On Finite Type Property of Étale Sites

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उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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Contents

Abstract

In this thesis, we consider the notion of finite type-ness or Postnikov completeness of a site introduced by Morel and Voevodsky. One of the important consequences of having a finite type site is the existence of an exact fibrant resolution functor which preserves fibrations. The finite type-ness of the Nisnevich site has crucial consequences in the development of $\mathbb{A}^1$-homotopy theory, in particular in obstruction theory.

With the motivation towards the development of étale $\mathbb{A}^1$-homotopy theory, we investigate the finite type-ness of the étale site $(\mathrm{Sm}/k)_{\text{ét}}$ of finite type smooth schemes over a field k. We conjecture that this étale site is of finite type if and only if k admits a finite extension L with finite cohomological dimension. Our main result proves this conjecture when the absolute Galois group G_k is first-countable, which holds, in particular, for countable fields. Additionally, we establish necessary conditions for the finite type-ness of this site by proving that if k has arbitrarily large order higher degree cohomologies, which includes the case when $cd_p(k)$ is infinite for infinitely many primes, then this site is not of finite type.

1

Introduction

1.1 Motivation

In modern homotopy theory in algebraic geometry (like in [MV]), one works with the category of simplicial sheaves (or presheaves) on a site (with Grothendieck topologies like the Nisnevich or étale) endowed with a local injective model structure.

The fibrant resolution functor (for the def. see 2.3.4) on a model category is a necessary tool to study its homotopy category. In the case of the category of simplicial sets, there exists Kan's fibrant resolution functor which is 'nice' in the sense that it preserves finite limits and fibrations [MV, Lemma 1.67, page 70]. However, in the category of simplicial sheaves (or presheaves) on a site, we currently lack a general construction of such a nice fibrant resolution functor.

One such construction is given by Morel and Voevodsky in [MV, Theorem 1.66, pages 69-70] such that, if the site T has enough points, the natural Godement resolution functor

$$Ex^{\mathcal{G}} : \Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T}) \to \Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$$

acts as such a nice fibrant resolution functor under the additional assumption that the site T is of finite type, *i.e.*, Postnikov complete. It is also remarked in [MV, page 66] that it is currently unknown if the assumption of being finite type/Postnikov complete is truly necessary for the existence of such a nice fibrant resolution functor.

We recall the notion of the finite type site by Morel and Voevodsky.

Definition 1.1.1. (Finite Type Site) [MV, Definition 1.31, page 58] A site T is called of finite type, if for each simplicial sheaf $\mathcal{X}$ on T, the canonical map

$$\mathcal{X} \longrightarrow \operatorname*{holim}_{n \geq 0} Ex(P^n(\mathcal{X}))$$

is a weak equivalence. Here, $P^n(\mathcal{X})$ is the n^{th} Postnikov term of $\mathcal{X}$ (see Definition 2.4.7) and Ex is a fibrant resolution functor as part of the local injective model structure on T.

Finite type-ness in the Nisnevich site

The Nisnevich site $(\mathrm{Sm/S})_{\mathrm{Nis}}$ of finite type smooth schemes over a Noetherian base scheme S, which is usually studied in $\mathbb{A}^1$-homotopy theory (as in [MV]), is of finite type (see Remark 4.1.3). This has various important consequences. For example,

1. There exists a simplicial fibrant replacement functor $Ex^{\mathcal{G}}$ on $\Delta^{\mathrm{op}}\mathrm{Shv}((\mathrm{Sm/S})_{\mathrm{Nis}})$ which commutes with finite limits. Furthermore, by [MV, Lemma 3.20, page 93] one can construct an $\mathbb{A}^1$-fibrant replacement functor $L_{\mathbb{A}^1}$ which commutes with finite limits. As a consequence, for any simplicial group $\mathcal{G}$, $L_{\mathbb{A}^1}\mathcal{G}$ is also a simplicial group, and for any $\mathcal{G}$-torsor $\mathcal{E}$, $L_{\mathbb{A}^1}\mathcal{E}$ is a $L_{\mathbb{A}^1}\mathcal{G}$-torsor with some additional assumptions (for details see [Mor, Definition 6.49, Theorem 6.50]). Further applications of the existence of such $L_{\mathbb{A}^1}$ can be seen in [Mor, Remark A.13, Theorem 6.46, Lemma B.7(2)].

2. The obstruction theory in [Mor, Appendix B] is developed particularly under the assumption that the site is of finite type, which allows us to construct maps in the homotopy category inductively through Postnikov towers. The reader can refer to [AHW1], [AHW2] for applications in this direction.

Note that, while using obstruction theory, the convergence of the Postnikov tower is required only for the particular object in question. But for the first point, *i.e.*, the existence of a nice $L_{\mathbb{A}^1}$, requires the site to be of finite type, in other words, the convergence of the Postnikov tower for every simplicial sheaf.

Towards étale $\mathbb{A}^1$-homotopy theory

Ideally, we would want to see the same consequences as above, in the case of the étale topology. Specifically, it would be beneficial to know if the category of smooth schemes over a field with the étale topology, is of finite type. We believe exploring the étale topology is valuable for at least two reasons, potentially leading to new applications not achievable with the Nisnevich topology.

1. Let X be a smooth projective variety over a field k. If k is an uncountable field of characteristic 0 and X is rationally connected, then X is étale-$\mathbb{A}^1$-connected. Conversely, if k is of characteristic 0 and X is étale-$\mathbb{A}^1$-connected, then X is rationally connected (see [DHP, Appendix] for details). This suggests that the étale version of $\mathbb{A}^1$-homotopy theory could be quite significant for understanding rationally connected varieties, especially over fields of characteristic zero.

2. The distinction between split and non-split algebraic groups is less critical in the étale topology. For example, according to [BHS, Corollary 3.4], for a reductive algebraic group G over a field k, $Sing_*^{\mathbb{A}^1} G$ is $\mathbb{A}^1$-(Nis)-local if and only if G is isotropic. In the étale version of this result, if $Sing_*^{\mathbb{A}^1} G$ was known to be $\mathbb{A}^1$-local for split groups G, it would automatically be known for all reductive algebraic groups, because every reductive algebraic group is étale-locally split.

1.2 Main Results

This thesis intends to contribute to the development of the étale version of $\mathbb{A}^1$-homotopy theory by investigating when the site $(\mathrm{Sm}/k)_{\text{ét}}$, of finite type smooth schemes over a field k, is of finite type (in the sense of 1.1.1). The main objective is to provide a criterion for this site to be of finite type. The results in this thesis are joint work with Amit Hogadi and Rakesh Pawar, and they appear in [DHP].

The first observation is the following conjecture.

Conjecture 1.2.1. [DHP, Conjecture 1.2] *Let k be any field. Let $\Delta^{\mathrm{op}}\mathrm{Shv}((\mathrm{Sm}/k)_{\text{ét}})$ denote the category of étale simplicial sheaves on $(\mathrm{Sm}/k)_{\text{ét}}$ with the local injective model structure. Then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is of finite type if and only if there exists a finite extension L/k such that $cd(L) < \infty$.*

Here, $cd(L)$ and $cd_p(L)$ denote the cohomological dimension and p-cohomological dimension of a field L and a prime p (see Definition 3.1.6).

The following is the main result of this thesis which proves the conjecture in the case of countable fields.

Theorem 1.2.2. [DHP, Theorem 1.3] *The Conjecture 1.2.1 holds, if the absolute Galois group $G_k = Gal(k^{sep}/k)$ is first-countable. In particular, if the field k is countable, then G_k is first-countable, hence Conjecture 1.2.1 holds.*

Furthermore, we prove that some cohomological boundedness conditions are necessary for the site $(\mathrm{Sm}/k)_{\text{ét}}$ to be of finite type. This is summarized in the following results.

Theorem 1.2.3. [DHP, Theorem 1.6] *Let k be any field which satisfies the condition that for each $n \in \mathbb{N}$, there exists a G_k-module A_n such that*

$$\limsup_{n\to\infty} exp\left(H^n(G_k, A_n)\right) = \infty$$

where for an abelian group B, the exponent of B is denoted by $exp\,(B)$. Then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type.

Corollary 1.2.4. [DHP, Proposition 1.7] *Let k be a field such that $\limsup_p cd_p(k) = \infty$ where the limit is over all primes p. Then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type.*

The Chapter 4 of this thesis includes proofs of all the above results and a detailed introduction to finite type sites. The Chapter 2 and Chapter 3 provide the prerequisites necessary for Chapter 4. All the results in Chapters 2 and 3 are standard and well-known, hence are only stated, except for some proofs that are included due to the unavailability of details in the literature.

2

Homotopy Theory of Simplicial Sheaves

In this chapter, we introduce foundational concepts–Grothendieck sites, sheaves, and model categories–that provide the framework needed to develop homotopy theory in the context of sheaves of simplicial sets.

2.1 Sites and Topologies

The usual concept of a topology on a space is generalized to a Grothendieck topology on a small category by defining coverings in a purely categorical way. The notion of a site is an abstraction of a topological space. We recall the definitions and examples from [Vis, Section 2.3] and [Jar, Section 3.1].

Definition 2.1.1. A **Grothendieck topology** τ on a small category T consists of a set $Cov(\mathrm{T})$ of families of morphisms $\{U_i \to U\}_{i\in I}$ for $U \in \mathrm{T}$, called the covering families of T, such that the following axioms hold:

1. If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.

2. If $\{U_i \to U\}_{i\in I}$ is a covering and $\{V_{ij} \to U_i\}_{j\in J_i}$ is a covering for each i, then the set of compositions $\{V_{ij} \to U\}_{i\in I, j\in J_i}$ is a covering.

3. If $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is any morphism in T, then the fibered products $\{U_i \times_U V\}_{i\in I}$ exist and the set of projections $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

A **site** is a small category T endowed with a Grothendieck topology τ, denoted as (T, τ).

Remark 2.1.2. The definition of a Grothendieck topology we stated above is called as a *pretopology* in [AGV]. A pretopology generates a Grothendieck topology. We work with the above definition since all the topologies we use in this thesis are generated from pretopologies. For more details, see [AGV, Exposé II].

Examples 2.1.3. Let X be a topological space and S be a scheme.

1. The site $\mathrm{op}|_X$ is the poset category of open subsets of X, where a covering family for an open subset U is an open cover $\{V_i \subset U\}_{i \in I}$ of U.

2. The *Zariski site* $S|_{\mathrm{Zar}}$ is the poset category of open subschemes $U \subset S$. A family $V_i \subset U$ is a covering if $\bigcup V_i = U$ as sets.

3. The *étale site* $S|_{\text{ét}}$ is the category whose objects are all étale maps $\phi : V \to S$ and all commutative diagrams

$$\begin{tikzcd} V \arrow[rr] \arrow[dr, "\phi"'] & & V' \arrow[dl, "\phi'"] \\ & S & \end{tikzcd}$$

as morphisms. A family $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ is a covering if all maps are étale and jointly surjective, *i.e.*, $U = \bigcup \phi_i(U_i)$ or equivalently every morphism $\operatorname{Spec} \Omega \to U$ lifts to some U_j whenever Ω is a separably closed field.

4. The *Nisnevich site* $S|_{\mathrm{Nis}}$ has same underlying category as the étale site. A family of étale maps $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ is a Nisnevich covering if every morphism $\operatorname{Spec} k \to U$ lifts to some U_j whenever k is any field.

5. There are corresponding big sites $(Sch/S)_{\mathrm{Zar}}$ and $(Sch/S)_{\text{ét}}$ where the underlying category is the category of schemes over S. (see [Vis, page 26] or [Sta, 03X7] for details)

2.2 Sheaves on a Site

After axiomatizing the notion of 'coverings' for a category, in this section we extend the notion of a sheaf in the context of sites. We recollect the definition of sheaf on a site and related notions from [Sta] and [Jar, Chapter 3].
Let T be a site and E be a category with all products.

Definition 2.2.1. A **presheaf** on the site T with values in E is an E-valued contravariant functor defined on T, or equivalently a functor $\mathrm{T}^{\mathrm{op}} \to E$ where T^{op} is the opposite category of T. The presheaves on T form a category, denoted as $\mathrm{PShv}_E(\mathrm{T})$, where the morphisms are natural transformations.

Definition 2.2.2. A presheaf $\mathcal{F}$ on the site T with values in E is said to be a **sheaf** if for every covering family $\{U_i \to U\}_{i \in I}$ in T the induced diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer diagram in E. The category $\mathrm{Shv}_E(\mathrm{T})$ of sheaves on the site T is a full subcategory of the presheaf category $\mathrm{PShv}_E(\mathrm{T})$. We denote the category of sheaves of sets simply by $\mathrm{Shv}(\mathrm{T})$.

It is easy to see that, for any topological space X the notion of sheaves on the site $\mathrm{op}|_X$ coincides with the usual notion of sheaves on a topological space.

2.2.1 Points of a site

In the context of sheaves on topological spaces, we study their local behavior at points of that space by defining stalks. A point of a site abstracts this idea.

Definition 2.2.3. Let T be a site. A **point** of a site (or a point of $\mathrm{Shv}(\mathrm{T})$) is a geometric morphism $p : \mathsf{Set} \to \mathrm{Shv}(\mathrm{T})$, by which we mean that it consists of an adjoint pair of functors

$$p^* : \mathrm{Shv}(\mathrm{T}) \rightleftarrows \mathsf{Set} : p_*$$

such that p^* is the left adjoint to p_* and p^* preserves finite limits.
The functor p^* is often called the stalk functor and for any sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{T})$, the stalk at the point p is given as $\mathcal{F}_p := p^*\mathcal{F}$.

Definition 2.2.4. We say that the site T (or Shv(T)) has **enough points** if there is a set of points $\{p_i\}_{i\in I}$ which determines isomorphisms of sheaves, *i.e.*, any map of sheaves $\mathcal{F} \to \mathcal{G}$ which is an isomorphism on all stalks $\mathcal{F}_{p_i} \to \mathcal{G}_{p_i}$, is an isomorphism. The set of points $\{p_i\}_{i\in I}$ with the above property is called as a conservative set of points.

