Extra twists of Siegel modular forms, the associated central simple algebras and dimensions of spaces of Siegel modular forms

विद्या िाचस्पवि की

उपावि की अपेक्षाओं की आंविक पविू में प्रस्िुि िोि प्रबंि

A thesis submitted in partial fulfillment of the requirements of the degree of Doctor of Philosophy

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भािीर् विज्ञान विक्षा एिं अनुसंिान संस्थान पुणे

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Dedicated to

Toni Kurz

I dedicate my thesis to Toni Kurz, who was born on 13 January 1913 in Berchtesgaden, Bavaria, Germany, where he was raised. He completed a brief apprenticeship as a pipefitter before joining the German Wehrmacht in 1934 as a professional soldier. Together with his childhood friend Andreas Hinterstoisser, he made numerous first ascents of peaks in the Berchtesgaden Alps, including some of the most difficult climbs of that time.

His tragic story of the second ascent of the Eiger became well known after publication of Heinrich Harrer's classic 1960 book The White Spider and was more recently covered by Joe Simpson's book (and Emmy-winning TV documentary), The Beckoning Silence, as well as the 2008 German dramatic movie North Face.

Certificate

Certified that the work incorporated in the thesis entitled "*Extra twists of Siegel modular forms, the associated central simple algebra and dimensions of spaces of Siegel modular forms*", submitted by *Ronit Debnath* was carried out by the candidate under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Debargha Banerjee

Date: March 17, 2025

Dr. Debargha Banerjee
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Declaration

Name of Student: Ronit Debnath
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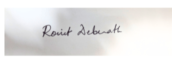
I declare that this written submission represents my ideas in my own words. Where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity. I have not misrepresented, fabricated, or falsified any idea, data, fact, or source in my submission. I understand that violation of the above will result in disciplinary action by the institute and may also lead to penal action if proper citation or permission has not been obtained from the sources used.

The work reported in this thesis is the original work done by me under the guidance of *Dr. Debargha Banerjee*.



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Contents

Abstract

Denote the Siegel upper half space as

$$\mathbb{H}_2 = \{X + iY \in M_2(\mathbb{C}) \mid X = X^t, Y = Y^t, Y > 0\}.$$

A holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ is called a Siegel modular form of weight k, if it satisfies certain conditions. Associated to a Siegel modular form we have the associated Galois representation :

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{K_\lambda}).$$

Assume $\mathfrak{g}_l$ to be the lie algebra of the image of the representation and

$$\mathfrak{a}_l = \{u \in \mathfrak{gsp}_4(K \otimes_{\mathbb{Q}} \mathbb{Q}_l) = \prod_{\lambda | l} \mathfrak{gsp}_4(K_\lambda) \mid \Lambda(u) \in \mathbb{Q}_l\}$$

where Λ denotes the symplectic similitude.

Let $\Gamma = \mathrm{Hom}(K, \mathbb{C}^\times)$. Then a Siegel modular form F is said to have an *extra twist* if there exists a tuple (γ, χ_γ) with $\gamma \in \Gamma$, and an associated χ_γ such that $\gamma(\rho_F) \cong \rho_F \otimes \chi_\gamma$.

One of the results if of the thesis shows that $\mathfrak{g}_l \subsetneq \mathfrak{a}_l$ if and only if F admits an extra twist.

While for a form with a non trivial character ϵ, the pair (c, ϵ^{-1}) is already an extra twist using Petersson inner product, we show examples in this thesis of how different ways of lifting classical modular forms with twists to Siegel modular forms gives rise to extra twists or not in the most natural sense. The lifts we cover include Yoshida lifts, Saito Kurokawa lifts and Sym^3 lifts.

Assume X to be the cross product algebra associated with the cocycle $c(\gamma, \delta)$ that is defined by

$$X = \bigoplus_{\gamma \in \Gamma} K \cdot x_\gamma$$

We introduce the intertwining opertator as well which is described as follows :

For a subgroup $U \subset G_{\mathbb{Q}}$ and a vector space V over K_λ, let us denote by

$$\iota_\lambda^U(V) := \{\Phi : (V, \Psi) \to (V, \Psi), \Phi \circ \mathfrak{g}_\lambda(h) = \mathfrak{g}_\lambda(h) \circ \Phi \text{ for all } h \in \log(U)\}.$$

Then, in one of our second major results we show that

$$\iota_l^{\overline{\mathfrak{g}_l}}(\overline{V_l}) \simeq X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l}.$$

We also share some results on the local Brauer classes of the central simple algebra X.

Due to Poor and Yuen, we have the restriction map $\phi_s^* : S_2^k(\Gamma_0^2(p)) \to S_1^{2k}(\Gamma_0(p))$. Since the right hand side of this equation is the space of classical modular forms, we derive information of Siegel modular forms using this map.

We have devised a program that gives the image of the restriction map of a Siegel modular form under arbitrary prime level.

We further compute an upper bound for the dimension of the space of Siegel modular form for a prime power.

Statement of originality

The main original research results presented in this thesis, along with their respective chapters and sections, are as follows:

In Chapter 4, Theorem 4.1.2 is discussed.

In Chapter 5, Theorem 5.1.8, Theorem 5.2.5, lemmas 5.2.7 and 5.2.9 are discussed.

In Chapter 6, Lemma 6.3.3 is discussed.

1

Introduction

Siegel modular forms, named after Carl Ludwig Siegel are a generalization of classical modular forms by modifying the domain to a higher dimension. Instead of the upper half plane, we look at functions into $\mathbb{C}$ from $n \times n$ complex matrices, that are symmetric with imaginary part positive definite which we denote by $\mathbb{H}_n$.

