dagger)}{\sqrt{2}} \frac{(a_{2p}^\dagger + ia_{2q}^\dagger)}{\sqrt{2}} \\
& \left. + a_{1V}^\dagger a_{2V}^\dagger \frac{(ia_{1m}^\dagger + a_{1n}^\dagger)}{\sqrt{2}} \frac{(ia_{2p}^\dagger + a_{2q}^\dagger)}{\sqrt{2}} \right] |0000\rangle
\end{aligned} \tag{5.10}$$

We are only interested in the possibilities where entanglement is swapped between the two Single Photon Entangled states. For that, one photon should be detected in one of the four detectors $D1, D2, D3, D4$, and the other one should be detected at the other four detectors $D5, D6, D7, D8$. Highlighting such terms in $|\Psi\rangle$;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{4} & \left[\textcolor{blue}{a_{1H}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2p}^\dagger} + ia_{1H}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2q}^\dagger + ia_{1H}^\dagger a_{2H}^\dagger a_{1n}^\dagger a_{2p}^\dagger - \textcolor{blue}{a_{1H}^\dagger a_{2H}^\dagger a_{1n}^\dagger a_{2q}^\dagger} \right. \\
& + ia_{1H}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2p}^\dagger + a_{1H}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2q}^\dagger - a_{1H}^\dagger a_{2V}^\dagger a_{1n}^\dagger a_{2p}^\dagger + \textcolor{blue}{ia_{1H}^\dagger a_{2V}^\dagger a_{1n}^\dagger a_{2q}^\dagger} \\
& + \textcolor{blue}{ia_{1V}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2p}^\dagger} - a_{1V}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2q}^\dagger + a_{1V}^\dagger a_{2H}^\dagger a_{1n}^\dagger a_{2p}^\dagger + \textcolor{blue}{ia_{1V}^\dagger a_{2H}^\dagger a_{1n}^\dagger a_{2q}^\dagger} \\
& \left. - \textcolor{blue}{a_{1V}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2p}^\dagger} + ia_{1V}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2q}^\dagger + ia_{1V}^\dagger a_{2V}^\dagger a_{1n}^\dagger a_{2p}^\dagger + \textcolor{blue}{a_{1V}^\dagger a_{2V}^\dagger a_{1n}^\dagger a_{2q}^\dagger} \right] |0000\rangle
\end{aligned} \tag{5.11}$$

As this state evolves further, a horizontal photon in n path mode goes to s mode, and a vertical photon goes to k mode. We can find similar transformation rules for the photon in p mode.

$$\begin{aligned}
a_{1H}^\dagger a_{1n}^\dagger & \rightarrow a_{1H}^\dagger a_{1s}^\dagger & a_{1V}^\dagger a_{1n}^\dagger & \rightarrow a_{1V}^\dagger a_{1k}^\dagger \\
a_{2H}^\dagger a_{2p}^\dagger & \rightarrow a_{2H}^\dagger a_{2k}^\dagger & a_{2V}^\dagger a_{2p}^\dagger & \rightarrow a_{2V}^\dagger a_{2s}^\dagger
\end{aligned}$$

Applying these transformation rules and separating the highlighted terms, we get;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{4} \Bigg[& a_{1H}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2k}^\dagger + ia_{1V}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2k}^\dagger + ia_{1V}^\dagger a_{2H}^\dagger a_{1k}^\dagger a_{2q}^\dagger + a_{1V}^\dagger a_{2V}^\dagger a_{1k}^\dagger a_{2q}^\dagger \\
& - a_{1H}^\dagger a_{2H}^\dagger a_{1s}^\dagger a_{2q}^\dagger + ia_{1H}^\dagger a_{2V}^\dagger a_{1s}^\dagger a_{2q}^\dagger + ia_{1H}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2s}^\dagger - a_{1V}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2s}^\dagger + \dots \Bigg] |0000\rangle
\end{aligned} \tag{5.12}$$

Projecting the photon present in k and s mode along the diagonal and antidiagonal polarizations, we get;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{4} \Bigg[& a_{1H}^\dagger a_{1m}^\dagger \frac{(a_{2+}^\dagger + a_{2-}^\dagger)}{\sqrt{2}} a_{2k}^\dagger + ia_{1V}^\dagger a_{1m}^\dagger \frac{(a_{2+}^\dagger + a_{2-}^\dagger)}{\sqrt{2}} a_{2k}^\dagger \\
& + ia_{2H}^\dagger a_{2q}^\dagger \frac{(a_{1+}^\dagger - a_{1-}^\dagger)}{\sqrt{2}} a_{1k}^\dagger + a_{2V}^\dagger a_{2q}^\dagger \frac{(a_{1+}^\dagger - a_{1-}^\dagger)}{\sqrt{2}} a_{1k}^\dagger \\
& - a_{2H}^\dagger a_{2q}^\dagger \frac{(a_{1+}^\dagger + a_{1-}^\dagger)}{\sqrt{2}} a_{1s}^\dagger + ia_{2V}^\dagger a_{2q}^\dagger \frac{(a_{1+}^\dagger + a_{1-}^\dagger)}{\sqrt{2}} a_{1s}^\dagger \\
& + ia_{1H}^\dagger a_{1m}^\dagger \frac{(a_{2+}^\dagger - a_{2-}^\dagger)}{\sqrt{2}} a_{2s}^\dagger - a_{1V}^\dagger a_{1m}^\dagger \frac{(a_{2+}^\dagger - a_{2-}^\dagger)}{\sqrt{2}} a_{1s}^\dagger + \dots \Bigg] |0000\rangle
\end{aligned} \tag{5.13}$$