Example 2.2.5. Given a topological space X, the site $\mathrm{op}|_X$ has enough points. Every point $x \in X$ determines a point p_x of the site $\mathrm{op}|_X$ and the stalk functor is the usual one given as $p_x^*\mathcal{F} = \varinjlim_{x\in U} \mathcal{F}(U)$ where the colimit is over the open subsets U containing x.

Definition 2.2.6. Let S be a scheme.

1. A **geometric point** of S is a morphism $\operatorname{Spec} k \xrightarrow{\overline{s}} S$, where k is a separably closed field and $s \in S$ is the image of the morphism.

2. An *étale neighborhood* of a geometric point $\overline{s}$ of S is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "\phi"] \\ \operatorname{Spec} k \arrow[ur] \arrow[r, "\overline{s}"] & S \end{tikzcd}$$

where ϕ is an étale morphism of schemes.

Example 2.2.7. The geometric points defined above induce points of the étale sites in the sense of Definition 2.2.3. Moreover, the collection of all geometric points is a conservative set of points in the sense of Definition 2.2.4.

1. Let $S|_{\text{ét}}$ be the small étale site of a scheme S. Every geometric point $\operatorname{Spec} k \xrightarrow{\overline{s}} S$ defines a point of the site $S|_{\text{ét}}$. Let $\mathcal{F}$ be a sheaf (or a presheaf) on $S|_{\text{ét}}$, then the stalk is given by $\mathcal{F}_{\overline{s}} = \operatorname{colim}_U \mathcal{F}(U)$, where the colimit is indexed by the étale neighborhoods of $\overline{s}$. Considering the collection of all geometric points of S, the site $S|_{\text{ét}}$ has enough points.

2. Let $(Sch/S)_{\text{ét}}$ be the big étale site of schemes over S. The geometric point $\operatorname{Spec} k \xrightarrow{\overline{t}} T$ of an object T/S induces a point of site $(Sch/S)_{\text{ét}}$ through the

inclusion of sites $T|_{\text{ét}} \hookrightarrow (Sch/S)_{\text{ét}}$. For any sheaf (or presheaf) $\mathcal{F}$ on $(Sch/S)_{\text{ét}}$, the stalk at $\bar{t}$ is given by $\mathcal{F}_{\bar{t}} := (\mathcal{F}|_{T|_{\text{ét}}})_{\bar{t}}$. Considering the collection of all geometric points of all objects in (Sch/S), the site $(Sch/S)_{\text{ét}}$ has enough points.

3. Similar to point 2., the big étale site $(\mathrm{Sm/S})_{\text{ét}}$ of smooth schemes over a scheme S has enough points, which are given as the geometric points of the objects $T/S \in \mathrm{Sm/S}$.

2.3 Model Categories

In this section we review the concept of a model structure on a category, which provides an axiomatic framework for homotopy theory. This framework allows us to formally invert our desired class of morphisms, called 'weak equivalences', while remaining in ordinary category theory by avoiding set theoretic issues. We recall the definition of a model category and some related notions from [Hov, Chapter 1] and [GJ, Chapter II].

Definition 2.3.1. A **model structure** on a category $\mathcal{M}$ consists of three classes of morphisms, called the cofibrations (C), the fibrations (F) and the weak equivalences (W), which together satisfy the following axioms:

1. (2-out-of-3) Consider the following commutative diagram in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z. & \end{tikzcd}$$

   If any two of f, g, and h are weak equivalences, then so is the third.

2. (Retracts) Let f and g be morphisms. If f is a retract of g and g is a weak equivalence, a fibration, or a cofibration, then so is f (for the definition of retract, see [Hov, Def. 1.1.1]).

3. (Lifting) Suppose we are given a commutative diagram of solid arrows:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

   The dotted arrow exists making the diagram commute if either of the following is satisfied.

   (a) i is a *trivial cofibration* (*i.e.*, $i \in C \cap W$) and p is a fibration.

   (b) i is a cofibration and p is a *trivial fibration* (*i.e.*, $p \in F \cap W$).

4. (Factorization) Any morphism $X \to Y$ has following *functorial factorizations*:

   - $X \xrightarrow{f} A \xrightarrow{g} Y$ where f is a trivial cofibration and g is a fibration.
   - $X \xrightarrow{i} B \xrightarrow{j} Y$ where i is a cofibration and j is a trivial fibration.

   (for details on factorization, see [Hov, Def. 1.1.1])

A **model category** is a category $\mathcal{M}$ with all small limits and colimits with a model structure on $\mathcal{M}$.

Remark 2.3.2. If $\mathcal{M}$ is a model category, then any two of the classes of cofibrations (C), fibrations (F), and weak equivalences (W) determine the third (see [Hir, Proposition 7.2.7]). Thus, it is enough to describe any two classes to determine a model structure. We list a few properties of these classes if they constitute a model structure:

- The classes C, F and W are closed under compositions.
- The classes C and $C \cap W$ are closed under pushouts and coproducts.
- The classes F and $F \cap W$ are closed under pullbacks and products.

Definition 2.3.3. Let $\mathcal{M}$ be a model category and X be an object of $\mathcal{M}$.

- X is called **cofibrant** if the map $0 \to X$ from the initial object 0 is a cofibration.
- X is called **fibrant** if the map $X \to *$ to the terminal object $*$ is a fibration.

Definition 2.3.4. Let $\mathcal{M}$ be a model category.

1. A **cofibrant resolution functor** on $\mathcal{M}$ is a pair (Q, θ) where Q is a functor $Q : \mathcal{M} \to \mathcal{M}$ and θ is a natural transformation $\theta : Q \to Id_{\mathcal{M}}$ such that for each object X, the map $\theta_X : QX \to X$ is a trivial fibration and QX is cofibrant.

2. A **fibrant resolution functor** on $\mathcal{M}$ is a pair (R, η) where R is a functor $R : \mathcal{M} \to \mathcal{M}$ and η is a natural transformation $\eta : Id_{\mathcal{M}} \to R$ such that for each object X, the map $\eta_X : X \to RX$ is a trivial cofibration and RX is fibrant.

Note that the functorial factorization axiom of a model category ensures that such resolution functors always exist.

Suppose $\mathcal{C}$ is a category with a class W of weak equivalences, then one can define the homotopy category $\mathrm{Ho}\,\mathcal{C}$ as the localization $\mathcal{C}[W]$ of $\mathcal{C}$ by formally inverting the weak equivalences (see [Hov, Def. 1.2.1] for details). In general, the collection $\mathrm{Ho}\,\mathcal{C}(X,Y)$ of morphisms in the homotopy category may not be a set. So $\mathrm{Ho}\,\mathcal{C}$ may not exist as a category in the usual sense (*i.e.*, locally small category).

A model structure on $\mathcal{C}$ with a class W of weak equivalences ensures that the corresponding homotopy category $\mathrm{Ho}\,\mathcal{C}$ exists, providing a concrete description of the morphism sets $\mathrm{Ho}\,\mathcal{C}(X,Y)$. More precisely,

Definition 2.3.5. Suppose $\mathcal{C}$ is a model category and $\mathcal{C}_{cf}$ is a full subcategory of $\mathcal{C}$ of the objects which are simultaneously cofibrant and fibrant. Then the **homotopy category** of $\mathcal{C}$, denoted as $\mathrm{Ho}\,\mathcal{C}$, is a category such that

1. The objects of $\mathrm{Ho}\,\mathcal{C}$ are same as the objects of $\mathcal{C}$.

2. Given two objects X, Y in $\mathcal{C}$, the set of morphisms is given by a natural isomorphisms

$$[X,Y] := \mathrm{Ho}\,\mathcal{C}(X,Y) \cong \mathcal{C}(QX, RY)/{\sim}$$

   where $\sim$ is the equivalence relation given by left or right homotopy between maps.

3. The inclusion $\mathcal{C}_{cf} \to \mathcal{C}$ induces the equivalence of categories

$$\mathcal{C}_{cf}/{\sim} \xrightarrow{\cong} \mathrm{Ho}\,\mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}.$$

(see [Hov, Theorem 1.2.10] for details).

Now, we introduce a canonical example of a model category: the category of simplicial sets. It lays the foundations for the modern homotopy theory of simplicial sheaves.

2.3.1 Simplicial sets

We recall the definition of a simplicial set and a few important examples from [Hov, Section 3.1] or [GJ, Section I.1].

Consider the simplex category Δ with finite ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ for $n \geq 0$ as objects. The set of morphisms $\Delta([n], [m])$ is the set of order preserving maps from $[n]$ to $[m]$.

Any morphism in Δ can be uniquely written as a compositions of morphisms of the form d^i followed by morphisms of the form s^j where, for each $n \geq 1$, $0 \leq i \leq n$ and $0 \leq j \leq n-1$:

$d^i : [n-1] \to [n]$ such that d^i is injective and its image does not contain i,

$s^j : [n] \to [n-1]$ such that s^j is surjective and s^j identifies i and $i+1$.

Definition 2.3.6. A **simplicial set** X is a functor

$$X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}.$$

We refer to sSet as the category of simplicial sets. We denote $X[n]$ by X_n and refer to X_n as the set of n-simplices. The 0-simplices are called as vertices of X.

Remark 2.3.7. The definition of a simplicial set X can be described as the collection of sets X_n for $n \geq 0$ and the collection of two types of maps between them as follows:

$d_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$ called the *face maps,*

$s_j : X_{n-1} \to X_n$ for $0 \leq j \leq n-1$ called the *degeneracy maps,*

such that these maps satisfy the simplicial identities.

Also, a map $f : X \to Y$ between simplicial sets is described as a collection of maps $f_n : X_n \to Y_n$ commuting with the face and degeneracy maps.

Now we recall a few examples of simplicial sets.

Example 2.3.8. A very important class of simplicial sets is given by the representable functors, $\mathrm{Hom}_\Delta(-, [n])$ typically denoted as Δ^n. By Yoneda lemma, for any simplicial set X, we have the natural bijection of sets $X_n \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X)$. So any n-simplex x can be considered as a map $x : \Delta^n \to X$.

Example 2.3.9. A *simplicial abelian group* X is a functor $X : \Delta^{\mathrm{op}} \to \mathsf{Ab}$, where Ab is the category of abelian groups. We denote the category of simplicial abelian groups by sAb.

Example 2.3.10. The simplicial set Δ^n has only one non-degenerate n-simplex given by $1_n \in \mathrm{Hom}_\Delta([n],[n])$. Then the *boundary* of Δ^n is the simplicial set $\partial\Delta^n \subset \Delta^n$ generated by all the faces $d_i(1_n)$, $0 \le i \le n$. Also, the k^{th} *horn* of Δ^n (for $0 \le k \le n$) is the simplicial set $\Lambda^n_k \subset \Delta^n$ generated by all the faces $d_i(1_n)$ except the k^{th} face $d_k(1_n)$.

The category sSet of simplicial sets acts as a combinatorial model for the category Top of topological spaces, which provides the computational techniques generally not available in Top. The bridge between the categories of simplicial sets and topological spaces is established via the realization and the singular functors between these two categories. We recall from [GJ, Section I.2].

Proposition 2.3.11. *There is an adjoint pair of functors*

$$| \ | : \mathsf{sSet} \rightleftarrows \mathsf{Top} : \mathcal{S}$$

where the realization functor $| \ |$ *is a left adjoint to the singular functor* $\mathcal{S}$*. Moreover, for any simplicial set* X*, the realization* $|X|$ *is a CW-complex.*

We describe the classical model structure on sSet and revisit some homotopy theory of simplicial sets from [GJ, Sections I.3, I.7, I.11].

Definition 2.3.12. A map $p : X \to Y$ of simplicial sets is said to be a **Kan fibration** if for every commutative diagram (of solid arrows) of simplicial sets

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there is a dotted arrow $\Delta^n \to X$ making the diagram commute.

Definition 2.3.13. A simplicial set Y is called a **Kan complex** if Y is fibrant, *i.e.*, the canonical map $Y \to *$ is a fibration, where $* = \Delta^0$ is the terminal object in sSet.

Equivalently, Y is a Kan complex if and only if any map $\Lambda^n_k \to Y$ can be extended to a map $\Delta^n \to Y$.

The singular complex $\mathcal{S}X$ for any topological space X is a Kan complex. Also, the underlying simplicial set of any simplicial abelian group K, given as composition $\Delta^{\mathrm{op}} \xrightarrow{K} \mathsf{Ab} \to \mathsf{Set}$, is a Kan complex.

The following model structure on sSet is originally given by Quillen [Qui].

Theorem 2.3.14. [GJ, Theorems I.11.3, I.11.4] *The category* sSet *of simplicial sets admits a model structure consisting of the following classes of morphisms:*

1. *The weak equivalences* (W) *are the morphisms* $f : X \to Y$ *such that the realization* $|f| : |X| \to |Y|$ *is a weak homotopy equivalence of topological spaces.*

2. *The cofibrations* (C) *are the monomorphisms of simplicial sets, i.e., the degreewise injections of sets.*

3. *The fibrations* (F) *are the Kan fibrations.*

Furthermore, the realization and singular functors induce an equivalence of the homotopy categories $\mathrm{Ho}\,\mathsf{sSet}$ *and* $\mathrm{Ho}\,\mathsf{Top}$*, where* $\mathrm{Ho}\,\mathsf{Top}$ *is obtained by formally inverting the weak homotopy equivalences.*

Henceforth, we always consider the category sSet endowed with this model structure and call it as the **classical model structure** on sSet.

Many fundamental tools of topological spaces, like the homotopy groups, are crucial for studying objects up to homotopy. We revisit their analogues in the simplicial sets.

Homotopy groups of simplicial sets

Definition 2.3.15. Let X be a Kan complex.

