Lévy Processes in finance

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Certificate

This is to certify that this dissertation entitled Lévy Processes in finance towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Siddhesh Sindar at Indian Institute of Science Education and Research Pune under the supervision of Dr. Indranil Sengupta, Department of Mathematics , during the academic year 2022-2023.

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Dr. Indranil Sengupta

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This thesis is dedicated to my family

Declaration

I hereby declare that the matter embodied in the report entitled Lévy Processes in finance are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Indranil Sengupta and the same has not been submitted elsewhere for any other degree.

Siddhesh

Siddhesh Sundar

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Abstract

This project aims to survey literature on lévy processes. We look at the stochastic calculus and the properties of jump processes and then look at applications of lévy processes in finance. These include markets driven by lévy processes and pricing of stock options under markets driven by lévy processes.

Contents

Introduction

Stochastic processes are being used since a long time to describe real world phenomena. In particular they have been extensively used to describe financial markets.These financial markets consist of many assets which keep changing in time with uncertainty. Hence, stochastic processes are useful tools to model market phenomena.

Black and Scholes [12] gave the price of an european call option under some very simplistic assumptions. Merton expanded upon this and developed a theory of rational option pricing in [13]. In this simplified black scholes model we have that the stock price of an asset under various assumptions follows a Geometric brownian motion i.e

$$dS_t/S_t = \mu t + \sigma W_t$$

Harrison and Pliska [3] gives a unified treatment of the requirement and pricing of european contingent claims.

However, as observed in the market ([8],[14]) the asset returns do not compute with what is observed from the black scholes model. So there was a need to remove strict assumptions the model. This led to the introduction of models in which σ varies with time. Essentially in such scenarios we deal with incomplete markets. A exposition in discrete time setting can be found in [15]. Stochastic volatility include those models in which the volatility is a stochastic process. See ([16], [17], [18]). Pricing of various options under these incomplete markets required the use of different hedging strategies including superhedging, quadratic hedging, etc. For a overview see Cont and Tankov[7].

Despite relaxing conditions and increasing complexity of the models we have that the stock prices are continuous which is not possible in reality as there are sudden drops or increases in stock prices and the risk due to that is not accounted for. Hence, jumps were introduced in the returns of stock prices. Merton [10] was the first one to introduce jumps in the model. Also see [19] where a jump diffusion model is used.

Non Gaussian Orsntein Uhlenbeck processes are used to model volatility as it gives rises as they many interesting properties which can be exploited to estimate volatility, returns, etc. This gives rise to the barndorff Nielsen and Shephard model. See [6, 9, 21]. Nicolato and Vernardos [22] under some assumptions gives a range of the option prices of an european call option

This thesis we will explain some of the literature mentioned above. The first chapter will just include some preliminaries required for the later chapters. Chapter 2 we will give elucidate on lévy process and their properties. In Chapter 3 we will introduce the non gaussian OU process and briefly talk about their properties. Also we will introduce the BNS model in this chapter. Chapter 4 will be about the basic terms and terminology required to understand option pricing. Chapter 5 will be about the Merton's jump diffusion model.

Original Contribution: There is no original contribution, rather this project is a literature survey of lévy process in finance, in particular we try to learn the BNS model. Also the stochastic calculus required was read from Protter[23].

Chapter 1

Preliminaries

We work with a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

Definition 1.0.1. *Consider an integrable process i.e* $E[|X_t|], \infty \forall t \geq 0$ *Then*

1. *It is a martingale if* $E[X_t|\mathcal{F}_s] = X_s$ *almost surely* $\forall 0 \leq s < t$
2. *It is a submartingale if* $E[X_t|\mathcal{F}_s] \leq X_s$ *almost surely* $\forall 0 \leq s < t$
3. *It is a supermartingale if* $E[X_t|\mathcal{F}_s] \geq X_s$ *almost surely* $\forall 0 \leq s < t$

Definition 1.0.2. *A stopping time* $T : \Omega \to [0, \infty]$ *is a measurable random variable(r.v) such that* $\{T \leq t\} \in \mathcal{F}_t \forall t \geq 0$

Definition 1.0.3. *A local martingale is a stochastic process such that there exists a sequence of stopping times* $T_n \to \infty$ *such that the stopped processes* $X_t^T(\omega) = X_{min(t,T(\omega))}(\omega)$ *are martingales* $\forall n \in \mathbb{N}$

Definition 1.0.4. *The variation of a function* $f : E \to \mathbb{R}$ *on an interval* $[a, b] \subset E$ *is given by*

$$V_b^a(f) = \sup_{\mathcal{P}} \sum_{i=1}^{n_P} (|f(x_i) - f(x_{i-1})|)$$

where $\mathcal{P}$ *is any partition of* $[a, b]$.

If any function has finite variation over any bounded interval then the function is said to have finite variation. If the variation process of a stochastic process X_t given by

$V_t^0(X(\omega)) = \sup_{\mathcal{P}} \sum_{i=1}^{n_P}(|X_i(\omega) - X_{i-1}(\omega)|)$ if finite almost surely $\forall t$ ten the process is said to be of finite variation.

Definition 1.0.5. *A poisson process X_t with intensity λ is a cadlag process with $X_0 = 0$ and $X_t - X_s$ is independent of $\mathcal{F}_s$ and $X_t - X_s \sim Poi(\lambda(t-s))$*

$Poi(a)$ is the poisson distribution with parameter a. It can be shown that every process can be written in the following form

$$X_t = \sum_{n \geq 1} \mathbf{1}_{T_n \leq t}$$

where $T_n = \sum_{i=1}^{n} \tau_i$ and τ_i are iid exponential λ variables. A poisson process is a markov process.

A compensated poisson process $\tilde{X}_t$ is defined as $\tilde{X}_t = X_t - \lambda t$ where X_t is a poisson process with intensity λ.

A radon measure is a measure which is finite on all compact measurable sets. Examples include lebesgue measure on the real number line.

Definition 1.0.6. *Given $(\Omega, \mathcal{F}, P)$ be a probability space, E is a set on some space S (can be $\mathbb{R}^d$) and μ is a positive radon measure on (E, ξ). Then $M(\omega, A), A \in \xi, \omega \in \mathcal{F}$ is a poisson random measure*

1. *$M(\omega, A)$ for fixed A is a poisson r.v in ω with parameter $\mu(A)$, for bounded measurable A, $M(A) < \infty$ is an integer valued r.v*
2. *$M(\omega, A)$ for fixed ω is an integer valued radon measure on E*
3. *For disjoint sets $A_1, A_2, ..., A_n$, $M(A_1), M(A_2), ..., M(A_n)$ are independent to each other.*

Chapter 2

Lévy processes

This chapter we are going to elaborate about lévy processes and their properties.

2.1 Lévy processes

Definition 2.1.1. *A lévy process X_t is a cadlag adapted $\mathbb{R}^d$ valued process starting from zero and satisfying the following properties*

1. *independent increments: for any $s < t \in \mathbb{R}_+$ we have that $X_t - X_s$ is independent of $\mathcal{F}_s$.*
2. *stationary increments: $\forall s < t, u \in \mathbb{R}_+$ we have that $X_{t+u} - X_{s+u}$ has the same distribution as $X_t - X_s$*
3. *X_t is continuous in probability.*

Examples of lévy processes include compound poisson processes, brownian motions, etc.

Definition 2.1.2. *An infinitely divisible distribution F is one in which for every n, $\exists$ iid r.vs X_i such that $X_1 + X_2 + .. + X_n$ has distribution F.*

For a lévy process X_t, marginal distribution of X_t is infinitely divisible as $X_t = (X_t - X_{t_{n-1}}) + (X_{t_{n-1}} - X_{tn-2}) + ... + (X_{t_1} - X_{t_0})$ where $t_k = kt/n$ and the increments are iid. Conversely, It can be shown that the every infinitely divisible distribution has a characteristic function equal to that of Z_1 of a lévy process. Actually every lévy process has a standard representation of its characteristic function. Therefore Z_1 has a standard representation. So the proof of the converse statement involves defining this representation directly and show that every infinitely divisible distribution has this representation. An analytic proof of this is given in [1].