The overlap of states in k and s modes makes the two photons indistinguishable, so we lose the information regarding the first and second photons. Taking this into consideration, we get;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{4} \Bigg[& a_+^\dagger a_k^\dagger \left(\frac{(a_H^\dagger + ia_V^\dagger)}{\sqrt{2}} a_m^\dagger + \frac{(ia_H^\dagger + a_V^\dagger)}{\sqrt{2}} a_q^\dagger \right) \\
& + a_-^\dagger a_k^\dagger \left(\frac{(a_H^\dagger + ia_V^\dagger)}{\sqrt{2}} a_m^\dagger - \frac{(ia_H^\dagger + a_V^\dagger)}{\sqrt{2}} a_q^\dagger \right) \\
& + a_+^\dagger a_s^\dagger \left(\frac{(ia_H^\dagger - a_V^\dagger)}{\sqrt{2}} a_m^\dagger - \frac{(a_H^\dagger - ia_V^\dagger)}{\sqrt{2}} a_q^\dagger \right) \\
& + a_-^\dagger a_s^\dagger \left(\frac{(-ia_H^\dagger + a_V^\dagger)}{\sqrt{2}} a_m^\dagger - \frac{(a_H^\dagger - ia_V^\dagger)}{\sqrt{2}} a_q^\dagger \right) + \dots \Bigg] |0000\rangle
\end{aligned} \tag{5.14}$$

Writing the polarization mode in left and right circular basis:

$$a_L^\dagger = \frac{(a_H^\dagger + ia_V^\dagger)}{\sqrt{2}} \quad \& \quad a_R^\dagger = \frac{(a_H^\dagger - ia_V^\dagger)}{\sqrt{2}}$$

$$\begin{aligned} |\Psi\rangle = \frac{1}{2\sqrt{2}} & \left[a_+^\dagger a_k^\dagger \frac{(a_L^\dagger a_m^\dagger + a_R^\dagger a_q^\dagger)}{\sqrt{2}} \right. \\ & + a_-^\dagger a_k^\dagger \frac{(a_L^\dagger a_m^\dagger - a_R^\dagger a_q^\dagger)}{\sqrt{2}} \\ & + a_+^\dagger a_s^\dagger \frac{(a_L^\dagger a_m^\dagger - a_R^\dagger a_q^\dagger)}{\sqrt{2}} \\ & \left. - a_-^\dagger a_s^\dagger \frac{(a_L^\dagger a_m^\dagger + a_R^\dagger a_q^\dagger)}{\sqrt{2}} + \dots \right] |0000\rangle \end{aligned} \quad (5.15)$$

Based on the outcome of one of the four detectors $D1, D2, D3, D4$, we get a Single Photon Entangled state in the $|m\rangle$ and $|q\rangle$ mode. Two of the four states are identical, but the initially separable modes $|m\rangle$ & $|q\rangle$ are now entangled, So this is equivalent to swapping entanglement between two EPR pairs.

After detecting a photon in one of the four detectors, a corresponding SPE state is manifested in the $|m\rangle$ & $|q\rangle$ modes. This state can be used to violate a CHSH inequality. This allows us to define a $CHSH_3$ equivalent for SPE states:

$$\begin{aligned} CHSH_3^{SPE} = & |(-1)^{b_2}(S_{11} + S_{12}) + (-1)^{b_1}(S_{21} - S_{22}) \\ & + (-1)^{b_2}(S_{13} + S_{14}) - (-1)^{b_1+b_2}(S_{33} - S_{34}) \\ & + (-1)^{b_1}(S_{25} + S_{26}) - (-1)^{b_1+b_2}(S_{35} - S_{36})| \end{aligned} \quad (5.16)$$

Where S_{xz} is again the expectation value of the joint projection operator of Alice and Charlie for their choices x and z , here, Bob's outcome, b , takes only two values.

$$S_{xz} = \langle \Psi | A_x \otimes C_z | \Psi \rangle$$

Here, Alice's observable is Path amplitude qubit, and Charlie's observable is polarization. Charlie has six projection choices as before, which he can project by using an appropriate half-wave plate, quarter-wave plate, and polarizing beam splitter. For Alice, she has three choices; the first one can be implemented by using the setup in the figure 5.5. For second and third choices, Alice has to use a modified setup of figure 5.5:

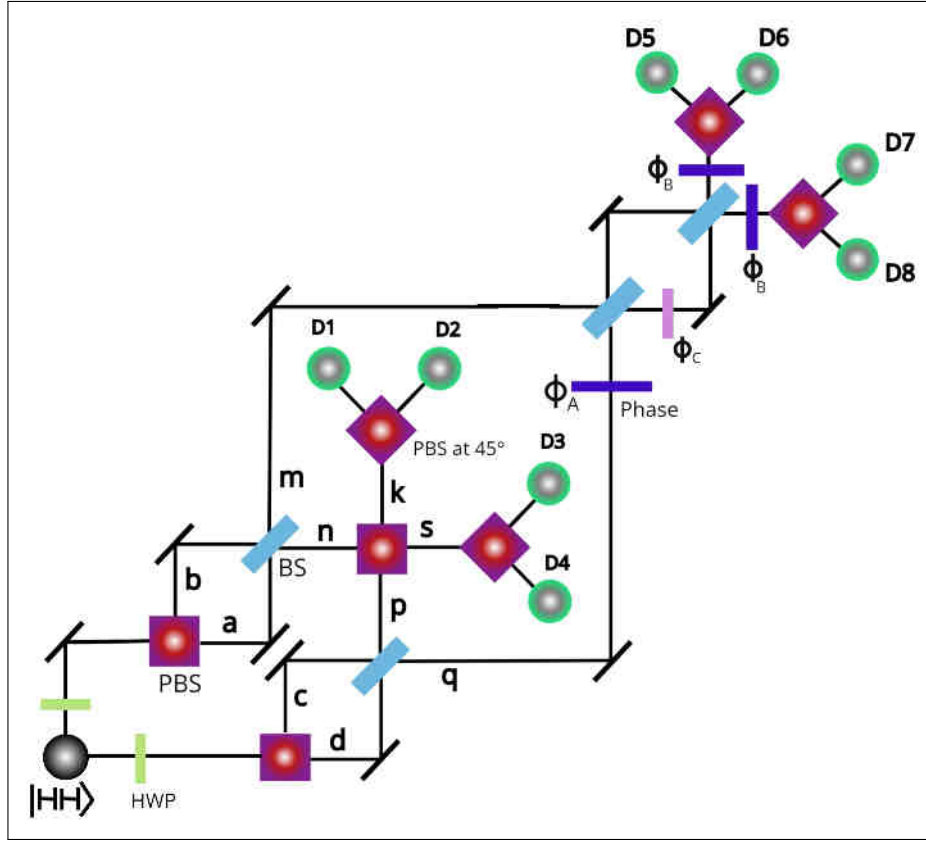


Figure 5.6: Modified Setup for Alice's 2^{nd} and 3^{rd} choices.

To choose between 2^{nd} and 3^{rd} choices, Alice has to change the value of $\Phi_c = \{0, \frac{\pi}{2}\}$.

We can write down the appropriate projection operators for Alice's and Charlie's choices as we did previously:

Alice	Basis	Charlie	Basis
x = 1	Π_z	z = 1	$\frac{\sigma_z + \sigma_x}{\sqrt{2}}$
x = 2	Π_x	z = 2	$\frac{\sigma_z - \sigma_x}{\sqrt{2}}$
x = 3	Π_y	z = 3	$\frac{\sigma_z + \sigma_y}{\sqrt{2}}$
		z = 4	$\frac{\sigma_z - \sigma_y}{\sqrt{2}}$
		z = 5	$\frac{\sigma_x + \sigma_y}{\sqrt{2}}$
		z = 6	$\frac{\sigma_x - \sigma_y}{\sqrt{2}}$

Here, Π_i operator corresponds to projection of Path amplitude qubit along following operators:

$$\Pi_z = |u'\rangle\langle u'| - |d'\rangle\langle d'|$$

$$\Pi_x = |u'\rangle\langle d'| + |d'\rangle\langle u'|$$

$$\Pi_y = -i|u'\rangle\langle d'| + i|d'\rangle\langle u'|$$

The maximum violation of CHSH_3^{SPE} will also be $6\sqrt{2}$ for the Complex Hilbert Space. But what can be said about the maximum violation provided by a Real Hilbert Space Formulation? Let us adopt the approach taken by the group that gave the CHSH_3 inequality[17]. Let us write the joint probability distribution for Alice's and Charlie's outcomes.

Once again, let $P(a, b, c|x, z)$ be the joint probability of getting the outcomes a,b, and c for Alice's and Charlie's choices.

$$P(a, b, c|x, z) = \sum_{\lambda} P(\lambda) \text{Tr} \left((\rho_{AB} \otimes \rho_{CD}) (A_{a|x} \otimes B_b \otimes C_{c|z}) \right) \quad (5.17)$$

Where ρ_{AB_1} and ρ_{B_2C} denote the density matrices of the two Single Photon Entangled states. $P(\lambda)$ denotes the probability distribution of a local hidden variable as in general Bell inequalities. $A_{a|x}$ and $C_{c|z}$ denote Alice and Charlies' projection operators for a and c outcomes. B_b is a projection operator that projects one of the photons in one of the four states. On projection, the other photon ends up in a Single Photon Entangled State.

$$B_b = \{a_+^\dagger a_k^\dagger | 00\rangle \langle 00|_{a_+ a_k}, \quad a_-^\dagger a_k^\dagger | 00\rangle \langle 00|_{a_- a_k}, \quad a_+^\dagger a_s^\dagger | 00\rangle \langle 00|_{a_+ a_s}, \quad a_-^\dagger a_s^\dagger | 00\rangle \langle 00|_{a_- a_s}\}$$

Since the Joint Probability Distribution in equation 5.17 for SPE states setup is equivalent to the Joint Probability Distribution of EPR pairs in equation 5.3, the bound for CHSH₃ for RHS formalism will also apply here. This means that if we can violate the CHSH₃^{SPE} inequality in equation 5.16 above 7.66, we have successfully falsified an RHS formalism.

There is something more we can ask from the SPE states setup. Remember that in the CHSH₃ inequality, the two EPR sources are independent of each other. They can be thought of to be correlated classically at maximum. Similarly, the two Single Photon Entangled states are prepared independently of each other. But it is also possible to use an entangled bipartite state instead of a separable bi-partite state. All we have to do is use an SPDC source that creates two entangled down-converted photons. The setup is given as below:

Let the creation operators for path and polarization modes be same as before; $a_a^\dagger, a_b^\dagger, a_c^\dagger, a_d^\dagger, a_m^\dagger, a_n^\dagger, a_p^\dagger, a_q^\dagger, a_k^\dagger, a_l^\dagger$ denote creation operators for the respective paths as labeled in the subscript and $a_H^\dagger, a_V^\dagger, a_+^\dagger, a_-^\dagger, a_L^\dagger, a_R^\dagger$ denote the creation operators for polarization mode as labeled in the subscript.