1. The **set of connected components** of X, denoted as $\pi_0(X)$, is defined as

$$\pi_0(X) = X_0/\sim$$

where, for any two vertices $x, y \in X_0$ we say $x \sim y$ (*i.e.*, x is homotopic to y) if there is a 1-simplex $v \in X_1$ such that $d_1 v = x$ and $d_0 v = y$. We say X is connected if $\pi_0(X)$ is trivial, *i.e.*, a singleton set.

2. For any vertex $x \in X_0$, the **simplicial homotopy groups**, denoted as $\pi_n(X, x)$ for $n \geq 1$, are defined as the set of simplicial homotopy classes of morphisms

$$\alpha : \Delta^n \to X \ (rel\ \partial\Delta^n)$$

for morphisms α which maps boundary $\partial\Delta^n$ to the vertex x (see [GJ, Section I.6] for the definition of simplicial homotopy of morphisms in sSet).

Remark 2.3.16. The constructions of the simplicial homotopy groups and $\pi_0(-)$ are functorial, and they match with the homotopy groups of the corresponding realization in the case of Kan complexes. More precisely, if X is a Kan complex and x is a vertex of X then there are natural isomorphisms

$$\pi_0(X) \cong \pi_0(|X|) \text{ and } \pi_n(X, x) \cong \pi_n(|X|, |x|) \text{ for each } n \geq 1.$$

Definition 2.3.17. Given any simplicial set Z, the set of connected components and the simplicial homotopy groups of Z are defined as that of its realization $|Z|$ or equivalently of its fibrant replacement RZ. More precisely,

- $\pi_0(Z) := \pi_0(|Z|) \cong \pi_0(RZ)$.
- $\pi_n(Z, z) := \pi_n(|Z|, |z|) \cong \pi_n(RZ, f(z))$ for each $n \geq 1$ and $z \in Z_0$.

Remark 2.3.18. A map $f : X \to Y$ of simplicial sets is a weak equivalence if and only if it induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and an isomorphism on all the simplicial homotopy groups.

The following property of the Kan complexes is well-established, but we'll include a proof since one isn't typically found in the literature.

Lemma 2.3.19. *The functor $\pi_0(-)$ commutes with the arbitrary products of Kan complexes. More precisely, for a family of Kan complexes $\{X_i\}_{i\in I}$ for some set I, the canonical map $\pi_0(\prod_{i\in I} X_i) \xrightarrow{\eta} \prod_{i\in I} \pi_0(X_i)$ is a bijection.*

Proof. It is clear that $\prod_{i\in I} X_i$ is a Kan complex. The map η is surjective, since product (over i) of vertices of X_i is a vertex of $\prod_{i\in I} X_i$ which represents some connected component. For injectivity, consider two vertices $x = (x_i)$ and $y = (y_i)$ of $\prod_{i\in I} X_i$ such that $\eta(x) = \eta(y)$, *i.e.*, for every i, $x_i = y_i$ in $\pi_0(X_i)$, *i.e.*, $x_i \sim y_i$. Then there is 1-simplex s_i

in each X_i such that $d_1 s_i = x_i$ and $d_0 s_i = y_i$. Then the tuple $(s_i)_{i\in I}$ is a 1-simplex of $\prod_{i\in I} X_i$ such that $d_1((s_i)) = (d_1 s_i) = x$ and $d_0((s_i)) = (d_0 s_i) = y$, hence $x \sim y$ and, x and y are in the same connected component of $\prod_{i\in I} X_i$. $\square$

The skeleton–coskeleton and the Postnikov tower constructions play a crucial role in obstruction theory by providing tools to approximate simplicial sets through limiting higher homotopical information (skeleton–coskeleton) and to decompose simplicial sets to distribute homotopical data step wise (Postnikov towers). We recall the same from [DK, 1.2.(vi)], [GJ, Section VI.3] and [May, Section 8].

Skeleton and coskeleton of simplicial sets

Definition 2.3.20. Let X be a simplicial set and $n \geq 0$ be an integer.

1. The **n-skeleton** of X, denoted as $sk_n X$, is the simplicial set generated from all j-simplices of X such that $j \leq n$.

2. The **n-coskeleton** of X, denoted as $cosk_n X$, is the simplicial set which has its j-simplices given by the maps $sk_n(\Delta^j) \to X$ for every j.

Remark 2.3.21. Let X be any simplicial set.

1. These skeleton and coskeleton constructions form an adjunction, see [Rie, Example 1.1.9] & [GJ, Section IV.3.2], such that

$$sk_n : \mathsf{sSet} \rightleftarrows \mathsf{sSet} : cosk_n$$

where the n-skeleton functor sk_n is left adjoint to the n-coskeleton functor $cosk_n$.

2. There is a canonical map $\eta : X \to cosk_n X$ such that for any m-simplex $x \in X_m$ given as a map $\Delta^m \to X$ we have

$$\eta(x) = x|_{sk_n(\Delta^m)} \text{ given as the map } sk_n(\Delta^m) \to X.$$

Remark 2.3.22. The canonical map $X \to cosk_n X$ induces bijection on the m-simplices for $m \leq n$. Also, for $m > n$, the simplices are described through adjunction as
$(cosk_n X)_m \cong \mathrm{Hom}_{\mathsf{sSet}}(sk_n \Delta^m, X) = \mathrm{Hom}_{\mathsf{sSet}}(sk_n \partial\Delta^m, X) \cong \mathrm{Hom}_{\mathsf{sSet}}(\partial\Delta^m, cosk_n X)$.

In other words, $cosk_nX$ extends X by adding all possible the m-simplices ($m > n$) which has n-dimensional faces in X.

As a consequence, there is an isomorphism between homotopy groups of X and $cosk_nX$ in dimension $< n$, and all the higher homotopy groups ($\geq n$) of $cosk_nX$ are trivial.

Postnikov towers of simplicial sets

Definition 2.3.23. Let X be a simplicial set. A **Postnikov tower** $\{X(n)\}_{n\geq 0}$ for X is a tower of simplicial sets satisfying the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & X(2) \arrow[r, "q_1"] & X(1) \arrow[r, "q_0"] & X(0) \\ & & & X \arrow[ull, "i_2"] \arrow[ul, "i_1"'] \arrow[u, "i_0"'] \end{tikzcd}$$

where for all n the maps satisfy $q_{n-1} \circ i_n = i_{n-1}$ and for all vertices $v \in X_0$ we have $\pi_k(X(n)) = 0$ for $k > n$ and $(i_n)_* : \pi_k(X) \xrightarrow{\cong} \pi_k(X(n))$ for $k \leq n$.

We may assume X be to fibrant, since if $Y \to X$ is a weak equivalence and X is fibrant then the Postnikov tower of X is the Postnikov tower for Y.

Now we recall a model for the Postnikov tower called the Moore-Postnikov tower.

Definition 2.3.24. Let X be a Kan complex. For each $n \geq 0$, define an equivalence relation $\sim_n$ on simplices of X as follows: two q-simplices $\alpha, \beta : \Delta^q \to X$ are equivalent, *i.e.*, $\alpha \sim_n \beta$ if $\alpha|_{sk_n\Delta^q} = \beta|_{sk_n\Delta^q} : sk_n\Delta^q \to X$. Define the simplicial sets $X^{(n)} := X/\sim_n$.

Alternatively, $X^{(n)}$ is also the image of X in n-coskeleton under the natural map given in Remark 2.3.21. Hence,

$$X^{(n)} = X/\sim_n = \mathrm{Im}\,(X \longrightarrow cosk_n(X)).$$

There are natural maps $i_n : X \to X^{(n)}$ and $q_n : X^{(n+1)} \to X^{(n)}$ such that $\{X^{(n)}\}_{n\geq 0}$ is a Postnikov tower for X. This tower is called as the **Moore-Postnikov tower** of X. For $n \geq 0$ we note that:

- This tower is functorial in X.
- The set of q-simplices is $X_q^{(n)} = X_q$ for each $q \leq n$.

- $X^{(n)}$ is a Kan complex.
- The maps $i_n : X \to X^{(n)}$ are surjective Kan fibrations.
- The natural maps $X^{(m)} \to X^{(n)}$ for $m > n$ are Kan fibrations.

Theorem 2.3.25. [GJ, Theorem VI.5.14] *Let X be a connected Kan complex. Then any two Postnikov towers for X are weakly equivalent as towers under X. More precisely, any Postnikov tower for X is weakly equivalent to the Moore-Postnikov tower for X_0, where $X_0 \subset X$ is a minimal subcomplex weakly equivalent to X.*

2.4 Simplicial Sheaves on a Site

The category of simplicial sheaves on an appropriate site lays the foundation for the motivic homotopy theory developed by Morel and Voevodsky. We recall some fundamental concepts mainly from [MV].

Let T be a site with enough points throughout this section. The category of simplicial sheaves on T is the category of sheaves on T with values in sSet, or equivalently, the category of simplicial objects in Shv(T). We use the following notation for the category of simplicial sheaves:

$$\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T}) := \mathrm{Shv}_{\mathsf{sSet}}(\mathrm{T}) = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Shv}(\mathrm{T})).$$

We use both definitions, in particular, for a simplicial sheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$, the sections on an object $U \in \mathrm{T}$ is a simplicial set $\mathcal{X}(U)$ and the n^{th}-component of $\mathcal{X}$ is the sheaf of sets $\mathcal{X}([n]) = \mathcal{X}_n$ satisfying $(\mathcal{X}(U))_n = \mathcal{X}_n(U)$.

Given a point p of a site T, we can define the stalk functor for simplicial sheaves such that, for any $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$ the stalk at p is a simplicial set $\mathcal{X}_p$ given by the composition $\Delta^{\mathrm{op}} \xrightarrow{\mathcal{X}} \mathrm{Shv}(\mathrm{T}) \xrightarrow{p^*} \mathsf{Set}$; that is, the n-simplices are $(\mathcal{X}_p)_n = p^*(\mathcal{X}_n) = (\mathcal{X}_n)_p$.

Theorem 2.4.1. [MV, Theorem 1.4, page 49] *Let* T *be a site with enough points. Then the category* $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$ *of simplicial sheaves on* T *admits a model structure consisting of the following classes of morphisms:*

1. *The weak equivalences are local weak equivalences, i.e., the morphisms $f : \mathcal{X} \to \mathcal{Y}$ such that for any point p of the site* T*, the corresponding morphism of simplicial sets $f_p : \mathcal{X}_p \to \mathcal{Y}_p$ is a weak equivalence.*

2. *The cofibrations are the monomorphisms of simplicial sheaves, i.e., objectwise monomorphisms of the simplicial sets.*

3. *The fibrations are the morphisms having right lifting property with respect to any trivial cofibration.*

Henceforth, we always consider the category $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$ endowed with this model structure and call it as the **local injective model structure** on $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$. We denote the corresponding homotopy category by $\mathscr{H}_s(\mathrm{T})$.

For an object $U \in \mathrm{T}$, the representable sheaf h_U is considered as a simplicial sheaf in the obvious way and denoted by the same letter U. The fibrant resolution functor on this model structure is denoted by the pair (Ex, θ). Also, each simplicial sheaf is a cofibrant object in this model structure.

Definition 2.4.2. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of simplicial sheaves.

- f is called a *local fibration* if for any point p of the site T the morphism of simplicial sets $f_p : \mathcal{X}_p \to \mathcal{Y}_p$ is a Kan fibration.

- f is called a *trivial local fibration* if it is a local fibration and a local weak equivalence.

- f is called an *objectwise weak equivalence* if for any object $U \in \mathrm{T}$ the map $f(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ is a weak equivalence of simplicial sets.

- f is called a *objectwise fibration* if for any object $U \in \mathrm{T}$ the map $f(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ is a Kan fibration.

Remark 2.4.3. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of simplicial sheaves.

- f is a objectwise weak equivalence $\implies$ f is a local weak equivalence.

- f is a local weak equivalence between fibrant objects $\implies$ f is a objectwise weak equivalence.

- f is a fibration $\implies$ f is a objectwise fibration $\implies$ f is a local fibration.

(see [Jar, Chapters 4 and 5] for details).

As remarked in [MV, page 49], the category $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$ is closed symmetric monoidal model category with respect to categorical product. As a consequence of this, the class of weak equivalences are closed under finite products. While this is a standard result in the literature, we present a brief proof for the reader's convenience.

Lemma 2.4.4. *Consider the category $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$ of simplicial sheaves on a site T with the local injective model structure. If $\{\mathcal{X}_i \longrightarrow \mathcal{Y}_i\}_{i=1}^n$ is a finite family of weak equivalences, then the product map $\prod_{i=1}^n \mathcal{X}_i \longrightarrow \prod_{i=1}^n \mathcal{Y}_i$ is also a weak equivalence.*

Proof. It is enough to prove this for the case $n = 2$. Since all objects are cofibrant, by applying the Ken Brown's lemma [Hov, Lemma 1.1.12] to the product functor $(- \times -)$: $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T}) \times \Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T}) \to \Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$, it is enough to prove that the product of trivial cofibrations is a weak equivalence.

Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $g : \mathcal{A} \longrightarrow \mathcal{B}$ be trivial cofibrations. We show that their product map $f \times g : \mathcal{X} \times \mathcal{A} \longrightarrow \mathcal{Y} \times \mathcal{B}$ is a weak equivalence. The map $f \times g$ can be expressed as the composition of two product maps as follows:

$$\mathcal{X} \times \mathcal{A} \xrightarrow{f \times id_{\mathcal{A}}} \mathcal{Y} \times \mathcal{A} \xrightarrow{id_{\mathcal{Y}} \times g} \mathcal{Y} \times \mathcal{B}.$$

By [Hov, Remark 4.2.3], the functors $(- \times \mathcal{A})$ and $(\mathcal{Y} \times -)$ are left Quillen, hence preserve trivial cofibrations. So the maps $f \times id_{\mathcal{A}}$ and $id_{\mathcal{Y}} \times g$ are trivial cofibrations, in particular weak equivalences. Hence their composition $f \times g$ is also a weak equivalence. □

Homotopy sheaves

The concept of simplicial homotopy groups of simplicial sets is extended to the sheaf theoretic setting in terms of the presheaves (or sheaves) of homotopy groups (or sets). These sheaves provide an alternate description for the local weak equivalences of simplicial sheaves.