2.2 Building processes from poisson random measures

Given any poisson random measure $M(\omega, x)$ we know there exists X_n such that $M = \sum \delta_{X_n}$. Now consider a poisson random measure on $\Omega \times E$ where $E = [0, T] \times \mathbb{R}^d$ there exists (T_n, Y_n) such that $M = \sum \delta_{(T_n, Y_n)}$. T_n are interpreted to be stopping times and Y_n is said to be $\mathcal{F}_{T_n}$ measurable Therefore T_n model the happening of some type of event and the value of some characteristic of that event corresponds to Y_n. Suppose M has intensity measure μ and take for any simple function of the form $f = \sum_{i=1}^n c_i \mathbf{1} A_i$ we can write

$$\int f dM = \sum i = 1^n c_i M(A_i) = \sum_{i=1}^{n} f(T_n, Y_n)$$

We can extend non negative functions by monotone convergence. That is for a non negative function f there are simple functions $f_n \uparrow f$, we have

$$M(f) = \int f dM = \lim_{n \to \infty} \int f_n dM$$

by monotone convergence. Now note that $E[\int f dM] = \int f d\mu$. So for a general real valued function f we can find it's integral wrt M by breaking $f = f^+ - f^-$ into two non negative parts. The integral will exist if

$$\int_0^T \int_{\mathbb{R}^d \backslash 0} |f(s, x)| \mu(ds, dx) < \infty$$

Thus for any function f we have

$$X_t(f) = \int_0^t \int_{\mathbb{R}^d \backslash 0} f(s, y) M(ds, dy) = \sum_{n, Tn \in [0,t]} f(T_n, Y_n)$$

. Thus this yields a jump process. The compensated process given by

$$X_t(f) = \int_0^t \int_{\mathbb{R}^d \setminus 0} f(s,y)(M(ds,dy) - \mu(ds,dy))$$

is a martingale.

Now if the (T_n, Y_n) are such T_n are stopping times that tend to infinity as n goes to infinity then (T_n, Y_n) are called marked point processes. We can construct the random measure to the similar way as before but it may not satisfy the properties of it being a poisson random measures. for example $M(\omega, A_i)$ for disjoint sets A_i may not be independent. However we just define integration of such processes in a similar vein as above.

Suppose for a given cadlag process $X_t, t \leq T$ we take (T_n, Y_n) to be the jump times of the process, i.e $T_n(\omega) =$ nth jump of $X_t(\omega), t \in \mathbb{R}_+$ and take $Y_n = \triangle X_{T_n}$. Note we can do this because the jumps of a cadlag process are countable. The measure called the jump measure of the process X is given by

$$J_X(\omega, A) = \sum_{n=1}^{\infty} \delta_{(T_n(\omega), Y_n(\omega))}(A) = \sum_{t \in [0,T], \triangle X_t \neq 0} \delta_{(t, \triangle X_t)}(A) \tag{2.1}$$

for a measurable subset $A \in \mathbb{R}^d$. We So we have that

$$J_X(\omega, [0,t] \times A) = \text{No of jumps in } [0,t] \text{with } \triangle X_t(\omega) \in A$$

Therefore $J(\omega, A) =$ No of $(t, \triangle X_t(\omega))$ such that $(t, \triangle X_t(\omega)) \in A$

2.3 Lévy ito decomposition

For a compound poisson process $X_t = \sum_{i=0}^{N_t} Y_i$ where N_t is a poisson process of intensity λ and Y_i are iid with distribution f, we have the characteristic function of X_t given by

$$E[e^{iuX_t}] = e^{t\lambda \int_{\mathbb{R}^d} (e^{iux} - 1) f(dx)} \tag{2.2}$$

To show this is very simple. Just condition LHS on the no of jumps N_t and since the Y_i are iid (2.2) follows.

Definition 2.3.1. *Lévy measure of a lévy process X_t on $\mathbb{R}^d$ is measure on $\mathbb{R}^d$.*

$$\nu(A) = E[No\ of\ t\ in\ [0,1] : \triangle X_t \neq 0, \triangle X_t \in A] \tag{2.3}$$

Therefore $\nu(A)$ gives the average no of jumps in one unit of time whose magnitude is in A. For a compound poisson process having intensity measure μ and intensity λ, we have $\mu(dt, dx) = dt\nu(dx) = dt\lambda f(dx)$.

As you can see the jump measure completely describes the jumps of the process and doesn't tell anything about the continuous part of the process.So we can decompose the continuous parts and the discontinuous parts separately. We know that processes of the form $X_t = bt + W_t + \sum_{s=0}^{t} \triangle X_s$ where the jumps are given by a compound poisson process are lévy processes. We will see subsequently that every lévy process can be decomposed into a definite term bt called the drift, a centered gaussian process with mean 0 and covariance matrix Σ and its jumps. Now take a look at it's jumps. For a cadlag process the no of jumps in a closed interval $[0, t]$ is countable and more importantly the set of jumps in $[0, t] \geq C, C > 0$ is finite for all possible C. However , the no of jumps in $[0, t]$ may be countably infinite. So for $\sum \triangle X_s$ to make sense the sum of jumps whose size is between 0 and C must converge and subsequently the lévy measure must satisfy certain properties for the decompsition to make sense.

We will introduce the lévy ito decomposition. Before that we see that the characteristic function of a lévy process is $\Phi_t(u) = E[e^{iuX_t}]$ is continuous in u and t. For a proof see [11]. This combined with fact that lévy processes have stationary and independent increments we can see that

$$\Phi_t(u) = E[e^{iuX_t}] = e^{t\psi(u)} \tag{2.4}$$

$\psi(u)$ is called the cumulant generating function of the lévy process or the characteristic exponent of the lévy process.

Theorem 2.3.1. ***Lévy ito decomposition:*** *Given a a lévy process X_t on $\mathbb{R}^d$ with jump measure $J_X(\omega, A), A \in \mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $\mu(dt, dx) = dt\nu(dx)$ where ν is the lévy measure $\exists\ b \in \mathbb{R}^d$ and a centered gaussian process W_t(i.e $W_t - W_s \sim N(0, \Sigma(t - s)$, $\Sigma_t ad \times dmatrix$(the covariance matrix) $W_t - W_s$ is independent of $\mathcal{F}_s$, it is continuous and* *starts from zero) such that*

$$\begin{aligned} X_t &= bt + W_t + X'_t + \tilde{X}_t \\ X'_t &= \int_{|x|\geq 1, s\in[0,t]} xJ_X\left(ds, dx\right) \\ \tilde{X}_t &= \int_{|x|<1, s\in[0,t]} x\tilde{J}_X\left(ds, dx\right) \end{aligned} \tag{2.5}$$

where $\tilde{J}$ *is the compensated jump measure and it is a poisson random measure on* $\mathbb{R}_+ \times \mathbb{R}^d$*. The last integral satisfies*

$$\int_{|x|<1, s\in[0,t]} x\tilde{J}_X\left(ds, dx\right) = \lim_{\epsilon\downarrow\downarrow 0} \int_{\epsilon\leq|x|<1, s\in[o,t]} x\tilde{J}_X\left(ds, dx\right)$$

. The convergence holds almost surely and it is uniform Also ν *is a positive radon measure on* $\mathbb{R}^d\backslash\{0\}$ *(called the lévy measure) which satisfies*

$$\int_{|x|\leq 1} |x|^2\nu\left(dx\right) < \infty \int_{|x|\geq 1} \nu\left(dx\right) < \infty \tag{2.6}$$

The terms in the sum are all independent to each other.