Initially, An EPR pair is released from the source towards the interferometer from the two ports;

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(a_{1H}^\dagger a_{2H}^\dagger + a_{1V}^\dagger a_{2V}^\dagger)|00\rangle$$

After passing through the polarizing beam splitter, the path and polarization modes become entangled, and we have a 4-partite system.

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(a_{1H}^\dagger a_{1a}^\dagger a_{2H}^\dagger a_{2c}^\dagger + a_{1V}^\dagger a_{1b}^\dagger a_{2V}^\dagger a_{2d}^\dagger \right) |0000\rangle$$

As this state passes through the non-polarizing beam splitters, it is transformed into;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{2\sqrt{2}} & \left[\textcolor{blue}{ia_{1H}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2p}^\dagger} + a_{1H}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2q}^\dagger - a_{1H}^\dagger a_{2H}^\dagger a_{1n}^\dagger a_{2p}^\dagger + \textcolor{blue}{ia_{1H}^\dagger a_{2H}^\dagger a_{1n}^\dagger a_{2q}^\dagger} \right. \\
& \left. + \textcolor{blue}{ia_{1V}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2p}^\dagger} + a_{1V}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2q}^\dagger + a_{1V}^\dagger a_{2V}^\dagger a_{1n}^\dagger a_{2p}^\dagger + \textcolor{blue}{ia_{1V}^\dagger a_{2V}^\dagger a_{1n}^\dagger a_{2q}^\dagger} \right] |0000\rangle
\end{aligned} \tag{5.19}$$

As this state evolves further, a horizontal photon in n path mode goes to s mode, and a vertical photon goes to k mode. We can find similar transformation rules for the photon in p mode.

$$\begin{aligned}
a_{1H}^\dagger a_{1n}^\dagger &\rightarrow a_{1H}^\dagger a_{1s}^\dagger & a_{1V}^\dagger a_{1n}^\dagger &\rightarrow a_{1V}^\dagger a_{1k}^\dagger \\
a_{2H}^\dagger a_{2p}^\dagger &\rightarrow a_{2H}^\dagger a_{2k}^\dagger & a_{2V}^\dagger a_{2p}^\dagger &\rightarrow a_{2V}^\dagger a_{2s}^\dagger
\end{aligned}$$

Applying these transformation rules and separating the highlighted terms, we get;

$$|\Psi\rangle = \frac{1}{2\sqrt{2}} \left[i \left(a_{1H}^\dagger a_{2H}^\dagger a_{1m}^\dagger a_{2k}^\dagger + a_{1H}^\dagger a_{2H}^\dagger a_{1s}^\dagger a_{2q}^\dagger + a_{1V}^\dagger a_{2V}^\dagger a_{1m}^\dagger a_{2s}^\dagger + a_{1V}^\dagger a_{2V}^\dagger a_{1k}^\dagger a_{2q}^\dagger \right) + \dots \right] |0000\rangle \tag{5.20}$$

Projecting the photon present in k and s mode along the diagonal and antidiagonal polarizations, we get;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{2\sqrt{2}} & \left[i \left(a_{1H}^\dagger a_{1m}^\dagger \frac{(a_{2+}^\dagger + a_{2-}^\dagger)}{\sqrt{2}} a_{2k}^\dagger + a_{2H}^\dagger a_{2q}^\dagger \frac{(a_{1+}^\dagger + a_{1-}^\dagger)}{\sqrt{2}} a_{1s}^\dagger \right. \right. \\
& \left. \left. + a_{1V}^\dagger a_{1m}^\dagger \frac{(a_{2+}^\dagger - a_{2-}^\dagger)}{\sqrt{2}} a_{2s}^\dagger + a_{2V}^\dagger a_{2q}^\dagger \frac{(a_{1+}^\dagger - a_{1-}^\dagger)}{\sqrt{2}} a_{1k}^\dagger \right) + \dots \right] |0000\rangle
\end{aligned} \tag{5.21}$$

The overlap of states in k and s modes makes the two photons indistinguishable, so we lose the information regarding the first and second photons. Taking this into consideration,

we get;

$$\begin{aligned}
|\Psi\rangle = \frac{1}{2\sqrt{2}} \Bigg[& i \left(a_+^\dagger a_k^\dagger \frac{(a_H^\dagger a_m^\dagger + a_V^\dagger a_q^\dagger)}{\sqrt{2}} \right. \\
& + a_-^\dagger a_k^\dagger \frac{(a_H^\dagger a_m^\dagger - a_V^\dagger a_q^\dagger)}{\sqrt{2}} \\
& + a_+^\dagger a_s^\dagger \frac{(a_H^\dagger a_q^\dagger + a_V^\dagger a_m^\dagger)}{\sqrt{2}} \\
& \left. + a_-^\dagger a_s^\dagger \frac{(a_H^\dagger a_q^\dagger - a_V^\dagger a_m^\dagger)}{\sqrt{2}} \right) + \dots \Bigg] |0000\rangle
\end{aligned} \tag{5.22}$$

Based on the outcome of one of the four detectors $D1, D2, D3, D4$, we get one of the four Single Photon Entangled states in the m and q mode. Notice here that all the Single Photon Entangled States are distinct from each other. This is slightly different than what we observed in the previous scenario. But it is also equivalent to swapping entanglement between two EPR pairs. Once again, We can formulate a CHSH₃ like inequality that depends on Bob's outcome. Again, CHS predicts the maximum violation of this inequality up to $6\sqrt{2}$. To calculate the maximum violation predicted by RHS formalism, let us first write down the expression for the Joint Probability distribution of Alice's, Bob's, and Charlie's outcome:

$$P(a, b, c|x, z) = \sum_{\lambda} P(\lambda) \text{Tr} \left((\rho_{Total})(A_{a|x} \otimes B_b \otimes C_{c|z}) \right) \tag{5.23}$$

Here, ρ_{Total} is the Total state of the system before the measurement. The state is not a separable one. Since this expression for Joint Probability Distribution is not the same as that of equation 5.17, we do not know what the maximum bound predicted by RHS for this system is. We leave this as an open question for future researcher to solve.