Definition 2.4.5. Let $\mathcal{X}$ be a simplicial sheaf (or presheaf).

1. The **sheaf of connected components** of $\mathcal{X}$, denoted as $\pi_0^s(\mathcal{X})$, is defined as the sheafification of the presheaf $\pi_0^{pre}(\mathcal{X})$ which is defined by either of the following:

   - $\pi_0^{pre}(\mathcal{X})(U) = \pi_0(\mathcal{X}(U))$ for each $U \in T$,
     where $\pi_0(\mathcal{X}(U))$ is the set of connected components of simplicial set $\mathcal{X}(U)$.

- $\pi_0^{pre}(\mathcal{X})(U) = \mathrm{Hom}_{\mathscr{H}_s(\mathrm{T})}(U, \mathcal{X}) = [U, \mathcal{X}]$ for each $U \in T$.

2. The **n$^{\text{th}}$-homotopy sheaf** of $\mathcal{X}$ at a base point $x \in \mathcal{X}_0(U)$, denoted as $\pi_n^s(\mathcal{X}, x)$ for $n \geq 1$, is defined as the sheafification of the presheaf $\pi_n^{pre}(\mathcal{X}, x)$ on a site T/U such that
$$\pi_n^{pre}(\mathcal{X}, x)(V) = \pi_n(\mathcal{X}(V), x|_V) \text{ for each } V \in \mathrm{T}/U$$
where $\pi_n(-,-)$ are the simplicial homotopy groups.

Remark 2.4.6. The map $f : \mathcal{X} \to \mathcal{Y}$ of simplicial sheaves is a local weak equivalence if and only if all induced maps on homotopy sheaves

$$\pi_0^s(\mathcal{X}) \to \pi_0^s(\mathcal{Y}) \text{ and } \pi_n^s(\mathcal{X}, x) \to \pi_n^s(\mathcal{Y}, f(x))$$

are isomorphisms, for all $n \geq 1$, all $U \in T$ and all $x \in \mathcal{X}_0(U)$.

Postnikov tower of simplicial sheaves

We revised the Moore-Postnikov tower for simplicial sets in Definition 2.3.24. This construction is extended in the sheaf-theoretic setup in objectwise sense. More precisely,

Definition 2.4.7. Let $\mathcal{X}$ be a simplicial sheaf. For each $n \geq 0$, the **n$^{\text{th}}$-Postnikov sheaf** associated to $\mathcal{X}$, denoted as $P^n(\mathcal{X})$, is the sheafification of the presheaf defined as

$$U \longmapsto \mathcal{X}(U)^{(n)}$$

where $\mathcal{X}(U)^{(n)}$ is the n$^{\text{th}}$ Moore-Postnikov term for $\mathcal{X}(U)$.
There are natural morphisms $P^{n+1}(\mathcal{X}) \to P^n(\mathcal{X})$ and $\mathcal{X} \to P^n(\mathcal{X})$ such that for each $n \geq 0$, the following diagram commutes:

$$\begin{tikzcd} P^{n+1}(\mathcal{X}) \arrow[r] & P^n(\mathcal{X}) \\ & \mathcal{X} \arrow[ul] \arrow[u] \end{tikzcd}.$$

2.4.1 Eilenberg-MacLane sheaves

We recall the classical Dold-Kan correspondence, *i.e.*, an equivalence between the category sAb of simplicial abelian groups and the category $\mathsf{Ch}_{\geq 0}$ of chain complexes (in non-negative degrees) of abelian groups (see [GJ, Corollary III.2.3]).

$$N : \mathsf{sAb} \rightleftarrows \mathsf{Ch}_{\geq 0} : \Gamma$$

According to [MV, Proposition 1.24, page 56], by applying the functors N and Γ objectwise, this equivalence extends to the equivalence between the category $\mathrm{Shv}_{\mathsf{sAb}}(\mathrm{T})$ of sheaves of simplicial abelian groups and the category $\mathrm{Shv}_{\mathsf{Ch}_{\geq 0}}(\mathrm{T})$ of sheaves of complexes (in non-negative degrees) of abelian groups. We have the following equivalence;

$$\widetilde{N} : \mathrm{Shv}_{\mathsf{sAb}}(\mathrm{T}) \rightleftarrows \mathrm{Shv}_{\mathsf{Ch}_{\geq 0}}(\mathrm{T}) : \widetilde{\Gamma}$$

$$\mathcal{F} \longmapsto \widetilde{N}\mathcal{F}$$

$$\widetilde{\Gamma}\mathcal{G} \longleftarrow\!\shortmid\ \mathcal{G}$$

where for any objects $U, V \in \mathrm{T}$

$$\widetilde{N}\mathcal{F}(U) = N(\mathcal{F}(U)) \quad \text{and} \quad \widetilde{\Gamma}\mathcal{G}(V) = \Gamma(\mathcal{G}(V)).$$

Note that, the category $\mathrm{Shv}_{\mathsf{Ch}_{\geq 0}}(\mathrm{T})$ is same as the category $\mathsf{Ch}_{\geq 0}(\mathrm{Shv}_{\mathsf{Ab}}(\mathrm{T}))$ of chain complexes (in non-negative degrees) of abelian sheaves. This equivalence given by functors $\widetilde{N}$ and $\widetilde{\Gamma}$ is called the *Dold-Kan correspondence for sheaves.*

Remark 2.4.8. The functors N and Γ, by applying objectwise, induce an equivalence of the corresponding presheaf categories. If $\mathcal{G}$ is a sheaf as above, then the presheaf $\widetilde{\Gamma}\mathcal{G}$ is already a sheaf, since being an equivalence Γ preserves all limits and Definition 2.2.2 is satisfied. The Same argument works for $\widetilde{N}\mathcal{F}$.

Definition 2.4.9. Let $\mathcal{A}$ be a sheaf of abelian groups on a site T. Then the **Eilenberg-MacLane sheaves** associated with $\mathcal{A}$ are defined as

$$\mathcal{K}(\mathcal{A}, n) := \widetilde{\Gamma}(\mathcal{A}[n]) \quad \text{for} \quad n \geq 0$$

where $\mathcal{A}[n]$ is the chain complex of sheaves with $\mathcal{A}$ in degree n as the only nontrivial term.

These Eilenberg-MacLane sheaves are objectwise Eilenberg-MacLane simplicial abelian group. More precisely, for an object $U \in \mathrm{T}$:

$$\mathcal{K}(\mathcal{A}, n)(U) = \widetilde{\Gamma}(\mathcal{A}[n])(U) = \Gamma(\mathcal{A}[n](U)) = \Gamma(\mathcal{A}(U)[n]) = K(\mathcal{A}(U), n)$$

where $K(\mathcal{A}(U), n)$ is the n^{th} Eilenberg-MacLane simplicial abelian group associated with abelian group $\mathcal{A}(U)$. The simplicial set $K(\mathcal{A}(U), n)$ has only one nontrivial homotopy group, $\pi_n(K(\mathcal{A}(U), n), x) = \mathcal{A}(U)$. Consequently, $\pi_n^s(\mathcal{K}(\mathcal{A}, n), x) = \mathcal{A}$ is the only non-trivial homotopy sheaf of $\mathcal{K}(\mathcal{A}, n)$.

3

Galois Cohomology

In this chapter, we briefly recall the theory of profinite group cohomology and its key tools, which are essential for Chapter 4. We are interested in the Galois cohomology of fields and its relationship with étale cohomology over a field.

3.1 Cohomology of Profinite Groups

Discrete group cohomology is generalized by incorporating the topological structure of the profinite group and considering modules over it with the discrete topology. The standard references are [Sha] and [RZ].

Definition 3.1.1. A topological group G is called a **profinite group** if it satisfies any of the following equivalent conditions:

1. G is the inverse limit of finite groups, each given the discrete topology.

2. G is a compact, Hausdorff, and totally disconnected group.

3. G is a compact, Hausdorff group in which the family of open normal subgroups forms a fundamental system of neighborhoods at 1.

Definition 3.1.2. Let G be a profinite group and let A be an abelian group on which G acts as a group of automorphisms. Then A is called a **discrete G-module** (or simply a G-module) if it satisfies any of the following equivalent conditions:

1. G acts continuously on A, *i.e.*, the map $G \times A \to A$ is continuous.

2. The stabilizer of each element of A is an open subgroup of G, *i.e.*, $U_a = \{g \in G \mid ga = a\}$ is open for all $a \in A$.

3. $A = \bigcup A^U$ where U runs over all open subgroups of G and A^U is the largest subgroup of A fixed by U, *i.e.*, $A^U = \{a \in A \mid ua = a \text{ for all } u \in U\}$.

Let G be a profinite group. The category of all G-modules with G-homomorphisms, denoted as $G\text{-}\mathsf{mod}$, forms an abelian category with enough injectives. We have the following fixed point functor:

$$(-)^G : G\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}.$$

The functor $(-)^G$ is left exact but not necessarily right exact. The **profinite group cohomology** of G is defined as the right derived functors of the fixed point functor $(-)^G$ given by, for $n \geq 0$

$$H^n(G, -) := R^n(-)^G(-) : G\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$$

such that $H^0(G, -) = (-)^G$. The cohomology groups $H^n(G, A)$ are called the profinite group cohomology of G with coefficients in the G-module A.

3.1.1 Functoriality and Shapiro's lemma

Let G and G' be profinite groups and modules $A \in G\text{-}\mathsf{mod}$ and $A' \in G'\text{-}\mathsf{mod}$. Consider a pair of maps (ϕ, f) where $\phi : G' \to G$ is a continuous homomorphism of profinite groups and $f : A \to A'$ is group homomorphism such that the diagram

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ {\scriptstyle\phi}\uparrow \;\; \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ G' \times A' & \longrightarrow & A' \end{array}$$

commutes in the obvious sense. Such a pair (ϕ, f) of compatible maps induces homomorphisms between the corresponding cohomology groups

$$(\phi, f)^n : H^n(G, A) \longrightarrow H^n(G', A') \text{ for all } n \geq 0.$$

We now recall some of the important homomorphisms of profinite cohomology groups.

Restriction maps

Let G be a profinite group and H be a closed subgroup of G. Then H is also a profinite group and any G-module A is naturally an H-module as well.

The pair of compatible maps given by the inclusion $H \hookrightarrow G$ and the identity map $A \to A$ induces homomorphisms of cohomology groups

$$\mathrm{Res}^G_H : H^n(G, A) \longrightarrow H^n(H, A) \text{ for all } n \geq 0.$$

called as the *restrictions.* For $n = 0$ the homomorphism is given by the inclusion $A^G \hookrightarrow A^H$. In fact, since $\{H^n(G, -)\}_{n\geq 0}$ is a universal δ-functor on $G\text{-}\mathsf{mod}$, the restriction maps form a morphism of δ-functors which is determined by $H^0(G, A) = A^G \hookrightarrow A^H = H^0(H, A)$.

Corestriction maps

Let G be a profinite group and H be an open subgroup of G. Then H is a closed subgroup of finite index in G.

Define a morphism of functors on the category $G\text{-}\mathsf{mod}$ as follows:

$$N_{G/H} : H^0(H, -) \longrightarrow H^0(G, -)$$

such that for any G-module A, the group homomorphism is given by

$$N_{G/H} : A^H = H^0(H, A) \longrightarrow H^0(G, A) = A^G \text{ such that } N_{G/H}(a) = \sum_t ta,$$

where $a \in A^H$ and t ranges over a left transversal of H in G.

The collection $\{H^n(H,-)\}_{n\geq 0}$ of functors is a universal δ-functor on $G\text{-}\mathsf{mod}$; hence, $N_{G/H}$ induces a unique morphism of δ-functors, called the corestriction map. In particular, for any $A \in G\text{-}\mathsf{mod}$, we have group homomorphisms:

$$\mathrm{Cor}_G^H : H^n(H, A) \longrightarrow H^n(G, A) \quad \text{for all} \quad n \geq 0.$$

which are called the *corestrictions*.

Theorem 3.1.3. [RZ, Theorem 6.7.3] *Let H be an open subgroup of profinite group G. Then the composition $\mathrm{Cor}_G^H \circ \mathrm{Res}_H^G$ is multiplication by the index $[G:H]$ of H in G, i.e.,*

$$\mathrm{Cor}_G^H \circ \mathrm{Res}_H^G : H^n(G,-) \to H^n(G,-) \quad \textit{such that} \quad \mathrm{Cor}_G^H \circ \mathrm{Res}_H^G = [G:H] \cdot \mathrm{id}$$

where id *is the identity on $H^n(G,-)$ and $n \geq 0$.*

Coinduced modules

Let G be a profinite group and H be a closed subgroup. For $A \in H\text{-}\mathsf{mod}$, consider the abelian group $\mathrm{Coind}_H^G(A)$ defined as:

$$\mathrm{Coind}_H^G(A) = \{f : G \to A \mid f \text{ is continuous and } f(hg) = hf(g) \text{ for all } h \in H,\ g \in G\}.$$

Then $\mathrm{Coind}_H^G(A)$ is a G-module with the action of G defined as:

$$(gf)(x) = f(gx) \text{ for any } g, x \in G,\ f \in \mathrm{Coind}_H^G(A).$$

The G-module $\mathrm{Coind}_H^G(A)$ is called a *coinduced* module. In fact, $\mathrm{Coind}_H^G(-)$ defines a functor from the category $H\text{-}\mathsf{mod}$ to $G\text{-}\mathsf{mod}$ and we have the following adjoint pair

$$i : G\text{-}\mathsf{mod} \rightleftarrows H\text{-}\mathsf{mod} : \mathrm{Coind}_H^G(-)$$

where $\mathrm{Coind}_H^G(-)$ is the right adjoint to the natural inclusion i of G-modules into H-modules, *i.e.*, $i(B) = B$ for any G-module B.