Proof. We will briefly describe the proof here. A detailed proof is given in [11]. First you must note that for any jump measure J, $J(\omega, [0,t] \times A)$ where A is bounded below is a poisson random measure and hence a poisson process in (t, ω). Also for disjoint sets A, B, the poisson processes $J(\omega, [0,t] \times A)$ and $J(\omega, [0,t] \times B)$ are independent. Then we check that $\int_{s\in[0,t], x\in A} xJ_x\left(ds, dx\right)$ and $\int_{s\in[0,t], x\in B} xJ_x\left(ds, dx\right)$ are independent poisson processes for disjoint sets A, B bounded below.

Now note that if the jumps of a lévy process are bounded, then all the moments of X_t $(E[|X_t|^n], n \in \mathbb{N})$ exist.
To show this we take a sequence of stopping times $T_{n+1} = inf\{t \geq 0, |X_t - X_{T_n}| > C\}$ where C is the constant te jumps are bounded above by. Then we can show that the stopped processes X^{T_n} are bounded above by $2nC$. We use this and the strong markov property to show that the moments exist.

This implies that is the jumps are bounded because all the moments exist, the characteristic function and so the characteristic exponent/ cumulant generating function of the lévy process is smooth. Now put

$$^{a}Y_t = X_t - \int_{s\in[0,t],|x|\geq a} xJ_X(ds,dx)$$

. $^{a}Y_t$ is a lévy process and a lévy process and it has bounded jumps iff it is of the form $^{a}Y_t$. Essentially, we are removing jumps of size at least a from the lévy process. The size here is arbitrary. Let us take $a = 1$ and let $^{1}Y_t = Y_t$ and $\hat{Y}_t = Y_t - E[Y_t]$.

We prove that $\hat{Y}_t = Y_t^c + Y_t^d$ where Y_t^c and Y_t^d are independent lévy process and Y_t^c is continuous. To do that we take $B_n = x, \epsilon_{n+1} \leq |x| < \epsilon_n$ and $A_n = \cup_{i=1}^{n} B_i$ and let

$$M_t(\omega, A_n) = \int_{s\in[0,t],x\in A_n} x\tilde{J}_x\,(\omega, ds, dx)$$

. If we know $M_t(\omega, A_n)$ converges almost surely, then Y_t^d is nothing the limit as $n \to \infty$ of $M_t(\omega, A_n)$. This will give rise to the last integral in the lévy ito decomposition. To do that we note that since B_i and B_j for $i \neq j$ are disjoint $M_t(., B_i)$ and $M_t(., B_j)$ are independent. Also $\hat{Y}_t - M_t(., A_n)$ and $M_t(., A_n)$ are independent. To prove this we use the property that two r.vs are independent iff $\Phi(uX + vY) = \Phi(uX)\Phi(vY)$ [11] $\forall u, v$. We can use these results to show that Y_t^d is a limit of $M_t(., A_n)$.

In fact, we will get that $M_t(., A_n)$ converges the $L2$ norm to Y_t^d. Since, we have the identity that $E[\sup_{s\leq t} M_s^2] \leq 4E[M_t^2]$ we get that $M_t(., A_n)$ converges a.s and uniformly to Y_t^d. Now to prove Y_t^d is a lévy process we know that since $M_t(., A_n)$ converges uniformly a.s it will converge in probability as well. There is a theorem[11] which says that if X^n is a sequence of lévy processes and X_t is another process such that $\lim_{n\to\infty} \limsup_{t\to 0} P(|X_t^n - X_t| > c) = 0$ $\forall\ c > 0$ then X_t is a lévy process.

In a similar fashion we can prove Y_t^c is a lévy process. Independence of Y_t^c and Y_t^d can be proved by looking at their characteristic functions as explained in the previous paragraph. We prove Y_t^c is continuous by contradiction. Also we can easily shows the conditions for the lévy measure. Finally to prove that X_t^c is a centered gaussian process, we look at the characteristic function of Y_t^c and see that it matches that of a centered gaussain process. Since, characteristic function completely determine the distribution we finally are done with the theorem. $\square$

Now because of this decomposition the characteristic function of any lévy process can be written in a standard manner.

Indeed noting that since all the terms in the decomposition are independent with each other we get that

$$E[e^{iuX_t}] = E[e^{iubt}]E[e^{iuW_t}]E[e^{iuX'_t}]E[e^{iu\tilde{X}_t}]$$

Hence, the characteristic function of a lévy process is given by

$$\begin{aligned} \Phi_t^X(u) &= E[e^{iuX_t}] = e^{t\psi^X(u)} \\ \psi^X(u) = ibu - \frac{1}{2}u^T\Sigma u &+ \int_{\mathbb{R}^d}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu\,(dx) \end{aligned} \tag{2.7}$$

where $b \in \mathbb{R}^d$, Σ is a positive symmetric matrix and ν is the lévy measure. As you can see from (2.7) (b, Σ, ν) completely determines ψ^X and hence it completely determines the c.f (characteristic function) and hence it completely determines X_t.

Now we will state some results without proofs.

2.4 Types of lévy processes and other properties

Lemma 2.4.1. *A lévy process is of finite variation iff $\Sigma = 0$ and $\int_{|x|\leq 1}|x|\nu(dx)$ is finite .*

The proof of this is given in [7]. We will just briefly explain how it comes about. Notice that in the ito lévy decomposition there is a gaussian term whose variation is infinite. So for finite variation the gaussian term should be zero. Hence, $\Sigma = 0$. So the lévy process is a pure jump process with a drift term. The drift term clearly is finite variation. Now for a pure jump process the variation up to a time t is nothing but the sum of jumps up to that time. Therefore the third term in the decomposition always has finite variation as the no of jumps in that term is finite. . For the fourth term, we need that $\sum_{s\in[0,t],\triangle X_s\leq 1}\triangle X_s$ to be finite. if $\int_{|x|\leq 1}|x|\nu(dx) = E\left[\sum_{s\in[0,t],\triangle X_s\leq 1}\triangle X_s\right]$ is finite we are done.

Subordinators are increasing lévy processes on $\mathbb{R}$. Hence, they are of finite variation.

Therefore $\Sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$. Also since, it is increasing we have positive drift b and no negative jumps. For that $\nu(-\infty, 0) = E[\text{No of jumps in}(-\infty, 0)] = 0$.In fact to check whether a lévy process is a subordinator you just need to check whether $X_t \geq 0$ a.s for some $t \geq 0$. This will be enough to guarantee that $X_t \geq 0$ a.s $\forall t \geq 0$ and the fact that it is a subordinator [7].

Subordinators are used as time changes of other lévy processes. Consider the following really simple example. Z_t is a poisson process then take $Y_t = X_{Z_t}$. Clearly this a pure jump process and jumps occur at the jump times of the poisson process. It is clearly a compound poisson process.

For a subordinator, we use the moment generating function instead of the characteristic function. we have

$$\begin{aligned} E[e^{uZ_t}] &= e^{t\psi((\frac{1}{i})u)} = e^{tl(u)} \\ l(u) &= au + \int_0^\infty (e^{ux} - 1)\nu(dx) \\ a &= b - \int_{|x|\leq 1} |x|\nu(dx) \end{aligned} \tag{2.8}$$

A lévy process time changed with subordinator is another lévy process. Consider the following theorem given below

Theorem 2.4.2. *Given a lévy process X_t with triplet (b, σ, ν) and a subordinator with triplet $(a, 0, \rho)$ then $Y_t = X_{Z_t}$ is a lévy process with triplet (b^Y, Σ^Y, ν^Y) such that*

$$\begin{aligned} \sigma^Y &= b\Sigma, \\ \nu^Y(A) &= b\nu(A) + \int_0^\infty p_s^X(A)\rho(ds), \quad \forall B \in \mathcal{B}(\mathbb{R}^d), \\ b^Y &= ab + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x p_s^X(dx), \end{aligned} \tag{2.9}$$

where p_t^X is the probability distribution of X_t.