We introduce the general symplectic group GSp_{2n}, its special subgroup Sp_{2n} for arbitrary genus n, and its important congruence subgroups $\Gamma^{(n)}(N), \Gamma_0^{(n)}(N), \Gamma_1^{(n)}(N)$, and $\Delta_0^{(n)}$ which will be useful throughout our thesis.

For a Siegel modular subgroup $\Gamma^{(n)}(N) \subset \Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$ and for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, a complex valued holomorphic function F is called a Siegel modular form if $F|_k[\gamma](\tau) = det(C\tau + D)^{-k} F((A\tau + B)(C\tau + D)^{-1})$.

Corresponding to any Siegel modular form F of genus 2, we have an associated representation

$$\rho_F : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{K}_\lambda)$$

where $K = \mathrm{Q}(t_p)$ is the number field we get by adjoining the Hecke eigenvalues, denoted by t_p of the form F for primes $p \nmid lN$.

Assume $\Gamma = \mathrm{Hom}(K, \mathbb{C}^\times)$.

In Chapter 2, we have provided a short overview of classical modular forms, while in Chapter 3, we delve into the definition of the Siegel upper half space, $\mathbb{H}_2$, and the definitions of scalar-valued Siegel modular forms and vector-valued Siegel modular forms.

In Chapter 4, we give explicit examples of extra twists. Specifically, we construct Siegel modular forms with extra twists by lifting classical modular forms through Yoshida, or Sym^3 lifts while also showing that lifts coming from CAP representations do not, in the most natural way necessarily give us Siegel modular forms with extra twist.

In Chapter 5, we discuss the image of the associated representation of the Siegel modular form and how this image is affected by the presence of an extra twist(s) in the form. Essentially what we prove is that $\mathfrak{g}_l \subsetneq \mathfrak{a}_l$ if and only if the form F has an extra twist.

In the same chapter, we also prove certain theorems regarding the central simple algebra associated with the image of the representation.

In Chapter 6, we show a program which gives us the expansion associated to the restriction map for a Siegel modular form of genus 2, for a general prime level p and weight 2.

In the first case described below is the Yoshida lift, we give two very explicit examples of the construction of Siegel modular forms with extra twists different from complex conjugation. The rest of the twist examples are more theoretical.

Theorem 1.0.1. 1. *If we start with two Siegel modular forms f, g with an extra twist associated to the same Dirichlet character, then if these two forms satisfy the conditions of an Yoshida lift, their lift which we denote by $Y(f \otimes g)$ happens to be a Siegel modular form with an extra twist.*

2. *For a classical modular form f equipped with an extra twist that is different from complex conjugation, its symmetric cube lift, which we denote by $Sym^3(f)$ contains an extra twist different from complex conjugation.*

3. *Beginning with a classical modular form f with an extra twist, if we consider its Saito-Kurakawa lift F_f might not necessarily be a Siegel modular form with an extra twist.*

For the theorem below, we define the intertwining operator. If we start with two symplectic vector spaces V and W and a subgroup H of $G_{\mathbb{Q}}$,

Definition 1.0.2. $I_\lambda^H(V,W) := \{\Phi : (V,\Psi) \to (W,\Psi), \Phi \circ \rho_F(h) = \rho_F(h) \circ \Phi \text{ for all } h \in H\}$ where Φ is a linear map.

We will also define the image of the Galois representation and the best estimate that it is compared against below.

Note for any matrix in $M \in \mathfrak{gsp}_4(F)$ (where F is any field), we have $M^tJ + JM = \Lambda(M)J$ and Λ denotes this scalar.

For a representation $\rho_F : G_\mathbb{Q} \to \mathrm{GSp}_4(K_\lambda)$, let $\mathfrak{g}_l = \mathrm{Lie}(\rho_F(G_\mathbb{Q}))$ which is the Lie algebra of the image. Also let $\mathfrak{a}_l = \{u \in \mathfrak{gsp}_4(K \otimes \mathbb{Q}_l) \mid \Lambda(u) \in \mathbb{Q}_l\}$. One of the major theorems of this thesis is stated below.

Definition 1.0.3. For a subgroup $U \subset G_\mathbb{Q}$ and a vector space V over K_λ, let us denote by

$$\iota_\lambda^U(V) := \{\Phi : (V,\Psi) \to (V,\Psi), \Phi \circ \mathfrak{g}_\lambda(h) = \mathfrak{g}_\lambda(h) \circ \Phi \text{ for all } h \in \log(U)\}.$$

Theorem 1.0.4. *The algebra $\iota_l^{\mathfrak{g}_l} = K_l$ if and only if $\mathfrak{g}_l = \mathfrak{a}_l$.*

The theorem above shows how the size of the image of the representation is affected by the presence of an extra twist and that its presence makes it strictly smaller than $\mathfrak{a}_l$.

We define the K-valued 2-cocycle c on Γ by

$$c(\gamma,\delta) = \frac{G(\chi_\delta^{-\gamma})G(\chi_\gamma^{-1})}{G(\chi_{\gamma\cdot\delta}^{-1})},$$

for γ, $\delta \in \Gamma$.

We introduce X to be the cross product algebra associated with the cocycle $c(\gamma,\delta)$ which is defined as

$$X = \bigoplus_{\gamma\in\Gamma} K \cdot x_\gamma.$$

. One may refer to 5.3 for the definition and a detailed description of the cross product algebra X.

Theorem 1.0.5. *With X as above, we also prove an equality $X \otimes_\mathbb{Q} \overline{\mathbb{Q}}_l \cong \iota_l^{\overline{\mathfrak{g}_l}}(\overline{V}_l)$.*

We also prove some lemmas regarding the local Brauer classes of the central simple algebra X.