The coinduced modules provide a way to transfer calculations between the cohomology of a profinite group and its closed subgroups using following result:

Theorem 3.1.4. (Shapiro's Lemma) [RZ, Theorem 6.10.5] *Let G be a profinite group, H be a closed subgroup of G and $A \in H\text{-}\mathsf{mod}$. Then there exist natural isomorphisms*

$$H^n(G, \mathrm{Coind}_H^G(A)) \xrightarrow[\textit{shapiro}]{\simeq} H^n(H, A) \quad \textit{for} \quad n \geq 0.$$

Remark 3.1.5. The Shapiro's isomorphisms above factors through the restriction and the counit map and we get the following commutative diagram: for all $n \geq 0$

$$\begin{tikzcd}
H^n(G, \mathrm{Coind}_H^G(A)) \arrow[rr, "\simeq", "shapiro"'] \arrow[dr, "\mathrm{Res}_H^G"'] & & H^n(H, A) \\
& H^n(H, \mathrm{Coind}_H^G(A)) \arrow[ur, "\nu"'] &
\end{tikzcd},$$

where ν is induced by the counit map $\mathrm{Coind}_H^G(A) \to A$. Hence, the restriction map in above diagram, *i.e.*, $H^n(G, \mathrm{Coind}_H^G(A)) \to H^n(H, \mathrm{Coind}_H^G(A))$ is injective for all $n \geq 0$.

3.1.2 Cohomological dimension

Recall that if B is an abelian group, then for any prime p, B_p denotes its p-primary component, *i.e.*, the subgroup of B consisting of elements whose order is some power of prime p. If $B = B_p$ then we say B is a p-primary abelian group.

Definition 3.1.6. Let G be a profinite group and p be a prime number.

1. The **p-cohomological dimension** of G, denoted as $cd_p(G)$, is the smallest non-negative integer n such that $H^i(G, A)_p = 0$ for all $i > n$ and all torsion G-modules A. If no such n exists, then we say that $cd_p(G) = \infty$.

2. The **cohomological dimension** of G, denoted as $cd(G)$, is defined as

$$cd(G) = \sup_p cd_p(G)$$

   where the supremum is over all prime numbers.

Note that, in the definition of p-cohomological dimension, it is enough to consider all p-primary torsion G-modules instead of all torsion G-modules.

3.2 Galois Cohomology

The amalgamation of profinite group cohomology and Galois theory is Galois cohomology. We briefly recall from [Ser, Chapter II] and [Sha, Chapter IV].

Let k be a field. For a Galois extension L/k, the Galois group $\mathrm{Gal}(L/k)$ is the group of all k-automorphisms of L. By construction, this Galois group is the inverse limit of the Galois groups of finite extensions L_i/k which are contained in L/k. Thus, $\mathrm{Gal}(L/k)$ is a profinite group with a canonical topology called the Krull topology. *The Galois cohomology is the profinite group cohomology of Galois groups.*

We recall the **fundamental theorem of infinite Galois theory** before further exposition into Galois cohomology.

Theorem 3.2.1. [Sta, 0BMI] *Let L/k be a Galois extension. Let $G = \mathrm{Gal}(L/k)$ be a Galois group viewed as a profinite group. Then there is following bijection*

$$\begin{aligned} \{\, \textit{closed subgroups of } G \,\} &\longleftrightarrow \{\, \textit{subextensions } L/M/k \,\} \\ H &\longmapsto L^H \\ \mathrm{Gal}(L/M) &\longleftarrow\!\shortmid M \end{aligned}$$

where L^H denotes the fixed field corresponding to the closed subgroup H of G. In particular, we have $L^G = k$. Furthermore, the following is satisfied:

- *The open subgroups H of G correspond exactly to the finite subextensions M/k.*
- *The normal closed subgroups H of G correspond exactly to the subextensions M Galois over k and there is short exact sequence of profinite groups*

$$1 \to \mathrm{Gal}(L/M) \to \mathrm{Gal}(L/k) \to \mathrm{Gal}(M/k) \to 1.$$

- *This bijection is inclusion reversing, i.e., for $H_1 \subset H_2 \subset G$ and $L/M/N/k$, we have correspondence $L/L^{H_1}/L^{H_2}/k$ and $\mathrm{Gal}(L/M) \subset \mathrm{Gal}(L/N)$ respectively.*

Case of absolute Galois group of a field

Let k be a field and let k^{sep} be a separable closure of k. For any intermediate extension L such that $k^{sep}/L/k$, the *absolute Galois group* of L is denoted as $G_L := \mathrm{Gal}(k^{sep}/L)$.

The profinite group G_k is described as

$$G_k = \varprojlim_{L} \mathrm{Gal}(L/k),$$

where L varies over all finite Galois extensions of k contained in k^{sep}. The respective collection of open normal subgroups $G_L \subset G_k$ acts as a fundamental system of neighborhoods at 1 in G_k. The **Galois cohomology groups of a field *k*** are defined as

$$H^n(k, A) := H^n(G_k, A) \;\; \text{for all} \;\; n \geq 0 \;\; \text{and} \;\; A \in G_k\text{-}\mathsf{mod}.$$

Note that, the cohomology groups $H^n(k, A)$ do not depend on the choice of the separable closure of k (see [Ser, Section II.1.1]).

The various notions of the cohomological dimension of a field k are defined as those of the profinite group G_k. We have following notations: for any prime number p,

$$cd_p(k) := cd_p(G_k) \;\; \text{and} \;\; cd(k) := cd(G_k).$$

Examples 3.2.2. We list cohomological dimension of some fields.

- If field k has non-zero characteristic p, then $cd_p(k) \leq 1$.
- If k is an algebraically closed field, then $cd(k) = 0$.
- If k is a finite field, then $cd_p(k) = 1$ for any prime p.
- If field $k = \mathbb{C}(x_1, x_2, \dots, x_n)$, then $cd_p(k) = n$ for any prime p.
- If field $k = \mathbb{C}(x_1, x_2, \dots)$, then $cd_p(k) = \infty$ for any prime p.
- $cd_2(\mathbb{R}) = \infty$ and $cd_p(\mathbb{R}) = 0$ for prime $p \neq 2$.
- If k is an algebraic number field, then $cd_p(k) \leq 2$ for prime $p \neq 2$.

3.2.1 Relation to étale cohomology

The Galois cohomology of a field k embeds into the broader framework of étale cohomology. More precisely, the category of G_k-modules is equivalent to the category of abelian sheaves on the étale site over $\operatorname{Spec} k$ and this equivalence preserves the respective cohomologies. We recall the details from [Poo, Section 6.4.3].

Let k be a field and fix a separable closure k^{sep} over k.

Consider the small étale site over $\operatorname{Spec} k$, *i.e.*, $(\operatorname{Spec} k)_{\text{ét}}$ (see Examples 2.1.3). The objects of the site $(\operatorname{Spec} k)_{\text{ét}}$ are of the form of a disjoint union $\bigsqcup_i \operatorname{Spec} L_i$ where each L_i is a finite separable extension of k.

For a sheaf of sets $\mathcal{F} \in \operatorname{Shv}((\operatorname{Spec} k)_{\text{ét}})$, define

$$\mathcal{F}(k^{sep}) := \varinjlim \mathcal{F}(\operatorname{Spec} L_i),$$

where the colimit is over all finite separable extensions L_i/k contained in k^{sep}. Note that, we can take only finite Galois extensions in the above colimit and still get the same colimit, since every finite separable extension is contained in some finite Galois extension.

Whenever $\mathcal{F}$ is an abelian sheaf, the étale sheaf cohomology groups of $\mathcal{F}$ for each $n \geq 0$ are denoted as $H^n_{\text{ét}}(\operatorname{Spec} k, \mathcal{F})$ (for the definition of sheaf cohomology see [Poo, Section 6.4.1]).

Theorem 3.2.3. [Poo, Theorem 6.4.6]

1. *There is an equivalence of categories*

$$\begin{array}{ccc} \operatorname{Shv}((\operatorname{Spec} k)_{\text{ét}}) & \rightleftarrows & G_k\text{-}\mathsf{set} \\ \mathcal{F} & \longmapsto & \mathcal{F}(k^{sep}) \\ \mathcal{F}_S & \longleftarrow\!\shortmid & S, \end{array}$$

where for each finite separable extension L/k, define $\mathcal{F}_S(\operatorname{Spec} L) = S^{G_L}$. For general object of site define $\mathcal{F}_S$ as corresponding product of such fixed point sets.

2. *The equivalence in part 1. restricts to an equivalence*

$$\operatorname{Shv}_{\mathsf{Ab}}((\operatorname{Spec} k)_{\text{ét}}) \xrightarrow{\simeq} G_k\text{-}\mathsf{mod}.$$

3. *Under the equivalence in part 1., the global section functor $\mathcal{F} \mapsto \mathcal{F}(k)$ corresponds to the functor that takes a G_k-set S to the fixed point set S^{G_k}.*

4. *There are natural isomorphisms*

$$H^n_{\text{ét}}(\operatorname{Spec} k, \mathcal{F}) \cong H^n(G_k, \mathcal{F}(k^{sep})) \quad \textit{for all} \quad n \geq 0.$$

4

Finite Type-ness of Étale Sites over Fields

In this chapter, we recall the developments related to the notion of finite type sites. Our main goals are to find an example of a non-finite type étale site and investigate the conditions on a field k under which the étale site of smooth k-schemes becomes of finite type.

4.1 A Primer on Finite Type Sites

In this section we introduce the notion of finite type sites and recall related current developments. The main reference for this section is [MV].

Let T be a site with enough points. Consider the category $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{T})$ of simplicial sheaves on T endowed with local injective model structure as stated in Theorem 2.4.1.

The Moore-Postnikov tower of simplicial set is a tower of fibrations of fibrant objects Definition 2.3.24. In the case of simplicial sheaves, the functors P^n do not necessarily take fibrant objects to fibrant objects, and the maps in the tower are not necessarily fibrations. Hence, the correct notion of the limit of Postnikov towers, *i.e.*, $\mathrm{holim}_{n\geq 0} Ex(P^n(\mathcal{X}))$ is not generally weakly equivalent to $\mathcal{X}$. This leads to the following definition from Chapter 1.

Definition 1.1.1 (Finite Type Site). A site T is called of finite type, if for each simplicial sheaf $\mathcal{X}$ on T, the canonical map

$$\mathcal{X} \longrightarrow \operatorname*{holim}_{n\geq 0} Ex(P^n(\mathcal{X}))$$

is a weak equivalence. Here, $P^n(\mathcal{X})$ is the n^{th} Postnikov term of $\mathcal{X}$ (see Definition 2.4.7), $Ex(-)$ is a fibrant resolution functor and holim$(-)$ is the homotopy limit of simplicial sheaves (for def. see [MV, page 54]).

In some cases the homotopy limit is easier to calculate. For example, if we consider a tower of fibration of fibrant simplicial sheaves, then the homotopy limit of the tower is weakly equivalent to the usual inverse limit of tower. This follows from [BK, XI.4.1] and the fact that a fibration of simplicial sheaves is an objectwise fibration of simplicial sets (see Remark 2.4.3).

We briefly recall the following example by Morel and Voevodsky of a site which is not of finite type.

Example 4.1.1. [MV, Example 1.30, page 58] Consider a group $G = \prod_{j>0} \mathbb{Z}/2$ and the site $(G\text{-}\mathsf{set})_{df}$ of finite G-sets with jointly surjective coverings. Let $\mathcal{X}$ be the constant simplicial sheaf corresponding to simplicial set $\prod_{i>0} K(\mathbb{Z}/2, i)$, where $K(\mathbb{Z}/2, i)$ is the i^{th} Eilenberg-MacLane simplicial abelian group corresponding to $\mathbb{Z}/2$.

Then, the sheaf $\mathrm{holim}_{n\geq 0} Ex(P^n(\mathcal{X}))$ is weakly equivalent to $\prod_{i>0} Ex(\mathcal{K}(\mathbb{Z}/2, i))$ and consequently the map $\mathcal{X} \to \mathrm{holim}_{n\geq 0} Ex(P^n(\mathcal{X}))$ is not weak equivalence. Hence, this site is not of finite type. This relies on the fact that $\mathbb{Z}/2$ has infinite group cohomological dimension.

The following result by Morel and Voevodsky gives a sufficient condition for a site to be of finite type by means of some cohomological boundedness.

Theorem 4.1.2. [MV, Theorem 1.37, page 60] *Let* T *be a site and suppose that there exists a family* $(A_d)_{d\geq 0}$ *of classes of objects of* T *such that the following conditions hold:*

1. *Any object* U *in* A_d *has cohomological dimension* $\leq d$ *i.e. for any sheaf* $\mathcal{F}$ *on* T/U *and any* $m > d$ *one has* $H^m(U, \mathcal{F}) = 0$.

2. *For any object* V *of* T *there exists an integer* d_V *such that any covering of* V *in* T *has a refinement of the form* $\{U_i \to V\}_{i\in I}$ *with each* U_i *being in* A_{d_V}.

Then T *is a site of finite type.*

Let S be a Noetherian scheme of finite dimension. In the motivic homotopy theory developed in [MV] one studies the category (Sm/S) of finite type smooth schemes over the scheme S. The topologies on this category are the Zariski, Nisnevich or étale topologies, and the corresponding sites are denoted as $(\mathrm{Sm}/\mathrm{S})_{\mathrm{Zar}}$, $(\mathrm{Sm}/\mathrm{S})_{\mathrm{Nis}}$ and $(\mathrm{Sm}/\mathrm{S})_{\text{ét}}$ respectively. Among these, the Nisnevich site $(\mathrm{Sm}/\mathrm{S})_{\mathrm{Nis}}$ is involved in the usual setting of $\mathbb{A}^1$-homotopy theory.