A proof of this is given in [1].

Lemma 2.4.3. *For a lévy process with triplet (b, σ, ν) on $\mathbb{R}$, the nth absolute moment*

$E[|X_t|^m]$ exists iff $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$. The moments can be got by differentiating the characteristic function of X_t. We get

$$E[X_t] = t\left(b + \int_{|x|\geq 1} x\nu(dx)\right) \; Var(X_t) = t\left(\sigma + \int_{\mathbb{R}} x^2\nu(dx)\right) \tag{2.10}$$

Chapter 3

Non gaussian OU process

In this chapter we introduce a type of stochastic process called the Non Gaussian Orsntein Uhlenbeck process. We elucidate some of their properties. We also introduce the BNS model and motivate why we were studying this process.

3.1 The non gaussian OU process

The non guassian OU process is the solution to the SDE

$$dX_t = -\lambda X_t dt + dZ_{\lambda t} \tag{3.1}$$

where Z_t is a lévy process. It is also called the *background driving lévy process(BDLP).*
In the above equation putting $Y_t = e^{\lambda t} X_t$ we get using product rule for stochastic processes and putting (3.1) in dY_t we get

$$dY_t = e^{\lambda t} dZ_{\lambda t}$$

Hence we get that

$$X_t = X_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dZ_s \tag{3.2}$$

If we take the following equation

$$X_t = \int_{-\infty}^0 e^s dZ_{\lambda t+s} \tag{3.3}$$

where Z_t is the same as above, we see by change of variables we get eqn 3.2. We will see later the reason for writing like this.

Lemma 3.1.1. *Take a function f defined on $[0,T]$ taking values on $\mathbb{R}$, then*

$$E\left[\exp\left(iu\int_0^T f(t)dZ_t\right)\right] = \exp\left\{\int_0^T \psi(uf(t))dt\right\}, \tag{3.4}$$

where $E\left[e^{iuZ_t}\right] = e^{t\psi(x)}$

Proof. The proof involves standard tricks employed in measure theory. First take a simple function of the form $f(t) = \sum_{i=1}^N f_i \mathbf{1}_{(t_{i-1},t_i)}(x)$. Then for this process we have $\sum_{i=1}^N f_i\left(Z_{t\wedge t_i} - Z_{t\wedge t_{i-1}}\right)$. Calculating the characteristic function of this integral and using the fact that increments are independent we get

$$\prod_{i=1}^N E[\exp\left(iuf_i Z_{t\wedge t_i} - Z_{t\wedge t_{i-1}}\right) = \prod_{i=1}^N \exp(f_i\left(t_i - t_{i-1}\right)$$

which gives the RHS of eqn (3.4). Once you have proved for simple functions note that for every non negative function f, there is a sequence of elementary simple functions f_n converging to f. The lemma holds for each fn. Then by monotone convergence, it holds for f. □

Given the above lemma, we can get the characteristic exponent(If $E\left[e^{iuZ_t}\right] = e^{t\psi(x)}$, then ψ is the cumulant generating function) of the non gaussian OU process. Comparing this the lévy khinchtine formula, we get that for a lévy process having triplet (Σ, ν, b), the triplet of the OU process is

1. $$\Sigma_t^X = \frac{\Sigma}{2\lambda}\{1 - e^{-2\lambda t}\} \tag{3.5}$$

2. $$b_t^X = \frac{b}{\lambda}\{1 - e^{-\lambda t}\} + X_0 e^{-\lambda t} \tag{3.6}$$

3. $$\nu_t^X = \int_1^{e^{\lambda t}} \nu(xB)\frac{dx}{\lambda x}, \quad \forall B \in \mathcal{B}(\mathbb{R}) \tag{3.7}$$

A lévy field $\mathbf{z}$ on E is a random measure on $(\Omega, \mathscr{F} X(E,\xi))$ such that for every set $B \in \xi$ $\mathbf{z}(\omega, B)$ is infinitely divisible and for disjoint sets $B_i \in \xi$, $\mathbf{z}(B_i)$ are independent . Thus, each $\mathbf{z}(A)$ has its own lévy khinchtine decomposition and triplet $(\Sigma(A), \nu(A), b(A))$. The lévy measure of $\mathbf{z}$ is given by $\nu(dx, da) = \nu(dx, a)\upsilon(da)$ we can get a formula similar to the one given in lemma (3.1.1), i.e

$$E\left[\exp\left(iu \int_E f(a)\mathbf{z}(da)\right)\right] = \exp\left(\int_{\mathbb{R}} \psi(uf(a))\upsilon(da)\right) \tag{3.8}$$

If we take $\kappa(u, X) = \log(E\left[e^{uX}\right])$ Then for a random field with triplet $(0, 0, \nu(dx, da)$ with a func f on E which is non negative admits a representation

$$\int f(a)\mathbf{dz}\,(\omega, a) = \int f(a)d\nu'\,(a) + \int f(a)M\,(\omega, da) \tag{3.9}$$

where $M(\omega, A)$ is a poisson random measure on $\Omega \times E$ with intensity measure $\nu(dx, da)$ and another measure ν' on E Then, we have

$$\psi(1, \int f\mathbf{dz}) = i\int f\nu'(da) + \int_{\mathbf{R}_+}\int_E \left\{e^{if(a)x} - 1\right\}\nu(dx, da). \tag{3.10}$$

Thus we can get

$$\kappa(1, \int f\mathbf{dz}) = \int f\nu'(da) + \int_{\mathbf{R}_+}\int_E \left\{e^{f(a)x} - 1\right\}\nu(dx, da). \tag{3.11}$$

3.2 Selfdecomposibility

A probability distribution P is self decomposible if it characteristic function can be decomposed in the following manner
for every positive λ $\exists$ probability distribution Q_λ such that $\Phi_P(u) = \Phi_P(e^{-\lambda}u)\Phi_{Q_\lambda}(u)$
Φ are the c.fs.
We have the cumulant function $\psi(u, X) = \log(E\left[e^{iu.X}\right])$, then a d dimensional r.v is self decomposible if $\forall\lambda > 0 \exists$ r.v X_λ s.t $\psi(u, X) = \psi(u, \lambda X) + \psi(u, X_\lambda)$

Theorem 3.2.1. *For a probability distribution which is infinitely divisible with lévy measure $\nu(dx)$, the following are equivalent:*

1. *It is self decomposible*

2. $\nu^+(e^s)$ *and* $\nu^-(-e^s)$ *are convex. Here* $\nu^+(x) = \nu([x, \infty))$ *and* $\nu^-(x) = \nu((-\infty, x])$

3. $\exists$ *lévy density* $u(x)$ *of the lévy measure associated with the infinitely divisible with* $|x|u(x)$ *increasing for* $x < 0$ *and decreasing for* $x > 0$.

A proof of this can be given [1].

Given the above theorem it can be easily shown that for a lévy process satisfying

$$\int_{|x|\geq 1} \log(|x|) \, \nu(dx) \tag{3.12}$$

then the non guassian OU process satisfying (3.3)(driven by the above the above lévy process has a stationary distribution μ which is infinitely divisible and self decomposible.

We can get the relation between the cumulant and te cumulant of the lévy process by

$$\psi(u) = u \frac{\partial \tilde{\psi}(u, X)}{\partial u} \tag{3.13}$$

where $\tilde{\psi}(u, X)$ is the cumulant function of a r.v with distribution μ as given above and ψ is the cumulant function of the lévy process. For a proof of this particlaur relation see [6]

Also, given a self decomposible and infinitely distribution μ, there is a lévy process Z, such that the non guassian OU process generated by this process has a stationary distribution μ. The lévy measure of μ can be got by taking $t \to \infty$ in (3.7). For a proof of hte above two statements see [7].