Chapter 6

Self-Certifying QRNG

Random Number Generators are used extensively in a number of classical as well as quantum protocols. A Quantum Random Number Generator extracts random numbers from the random collapse of a wave function. Since the collapse is inherently unpredictable, the random numbers are said to be truly random. There is no way to predict the outcome of these random number generators with certainty. This can be used to ensure the safety of security protocols that need random numbers in their implementation. A Self-Certifying QRNG is slightly different than a QRNG. The latter one cannot prove that the random numbers it has produced are quantum random. To certify that the random numbers are obtained from the collapse and not from a classical process which we are ignorant about, one can use an entangled state that can show correlations that are not possible to exist because of classical processes. A CHSH like inequality has to be violated to prove that entanglement is present. So, a Self-Certifying QRNG requires an entangled pair generator. For photons, a high intensity laser beam has to be incident on a non-linear crystal. By Spontaneous Parametric Down Conversion Process, we can have polarization entangled photon pairs. But the SPDC process is highly inefficient and requires a lot of energy and resources which makes it less preferable for manufacturing QRNGs.

Here, we will look at how Single Photon Entangled States can be used instead of using bipartite entangled photon pairs. This will allow us to make a QRNG that can certify that the random numbers are obtained from the random collapse of the wave function. The setup looks similar to Figure 2.2: Notice that we are using a Time Tagger here; it is to note down

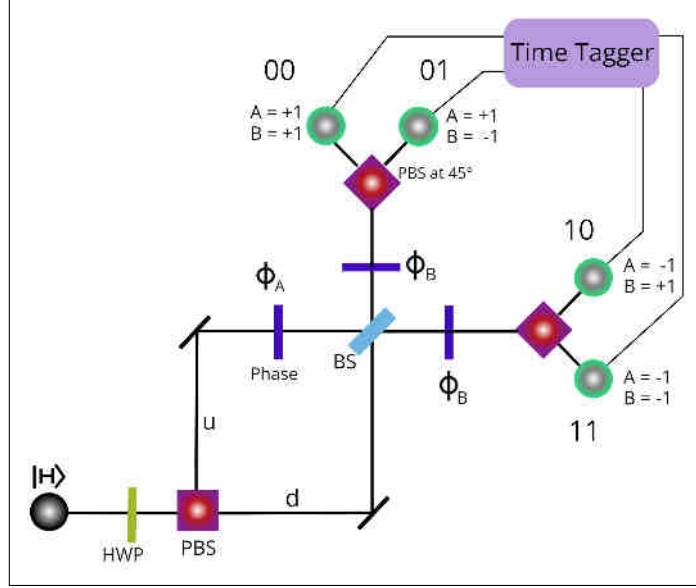


Figure 6.1: Self-Certifying QRNG Setup

which detector clicked in a given time interval of nanoseconds. It is implicit that we need a Single Photon source to make the QRNG. Otherwise, more than one detector will click in a given time window and we'll not know which value to consider and which to discard. Since the source may not always emit anti-bunched photons, we can post select the time intervals where only single detector counts were detected and no coincidence was observed between any of the four detectors. Before going any further, let's take a look at how we can derive a CHSH like inequality for the expectation value of Alice's and Bob's observables. Remember that Alice measures the path amplitude qubit and Bob measures the polarization. Classically, Alice's observable should only depend on the value of Φ_a and a hidden variable λ ; Similarly, Bob's observable should depend only on Φ_b and λ . The derivation will be identical to the derivation in [3]:

Let the Alice's and Bob's observable be represented by $A(\Phi_a, \lambda)$ & $B(\Phi_b, \lambda)$. Let $\rho(\lambda)$ represent the probability distribution of the hidden variable. Where, $a, b \in \{1, -1\}$. The expectation value of ab for choices of Φ_a & Φ_b will be given as:

$$E(a, b) = \int_{\lambda} A(\Phi_a, \lambda) B(\Phi_b, \lambda) \rho(\lambda) d\lambda$$

Adopting the derivation from [3]:

$$\begin{aligned}
E(a_1, b_1) - E(a_1, b_2) &= \int_{\lambda} A(\Phi_{a_1}, \lambda) B(\Phi_{b_1}, \lambda) \rho(\lambda) d\lambda - \int_{\lambda} A(\Phi_{a_1}, \lambda) B(\Phi_{b_2}, \lambda) \rho(\lambda) d\lambda \\
&= \int_{\lambda} \left[A(\Phi_{a_1}, \lambda) B(\Phi_{b_1}, \lambda) (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_2}, \lambda)) \right. \\
&\quad \left. - A(\Phi_{a_1}, \lambda) B(\Phi_{b_2}, \lambda) (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_1}, \lambda)) \right] \rho(\lambda) d\lambda
\end{aligned} \tag{6.1}$$