Remark 4.1.3. With notations as above, we have the following:

- The site $(\mathrm{Sm}/\mathrm{S})_{\mathrm{Zar}}$ is of finite type, since Zariski cohomological dimension of a Noetherian scheme is bounded by its usual dimension [Har, Chapter III, Theorem 2.7] and hence it follows from Theorem 4.1.2.
- The site $(\mathrm{Sm}/\mathrm{S})_{\mathrm{Nis}}$ is of finite type. This follows from [MV, Proposition 1.8, page 98] and above Theorem 4.1.2.

4.2 Investigating Finite Type-ness of $(\mathrm{Sm}/k)_{\text{ét}}$

In this section, as a view towards the development of étale $\mathbb{A}^1$-homotopy theory, we discuss the conditions for the site $(\mathrm{Sm}/k)_{\text{ét}}$ to be of finite type. More precisely, we explore the relation between the cohomological dimension of a field k and the finite type-ness of the site $(\mathrm{Sm}/k)_{\text{ét}}$.

First, observe that, unlike the case of the Zariski or Nisnevich sites, we cannot apply Theorem 4.1.2 to the étale site of every field, since the sheaf cohomological dimension of the étale site of a field could be infinite and condition 1. in Theorem 4.1.2 may not be satisfied. We see such class of examples in Section 4.2.3.

4.2.1 Conjecture

The Theorem 4.1.2 gives a sufficient criterion for a site to be of finite type in terms of bounded cohomology. We conjecture that, in the case of $(\mathrm{Sm}/k)_{\text{ét}}$, some bound on the cohomological dimension is also necessary. We recall the statement of the conjecture from Chapter 1.

Conjecture 1.2.1. *Let k be any field. Let $\Delta^{\mathrm{op}}\mathrm{Shv}((\mathrm{Sm}/k)_{\text{ét}})$ denote the category of étale simplicial sheaves on $(\mathrm{Sm}/k)_{\text{ét}}$ with local injective model structure. Then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is of finite type if and only if there exists a finite extension L/k such that $cd(L) < \infty$.*

The **'if part'** of the conjecture follows easily from Theorem 4.1.2. This is a well-known result in the literature; we provide a brief proof for completeness. More precisely, we have the following:

Proposition 4.2.1. *Let k be any field. If there exists a finite extension L/k such that $cd(L) < \infty$, then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is of finite type.*

Proof. We prove this using Theorem 4.1.2, for that, we show that for any given object V in $(\mathrm{Sm}/k)_{\text{ét}}$, there exists an integer d_V such that any covering $\{V_i \to V\}_{i\in I}$ has a refinement by objects whose étale cohomology is bounded by d_V.

We claim that $d_V = 2\dim(V) + cd(L) + 1$ and the covering $\{V_i \times_k L \to V\}_{i\in I}$ is the required refinement of $\{V_i \to V\}_{i\in I}$.

We can consider L/k to be a finite Galois extension such that $cd(L) < \infty$. For each $i \in I$, the pullback $V_i \times_k L$ is a finite type smooth scheme over L and the map $V_i \times_k L \to V_i$ is a covering of V_i. Hence, being a composition of covers, $\{V_i \times_k L \to V\}_{i\in I}$ is a refinement of $\{V_i \to V\}_{i\in I}$. Now we show that the étale sheaf cohomology of each $V_i \times_k L$, for $i \in I$, is bounded by d_V, *i.e.*, for any abelian sheaf $\mathcal{F}$ on $(\mathrm{Sm}/k)_{\text{ét}}$, we want

$$H^j(V_i \times_k L, \mathcal{F}) = 0 \quad \text{for} \quad j > d_V$$

We do this by cases as follows:

<u>Case 1</u>: Let $\mathcal{F}$ be a torsion abelian sheaf.

By [Mil, Chapter VI, 1.4 and 1.5], the étale cohomological dimension of $V_i \times_k L$, *i.e.*, $cd((V_i \times_k L)_{\text{ét}})$ is finite, since $cd((V_i \times_k L)_{\text{ét}}) \leq 2\dim(V_i \times_k L) + cd(L)$. The maps $V_i \times_k L \to V_i$ and $V_i \to V$ are étale, hence $\dim(V_i \times_k L) = \dim(V_i) = \dim(V)$. We have,

$$H^j(V_i \times_k L, \mathcal{F}) = 0 \quad \text{for} \quad j > 2\dim(V) + cd(L), \quad \text{hence for} \quad j > d_V.$$

<u>Case 2</u>: Let $\mathcal{F}$ be a torsion-free abelian sheaf.

Consider the following short exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Since $\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ is a torsion sheaf, by case 1 and long exact sequence of cohomologies, we have the following isomorphisms

$$H^j(V_i \times_k L, \mathcal{F}) \cong H^j(V_i \times_k L, \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}) \quad \text{for} \quad j > d_V. \tag{4.1}$$

The sheaf $\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a sheaf of $\mathbb{Q}$-vector spaces. By [MVW, Proposition 14.23] the Nisnevich and étale cohomologies coincide for sheaf $\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}$ and by [MV, Proposition 1.8, page 98] the Nisnevich cohomology vanishes such that

$$H^j(V_i \times_k L, \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}) = 0 \quad \text{for} \quad j > \dim(V_i \times_k L), \quad \text{hence for} \quad j > d_V. \tag{4.2}$$

By (4.1) and (4.2), we get required result.
Case 3: Let $\mathcal{F}$ be any abelian sheaf.

Consider the following short exact sequence:

$$0 \longrightarrow \mathcal{F}_{tors} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}_{tors} \longrightarrow 0$$

where $\mathcal{F}_{tors}$ is a torsion subsheaf of $\mathcal{F}$ and $\mathcal{F}/\mathcal{F}_{tors}$ is a torsion-free quotient sheaf. By long exact sequence of cohomologies and by vanishing of cohomology for $\mathcal{F}_{tors}$ and $\mathcal{F}/\mathcal{F}_{tors}$ for $j > d_V$ from case 1 and case 2, we have

$$H^j(V_i \times_k L, \mathcal{F}) = 0 \quad \text{for} \quad j > d_V.$$

Hence, the étale cohomology of each $V_i \times_k L$ for $i \in I$ is bounded by d_V and the result follows from Theorem 4.1.2. $\square$

Now, the **'only if'** part of the Conjecture 1.2.1 remains open, which we investigate in the next few sections. More precisely, we are concerned about proving the following implication:

"If k is a field such that every finite extension L/k has $cd(L) = \infty$, then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type."

4.2.2 Condition for non-finite type-ness

The following Lemma 4.2.2 provides a sufficient condition for the site $(\mathrm{Sm}/k)_{\text{ét}}$ to be not of finite type. This lemma is crucial in proving all results in later sections. With certain conditions, we construct a simplicial sheaf which acts as an indicator of non-finite type-ness. The proof of this lemma is broadly motivated from Example 4.1.1.

Lemma 4.2.2. *Let k be a field. Assume that for each $i \in \mathbb{N}$, there is a continuous G_k-module A_i and classes*

$$\{\alpha_i \in H^i(G_k, A_i)\}_{i\in\mathbb{N}}$$

such that for each L/k finite Galois extension, there is $m \in \mathbb{N}$ (possibly depending on L) such that $(\alpha_m)_{|L} \neq 0$ in $H^m(G_L, A_m)$. Then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type.

Proof. To prove the lemma we need to construct a simplicial sheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}((\mathrm{Sm}/k)_{\text{ét}})$ such that the map $\mathcal{X} \to \operatorname{holim}_{n\geq 0} Ex(P^n(\mathcal{X}))$ is not a weak equivalence. We do this in several steps.

<u>*Step 1*</u> : In this step we construct the required simplicial sheaf $\mathcal{X}$.

Given each G_k-module A_i, consider the associated abelian sheaf by the Galois-étale correspondence from Theorem 3.2.3, denoted as $\widetilde{\mathcal{A}_i} \in \mathrm{Shv}_{\mathsf{Ab}}((\operatorname{Spec} k)_{\text{ét}})$.

Let us denote the small étale site at an object $\operatorname{Spec} k \in (\mathrm{Sm}/k)_{\text{ét}}$ by $(\operatorname{Spec} k)_{\text{ét,ft}}$, since it contains only finite type étale schemes over $\operatorname{Spec} k$ unlike the site $(\operatorname{Spec} k)_{\text{ét}}$. We have the following inclusion functors:

$$(\operatorname{Spec} k)_{\text{ét}} \xhookleftarrow{g} (\operatorname{Spec} k)_{\text{ét,ft}} \xhookrightarrow{f} (\mathrm{Sm}/k)_{\text{ét}}.$$

These inclusions induces the adjoint pairs of functors between respective categories of abelian sheaves by [Tam, Section I.3.6] as follows:

$$\begin{array}{ccccc} \mathrm{Shv}_{\mathsf{Ab}}((\operatorname{Spec} k)_{\text{ét}}) & \underset{g^{-1}}{\overset{g_*}{\rightleftarrows}} & \mathrm{Shv}_{\mathsf{Ab}}((\operatorname{Spec} k)_{\text{ét,ft}}) & \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} & \mathrm{Shv}_{\mathsf{Ab}}((\mathrm{Sm}/k)_{\text{ét}}) \\ \widetilde{\mathcal{A}_i} & \longmapsto & g_*\widetilde{\mathcal{A}_i} & \longmapsto & f^{-1}g_*\widetilde{\mathcal{A}_i}, \end{array}$$

where the functors g^{-1} and f^{-1} are left adjoint to g_* and f_* respectively. Moreover, the adjoint pair (g^{-1}, g_*) is equivalence of categories by [Tam, Theorem I.3.9.1].

We denote sheaf $\mathcal{A}_i := f^{-1}g_*\widetilde{\mathcal{A}}_i$ for all $i > 0$.

Recall the Eilenberg-MacLane sheaves from Definition 2.4.9, we consider the i^{th} Eilenberg-MacLane sheaf corresponding to $\mathcal{A}_i$ for each $i > 0$, *i.e.*,

$$\mathcal{K}(\mathcal{A}_i, i) := \widetilde{\Gamma}(\mathcal{A}_i[i]) \in \Delta^{\mathrm{op}}\mathrm{Shv}((\mathrm{Sm}/k)_{\text{ét}}),$$

which are described as

$$\mathcal{K}(\mathcal{A}_i, i)(U) = K(\mathcal{A}_i(U), i) \quad \text{for any} \quad U \in (\mathrm{Sm}/k)_{\text{ét}}.$$

We define $\mathcal{X} := \prod_{i>0} \mathcal{K}(\mathcal{A}_i, i) \in \Delta^{\mathrm{op}}\mathrm{Shv}((\mathrm{Sm}/k)_{\text{ét}})$ as the product of simplicial sheaves.

Step 2 : In this step we show that the simplicial sheaf $\mathcal{X}$ is connected.

Recall the definition of homotopy presheaf $\pi_0^{pre}(\mathcal{X})$ from Definition 2.4.5. For each $U \in (\mathrm{Sm}/k)_{\text{ét}}$ we have,

$$\begin{aligned} \pi_0^{pre}(\mathcal{X})(U) = \pi_0(\mathcal{X}(U)) &= \pi_0(\prod_{i>0} \mathcal{K}(\mathcal{A}_i, i)(U)) \\ &= \pi_0(\prod_{i>0} K(\mathcal{A}_i(U), i)) \\ &= \prod_{i>0} \pi_0(K(\mathcal{A}_i(U), i)). \end{aligned}$$

The last equality follows since each $K(\mathcal{A}_i(U), i)$ is a simplicial abelian group, hence a Kan complex by [GJ, Lemma I.3.4], and the functor $\pi_0(-)$ commutes with a infinite product of Kan complexes by Lemma 2.3.19.

Since each $K(\mathcal{A}_i(U), i)$ is connected, we have $\pi_0^{pre}(\mathcal{X})(U) = *$ for each $U \in (\mathrm{Sm}/k)_{\text{ét}}$. Hence, the presheaf $\pi_0^{pre}(\mathcal{X}) = *$, and hence the associated sheaf $\pi_0^s(\mathcal{X}) = *$. Thus, the simplicial sheaf $\mathcal{X}$ is connected.

Step 3 : In this step we determine the Postnikov sheaves $\{P^n(\mathcal{X})\}_{n>0}$ associated to $\mathcal{X}$.

Let $n > 0$ be an integer. Recall from Definition 2.4.7 that $P^n(\mathcal{X})$ is the sheaf associated to the presheaf

$$P^n_{pre}(\mathcal{X})(U) = \mathcal{X}(U)^{(n)} = \mathrm{Im}\,(\mathcal{X}(U) \xrightarrow{\eta} cosk_n(\mathcal{X}(U))),$$

where η is the canonical map such that for any m-simplex x given as a map $\Delta^m \to \mathcal{X}(U)$

$$\eta(x) = x|_{sk_n(\Delta^m)} \text{ given as the map } sk_n(\Delta^m) \to \mathcal{X}(U).$$

Recall that the functor $cosk_n(-)$ is right adjoint (see Remark 2.3.21), hence it commutes with the limits and we have following:

$$cosk_n(\mathcal{X}(U)) = cosk_n(\prod_{i>0} \mathcal{K}(\mathcal{A}_i, i)(U)) = cosk_n(\prod_{i>0} K(\mathcal{A}_i(U), i)) = \prod_{i>0} cosk_n(K(\mathcal{A}_i(U), i)).$$

Hence, the map η is given as

$$\mathcal{X}(U) = \prod_{i>0} K(\mathcal{A}_i(U), i) \xrightarrow{\eta = \prod_i \eta_i} \prod_{i>0} cosk_n(K(\mathcal{A}_i(U), i))$$

where each $\eta_i : K(\mathcal{A}_i(U), i) \to cosk_n(K(\mathcal{A}_i(U), i))$ is the canonical map to coskeleton. Then, the image of the map η also splits as the product of images of η_i and we obtain the following:

$$\begin{aligned} P^n_{pre}(\mathcal{X})(U) &= \mathrm{Im}\,(\prod_{i>0} K(\mathcal{A}_i(U), i) \xrightarrow{\eta} \prod_{i>0} cosk_n(K(\mathcal{A}_i(U), i))) \\ &= \prod_{i>0} \mathrm{Im}\,(K(\mathcal{A}_i(U), i) \xrightarrow{\eta_i} cosk_n(K(\mathcal{A}_i(U), i))) \\ &= \prod_{i>0} K(\mathcal{A}_i(U), i)^{(n)}. \end{aligned}$$

Now we determine the Moore-Postnikov tower of $K(\mathcal{A}_i(U), i)$ for each $i > 0$. Recall that each $K(\mathcal{A}_i(U), i)$ is Eilenberg-MacLane simplicial abelian group having only one nontrivial homotopy group $\pi_i(K(\mathcal{A}_i(U), i), x)$.