A prime p is said to be a good prime if $(p, N) = 1$. At infinite places v, the X_v is a central simple algebra. Also consider $X_v = X \otimes_v F_v$ and ϵ be the nebentypus character.

Lemma 1.0.6. *For good primes the following holds true :*

- *Given* $\gcd(p, N) = 1$ *and* $t_p \neq 0$. *Assuming* v *is a place of* F *lying over* p, X_v *is a matrix algebra over* F_v, *if and only if the slope* $m_v := [F_v : \mathbb{Q}_p] \cdot v(t_p^2/\epsilon(p)) \in \mathbb{Z}$ *where* v *is so normalized such that* $v(p) = 1$.
- *Also if* $\gcd(p, N) = 1$, $t_p = 0$, *and* v *is a place of* F *lying over* p, *then* X_v *is a matrix algebra over* F_v *if and only if* $m_v^{\dagger} := [F_v : \mathbb{Q}_p] \cdot v(t_p^2/\epsilon(p^{\dagger})) \in \mathbb{Z}$ *is even.*

In the latter chapter of this thesis, we also talk about the image of the restriction map of a Siegel modular form. From Poor and Yuen we have the following map from the space of Siegel modular forms to the space of classical modular forms.

$$\phi_s^*(f) : S_2^k(\Gamma_0^2(p)) \to S_1^{2k}(\Gamma_0(p)).$$

For the definition of the map ϕ_s^*, we refer to [24, Section 4]. We use the fact that much is known about the right hand side as they are the space of classical modular forms for many of our calculations. We have written down a program that gives the right hand side of this expansion of different choices of matrices s, and primes p.

Using the fact that we have better information about the bounds for the reference matrices X_2 even for a prime power now, due to Klein, we use that to give an upper bound for the dimensions of the space of Siegel modular forms for a prime power.

2

The Classical Modular Forms

The group $\mathrm{SL}_2(\mathbb{Z})$ also known as the *Full Modular Group*, is the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ consisting of matrices $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \mid ad - bc = 1 \right\}$.

$$\text{Let } \mathbb{H} := \{\tau = x + iy \in \mathbb{C} \mid y > 0\}$$

be the upper half plane in $\mathbb{C}$.

For $\tau \in \mathbb{H}$ and the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Definition 2.0.1. For an integer $k \in \mathbb{Z}$, a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *weakly modular* of weight k if for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tau \in \mathbb{H}$ and

$$f(\alpha.\tau) = (c\tau + d)^k f(\tau). \tag{2.1}$$

One can we consider the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, to prove that any weakly modular function of weight k becomes a periodic function with period 1 (that is $f(\tau) = f(\tau + 1)$), and satisfies $f\left(\frac{-1}{\tau}\right) = \tau^k f(\tau)$.

Further, because $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, plugging in this matrix to the equation 2.1 gives $f = (-1)^k f$ which when k is odd, gives $f \equiv 0$.

Consider the domain $D = \left\{\tau \in \mathbb{C} \mid |\tau| < 1\right\}$. The upper half-plane $\mathbb{H}$ is bijective with the punctured open unit disk $D' = D - \{0\}$ using the map $\tau \mapsto e(\tau) = e^{2\pi i \tau}$ and $q = e(\tau)$.

For all functions f defined on $\mathbb{H}$, they can be represented as g on D', where $g(q) = f\left(\frac{\log q}{2\pi i}\right)$. Since the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the defined function g is well defined even as $\log$ is determined up to $2\pi i\mathbb{Z}$.

For a weakly modular function f of weight k, we say f is holomorphic at ∞ if g extends analytically to $0 \in \mathbb{C}$. In such a case, f admits a Fourier expansion $f(\tau) = \sum\limits_{n \geq 0} a_n(f) q^n$, where $q = e(\tau)$.

Definition 2.0.2. (**Modular form**). A function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form* for $\mathrm{SL}_2(\mathbb{Z})$ of weight k if :

(i) f is weakly modular of weight k,

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at ∞.

The set of such modular forms of weight k is denoted by $M_1^k(\mathrm{SL}_2(\mathbb{Z}))$.

Definition 2.0.3 (Cusp form). A modular form f of weight k is called a cusp form of weight k if the zeroeth Fourier coefficient of f vanishes at ∞.

For a natural number $N \in \mathbb{N}$, let

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

As the kernel of the natural morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, we see that $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$, called the principal congruence subgroup of level N.

Definition 2.0.4. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if there exists a $N \in \mathbb{N}$ such that $\Gamma(N) \subset \Gamma$, and the smallest such N satisfying this condition is called the level of Γ.

Below we list two of the most used congruence subgroups. Let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}.$$

The subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$ are congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of level N.

From the definitions, it is evident following the inclusion conditions that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$. Being the kernel of the natural homomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}),$$

the subgroup $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. This map is a surjection, inducing the isomorphism

$$\frac{\mathrm{SL}_2(\mathbb{Z})}{\Gamma(N)} \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}),$$

and so

$$\left|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Hence any congruence subgroup has a finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Consider $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\tau \in \mathbb{H}$. The factor of automorphy at γ is defined as $j(\gamma, \tau) = c\tau + d$.

For $f : \mathbb{H} \to \mathbb{C}$, we define the weight-$k$ operator $[\gamma]_k$ such that

$$(f[\gamma]_k)(\tau) = (\det(\gamma))^{k-1} j(\gamma, \tau)^{-k} f(\gamma(\tau)).$$

Since the factor of automorphy is neither zero nor infinity, f is meromorphic if and only if $f[\gamma]_k$ is meromorphic.

Definition 2.0.5. We define a function $f : \mathbb{H} \to \mathbb{C}$ to be a *weakly modular form* of weight k with respect to Γ if $f[\gamma]_k = f$ for all $\gamma \in \Gamma$.