From (3.13) with a change of variables and differentiating, we get A r.v X is said to belong of class L if it is infinitely divisible and self decomposible. If the lévy measure of the distribution μ given by $\tilde{\nu}$ has a density $\tilde{u}$ then the lévy density of the lévy process Z is given by

$$u(x) = -\tilde{u}(x) - x\tilde{u}'(x) \tag{3.14}$$

From the above it is clear that for every lévy process, under certain conditions, it's non Gaussian OU process is a stationary process with the marginal distribution of the OU process (distribution of X_t) given by a self decomposible and infinitely divisible distribution. This property is very useful because any system governed by a non Gaussian OU process can be estimated/simulated by using the marginal distribution of the OU process. Two examples of self decomposible class L include the Generalized Inverse Gaussian distribution(GIG$(\lambda, \delta, \gamma)$) and the Normal Inverse Gaussian distribution(NIG $\alpha, \beta, \mu, \delta$). their properties can be found in [6]. We will just list their densities here.

For GIG$(\lambda, \delta, \gamma)$, the probability distribution is

$$p(x, \lambda, \gamma, \delta) = \frac{(\gamma/\delta)^{\lambda}}{2K_{\lambda}(\delta, \gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{x}+\gamma^2 x\right)} \tag{3.15}$$

where λ is real and the other constants are non negative. K_{λ} is the bessel function of the third kind with index λ.

For NIG$(\alpha, \beta, \mu, \delta)$ we have

$$p(x, \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q\left(\frac{x-\mu}{\delta}\right)^{-1} K_1\left(\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right) e^{\beta x} \tag{3.16}$$

where $q\left(x\right) = \sqrt{1+x^2}$ and $\alpha = \sqrt{\beta^2+\gamma^2}$ and

$$a = \frac{1}{\pi}\alpha e^{\delta\sqrt{\alpha^2-\beta^2}-\beta\mu}$$

and K_1 is the bessel function of third kind with index 1

3.3 The BNS model

Let us now motivate the use of the BNS model which we state below. The stock price in a market following a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ follows

$$\begin{aligned} S_t &= S_0 e^{X_t}, \\ dX_t &= (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho d\bar{Z}_t, \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + d\bar{Z}_t, \quad \sigma_0^2 > 0, \end{aligned} \tag{3.17}$$

where W_t is a standard brownian motion and $d\bar{Z}_t = dZ_t - E[Z_t]$ and Z_t is a lévy subordinator with no drift starting from zero. This is called the background driving lévy process(BDLP). ρ is called the leverage effect and is strictly negative. As you can see the stochastic volatility follows as a non gaussian OU process. So the process has negative jumps only.

There are certain properties of stock returns that motivate the use of modelling stock prices as something other than geometric brownian motions. If we measure log returns over a period $\triangle$, divide the timeline into $\triangle$ time periods , then the log return over $\triangle$ is given by $X(n\triangle) - X((n-1)\triangle)$. You should note that stock prices are reported in a discrete manner(could be in terms of hours, days, etc.) So that's we need to discretize our model. When these returns were analysed [14] they had the following properties

1. For small $\triangle$ they were heavy tailed in the sense the distribution of the returns decayed slower than an exponential distribution . As $\triangle$ became bigger it gradually weaned towards the normal distribution.

2. They were highly skewed.

3. Estimates of volatility decreased with the increase in returns

However for a GBM we have that the returns over a particular time period is given $X(n\triangle) - X((n-1)\triangle) = \mu\triangle + \sigma\ (W(n\triangle) - W((n-1)\triangle)$ we get that the return is normally distributed which is not heavy tailed, not skewed.

Now let us consider the BNS model. (3.15) gives rise to

$$X_t = \mu t + \beta \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s W_s + \rho \overline{Z}_{\lambda t} \tag{3.18}$$

The second term is called the integrated variance and we denote it by $(\sigma^2)_t^* = \int_0^t \sigma_s^2 ds$, the same as it was in [9]. Now since σ^2 is an inverse gaussian OU process from (3.2), it is easy to find the integrated variance. The integrated variance is given by

$$(\sigma^2)_t^* = \frac{1}{\lambda}\big(1 - e^{-\lambda t}\big)\sigma_t^2 + \int_0^t \frac{1}{\lambda}\big(1 - e^{-\lambda t}\big) dZ_{\lambda s}. \tag{3.19}$$

From the representation introduced before let us take for a r.v X

$$\kappa(\theta, X) = \log E[e^{\theta X}], \overline{\kappa}(\theta, X) = \log E[e^{-\theta X}] \tag{3.20}$$

κ is taken as the cumulant transform for the subordinator. Since we know that σ^2 is a non gaussian OU process under some conditions, it will be a stationary with its marginal distribution of class L. There are various class L distributions including GIG and NIG. Using this we can find the distribution of the BDLP and the integrated variance. Thus, we can estimate various quantities.

From ito's formula we easily get that

$$dS_t = S_{t-}(b_t dt + \sigma_t dW_t + \rho dZ_{\lambda t}) \tag{3.21}$$

where

$$b_t = \lambda + \kappa(\rho) + (\beta + 1/2)\, \sigma_t^2 \tag{3.22}$$

b_t is called the appreciation rate and M_t is given by

$$M_t = \sum_{0 \le s \le t} \left(e^{\rho \triangle_{\lambda s}} - 1 \right) - \lambda \kappa(\rho) t \tag{3.23}$$

Chapter 4

Option pricing fundamentals

This chapter we will introduce pricing in the BNS model

4.1 Basic financial terms and theorems

Consider a filtered probability space (ω, F, F_t, P). An asset is an entity which holds economic value which can be traded in the market. For eg, stocks and bonds are examples of assets. A derivative is any contract that gets value from the underlying asset. For example, an European option is a contract that gives the buyer of the contract the right to buy 100 stocks of a particular company at a later time at a fixed time decided beforehand.

Now consider that you can trade in stocks and bonds only. Let S^i $i \geq 1$ be the ith asset(eg stock) and S^0 be the price of a bond. A market is described by these assets. A contingent claim is any random variable X that is $\sigma(S^i_u, u \in [0,T], 1 \leq i \leq d)$ measurable, has finite expectation and variance and give the owner of the claim X units of money at time T, i.e a payoff at time T. For an european call option as explained above if the stock price underlying the option at time T (called as the time to maturity) is S_T and the stock to be bought at K, called the strike price then, the payoff at time T is $S_T - K$.

A person generally invests in stocks, options, bonds, etc. so that a later time he can reap profits. A trading strategy is all he investments and selling and trading of stocks, options, bonds, etc. that he does so as to reap a profit at a later time. A portfolio is the set of all assets held by an investor. If he holds, π_t^i amount of stock S^i at time t, The valuation of the portfolio is $V_t(\theta) = \sum_{i=0}^{d} \pi_t^i S_t^i$ where $\theta = (\pi^0, \pi^1, , \pi^j)$ is the trading strategy. Clearly as you cannot look into the future π_t^i is F_t measurable.

If suppose we can take and deposit money at the interest rate r, then a loan or deposit of A units will grow to Ae^{rt} in time t. Now consider the following scenario:
Suppose at time we have one stock S_0. Suppose at time t, becomes S_t. If we say that no matter what we know that $S_t > S_0 e^{rt}$, then I can in theory, borrow S_0 from the bank, buy one unit of stock and sell the stock at time t, and close the loan I will make a profit without investing anything. This is called an arbitrage opportunity. More formally, an arbitrary opportunity is:
$\exists$ a strategy θ where the initial investment π_0^i is zero but the we have that

$$\begin{aligned} V_t(\theta) \geq 0 \text{ almost surely and} \\ P(V_T(\theta) > 0) > 0 \end{aligned} \tag{4.1}$$

If there is no such strategy then it is said to satisfy no arbitrage. The discounted stock and bond prices denote $\tilde{S}_t^i = S_t e^{-rt}$ and the discounted valuation is $\tilde{V}_t(\theta) = V_t e^{-rt}$.