Thus, there is a Postnikov tower of $K(\mathcal{A}_i(U), i)$, denoted as $\{K(\mathcal{A}_i(U), i)(m)\}_{m \geq 0}$, such that its terms are trivial for $m < i$ and equal to $K(\mathcal{A}_i(U), i)$ for $m \geq i$. By the uniqueness of Postnikov towers as in [GJ, Theorem VI.5.14], this tower is weakly equivalent to the Moore-Postnikov tower of $K(\mathcal{A}_i(U), i)$. More precisely, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & K(\mathcal{A}_i(U),i)^{(i)} \arrow[r] & K(\mathcal{A}_i(U),i)^{(i-1)} \arrow[r] & \cdots \arrow[r] & K(\mathcal{A}_i(U),i)^{(1)} \arrow[r] & K(\mathcal{A}_i(U),i)^{(0)} \\
\cdots \arrow[r] & K(\mathcal{A}_i(U),i) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
 & & & & & K(\mathcal{A}_i(U),i). \arrow[u] \arrow[ul] \arrow[ulll] \arrow[ullll, "id"]
\end{tikzcd}$$

We have the following map of towers over $K(\mathcal{A}_i(U),i)$

$$\{K(\mathcal{A}_i(U),i)(m)\}_{m\geq 0} \longrightarrow \{K(\mathcal{A}_i(U),i)^{(m)}\}_{m\geq 0},$$

such that each $K(\mathcal{A}_i(U),i)(m) \to K(\mathcal{A}_i(U),i)^{(m)}$ is a weak equivalence.

Since the infinite product of weak equivalences between Kan complexes is again a weak equivalence, above observation induces the following weak equivalence:

$$\prod_{i=1}^{n} K(\mathcal{A}_i(U),i) \longrightarrow \prod_{i>0} K(\mathcal{A}_i(U),i)^{(n)} = P^n_{pre}(\mathcal{X})(U) \quad \text{for each} \quad U \in (\mathrm{Sm}/k)_{\text{ét}}.$$

It is easy to see that this is a morphism of presheaves, and hence by composing with sheafification map we have following:

$$\prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i,i) \xrightarrow{\phi} P^n_{pre}(\mathcal{X}) \xrightarrow{sh} P^n(\mathcal{X}).$$

Here, the map ϕ is an objectwise weak equivalence, hence a local weak equivalence by [Jar, Lemma 4.1]. Also the sheafification map, sh, is a local weak equivalence by [Jar, Corollary 4.21]. Thus, we have determined the Postnikov sheaves $P^n(\mathcal{X})$ upto weak equivalence as follows:

$$\prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i,i) \xrightarrow[equivalence]{weak} P^n(\mathcal{X}) \quad \text{for each} \quad n > 0.$$

Moreover, by construction, these maps are compatible with n inducing a map of towers

$$\{\prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i,i)\}_{n>0} \longrightarrow \{P^n(\mathcal{X})\}_{n>0},$$

where the maps $\prod_{i=1}^{m} \mathcal{K}(\mathcal{A}_i,i) \to \prod_{i=1}^{m-1} \mathcal{K}(\mathcal{A}_i,i)$ are projections.

<u>*Step 4*</u> : In this step, we simplify the sheaf $\operatorname{holim}_n Ex(P^n(\mathcal{X}))$ upto weak equivalence, since to show that the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type, it is enough to show that the simplicial sheaf $\operatorname{holim}_n Ex(P^n(\mathcal{X}))$ is not connected.

Observe that, each map $Ex(\prod_{i=1}^n \mathcal{K}(\mathcal{A}_i, i)) \to Ex(P^n(\mathcal{X}))$ is a local weak equivalence between fibrant sheaves, hence an objectwise weak equivalence between Kan complexes by [Jar, Corollary 5.12 and Corollary 5.13]. By the Homotopy lemma [BK, XI.5.6], for any $U \in (\mathrm{Sm}/k)_{\text{ét}}$ we have the following weak equivalence:

$$\operatorname*{holim}_n(Ex(\prod_{i=1}^n \mathcal{K}(\mathcal{A}_i, i))(U)) \longrightarrow \operatorname*{holim}_n(Ex(P^n(\mathcal{X}))(U)).$$

The homotopy limit of simplicial sheaves is defined as an objectwise homotopy limit of simplicial sets (see [MV, page 54]). Thus we have an objectwise weak equivalence, hence a local weak equivalence of simplicial sheaves

$$\operatorname*{holim}_n Ex(\prod_{i=1}^n \mathcal{K}(\mathcal{A}_i, i)) \longrightarrow \operatorname*{holim}_n Ex(P^n(\mathcal{X})). \tag{4.3}$$

Further, observe that, we have the following maps for each $n > 0$:

$$Ex(\prod_{i=1}^n \mathcal{K}(\mathcal{A}_i, i)) \xleftarrow{\phi_n} \prod_{i=1}^n \mathcal{K}(\mathcal{A}_i, i) \xrightarrow{\psi_n} \prod_{i=1}^n Ex(\mathcal{K}(\mathcal{A}_i, i)), \tag{4.4}$$

where both maps are weak equivalences. The map ϕ_n is a trivial cofibration as part of definition of resolution functor Ex. The map ψ_n is a weak equivalence by Lemma 2.4.4, since it is finite product of weak equivalences $\mathcal{K}(\mathcal{A}_i, i) \to Ex(\mathcal{K}(\mathcal{A}_i, i))$.

Using the lifting properties of model structure, we induce weak equivalences

$$Ex(\prod_{i=1}^n \mathcal{K}(\mathcal{A}_i, i)) \xrightarrow{f_n} \prod_{i=1}^n Ex(\mathcal{K}(\mathcal{A}_i, i)) \ \text{ for each } \ n > 0,$$

which are compatible with n. We do this by induction.
For $m = 1$: we can clearly set $f_1 = id$.

For $m = n+1$: Using (4.4), we have following diagram:

$$\begin{tikzcd} \prod_{i=1}^{n+1} \mathcal{K}(\mathcal{A}_i, i) \arrow[rr, "\psi_{n+1}"] \arrow[d, "\phi_{n+1}"'] & & \prod_{i=1}^{n+1} Ex(\mathcal{K}(\mathcal{A}_i, i)) \arrow[d, "fibration"] \\ Ex(\prod_{i=1}^{n+1} \mathcal{K}(\mathcal{A}_i, i)) \arrow[r] \arrow[urr, dashed, "f_{n+1}"] & Ex(\prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i, i)) \arrow[r, "f_n"] & \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i)), \end{tikzcd} \tag{4.5}$$

where right vertical map is a fibration since it is a projection. The outer square commutes because it is given as the outer square of the following diagram:

$$\begin{tikzcd} Ex(\prod_{i=1}^{n+1} \mathcal{K}(\mathcal{A}_i, i)) \arrow[d] & \prod_{i=1}^{n+1} \mathcal{K}(\mathcal{A}_i, i) \arrow[l, "\phi_{n+1}"'] \arrow[r, "\psi_{n+1}"] \arrow[d] & \prod_{i=1}^{n+1} Ex(\mathcal{K}(\mathcal{A}_i, i)) \arrow[d] \\ Ex(\prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i, i)) \arrow[rr, bend right, "f_n"'] & \prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i, i) \arrow[l, "\phi_n"'] \arrow[r, "\psi_n"] & \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i)), \end{tikzcd}$$

where vertical maps are projections and Ex applied to projections. The left square commutes by a natural transformation that is given with Ex (see Definition 2.3.4). The right square commutes by construction of the product of maps. The lower triangle commutes by induction hypothesis for the map f_n. Thus, collectively the outer square commutes.

Then, using the axiom of model structure by which a fibration has right lifting property with respect to a trivial cofibration, we have a lift f_{n+1} as required. By the 2-out-of-3 property of weak equivalences, f_{n+1} is a weak equivalence.

The lower triangle in commutative diagram 4.5 is exactly the compatibility of collection of maps $\{f_n\}_{n>0}$. Thus, we have a weak equivalence of towers of simplicial sheaves

$$\{Ex(\prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i, i))\}_{n>0} \xrightarrow{\{f_n\}} \{\prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i))\}_{n>0}.$$

Then, using the same argument that we used to obtain (4.3) we get the following weak equivalence between the homotopy limits of above towers:

$$\operatorname*{holim}_n Ex(\prod_{i=1}^{n} \mathcal{K}(\mathcal{A}_i, i)) \longrightarrow \operatorname*{holim}_n \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i)). \tag{4.6}$$

Observe that the tower $\left\{ \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i)) \right\}_{n>0}$ is a tower of fibrant objects and maps are projections hence fibrations. Then objectwise we get a tower of Kan fibration between Kan complex by [Jar, Lemma 5.12]. Then using [BK, XI.4.1] the natural map

$$\lim_n \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i))(U) \longrightarrow \operatorname*{holim}_n \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i))(U)$$

is a weak equivalence for each $U \in (\mathrm{Sm}/k)_{\text{ét}}$. Since the limits and homotopy limits of simplicial sheaves are defined objectwise, by [Jar, Lemma 4.1], the natural map

$$\lim_n \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i)) \longrightarrow \operatorname*{holim}_n \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i)) \tag{4.7}$$

is a weak equivalence of simplicial sheaves. Let us denote

$$\widetilde{\mathcal{X}} := \lim_n \prod_{i=1}^{n} Ex(\mathcal{K}(\mathcal{A}_i, i)) = \prod_{i>0} Ex(\mathcal{K}(\mathcal{A}_i, i)).$$

Then by (4.3), (4.6) and (4.7), the sheaf $\operatorname{holim}_n Ex(P^n(\mathcal{X}))$ is weakly equivalent to $\widetilde{\mathcal{X}}$.

Step 5 : We show that simplicial sheaf $\operatorname{holim}_n Ex(P^n(\mathcal{X}))$ is not connected. Since it weakly equivalent to $\widetilde{\mathcal{X}}$, it is enough to show that $\pi_0^s(\operatorname{holim}_n Ex(P^n(\mathcal{X}))) \cong \pi_0^s(\widetilde{\mathcal{X}}) \neq *$.

For each $U \in (\mathrm{Sm}/k)_{\text{ét}}$, the homotopy presheaf is given as

$$\begin{aligned} \pi_0^{pre}(\widetilde{\mathcal{X}})(U) &= \operatorname{Hom}_{\mathscr{H}_s}(U, \prod_{i>0} Ex(\mathcal{K}(\mathcal{A}_i, i))) \\ &= \prod_{i>0} \operatorname{Hom}_{\mathscr{H}_s}(U, Ex(\mathcal{K}(\mathcal{A}_i, i))) \\ &= \prod_{i>0} H^i(U, \mathcal{A}_i), \end{aligned}$$

where the last equality follows from [MV, Proposition 1.26, page 57]. Recall the stalk of étale sheaves from Example 2.2.7. At the geometric point $\bar{s} = \operatorname{Spec}(k^{sep})$, the stalk of the homotopy sheaf $\pi_0^s(\widetilde{\mathcal{X}})$ is given by

$$\begin{aligned}\pi_0^s(\widetilde{\mathcal{X}})_{\bar{s}} = \pi_0^{pre}(\widetilde{\mathcal{X}})_{\bar{s}} &= \varinjlim_{U} \pi_0^{pre}(\widetilde{\mathcal{X}})(U) \\ &= \varinjlim_{U} \prod_{i>0} H^i(U, \mathcal{A}_i) \\ &= \varinjlim_{k\subset L} \prod_{i>0} H^i(\operatorname{Spec} L, \mathcal{A}_i),\end{aligned}$$

where the colimit runs over the étale neighborhoods U of $\bar{s}$, which are given by the finite Galois extensions L/k such that $k \subset L \subset k^{sep}$.

By our choice of sheaves $\mathcal{A}_i$, the sheaf cohomology of $\mathcal{A}_i$ is isomorphic to Galois cohomology of A_i. More precisely, we have the following:

$$H^i(\operatorname{Spec} L, \mathcal{A}_i) \cong H^i(\operatorname{Spec} L, \widetilde{\mathcal{A}}_i) \cong H^i(G_L, A_i) \quad \text{for all} \quad i > 0,$$

where first isomorphism follows from [Tam, Chatper I, Corollary 3.9.3] and second isomorphism follows from Theorem 3.2.3. Hence, we have

$$\pi_0^s(\widetilde{\mathcal{X}})_{\bar{s}} \cong \varinjlim_{k\subset L} (\prod_{i>0} H^i(G_L, A_i)).$$

Let $\alpha := \prod_i \alpha_i \in \prod_{i>0} H^i(G_k, A_i)$. Then $\alpha|_L \neq 0$ in $\prod_{i>0} H^i(G_L, A_i)$ for each finite Galois extension L/k. Thus, α survives in above colimit, hence $\pi_0^s(\widetilde{\mathcal{X}})_{\bar{s}} \neq *$. This implies that the homotopy sheaf $\pi_0^s(\widetilde{\mathcal{X}})$ is nontrivial and the simplicial sheaf $\operatorname{holim}_n Ex(P^n(\mathcal{X}))$ is not connected.