Definition 2.0.6. Let Γ be a congruence subgroup and k be an integer. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ if

(i) The function f is holomorphic,

(ii) It has weight k invariance under Γ, which means that it satisfies equation 2.1

(iii) $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 2.0.7. A modular form f of weight k with respect to Γ is termed a *cusp form* of weight k with respect to Γ if $f[\gamma]_k$ vanishes at infinity for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 2.0.8 (Fundamental domain). A fundamental domain for a group Γ acting on upper half space $\mathbb{H}$ is a region in the upper half-plane $\mathbb{H}$ that contains exactly one point from each Γ-orbit.

Proposition 2.0.9. *Let $\mathcal{D} = \{\tau \in \mathbb{C} \mid |\tau| \geq 1, |Re(\tau)| \leq 1\}$. Then, the set $\mathcal{D}$ is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.*

Let Γ be a congruence subgroup. Then the modular curve for Γ, denoted by $Y(\Gamma)$, is the set $\{\Gamma\tau \mid \tau \in \mathbb{H}\} = \Gamma\backslash\mathbb{H}$.

We define $X(\Gamma) = \{\Gamma\tau \mid \tau \in \overline{\mathbb{H}}\}$, where $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.

For the topology on $X(\Gamma)$, we define a map $\phi : \mathbb{H} \to Y(\Gamma)$ given by $\tau \mapsto \Gamma\tau$. The topology on $Y(\Gamma)$ is defined using the quotient topology, where $\mathbb{H}$ inherits the subspace topology of the Euclidean topology on $\mathbb{C}$.

We define the topology on $X(\Gamma)$ in such a way that $Y(\Gamma)$ becomes a dense subset.

Since the action of the congruence subgroup Γ on $\mathbb{H}$ is properly discontinuous, $Y(\Gamma)$ is a Hausdorff space. For a more detailed view on this, one can refer to [10, Chapter 2].

The subgroup $\Gamma_\tau = \{\gamma \in \Gamma \mid \gamma(\tau) = \tau\}$ is known as the isotropy subgroup.

Definition 2.0.10. An element $\tau \in \mathbb{H}$ is termed an *elliptic point* if the isotropy subgroup Γ_τ is non-trivial as a group of transformations, that is if the containment $\{\pm I\}\Gamma_\tau \supset \{\pm I\}$ of matrix groups is proper. The corresponding point $\Gamma\tau \in Y(\Gamma)$ is also referred to as an elliptic point.

Remark 2.0.11. *[10, Corollary 2.2.3] For any congruence subgroup Γ and any $\tau \in \mathbb{H}$, there exists a neighborhood $U \subset \mathbb{C}$ of τ such that for any $\gamma \in \Gamma$, if $\gamma(U) \cap U \neq \emptyset$, then $\gamma \in \Gamma_\tau$. Additionally, U contains no other elliptic points.*

Definition 2.0.12. For $\tau \in \mathbb{H}$, *the period of τ*, denoted by h_τ, is defined as $|\frac{\{\pm I\}\Gamma_\tau}{\{\pm I\}}|$.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ (where $\gamma \neq \pm I$) and $\tau \in \mathbb{H}_2$. Then $\gamma(\tau) = \tau$ if and only if the equation $c\tau^2 - (a-d)\tau - b = 0$ has a solution. The fact that $\tau \in \mathbb{H}_2$ implies that the characteristic polynomial of the matrix is either x^2+1 or $x^2 \pm x + 1$.

Furthermore, we observe that $\gamma^3 = I$, $\gamma^4 = I$, or $\gamma^6 = I$. Hence, h_τ is finite. Moreover, it can be verified that the isotropy subgroups are cyclic.

Additionally, $h_\tau = h_{\gamma(\tau)}$ for any $\gamma \in \Gamma$. It is straightforward to show that the imaginary part of an elliptic point is strictly less than 2.

Now we can describe the local structure at any point $\Gamma\tau \in Y(\Gamma)$ (which is well-defined). Let $h = h_\tau$. Consider $\delta = \delta_\tau = \begin{pmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$.

It follows that $\delta(\tau) = 0$ and $\delta(\overline{\tau}) = \infty$. Evidently, $h_{\delta(\tau)} = h$. We choose a neighborhood U of τ as in Corollary 2.0.11, and define $\psi : U \to \mathbb{C}$ by $\psi = \rho \circ \delta$, where $\rho(z) = z^h$.

Now we define $\varphi : \phi(U) \to \psi(U)$ such that $\varphi(\phi(z)) = \psi(z)$. This map is a homeomorphism because fractional linear transformations are homeomorphisms. It can be verified that if U_1 and U_2 are two neighborhoods of τ and φ_1, φ_1 are two maps like φ above, the map between $\varphi_1(\phi(U_1) \cap \phi(U_2))$ and $\varphi_2(\phi(U_1) \cap \phi(U_2))$ is analytic. Thus, this provides a manifold structure on $Y(\Gamma)$.

Remark 2.0.13. *Let $\mathcal{D} = \{\tau \in \mathbb{C} \mid |\tau| \geq 1, |Re(\tau)| \leq 1\}$. Then the map $f : \mathcal{D} \to Y(\mathrm{SL}_2(\mathbb{Z}))$ defined by $f(\tau) = \mathrm{SL}_2(\mathbb{Z})\tau$ is a surjective mapping.*

Remark 2.0.14. *The modular curve $Y(\Gamma)$ is a connected but non-compact complex manifold* *of dimension 1. To render it compact, it is enough to compactify the domain of the map φ, as the continuous. Hence, we include $\mathbb{Q} \cup \{\infty\}$ in $\mathbb{H}_2$. It is evident that $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and then the extended quotient $\Gamma \backslash \overline{\mathbb{H}}$ is compact.*

Definition 2.0.15. An element in the set $\Gamma \backslash (\mathbb{Q} \cup \{\infty\})$ is called a *cusp* for the congruence subgroup Γ.