Pricing refers to finding the current price of a derivative or contingent claim if it gives a payoff at later time.
. There are two generally two methods to price derivatives. Before that we will give the following statements without proof. A rigorous treatment is given in [3].

1. An equivalent martingale measure is a measure Q that is equivalent to P such that the discounted assets(e.g discounted stock prices) are local martingales with respect to Q.

2. An attainable claim is a claim in which there exists a strategy θ and a portfolio V such that $V_T = X$

3. A market is said to be complete if every claim is attainable.

4. A market satisfies NA(no arbitrage) opportunity iff there exists an EMM wrt P

5. The market is complete iff all Q martingales M_t can be represented in the following

manner

$$M_t = E^Q(M_T) + \sum_{i=1}^{d} f^i d\tilde{S}^i \tag{4.2}$$

for predictable processes satisfying

$$\sum_{i=1}^{d} \int_0^t \left|f_s^i\right|^2 d\left[\tilde{S}^i, \tilde{S}^i\right] \text{ a.s}Q \tag{4.3}$$

Now let us formalise these concepts. Given a market (ω, F, F_t, P) where F_t is the information of the market gathered upto time t. We have

1. Let the market consists of d assets. S^0 representing the bond. $V_0(\theta)$ is the investment made at time zero.

2. A simple trading strategy involves trading of assets at finite times $0 < t_1 < t_2 ... < t_{n+1} = T$, i.e π_t^i is a piece wise constant function.Let the investor hold $\pi_{t_j}^i$ amount of asset between times (t_j, t_{j+1}) then clearly $\pi_t^i = \pi_{t_j}^i$ for times between t_j and t_{j+1}. He makes changes in the portfolio at time t_i.

3. The gains process $G_t(\theta)$ is defined as the total profits or losses accumulated upto time t.

$$G_t(\theta) = \sum_{i=0}^{d} \pi_{t_i}\left(S_{t_i} - S_{t_{i-1}}\right) \tag{4.4}$$

and hence the discounted gains process is given by

$$G_t(\theta) = \sum_{i=0}^{d} \pi_{t_i}\left(\tilde{S}_{t_i} - \tilde{S}_{t_{i-1}}\right) \tag{4.5}$$

4. A self financing trading strategy is one in which after an initial investment there is no further investment or consumption(selling of assets) up to time T. Hence for a self financing strategy, we have that

$$\begin{aligned} V_t(\theta) &= V_0(\theta) + \sum_{i=0}^{d} \pi_{t_i}\left(S_{t_i} - S_{t_{i-1}}\right) \\ \tilde{V}_t(\theta) &= V_0(\theta) + \sum_{i=0}^{d} \pi_{t_i}\left(\tilde{S}_{t_i} - \tilde{S}_{t_{i-1}}\right) \end{aligned} \tag{4.6}$$

5. We now assume that the assets are continuously traded in time. Then we define the gains process and the discounted gains process as follows:

$$\begin{aligned} G_t(\theta) &= \sum_{i=0}^{d} \int_0^t \pi_u^i dS_u \\ \tilde{G}_t(\theta) &= \sum_{i=0}^{d} \int_0^t \pi_u^i d\tilde{S}_u \end{aligned} \tag{4.7}$$

6. Hence, for a self financing strategy, we have that

$$\tilde{V}_t(\theta) = V_0(\theta) + \sum_{i=0}^{d} \int_0^t \pi_u^i d\tilde{S}_u \tag{4.8}$$

The notion of completeness and NA arbitrage opportunity is really important. If we have a complete market and a contingent claim X, we know that there exists a strategy (V_t, θ) such that $X_T = V_T(\theta)$. Then for the market to satisfy NA we require that the current price of the claim to be the expectation of the discounted Claim under the EMM, ie.

$$X_t = E^Q\left[e^{-r(T-t)} X_T\right] \tag{4.9}$$

where X_T is the payoff at time t. Also the discounted valuation process is a Q local martingale.

So pricing of a derivative involves finding an EMM which gives rise to NA then trying to prove completeness by showing (4.2) is true. Then if we know the payoff function of the derivative (4.9) gives the current price of the derivative. The measure Q is called the risk neutral measure.

4.2 Measure transformations of lévy processes

Girsanov's theorem tells us how a brownian motion under a particular measure can be transformed to a brownian motion under another equivalent measure. Similar thing can be done for lévy processes. Sato[1] talks about the conditions for which lévy processes under two different probability measures can be equivalant. It gives a way to convert a lévy process with a particular lévy measure to be a lévy process under an equivalent lévy measure. We will talk about a particular transformation called Escher's transform.

Let X be a lévy process with triplet (σ^2, ν, b) and asssume that we can find a θ s.t $\int_{-1}^{1} e^{\theta x}\nu\left(dx\right) < \infty$, then define

$$\begin{aligned}\tilde{\nu}(x) &= e^{\theta x}\nu(x)\\ \tilde{b} &= b + \int_{-1}^{1} x(e^{\theta x} - 1)\nu(dx)\end{aligned} \tag{4.10}$$

Then X with (σ^2, ν, b) and $\tilde{X}$ with $(\sigma^2, \tilde{\nu}, \tilde{b})$ are equivalent lévy processes with their probability distributions defined from the processes equivalent to each other. This transformation is called the escher's tranform of X.Suppose (X, P) and $(\tilde{X}, Q)$ are escher transforms of each other with P and Q equivalent. Then

$$\frac{dQ|_{F_t}}{dQ|_{F_t}} = \frac{e^{\theta X_t}}{E\left[e^{\theta X_t}\right]} \tag{4.11}$$

which comes out to be

$$\frac{dQ|_{F_t}}{dQ|_{F_t}} = \exp\left(\theta X_t - bt - t\left(\int_{\mathbb{R}}\left(e^{\theta x} - 1 - \theta x\mathbf{1}_{|x|\leq 1}\right)\right)\right) \tag{4.12}$$

Theorem 4.2.1. *Let (X, P) on $\mathbb{R}$ be a lévy process. Then if it neither as increasing nor as decreasing sample paths, then a market consisting of an asset of the form $S_t = e^{rt+X_t}$ satisfies NA opportunity.*

Proof. So we have to provide a measure in which e^{X_t} is a Q-local martingale. Then an EMM exists hence NA is satisfied. If the triplet is given by (σ^2, ν, b), the proof can be reduced to the case when σ is zero. Since brownian motion can transformed by subtracting a drift term to get an EMM. Therefore let us consider the case when σ is zero.

Now for the transformation let us consider the escher transform of X. Therfore, $\tilde{\nu}(x) = e^{\theta x}\nu(x)$ is equivalent measure to ν. Let us prove that this is an EMM. Now we want $\tilde{E}\left[\mathbf{1}_A e^{X_t}\right] = \tilde{E}\left[\mathbf{1}_A e^{X_s}\right] \forall A \in F_s$. So in general we want that $\tilde{E}\left[e^{X_t}\right] = \tilde{E}\left[e^{X_s}\right]$. Putting

$u = -(1/i)$ in the c.f of $\tilde{X}$ we get from the above equation that

$$(t-s)(\tilde{b} + \int_{-\infty}^{\infty} (e^x - 1 - x_{|x|\leq 1})\tilde{\nu}(dx) = 0) \ \forall t, s > 0$$

We need to find a solution to the above equation. Putting the value of $\tilde{b}$, we get that

$$f(\theta) = \int_{-\infty}^{\infty} (e^x - 1 - x_{|x|\leq 1}) e^{\theta x} \nu(dx) + \int_{-1}^{1} x(e^{\theta x} - 1)\nu(dx) = -b$$

Differentating f we get that

$$f'(\theta) = \int x \left(e^{\theta x} - 1\right) \nu\left(dx\right)$$

. Clearly since e^x dominates x, suppose that $\theta > 0$. In this case on $\mathbb{R}_+$ in the integrand $xe^{\theta x}$ dominates x and the integral will be a lrge positive value. When get that $f' \geq 0$. Now assume the conditions of the theorem. This can happen when:

Both $\nu(-\infty, 0)$ and $\nu(0, \infty)$ are positive, (+ve and -ve jumps). Break this case into two parts: when $\theta > 0$. Here the integral on $(0, \infty)$ will clearly dominate over the integral on $(-\infty, 0)$. Hence, the integral is strictly positive. Same can be done when $\theta < 0$. Hence, we have $f' > 0$. Therefore where f cuts the line $y = -b$ gives the value of θ.