Taking the absolute values of both side;

$$\begin{aligned}
|E(a_1, b_1) - E(a_1, b_2)| &= \int_{\lambda} \left| A(\Phi_{a_1}, \lambda) B(\Phi_{b_1}, \lambda) (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_2}, \lambda)) \right. \\
&\quad \left. - A(\Phi_{a_1}, \lambda) B(\Phi_{b_2}, \lambda) (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_1}, \lambda)) \right| \rho(\lambda) d\lambda
\end{aligned} \tag{6.2}$$

Using the inequality:

$$|a - b| \leq |a| + |b|$$

$$\begin{aligned}
|E(a_1, b_1) - E(a_1, b_2)| &\leq \int_{\lambda} \left\{ \left| A(\Phi_{a_1}, \lambda) B(\Phi_{b_1}, \lambda) \right| \left| (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_2}, \lambda)) \right| \right. \\
&\quad \left. + \left| A(\Phi_{a_1}, \lambda) B(\Phi_{b_2}, \lambda) \right| \left| (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_1}, \lambda)) \right| \right\} \rho(\lambda) d\lambda
\end{aligned} \tag{6.3}$$

Since $|A(\Phi_a, \lambda) B(\Phi_b, \lambda)| \leq 1$, we can rewrite equation 6.3 as:

$$\begin{aligned}
|E(a_1, b_1) - E(a_1, b_2)| &\leq \int_{\lambda} \left\{ (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_2}, \lambda)) \right. \\
&\quad \left. + (1 \pm A(\Phi_{a_2}, \lambda) B(\Phi_{b_1}, \lambda)) \right\} \rho(\lambda) d\lambda
\end{aligned} \tag{6.4}$$

Rewriting this in terms of expectation values:

$$|E(a_1, b_1) - E(a_1, b_2)| \leq 2 \pm (E(a_2, b_2) - E(a_2, b_1))$$

The terms on RHS is less than or equal to both $2 + (E(a_2, b_2) + E(a_2, b_1))$ & $2 - (E(a_2, b_2) - E(a_2, b_1))$. Taking this into consideration and using inequality $|a + b| \leq |a| + |b|$:

$$|E(a_1, b_1) + E(a_2, b_1) + E(a_2, b_2) - E(a_1, b_2)| \leq 2 \quad (6.5)$$

Hence, we have obtained an upper bound on the maximum possible correlation between the observables of Alice's and Bob's measurement. In equation 5.8 in section 2.2, we saw that this inequality can reach a violation up to $2\sqrt{2}$ for choices of $\Phi_a = 0, \frac{\pi}{2}$ and $\Phi_b = \frac{\pi}{4}, \frac{3\pi}{4}$.

Now let us see how we can calculate the expectation value for a given choice of Φ_a & Φ_b from the string of random numbers obtained from the Time Tagger. In a given time interval of the order of pico seconds, the time dagger will register a click between one of the four detectors. Let's label the detectors as 00, 01, 10 & 11. Each detector click will provide the values of Alice's and Bob's observables. For some chosen Φ_a & Φ_b , let the following be the string of single detector clicks obtained from the Time Tagger.

$$(\Phi_a, \Phi_b) \Rightarrow (00)(11)(10)(01)(00)(00)(10)(11)(10)(00)(01)(11)...$$

A click on 00 & 11 means the Alice and Bob's observables are identical, 1, 1 or $-1, -1$, and a click on 01 & 10 means the observables are distinct, 1, -1 or $-1, 1$. The expectation value can be calculated in terms of number of clicks for identical and distinct observables as:

$$E(a, b) = \frac{N_{Same} - N_{Opposite}}{N_{Total}}$$

Writing in terms of number of clicks for each detector:

$$E(a, b) = \frac{N_{00} + N_{11} - N_{01} - N_{10}}{N_{00} + N_{11} + N_{01} + N_{10}}$$

Similarly, we can obtain the expectation values for other choices of Φ_a & Φ_b and check if it violates the inequality in equation 6.5. If the inequality has found to be violated, we have successfully certified that the numbers are obtained from the random collapse of the wave function and hence are truly random.

Chapter 7

Summary and Conclusion

- In this thesis, we saw how Bell State Measurement is performed and what is the underlying effect behind it. BSM can be used to implement sophisticated protocols in Quantum Information Theory as well as can help in addressing the foundational issues in Quantum Mechanics.
- In order to experimentally verify that our source generating a quantum state does generate the state it is supposed to generate, we performed Quantum State Tomography on the source and reconstructed the density matrix of the ensemble.
- We experimentally tested the CHSH inequality and obtained a value of 2.287 and saw a violation by around 14 standard deviations. The results are found to be consistent with the predictions of Quantum Mechanics. We should, however, mention that the conducted experiment was not a loophole free Bell test. As a part of the present work, a high precision type-1 SPDC was built by me, but the characterization of the source could not be completed due to lack of time. Improvement over the standard system was done by using high precision optics, filters, and high efficiency detectors with an efficiency of 60% at 810nm and dark count rate of 200 cps as against the commercial system which had a detector efficiency of 30% and dark count rate of 500 cps. While the commercial system only used a long pass filters, band pass filters at 810 ± 10 nm were used to improve the visibility.
- We examined Single Photon path-polarization hyper-entangled states and investigated how they can be used to distinguish between RHS and CHS. The original proposal

for the experimental of the difference between RHS and CHS entails the use of two entanglement sources with swapping. We propose an experimental arrangement that requires only a single hyper-entangled bipartite source.

- We also proposed a Self-Certifying Quantum Random Number Generator that uses Single Photon Entangle States.