Theorem 2.0.16. *For any congruence subgroup Γ, $X(\Gamma)$ is a compact, connected, and Hausdorff space. Furthermore, $X(\Gamma)$ is a complex manifold of dimension 1, making it a Riemann Surface.*

Remark 2.0.17. *For the classical case, where $X(\Gamma)$ is a Riemann surface, we can utilize the Riemann-Hurwitz formula, genus, and Riemann-Roch theorem to determine the dimension of the set of all modular forms, the set of all cusp forms, and the set of all Eisenstein forms.*

Definition 2.0.18 (Differential Forms). When k is even, the weight k modular form $f(\tau)$ can be interpreted as the differential form $f(\tau)(d\tau)^{k/2}$.

This form is fixed under the action of every element of Γ, meaning it transforms in a way that maintains its overall structure. This invariance under Γ indicates that $f(\tau)(d\tau)^{k/2}$ is a Γ-invariant differential form.

Let M_k denote the set of all modular forms of weight k, and S_k denote the set of all cusp forms of weight k. Both sets are vector spaces over $\mathbb{C}$.

Definition 2.0.19 (Petersson inner product). Let f and g be modular forms of weight k for $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, where at least one is a cusp form. Let $\mathcal{D}$ be a fundamental domain for Γ. The function

$$\langle , \rangle : M_k \times S_k \longrightarrow \mathbb{C},$$

$$\langle f, g \rangle = \iint\limits_{\mathcal{D}} f(\tau)\overline{g(\tau)} y^{k-2} dx dy$$

is called the Petersson inner product.

Remark 2.0.20. *The differential $f(\tau)\overline{g(\tau)} y^{k-2} dx dy$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, which means that the inner product is independent of the fundamental domain chosen*

The set of all Eisenstein series $E_k(\Gamma)$ of weight k is the orthogonal complement of $S_k(\Gamma)$ inside $M_k(\Gamma)$. In other words, we have the decomposition

$$M_k(\Gamma) = S_k(\Gamma) \oplus E_k(\Gamma).$$

For proofs and more detailed theory of classical modular forms, we refer to [10].

3

Siegel Modular Forms

3.1 Siegel modular forms

Siegel modular forms are holomorphic functions on the set of symmetric $n \times n$ matrices with positive definite imaginary part. The forms satisfy an automorphy condition. Siegel modular forms can be thought of as multivariable modular forms, that is the generalization of classical modular forms to a domain of higher genus.

3.1.1 The Symplectic Group

Let us start with a natural number $n \in \mathbb{N}$ and a commutative ring R with 1. We denote by $\mathrm{GSp}_{2n}(R)$, the general symplectic group of similitudes which is defined by :

$$\mathrm{GSp}_{2n}(R) = \{M \in \mathrm{GL}_{2n}(R) \mid M^t J M = \mu(M) J\}$$

where $\mu(M) \in R^\times$ and $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$.

We also have a homomorphism from the multiplier map $\mu : \mathrm{GSp}_{2n}(R) \to R^\times$. The kernel of this map is called the special symplectic group given as :

$$\mathrm{Sp}_{2n}(R) = \{M \in \mathrm{GSp}_{2n}(R) \mid \mu(M) = 1\}$$

Lemma 3.1.1. *The special symplectic group over real numbers,* $\mathrm{Sp}_{2n}(\mathbb{R})$ *is a subset of the special linear group,* $\mathrm{SL}_{2n}(\mathbb{R})$*.*

Proof. By the *Iwasawa* decomposition [23, Exercise 1.4], any matrix $M \in \mathrm{Sp}_{2n}(\mathbb{R}) = KAN$ where K is the orthogonal subgroup, A is the positive diagonal subgroup and N is the unipotent subgroup as described below. More explicitly, the relevant sets are

- $K = \left\{ \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \mid X^tY = Y^tX, X^tX + Y^tY = I_n \right\}$
- $A = \left\{ \begin{bmatrix} I_n & X \\ 0_n & I_n \end{bmatrix} \mid X = X^t \right\}$
- $N = \left\{ \begin{bmatrix} g & 0_n \\ 0_n & {}^tg^{-1} \end{bmatrix} \mid g \in \mathrm{GL}_n(\mathbb{R}) \right\}$

Since for each of these subgroups, the matrices of K, N, and A are in $\mathrm{SL}_n(\mathbb{R})$ as they are determinant 1 matrices, their product is also in $\mathrm{SL}_n(\mathbb{R})$.

□

Below we will describe the generalization of the upper half plane for higher genus which will be the domain for our forms.

3.1.2 The Siegel upper half space

For a genus $n \in \mathbb{N}$, the Siegel upper half-space is defined as :

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) \mid Z = Z^t \text{ and } Im(Z) > 0\},$$

where by $Im(Z) > 0$, we mean that if $Z = X + iY$, then Y is positive definite.

Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_{2n}(\mathbb{R})^+ = \{g \in \mathrm{GSp}_{2n}(\mathbb{R}) \mid \mu(g) > 0\}$.