Thus, the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type. $\square$

Remark 4.2.3. The hypothesis of Lemma 4.2.2 also implies that the small étale site $(\operatorname{Spec} k)_{\text{ét}}$ is not of finite type.

Remark 4.2.4. Observe that, under the hypothesis of Lemma 4.2.2, the cohomological dimension $cd(L) = \infty$ for any finite extension L/k; in particular the hypothesis for Lemma 4.2.2 is stronger that the hypothesis for 'only if' part of Conjecture 1.2.1.

4.2.3 Case when G_k is first-countable

In this section we prove the main result of this thesis which proves the Conjecture 1.2.1 for countable fields or, more generally, when absolute Galois group G_k is first-countable. The main result is introduced in Chapter 1 as the following theorem.

Theorem 1.2.2. *The Conjecture 1.2.1 holds, if the absolute Galois group $G_k = Gal(k^{sep}/k)$ is first-countable. In particular, if the field k is countable, then G_k is first-countable, hence Conjecture 1.2.1 holds.*

Proof. The absolute Galois group G_k is first-countable, so by Definition 3.1.1 and Galois correspondence Theorem 3.2.1, we can choose a countable nested local basis at identity as the open normal subgroups $\{G_{L_i}\}_{i\in\mathbb{N}}$ of G_k which correspond to sequence of finite Galois extensions

$$k \subset L_1 \subset L_2 \subset \cdots \subset L_n \subset \cdots \subset k^{sep}.$$

Since this collection is a basis, for any finite Galois extension E/k, $E \subset L_m$ for some m.

The one side implication of the conjecture follows from Proposition 4.2.1.

For converse part, we assume that any finite extension L/k has infinite cohomological dimension. In particular, $cd(L_n) = \infty$ for each $n \in \mathbb{N}$. We choose non-zero cohomology classes for each $n \in \mathbb{N}$ as follows:

$$\alpha_n \in H^n(G_{L_n}, A_n) \text{ where } A_n \text{ is some } G_{L_n}\text{-module.}$$

Recall the coinduced modules from Section 3.1.1. The coinduced modules corresponding to each A_n are denoted as

$$M(A_n) := \mathrm{Coind}_{G_{L_n}}^{G_k}(A_n) \in G_k\text{-}\mathsf{mod}.$$

The classes corresponding to α_n, through Shapiro's isomorphism (see Theorem 3.1.4), in $H^n(G_k, M(A_n))$ are denoted by same notation, *i.e.*, $\alpha_n \in H^n(G_k, M(A_n))$.

For each $n \in \mathbb{N}$, the restrictions $(\alpha_n)|_{L_n} \neq 0$ in $H^n(G_{L_n}, M(A_n))$, since the restriction map $H^n(G_k, M(A_n)) \xrightarrow{res} H^n(G_{L_n}, M(A_n))$ is injective by Remark 3.1.5. Furthermore, given any finite Galois extension E/k, there is m such that $E \subset L_m$. We have following composition of restriction maps:

$$\begin{array}{ccccc} H^m(G_k, M(A_m)) & \longrightarrow & H^m(G_E, M(A_m)) & \longrightarrow & H^m(G_{L_m}, M(A_m)) \\ \alpha_m & \longmapsto & (\alpha_m)|_E & \longmapsto & (\alpha_m)|_{L_m}. \end{array}$$

Since the restriction $(\alpha_m)|_{L_m} \neq 0$, the restriction $(\alpha_m)|_E \neq 0$.

Thus the collection of cohomology classes $\{\alpha_n \in H^n(G_k, M(A_n)\}_{n\in\mathbb{N}}$ satisfies the hypothesis of Lemma 4.2.2, hence the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type. □

Remark 4.2.5. The above Theorem 1.2.2 provides a whole class of fields for which the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type. One such field is $\overline{\mathbb{Q}}(x_1, x_2, \dots)$.

4.2.4 Case when k has large order cohomologies

In this section we show that certain cohomological boundedness conditions on the field k are necessary for the site $(\mathrm{Sm}/k)_{\text{ét}}$ to be of finite type. In particular, we prove 'only if' part of the Conjecture 1.2.1 with stronger assumptions. We prove the following theorem from Chapter 1.

Theorem 1.2.3. *Let k be any field which satisfies the condition that for each $n \in \mathbb{N}$, there exists G_k-module A_n such that*

$$\limsup_{n\to\infty} exp\,(H^n(G_k, A_n)) = \infty$$

where for an abelian group B, the exponent of B is denoted by $exp\,(B)$. Then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type.

Proof. According to the hypothesis, there is a subsequence $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\lim_i exp(H^{n_i}(G_k, A_{n_i})) = \lim_i c_i = \infty \quad \text{with} \quad c_i := exp(H^{n_i}(G_k, A_{n_i})),$$

where the sequence $\{c_i\}_{i\in\mathbb{N}}$ either consists of $\infty's$ only or is strictly monotonically increasing. For each i, either $c_{i+1} > c_i$ or $c_{i+1} = c_i = \infty$. In both cases we are able to choose cohomology classes

$$\{\alpha_{n_i} \in H^{n_i}(G_k, A_{n_i})\}_{i\in\mathbb{N}} \text{ such that } |\alpha_{n_{i+1}}| > |\alpha_{n_i}| \text{ for each } i,$$

where for an abelian group B and an element $\beta \in B$, the order of β is denoted by $|\beta|$. Hence for this collection of classes, $\lim_i |\alpha_{n_i}| = \infty$.

Consider any finite Galois extension L/k, then L corresponds to open normal subgroup G_L of G_k with the index $[G_k : G_L] = m$ for some $m \in \mathbb{N}$. Consider n_j such that $|\alpha_{n_j}| > m$. By Theorem 3.1.3, the composition of restriction and corestriction maps is given as

$$\begin{tikzcd} H^{n_j}(G_k, A_{n_j}) \arrow[r, "res"] & H^{n_j}(G_L, A_{n_j}) \arrow[r, "cor"] & H^{n_j}(G_k, A_{n_j}) \\ \alpha_{n_j} \arrow[r, mapsto] & (\alpha_{n_j})|_L \arrow[r, mapsto] & cor((\alpha_{n_j})|_L) = m\alpha_{n_j}. \end{tikzcd}$$

Since $m\alpha_{n_j} \neq 0$, the composition remains nonzero and hence $(\alpha_{n_j})|_L \neq 0$.

Thus, we have a collection of cohomology classes $\{\alpha_n \in H^n(G_k, A_n)\}_{n\in\mathbb{N}}$, possibly with $\alpha_n = 0$ and $A_n = 0$ whenever $n \neq n_i$ for any i, which satisfies hypothesis of Lemma 4.2.2. Hence the site $(\mathrm{Sm/k})_{\text{ét}}$ is not of finite type. □

The above Theorem 1.2.3 proves that if field k has arbitrarily large order cohomology classes in arbitrarily large degree, then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type.

Relation to p-cohomological dimension

Let k be a field such that $cd(k) = \infty$. Since $cd(k) = \sup_p cd_p(k)$, there are following three cases:

$$\begin{tikzcd} & (1)\ cd_p(k) = \infty \text{ for infinitely many primes.} \\ cd(k) = \infty \arrow[ur] \arrow[r] \arrow[dr] & (2)\ cd_p(k) < \infty \text{ for all primes.} \\ & (3)\ cd_p(k) = \infty \text{ for finitely many primes, else its bounded.} \end{tikzcd}$$

Observe that, in cases (1) and (2), we obtain arbitrarily large order cohomology classes and hypothesis of Theorem 1.2.3 is satisfied. We summarize this in the following corollary of Theorem 1.2.3 from Chapter 1.

Corollary 1.2.4. *Let k be a field such that $\limsup_p cd_p(k) = \infty$ where limit is over all primes p. Then the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type.*

Proof. According to the hypothesis, there is a strictly monotonically increasing subsequence of primes $\{p_i\}_{i\in\mathbb{N}}$ such that $\lim_i cd_{p_i}(k) = \infty$ and the sequence $\{cd_{p_i}(k)\}_{i\in\mathbb{N}}$ either consists of $\infty's$ only or is strictly monotonically increasing. We choose cohomology degrees for each p_i inductively.

By Definition 3.1.6 of p-cohomological dimension, for a prime p_1, we can choose $n_1 \geq 1$ and G_k-module A_{n_1} such that p_1-primary component $H^{n_1}(G_k, A_{n_1})_{p_1} \neq 0$ and hence the exponent $exp\,(H^{n_1}(G_k, A_{n_1})) \geq p_1$.

Inductively for a prime p_i, since $cd_{p_i}(k) = cd_{p_{i-1}}(k) = \infty$ or $cd_{p_i}(k) > cd_{p_{i-1}}(k)$, we can choose $n_i > n_{i-1}$ and G_k-module A_{n_i} such that p_i-primary component $H^{n_i}(G_k, A_{n_i})_{p_i} \neq 0$ and hence the exponent $exp\,(H^{n_i}(G_k, A_{n_i})) \geq p_i$.

Thus, we have a collection of G_k-modules $\{A_{n_i}\}$ for the sequence $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ which is strictly monotonically increasing such that $exp\,(H^{n_i}(G_k, A_{n_i})) \geq p_i$. Hence

$$\lim_i exp\,(H^{n_i}(G_k, A_{n_i})) = \infty, \text{ which implies, } \limsup_{n\to\infty} exp\,(H^n(G_k, A_n)) = \infty,$$

where $A_n = 0$ whenever $n \neq n_i$ for any i. Thus the hypothesis of Theorem 1.2.3 is satisfied and the site $(\mathrm{Sm}/k)_{\text{ét}}$ is not of finite type. □

Remark 4.2.6. The Corollary 1.2.4 provides non-finite type-ness of the site $(\mathrm{Sm}/k)_{\text{ét}}$ for the fields like $\mathbb{C}(x_1, x_2, \dots)$ or more generally $k(x_1, x_2, \dots)$ for an algebraically closed field k. It also implies that, with $cd(k) = \infty$, the possibility of site $(\mathrm{Sm}/k)_{\text{ét}}$ to be of finite type is only in case of fields k which has bounded p-cohomological dimension, except for some finitely many primes.

Bibliography

[AGV] M. Artin, A. Grothendieck, and J.-L. Verdier. *Théorie des Topos et Cohomologie Etale des Schémas (SGA4)*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer, 1972-73. 6

[AHW1] A. Asok, M. Hoyois, and M. Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory, I: Vector bundles. *Duke Mathematical Journal*, 166(10):1923–1953, 2017. 2

[AHW2] A. Asok, M. Hoyois, and M. Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory, II: Principal bundles and homogeneous spaces. *Geometry & Topology*, 22(2):1181–1225, 2018. 2

[BHS] C. Balwe, A. Hogadi, and A. Sawant. Strong $\mathbb{A}^1$-invariance of $\mathbb{A}^1$-connected components of reductive algebraic groups. *Journal of Topology*, 16(2):634–649, 2023. 3

[BK] A.K. Bousfield and D.M. Kan. *Homotopy Limits, Completions and Localizations*, volume 304 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972. 34, 42, 44

[DHP] S. Dhamore, A. Hogadi, and R. Pawar. Non-finite type étale sites over fields. *Journal of Algebra*, 668:265–277, 2025. 3, 4

[DK] W.G. Dwyer and D.M. Kan. An obstruction theory for diagrams of simplicial sets. *Indagationes Mathematicae (Proceedings)*, 87(2):139–146, 1984. 16

[GJ] P.G. Goerss and J.F. Jardine. *Simplicial Homotopy Theory*. Modern Birkhäuser Classics. Birkhäuser Basel, 2009. 9, 12, 13, 14, 15, 16, 18, 22, 39, 40

[Har] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer New York, NY, 1977. 35

[Hir] P.S. Hirschhorn. *Model Categories and Their Localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003. 10

[Hov] M. Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. 9, 10, 11, 12, 20

[Jar] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer, New York, 2015. 5, 7, 19, 41, 42, 44

[May] J.P. May. *Simplicial Objects in Algebraic Topology*. Van Nostrand Mathematical Studies, No. 11. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967. 16

[Mil] J. S. Milne. *Étale Cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1980. 36

[Mor] F. Morel. $\mathbb{A}^1$-*Algebraic Topology over a Field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012. 2

[MV] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 90:45–143, 1999. 1, 2, 18, 20, 22, 33, 34, 35, 37, 42, 44

[MVW] C. Mazza, V. Voevodsky, and C. Weibel. *Lecture Notes on Motivic Cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. 37

[Poo] B. Poonen. *Rational Points on Varieties*, volume 186 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017. 31

[Qui] D. Quillen. *Homotopical Algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer, Berlin-Heidelberg, 1967. 14

[Rie] E. Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, Cambridge, 2014. 16

[RZ] L. Ribes and P. Zalesskii. *Profinite Groups*, volume 40 of *A Series of Modern Surveys in Mathematics*. Springer, Berlin, 2010. 24, 27, 28

[Ser] Jean-Pierre Serre. *Galois Cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author. 29, 30

[Sha] S. S. Shatz. *Profinite Groups, Arithmetic, and Geometry*, volume 67 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 1972. 24, 29

[Sta] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2025. 6, 7, 29

[Tam] G. Tamme. *Introduction to Étale Cohomology*. Universitext. Springer, Berlin, 1994. Translated from the German by Manfred Kolster. 38, 45

[Vis] A. Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental Algebraic Geometry*, volume 123 of *Mathematical Surveys and Monographs*, pages 1–104. American Mathematical Society, Providence, RI, 2005. 5, 6