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Appendix A

Python Codes for Quantum State Tomography

A.1 Single Photon QST

The python codes for Linear Inversion and Maximum Likelihood Estimation for Single photon QST are given below:

Linear Inversion

The code for linear inversion will is:

```
1 from numpy.linalg import eig
2 import numpy as np
3
4 print('Input no. of counts in horizontal basis')
5 Nh = int(input())
6
7 print('Input no. of counts in vertical basis')
8 Nv = int(input())
9
10 print('Input no. of counts in diagonal basis')
11 Nd = int(input())
```

```

12
13 print('Input no. of counts in right-circular basis')
14 Nr = int(input())
15
16 Nt = Nv + Nh
17 Na = Nt - Nd
18 Nl = Nt - Nr
19
20 Density = [[Nh/Nt, complex((Nd-Na)/(2*Nt), (Nr-Nl)/(-2*Nt))],
21             [complex((Nd-Na)/(2*Nt), (Nr-Nl)/(2*Nt)), Nv/Nt]]
22
23 print('The Density Matrix is:', Density)
24
25 w, v = eig(Density)
26 print('Eigenvalues are : ', w)
27
28 Density_square = np.dot(Density, Density)
29 k = 0
30 p = 0
31 for i in range (2):
32
33     k = k + Density_square[i][i]
34     p = p + Density[i][i]
35
36 print('The trace of square density matrix is:', k, "The trace of density
    matrix is:", p)

```

Maximum Likelihood Estimation

```

1 import numpy as np
2 import math
3 from numpy.linalg import eig
4 from scipy.optimize import fmin_bfgs
5
6 def f1(t):
7     Ta = np.array([[t[0], 0], [t[1] + complex(0, 1)*t[2], t[3]]])
8     T_dagger = Ta.conj().T
9     A = np.dot(T_dagger, Ta)
10    k = 0
11    for i in range (2):

```

```

12     k = k + A[i][i]
13
14     Density_2 = (1/k)*A
15
16     eig_value,eig_vec = eig(Density_2)
17
18     coincidence_counts = np.array([4894,5102,9987,5323])
19
20     h = np.array([[1],[0]])
21     v = np.array([[0],[1]])
22     d = np.array([[1/(2**(1/2))],[1/(2**(1/2))]])
23     l = np.array([[1/(2**(1/2))],[complex(0,1)*1/(2**(1/2))]])
24
25     ket = [h,v,d,l]
26     Norm = coincidence_counts[0] + coincidence_counts[1]
27
28     def f2(i):
29         bra= ket[i].conj().T
30         p = np.dot(bra,Density_2)
31         q = np.dot(p,ket[i])
32         y = q.real
33
34         return y
35     Fun2 = 0
36
37     for i in range(4):
38
39         Fun2 = Fun2 + ((Norm*f2(i) - (coincidence_counts[i]))**2)/(Norm
40         *2*f2(i)) + 0.5*math.log(Norm*f2(i))
41     return Fun2.ravel()
42
43 x1 = np.array([0.1,0.1,0.1,0.1])
44 xopt = fmin_bfgs(f1,x1 )
45 print(xopt)
46
47 Ta = np.array([[xopt[0],0],[xopt[1] + complex(0,1)*xopt[2],xopt[3]]])
48 T_dagger = Ta.conj().T
49
50 A = np.dot(T_dagger,Ta)
51 k = 0
52 for i in range (2):

```

```

52     k = k + A[i][i]
53
54 Density = (1/k)*A
55
56 print(Density)
57
58 w,v = eig(Density)
59 print('Eigenvalues are : ',w)
60
61 Density_square = np.dot(Density,Density)
62 k = 0
63 p = 0
64 for i in range (2):
65
66     k = k + Density_square[i][i]
67     p = p + Density[i][i]
68
69 print('The trace of square density matrix is:',k,"The trace of density
    matrix is:",p)

```

A.2 Two Photon QST

The python codes for Linear Inversion and Maximum Likelihood Estimation for Two photon QST are given below:

A.2.1 Linear Inversion

```

1 import numpy as np
2 import math
3 from numpy.linalg import eig
4 from scipy.optimize import fmin_bfgs
5
6 def f1(t):
7     Ta = np.array([[t[0],0],[t[1] + complex(0,1)*t[2],t[3]]])
8     T_dagger = Ta.conj().T
9     A = np.dot(T_dagger,Ta)
10    k = 0

```

```

11     for i in range (2):
12         k = k + A[i][i]
13
14     Density_2 = (1/k)*A
15
16     eig_value,eig_vec = eig(Density_2)
17
18     coincidence_counts = np.array([4894,5102,9987,5323])
19
20     h = np.array([[1],[0]])
21     v = np.array([[0],[1]])
22     d = np.array([[1/(2**(1/2))],[1/(2**(1/2))]])
23     l = np.array([[1/(2**(1/2))],[complex(0,1)*1/(2**(1/2))]])
24
25     ket = [h,v,d,l]
26     Norm = coincidence_counts[0] + coincidence_counts[1]
27
28     def f2(i):
29         bra= ket[i].conj().T
30         p = np.dot(bra,Density_2)
31         q = np.dot(p,ket[i])
32         y = q.real
33
34         return y
35     Fun2 = 0
36
37     for i in range(4):
38
39         Fun2 = Fun2 + ((Norm*f2(i) - (coincidence_counts[i]))**2)/(Norm
40         *2*f2(i)) + 0.5*math.log(Norm*f2(i))
41     return Fun2.ravel()
42
43 x1 = np.array([0.1,0.1,0.1,0.1])
44 xopt = fmin_bfgs(f1,x1 )
45 print(xopt)
46
47 Ta = np.array([[xopt[0],0],[xopt[1] + complex(0,1)*xopt[2],xopt[3]]])
48 T_dagger = Ta.conj().T
49
50 A = np.dot(T_dagger,Ta)
51 k = 0