For a $Z \in \mathbb{H}_n$, we aim to define the action of $\mathrm{GSp}_{2n}(\mathbb{R})^+$ on $\mathbb{H}_n$ as follows :

$$g < Z >= (AZ + B)(CZ + D)^{-1}.$$

The above association satisfies the following properties [23, Theorem 1.5] :

1. Denote by $J(g, Z) = CZ + D$. Then the matrix $J(g, z)$ is non singular and for two matrices $g_1, g_2 \in \mathrm{GSp}_{2n}(R)^+$, we have the equation

$$J(g_1 g_2, Z) = J(g_1, g_2 < Z >)J(g_2, Z).$$

2. The matrix $g < Z >$ is symmetric and we have

$$Im(g < Z >) = \mu(g)^t(C\overline{Z} + D)^{-1}(ImZ)(CZ + D)^{-1}.$$

3. The function $Z \to g < Z >$ is an action of $\mathrm{GSp}_{2n}(\mathbb{R})^+$ on $\mathbb{H}_n$.

The aforementioned list proves that the association is an action. One can refer to [23, Section 1.3] for the proofs.

3.2 Scalar valued Siegel modular forms

Let Γ_n denote the group $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Definition 3.2.1. [23, Definition 1.8] A function $F : \mathbb{H}_n \to \mathbb{C}$ is called a scalar valued Siegel modular form of weight k if the following conditions are satisfied :

1. F is holomorphic.

2. F satisfies the conditions

$$F((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^k F(Z),$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$.

3. For $n = 1$, F is bounded in $Y \geq Y_0$ for any $Y_0 > 0$.

The space of Siegel modular forms of weight k and degree n, thus form a complex vector space which we denote by $M_k(\Gamma_n)$.

Remark 3.2.2. *If kn is odd, then the space $M_k(\Gamma_n)$ is the zero space.*

Proof. We use the matrix $\begin{pmatrix} -I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and use it in the equation $F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z)$, to get $F = -F$ which implies $F \equiv 0$. □

Definition 3.2.3. [23, Definition 1.11] We define the Siegel Φ operator which acts on $F \in M_k(\Gamma_n)$ as follows :

$$(\Phi F)(Z') = lim_{t \to \infty} F(\begin{pmatrix} Z' & 0 \\ 0 & it \end{pmatrix}),$$

with $Z' \in \mathbb{H}_{n-1}$ and $t \in \mathbb{R}$.

This operator is necessary to define what would be a Siegel cusp form which we define below.

Definition 3.2.4. [23, Definition 1.13] A Siegel modular for $F \in M_k(\Gamma_n)$ is called a Siegel cusp form if F lies in the kernel of the Φ operator described above. We denote the space of cusp forms as $S_k(\Gamma_n)$.

3.3 Vector valued Siegel modular forms

The definition of vector valued Siegel modular forms as stated below is mainly taken from [15].

We start with a pair of non-negative integers (k, j) and let $\rho_{k,j} : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{j+1}(\mathbb{C})$ be the irreducible rational representation of weight $(j+k, k)$ that is

$$\rho_{k,j} := \det{}^k \otimes \mathrm{Sym}(j),$$

where $\mathrm{Sym}(j)$ is the symmetric tensor representation of $\mathrm{GL}_2(\mathbb{C})$ of degree j. For a $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})$, and a $\mathbb{C}^{j+1}$ valued function F on $\mathbb{H}_2$,we define the following :

$$(F|_{k,j}[\gamma])(Z) = \rho_{k,j}(CZ+D)^{-1} F(\gamma Z)$$

for $Z \in \mathbb{H}_2$.

Definition 3.3.1. A holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}^{j+1}$ is called a vector valued Siegel modular form of weight $\rho_{k,j}$ with respect to the group $\mathrm{Sp}_4(\mathbb{Z})$ if $F|_{k,j}[\gamma] = F$ for all $\gamma \in \mathrm{Sp}_4(\mathbb{Z})$.

A vector valued Siegel modular form is scalar valued if $j = 0$ and in that case we write the slash operator as just $|_k$.

3.4 Congruence subgroups

Let us start with a positive integer $N \in \mathbb{N}$. The principal congruence subgroup for the level N and a genus n is defined as follows :

$$\Gamma_n(N) := \{g \in \Gamma_n : g \equiv 1_n \pmod{N}\}.$$

A subgroup of the group $\mathrm{Sp}_{2n}(\mathbb{Q})$ is called a congruence subgroup if it contains some principal congruence subgroup of a certain level.

We will list the important congruence subgroups that concern us. The list is taken from [23, Section 2.2.1].

- The Siegel congruence subgroup :

$$\Gamma_0^n(\mathbb{Z}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) \mid C \equiv 0 \ (\mathrm{mod} N) \right\}$$

.

- The Γ_1 subgroup:

$$\Gamma_1^n(\mathbb{Z}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) \mid C \equiv 0 \ (\mathrm{mod}\ N) \text{ and } A, D \equiv I_n (\mathrm{mod} N) \right\}$$

.

- The Borel congruence subgroup :

$$B(N) = \mathrm{Sp}_4(\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

- The Klingen congruence subgroup :

$$Q(N) = \mathrm{Sp}_4(\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

- The Paramodular subgroup :

$$K(N) = \mathrm{Sp}_4(\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

Apart from that we also have the delta subgroup which is not a congruence subgroup as mentioned below :

$$\Delta_0^n(\mathbb{Z}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) \mid C = 0 \right\}$$

3.4.1 Siegel modular form for a subgroup

Let k be a positive integer and Γ be a congruence subgroup which means $\Gamma \supset \Gamma(N)$ for N. The largest such N for which so is true is the level of the subgroup Γ. Let $F \in S_k(\Gamma)$. Then F is said to be a Siegel modular form of level N and weight k for the subgroup Γ if $F|_k[\gamma] = F$ for all $\gamma \in \Gamma$.

3.5 Siegel modular forms with character

In this section, we will describe how Siegel modular forms with character are defined, analogous to the case of classical modular forms.