Now consider the case when ν is concentrated on $(0, \infty)$. In this case for $\theta \to \infty$ $f \to \infty$ but for $\theta \to -\infty$ we have $f = -\int_0^1 x d\nu\left(x\right)$, so $f < 0$ but $b < 0$ and hence there exists a solution. $\square$

4.3 NA of BNS model

In this section we will show the NA for the BNS model. We take the simple case of $\mu = 0$ as given in [9]. The model we have is given by

$$\begin{aligned} X_t &= \beta \int_0^t \sigma_s^2 dt + \int_0^t \sigma_s dW_s + \rho \bar{Z}_{\lambda t} \\ \sigma_t^2 &= e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} dZ_{\lambda s} \end{aligned} \tag{4.13}$$

We have

$$\bar{Z}_t = Z_t - E[Z_t] = Z_t - t\left(b + \int_{|x|\geq 1} x\nu(dx)\right) = Z_t - t\xi$$

let $\kappa(\theta) = \log(E[e^{-\theta Z_1}])$ is the cumulant function of Z_1. Then $\bar{\kappa}(\theta) = \kappa(\theta) - \xi\theta$ is the cumulant function of Z_1. Let θ' be the solution to the equation

$$\kappa(\rho + \theta) - \kappa(\theta) = \xi\rho$$

Again we assume W_t and Z_t are independent. From (4.13) it is clear that σ^2 is $\sigma(Z)$ measurable and independent of W.

Take $\phi = \beta + 1/2$. $Y_t = \int_0^t \sigma_s dW_s$. We know the integrated variance $(\sigma^2)_t^* = \int_0^t \sigma_s^2 ds$ now and let

$$U_t = -\phi Y_t - \frac{1}{2}\phi^2(\sigma^2)_t^* + \theta' \bar{Z}_{\lambda t} - \lambda t \bar{\kappa}(\theta') \tag{4.14}$$

Now take

$$P'(A) = E[\mathbf{1}_A e^{U_t}]$$

Then P' will be a probability measure and X_t will be a martingale under P'.

Chapter 5

Merton's jump diffusion model

This chapter we will discuss Merton's jump diffusion model[10]. This is just a simple extension to the black scholes model. We will see just by addition of simple jumps leads to quite a lot of complications.

5.1 Model description

Given a market described by a complete filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}, \mathbb{P})$. Consider this market consists of stocks and bonds. The assumptions on the market include that there are no transaction costs involved. Also, assume that any trading strategy can be continuously changed in time. The lending and borrowing rates are assumed to be a risk free interest rate r. Then the evolution of a particular stock in the market is given by a geometric brownian motion:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)\, dt + \sigma dW_t + dM_t \tag{5.1}$$

where S_t is the stock price at time t, W_t is a standard brownian motion and

$$M_t = \sum_{i=1}^{N_t} Y_i \tag{5.2}$$

M_t is a compound poisson process, Y_i are iid r.vs and N_t is a poisson r.v with intensity λ.

When we mean instantaneous it usually means quantities involving 'dS'. So the instantaneous rate of return of the stock is given by $E[dS/S]$ which is α and wen the stock price doesn't jump the instantaneous variance of the return is given by σ^2. You can see the stock is continuous in intervals between jumps. Hence, this poisson process could be indicate the arrival of vital instantaneous information that causes the stock price to jump or down. The Y_i are assumed to follow a $N(\mu, \gamma^2)$ distribution. Now Note that eqn (5.1) can be written in more cleaner manner

$$dS_t/S_t = \begin{cases} (\alpha - \lambda k)\, dt + \sigma dW_t & \text{if the jump event doesn't occur} \\ \\ (\alpha - \lambda k)\, dt + \sigma dW_t + Y_t - 1 & \text{if the jump event occurs} \end{cases} \tag{5.3}$$

If suppose at time dt the price jumps from S_t to $S_t y$. Then we have the rate to change to be $(S_t y_t - S_t)/S = y_t - 1$. Here $k = E[y - 1]$. Clearly $y = e^Y$

5.2 Solution of the SDE

We will give a way to derive the solution of the sde (5.1). Using the Ito's formula for the jump diffusion and use it for $f(x) = log(x)$ to get

$$d \ln S_t = (\alpha - \lambda k) S_t \frac{1}{S_t} dt + \frac{\sigma^2 S_t^2}{2} \left(-\frac{1}{S_t^2} \right) dt + \sigma S_t \frac{1}{S_t} dW_t + [\ln y_t + \ln S_t - \ln S_t]$$

Here note that $S_{t-} + \triangle S_t = S_t y_t$ if it jumps at that time. If it doesn't jump $y_t = 0$. From this we get the solution of the sde to be

$$S_{_t} = S_{_0} e^{(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i} \tag{5.4}$$

5.3 Absence of perfect hedge

We will show that whatever happens you cannot find a trading strategy to eliminate the risk completely consisting of the stock and bond using options. For that consider a self financing strategy (π_t^0, π_t^1) which $\pi 0_t$ is the amount of bond held by the investor at time t at price B_t

and π_t^1 is the amount of stock S held at time t. We know that $dB_t = rB_t dt$. Therefore we have

$$V_t = \pi_t^0 B_t + \pi_t^1 S_t \tag{5.5}$$

Since the strategy is self financing we have that

$$dV_t = \pi_t^0 dB_t + \pi_t^1 dS_t \tag{5.6}$$

We will hedge by considering edging against one European call option $C(t, S_t)$. Therefore we have that

$$dV_t = dC(t, S_t) \tag{5.7}$$

The price of the European call option is assumed to be twice continuously differentiable function. So we again apply ito's formula on the both sides of (5.7). So we get,

$$\begin{aligned} dC\left(t,S_t\right) = \left(\frac{\partial C}{\partial t}\left(t,S_t\right)dt + \left(\alpha - \lambda k\right)\frac{\partial C\left(t,S_t\right)}{\partial x}dt + \frac{\sigma^2}{2}\frac{\partial^2 C\left(t,S_t\right)}{\partial^2 x^2}dt\right) + \\ \sigma\frac{\partial C\left(t,S_t\right)}{\partial x}dW_t + \left[C(S_{t-} + \triangle S_t\right] - C(S_{t-})] \end{aligned} \tag{5.8}$$

Now substituting en (5.1) in (5.6)

$$dV_t = r\pi_t^0 B_t dt + \pi_t^1\left(\alpha - \lambda k\right)dt + \pi_t^1 \sigma S_t dW_t + \pi_t^1(S_t\left(y_t - 1\right)) \tag{5.9}$$

Now putting (5.9) and (5.8) in (5.7) and equating all the dt, dW_t we get

$$\pi_t^1 = \frac{\partial C\left(t,S_t\right)}{\partial x} \tag{5.10}$$

and also we get

$$\begin{aligned} \pi_t^0 n B_t + \pi_t^1\left(\alpha - \lambda k\right)S_t = \frac{\partial C}{\partial t}\left(t,S_t\right) + \frac{\partial C\left(t,S_t\right)}{\partial x}\left(\alpha - \lambda k\right)S_t \\ + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C\left(t,S_t\right)}{\partial x^2} \end{aligned} \tag{5.11}$$

To eliminate risk completely we also need

$$\pi_t^1 S_t\left(y_t - 1\right) = C(S_t y_t) - C(S_t) \tag{5.12}$$

Now take a look at equation 5.10 and 5.12. Suppose that $C(t, S_t)$ is a linear function in x, i.e we have

$$C(t, S_t) = \frac{\partial C\,(t, S_t)}{\partial x} S_t + b \tag{5.13}$$

Then π_t^1 will satisfy (5.12). However, the price of European call options are non linear continuously differentiable convex functions on the positive real number line. Since C is convex LHS will always be smaller than RHS unless C is a linear function. Therefore, regardless of the strategy you cannot eliminate all risk.