```

```

51 for i in range (2):
52     k = k + A[i][i]
53
54 Density = (1/k)*A
55
56 print(Density)
57
58 w,v = eig(Density)
59 print('Eigenvalues are : ',w)
60
61 Density_square = np.dot(Density,Density)
62 k = 0
63 p = 0
64 for i in range (2):
65
66     k = k + Density_square[i][i]
67     p = p + Density[i][i]
68
69 print('The trace of square density matrix is:',k,"The trace of density
    matrix is:",p)

```

A.2.2 Maximum Likelihood Estimation

```

1
2 import numpy as np
3 from numpy.linalg import eig
4 from scipy.optimize import fmin_bfgs
5 import math
6
7 def f1(v):
8
9     Ta = np.array([[v[0], v[4]+complex(0,1)*v[5], v[6]+complex(0,1)*v[7],
10                     v[10]+complex(0,1)*v[11]], [0, v[1], v[8]+complex(0,1)*v
11                     [9],
12                     v[12]+complex(0,1)*v[13]], [0,0, v[2],
13                     v[14]+complex(0,1)*v[15]], [0,0,0, v[3]]])
14
15     T_dagger = Ta.conj().T
16
17     A = np.dot(T_dagger, Ta)

```

```

17
18 k = 0
19 for i in range (4):
20     k = k + A[i][i]
21
22 Density_2 = (1/k)*A
23
24
25 eig_value,eig_vec = eig(Density_2)
26
27 matrix = np.array([[40230, 150 ,220, 39094],
28                    [19970, 20533 ,18542, 20050],
29                    [21988 ,18910, 18003, 21054],
30                    [20556, 19706, 20010 ,18629],
31                    [333 ,39918, 38011, 440],
32                    [20475 ,19975, 18931, 19544],
33                    [19900, 19649, 20761, 19330],
34                    [17423, 21755 ,21207 ,18804],
35                    [38717, 401, 483, 39382]])
36
37 p = 0
38 Norm = 0
39 for i in range(4):
40     for j in range(4):
41         p = p + matrix[i][j]
42         Norm = 0.25*p
43
44 h = np.array([[1],[0]])
45 v = np.array([[0],[1]])
46 d = np.array([[1/(2**(1/2))],[1/(2**(1/2))]])
47 a = np.array([[1/(2**(1/2))],[-1/(2**(1/2))]])
48 l = np.array([[1/(2**(1/2))],[complex(0,1)*1/(2**(1/2))]])
49 r = np.array([[1/(2**(1/2))],[-complex(0,1)*1/(2**(1/2))]])
50
51 ket = np.array([[np.kron(h,h),np.kron(h,v),np.kron(v,h),np.kron(v,v)],
52                [np.kron(h,d),np.kron(h,a),np.kron(v,d),np.kron(v,a)],
53                [np.kron(h,r),np.kron(h,l),np.kron(v,r),np.kron(v,l)],
54                [np.kron(d,h),np.kron(d,v),np.kron(a,h),np.kron(a,v)],
55                [np.kron(d,d),np.kron(d,a),np.kron(a,d),np.kron(a,a)],
56                [np.kron(d,r),np.kron(d,l),np.kron(a,r),np.kron(a,l)],
57                [np.kron(r,h),np.kron(r,v),np.kron(l,h),np.kron(l,v)],
58                [np.kron(r,d),np.kron(r,a),np.kron(l,d),np.kron(l,a)],

```

```

58     [np.kron(r,r),np.kron(r,l),np.kron(l,r),np.kron(l,l)]]))
59
60     def f2(i,j):
61         bra= ket[i][j].conj().T
62         p = np.dot(bra,Density_2)
63         q = np.dot(p,ket[i][j])
64         y = q.real
65
66         return y
67     Fun2 = 0
68
69     for i in range(9):
70         for j in range (4):
71             Fun2 = Fun2 + ((Norm*f2(i,j) - matrix[i][j])**2)/(2*Norm*f2(i,
72             j)) + 0.5*math.log(Norm*f2(i,j))
73
74     return Fun2.ravel()
75
76 v1 = np.array
77     ([0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,
78     0.01,0.01,0.01,0.01])
79 v = fmin_bfgs(f1,v1 )
80
81 Ta = np.array([[v[0],v[4]+complex(0,1)*v[5],v[6]+complex(0,1)*v[7],v[10]+
82     complex(0,1)*v[11]], [0,v[1],v[8]+complex(0,1)*v[9],v[12]+complex(0,1)*v
83     [13]], [0,0,v[2],v[14]+complex(0,1)*v[15]], [0,0,0,v[3]]])
84
85 T_dagger = Ta.conj().T
86
87 A = np.dot(T_dagger,Ta)
88 k = 0
89 for i in range (4):
90     k = k + A[i][i]
91     Density_2 = (1/k)*A
92
93 print(Density_2)
94 w,v = eig(Density_2)
95 print('Eigenvalues are : ',w)

```

```
95 Density_square = np.dot(Density_2,Density_2)
96 k = 0
97 p = 0
98 for i in range (4):
99
100     k = k + Density_square[i][i]
101     p = p + Density_2[i][i]
102
103 print('The trace of square density matrix is:',k,"The trace of density
    matrix is:",p)
```