Let us start with a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}^{\times}$. We extend this character to the group of matrices and write $\chi(M) = \chi(\det D)$ where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Definition 3.5.1. A holomorphic function $F : \mathbb{H}_n \to \mathbb{C}$ is said to be a Siegel modular form with the character χ, weight k and level N if

$$F|_k[M](Z) = \chi(M)F(Z) \text{ for all } M \in \Gamma_0^n(N).$$

The set of such functions is denoted by $M_k^n(N,\chi)$ and it forms a finite dimensional complex vector space. We also say that an $F \in M_k^n(N,\chi)$ is cuspidal if it vanishes at every cusp of $\Gamma_0^n(N)$. The set of Siegel cusp forms of $M_k^n(N,\chi)$ is denoted as $S_k^n(N,\chi)$.

Lemma 3.5.2. *The space* $S_n^k(N,\chi) \neq 0$ *if and only if* $(-1)^{(nk)} = \chi((-1)^n)$.

Proof. Consider the matrix $\begin{pmatrix} -I_n & 0 \\ 0 & -I_n \end{pmatrix} \in \Gamma_0^n(N)$. The relation above gives that $\det(-I_n)^{-k}F(Z) = \chi(\det(-I_n))F(Z)$. Hence this gives $(-1)^{nk}F(Z) = \chi((-1)^n)F(Z)$. Hence for the space $S_n^k(N,\chi) \neq 0$, we require $(-1)^{nk} = \chi((-1)^n)$. □

Most of the discussions in this thesis will be that of the genus 2 case, hence we write the expansion in that case below as it will be useful later.

Any given $F \in S_k^2(N,\chi)$ has the Fourier expansion :

$$F(Z) = \Sigma_{T\in\Delta} a_F(T)e(TZ), \tag{3.1}$$

where the coefficients $a_F(T) \in \mathbb{C}$ and Δ is the set of half integral symmetric 2×2 matrices.

4

Extra Twists of Siegel modular forms

4.1 The Representation associated to a Siegel modular form

Let F be a cuspidal vector valued genus two Siegel modular form of weight (k_1, k_2) with $k_1 \geq k_2 \geq 2$, and level N and nebentypus ϵ, which is also a Hecke eigenform. This means that the weights of these modular forms are $\mathrm{Sym}^{k_1-k_2} \otimes \det^{k_2}$ and $T_pF = t_pF$ for all $p \nmid N$. Siegel modular forms are of two types, namely those that are lifted from classical elliptic forms and those that are non lifted.

Analogous to the classical case, we may associate a compatible system of λ-adic Galois representations to reach Siegel modular form F of genus $g = 2$, from an amazing construction of Laumon, Taylor and Weissauer :

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{K}_\lambda).$$

The construction can be seen in [39, Theorem 3.1]. Here $K = \mathbb{Q}(\{t_p\})$ is the number field obtained by adjoining the Hecke eigenvalues t_p of the Siegel modular form F for all primes for which $p \nmid N$. This Galois representation has the characteristic property :

- $\mathrm{Trace}(\rho_{F,\lambda}(\mathrm{Frob}_p)) = t_p$

- $\mathrm{Sim}(\rho_{F,\lambda}(\mathrm{Frob}_p)) = \epsilon(p)p^{k_1+k_2-3}$, for all primes sch that $p \nmid lN$ where "Sim" denotes the similitude of the matrix.

The Langlands dual group of GSp_4 is still the group GSp_4. Denote the l-adic Galois representation by $\rho_{F,l} := \prod_{\lambda|l} \rho_{F,\lambda}$. The images of these Galois representations are large in the sense as stated in [17, Theorem 2.1].

By [18, Corollary 2.2], for all but a finite set of primes in a set of primes of density 1, and for each $\lambda|l$ in that set, we have

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(K_\lambda).$$

Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_l}$ and let $\Gamma = \mathrm{Hom}(K, \mathbb{C})$ be the set of homomorphisms. We define extra twists of Siegel modular forms similar to the elliptic modular forms.

Definition 4.1.1. [3] A Siegel modular form F is said to have an *extra twist* if there exists a tuple (γ, χ_γ) with $\gamma \in \Gamma$, and an associated χ_γ such that $\gamma(\rho_F) \sim \rho_F \otimes \chi_\gamma$.

As usual, two representations are said to be equivalent $\sim$ if they are conjugate of each other. The notation $\gamma(\rho_F)$ means that the embedding acts on the matrices entry wise. Very little is known about actual Fourier coefficients and hence Hecke eigenvalues of Siegel modular form. As a consequence, very less is known about the the coefficient field K. The aim of this paper is about the *existence* of extra twists for automorphic forms for the group $\mathrm{GSp}_4/\mathbb{Q}$. In our first theorem, we give examples of *Siegel* modular forms with extra twists different from complex conjugation coming from lifted modular forms. We extend a result of Ribet [1, pg. 49 Theorem 5.7] to automorphic forms on the group $\mathrm{GSp}_4/\mathbb{Q}$ for which the corresponding Galois representation is irreducible.

Let c be the complex conjugation and ϵ be the nebentypus character. From the Petersson inner product, the tuple (c, ϵ^{-1}) is always an extra twist for the Siegel modular forms [18, Remark 2.7] if the nebentypus is a non-quadratic Dirichlet character since they take value in K. Somehow, complex conjugation produces a trivial extra twist. It is natural to ask if there are extra twists of Siegel modular forms *different* from complex conjugation.

In our first theorem, we explicitly produce Siegel modular forms with extra twists different from complex conjugation. Start with two classical elliptic modular forms

(f, g) with $f \in S^1_{k_1}(N, \epsilon)$, $g \in S^1_{k_2}(N, \epsilon)$. Using a suitable embedding $\mathrm{GL}_2 \times \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4$, it is possible to produce a Siegel modular form $Y(f \otimes g)$ that is a Yoshida lift of (f, g). However, these are endoscopic lifts and the corresponding Galois representations are reducible [37, Theorem 3.2.1].