5.4 Option pricing of the model

The model given above is incomplete as there are many EMMs. However, we will take a single EMM and finding the expectation of the discounted payoff of the European call option under the EMM.

Take

$$U_T = \exp\left(\frac{r-\alpha}{\sigma} W_T - \frac{1}{2}\left(\frac{r-\alpha}{\sigma}\right)^2 T\right) \tag{5.14}$$

Then from Girsanov's theorem we can say that $dQ = U_T dP$ are equivalent.

$W^{\mathbb{Q}} = W - \left(\frac{r-\alpha}{\sigma}\right)$ is $\mathbb{Q}$ local martingale.

Note that the compound Poisson process remains unchanged under this transformation. Under this transformation we have

$$S_t = S_0 \exp\left((r - \sigma^2/2 - \lambda k)t + \sigma W_t^{\mathbb{Q}} + \sum_{i=0}^{N_t} Y_i\right) \tag{5.15}$$

S_t is a markov process. IF $\mathscr{F}_t$ is the natural filtration generated by (W_t, M_t) then it is clear from (5.14) that $\sigma(S_s, s \leq t) = \sigma((W_s, M_s), s \leq t)$. Given this we have

$$E[f(S_t)|\mathscr{F}_s] = E[f(S_0 e^{r-\sigma^2/2-\lambda kt} e^{\sigma(W_t - W_s)} e^{M_t - M_s} e^{\sigma W_s + M s})|S_s] \tag{5.16}$$

Since $W_t - W_s$ and $M_t - M_s$ is independent of $\mathscr{F}_s$. We can take

$$E[f(S_t)|\mathscr{F}_s] = E[f(Ae^{\sigma W_s + M_s})|\mathscr{F}_s] = h(Ae^{\sigma W_s + M_s})$$

where A is independent of $\mathscr{F}_s$. Also note that h is S_s measurable. By iterative conditioning we have

$$E[f(S_t)|S_s] = E[E[f(Ae^{\sigma W_s + M_s})|\mathscr{F}_s)]|S_s] = E[h(Ae^{\sigma W_s + M_s})|S_s] = h(Ae^{\sigma W_s + M_s})$$

.

(5.14) gives the dynamics under $\mathbb{Q}$. If the payoff of the call option at time T is $X(S_T)$, then the option pricing formula at time T is given by

$$C_t^{\mathbb{Q}} = e^{-r(T-t)} E^{\mathbb{Q}}[(S_T - K)^+ |\mathcal{F}_t]. \tag{5.17}$$

This will be true if we assume that the risk is only due to the diffusion part i.e we just take (5.10) to be true and try to price the option without worrying about the risk due to jumps. Merton argued that in a large portfolio with many assets this is possible since the the diffusion (due to the brownian motion part) of diffusion assets are correlated but the jumps of different assets are not and just a feature of that asset. Therefore there is no 'systemic risk' due to the jumps. First note that since S_t is markov, we have that

$$C_t^{\mathbb{Q}} = e^{-r(T-t)} E^{\mathbb{Q}}[(S_T - K)^+ |S_t = S].$$

also noting that

$$\frac{S_T}{S_t} = \exp\left(\left(r - \sigma^2/2 - k\lambda\right)(T-t) + \sigma(W_T^{\mathscr{Q}} - W_t^{\mathscr{Q}}) + \sum_{i=1}^{N_T} Y_i - \sum_{i=1}^{N_t} Y_i\right)$$

we get that

$$C_t^{\mathbb{Q}} = e^{-r(T-t)} E^{\mathbb{Q}}\left[\left(S\exp\left(\left(r - \sigma^2/2 - k\lambda\right)(T-t) + \sigma(W_T^{\mathscr{Q}} - W_t^{\mathscr{Q}}) + \sum_{i=1}^{N_T} Y_i - \sum_{i=1}^{N_t} Y_i\right) - K\right)^+ \middle| S_t = S\right]$$

Now since brownian motion and the compound poisson measure have independent and stationary we have

$$C_t^{\mathbb{Q}} = e^{-r(T-t)}E^{\mathbb{Q}}\left[\left(S\exp\left(\left(r-\sigma^2/2-k\lambda\right)(T-t)+\sigma W_{T-t}^{\mathscr{Q}}+\sum_{i=1}^{N_{T-t}}Y_i\right)-K\right)^+\middle|S_t=S\right]$$

Take $\tau = T-t$. Condition of the no of jumps N_τ and the fact that the compound poisson measure is unchanged under change of measure.

$$C_t^{\mathbb{Q}} = e^{-r\tau}\sum_{i=0}^{\infty}\frac{e^{-r\tau}(\lambda\tau)^n}{n!}E^{\mathbb{Q}}\left[\left(S\exp\left(\left(r-\sigma^2/2-k\lambda\right)\tau+\sigma W_{\tau}^{\mathscr{Q}}+\sum_{i=1}^{n}Y_i\right)-K\right)^+\middle|N_\tau=n\right]$$

$\sigma W_{T-t}^{\mathscr{Q}}$ follows a $N(0,\sigma^2\tau)$ distribution. We assumed each jump of the compound poisson process has a $N(\mu,\gamma^2)$. Therefore the compound poisson process follows $N(n\mu,n\gamma^2)$. So we can see that

$$\sqrt{\sigma^2+n\gamma^2/\tau}W_\tau+n\mu=\sigma W_\tau+\sum_{i=0}^{n}Y_i \text{ in distribution}$$

SO under expectations they will give the same value. Putting $\sigma_n^2=\sigma^2+n\gamma^2/\tau$ and putting the value of $k=E[e^Y-1]=e^{\mu+\gamma^2/2}-1$ we get

$$E^{\mathbb{Q}}\left[\left(S\exp\left(r\tau-\frac{\sigma^2}{2}\tau-(e^{\mu+\gamma^2/2}-1)\lambda\tau+n\gamma^2/2-n\gamma^2/2+n\mu+\sigma_nW_\tau^{\mathscr{Q}}\right)-K\right)^+\middle|N_\tau=n,S_t=S\right]$$

$$E^{\mathbb{Q}}\left[\left(S\exp\left(n\mu+n\gamma^2/2-e^{\mu+\gamma^2/2}\lambda\tau-\lambda\tau\right)\exp\left(r\tau-\frac{\sigma_n^2}{2}\tau+\sigma_nW_\tau^{\mathscr{Q}}\right)-K\right)^+\middle|N_\tau=n,S_t=S\right]$$

If you take $S_n=S\exp\left(n\mu+n\gamma^2/2-e^{\mu+\gamma^2/2}\lambda\tau-\lambda\tau\right)$ then we get

$$C_t^{\mathbb{Q}} = e^{-r\tau}\sum_{i=0}^{\infty}\frac{e^{-r\tau}(\lambda\tau)^n}{n!}E^{\mathbb{Q}}\left[\left(S_ne^{(r-\sigma_n^2/2)\tau+\sigma_nW_\tau^{\mathbb{Q}}}-K\right)^+\middle|N_\tau=n,S_t=S\right]$$

but

$$e^{-r\tau}E^{\mathbb{Q}}\left[\left(S_ne^{(r-\sigma_n^2/2)\tau+\sigma_nW_\tau^{\mathbb{Q}}}-K\right)^+\middle|N_\tau=n,S_t=S\right]$$

is the black scholes price of a european call option which of time to maturity τ, strike price K and stock price S_n and volatility σ.

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