For two carefully chosen classical modular forms f and g, their Yoshida lift $Y(f \otimes g)$ is a Siegel modular form with extra twists different from complex conjugation. On the other hand, for a classical modular form with an extra twist different from complex conjugation, consider the automorphic representation $F := \mathrm{Sym}^3(f)$. Now F is a Siegel modular form with an irreducible Galois representation. Our theorem says that F is also a Siegel modular form with an extra twist different from complex conjugation.

However, we also observe that classical elliptic modular forms with extra twists *do not necessarily* lift to Siegel modular form with an extra twist.

Let f and g be two classical non-CM elliptic modular forms with extra twist associated to the same Dirichlet character χ different from complex conjugation. Assume that the Yoshida lift $Y(f \otimes g)$ of (f, g) exists, then $Y(f \otimes g)$ is a Siegel modular form that contains extra twists different from complex conjugation.

However, we also observe that classical elliptic modular forms with extra twists *do not neccessarily* lift to Siegel modular form with an extra twist.

We have defined Yoshida lifts in § 4.3.1. Also recall the definition of Saito-Kurakawa lifts from [32, Section 1.3].

Theorem 4.1.2. 1. *Let f and g be two classical non-CM elliptic modular forms with extra twist associated to the same Dirichlet character χ different from complex conjugation. Assume that the Yoshida lift $Y(f \otimes g)$ of (f, g) exists, then $Y(f \otimes g)$ is a Siegel modular form that contains extra twists different from complex conjugation.*

2. *Let f be a classical elliptic modular form of level N such that $3 \nmid \phi(N)$ that contains an extra twist other than the complex conjugation. Then the lift $\mathrm{Sym}^3(f)$ is a Siegel modular form that contains an extra twist other than the complex conjugation.*

3. *Let F_f be a Siegel modular form of Saito-Kurakawa type coming from classical modular form f that contains an extra twist other than complex conjugation, then the non trivial extra twist of f will not produce extra twist for F_f.*

The above theorem says that the group of extra twists can be really large. The method of this theorem also provides a systematic way of producing a large class of examples of Siegel modular forms with extra twists. We give a few examples of explicit Siegel modular forms with extra twists. We construct these examples by taking the Yoshida lift of two classical modular forms of the same level and the same central nebentypus character. For the families of modular forms, the group of extra twists is studied by Conti [9] in a recent paper.

Kumar et al use the extra twists for Siegel modular forms with arithmetic applications in the context of Lang-Trotter conjecture [17]. It is worth mentioning that as of now it is very hard to compute Hecke eigenvalues for Siegel modular forms for congruence subgroups even for genus two. It will be very intriguing to find *extra twists* of non lifted Siegel modular forms that do not come from complex conjugation.

In § 5.1.1, we give a necessary and sufficient condition for the existence of an extra twist similar to the classical elliptic modular forms. This is a generalization of Ribet's result on a large image of the Galois representation associated with elliptic modular forms (cf. [26], [27] and [29]). This theorem is about the image of the Siegel modular form and how it is affected by the presence of a twist.

We can associate a central simple algebra X associated to the extra twists of Siegel modular forms. For the classical modular forms, local Brauer classes of these central simple algebras are studied by Eknath Ghate and his collaborators (cf. [12], [7], [5], [6]). We determine the local Brauer classes of these central simple algebras associated to Siegel modular forms of genus 2. As it is 2 torsion in the local Brauer group, so either it is a matrix algebra (split case) or not a matrix algebra (non-split case). We say $X_v \sim 1$ if it is a matrix algebra and $X_v \sim -1$ otherwise. Similar to the classical case if $t_p \neq 0$, consider the quantity

$$m_v := [F_v : \mathbb{Q}_p] \cdot v(\frac{t_p^2}{\epsilon(p)}) \in \mathbb{Z}.$$

For the cases where $t_p = 0$, we have the following criterion similar to the elliptic modular forms. Let $p^\dagger \nmid N$ be a prime such that $p^\dagger \equiv p \bmod N$ and $t_{p^\dagger} \neq 0$. Let

$$m_v^\dagger := [F_v : \mathbb{Q}_{p^\dagger}] \cdot v(\frac{t_{p^\dagger}^2}{\epsilon(p^\dagger)}) \in \mathbb{Z},$$

where v is normalized so that $v(p) = 1$. The local Brauer classes are given as in the theorem below.

Theorem 4.1.3. *The local Brauer classes X_v are given at all places $p \nmid N$ as follows:*

$$X_v \sim \begin{cases} 1 & \text{if } v \mid \infty, \\ (-1)^{m_v} & \text{if } p \nmid N \text{ and } t_p \neq 0 \\ (-1)^{m_v^\dagger} & \text{if } p \nmid N \text{ and } t_p = 0. \end{cases}$$

4.2 Extra twists for Siegel modular forms

Before we talk about extra twists for Siegel modular forms, we first talk about extra twists for classical modular forms and the result from Ribet which we generalize here.

4.2.1 Extra twists for classical modular forms

This subsection and its outline is mainly derived from [35, Section V, Chapter 2]. Let us start with a classical newform f of level k with weight N. For a prime l, let

$$\rho_l : G_{\mathbb{Q}} \to \mathrm{GL}_2(K \otimes \mathbb{Q}_l)$$

be the l-adic representation attached to f. Here $\mathfrak{g}_l$ is the Lie algebra of the image of the representation.

We also let $\mathfrak{a}_l$ be the Lie algebra

$$\mathfrak{a}_l = \{u \in \mathfrak{gl}_2(K \otimes \mathbb{Q}_l) \mid tr(u) \in \mathbb{Q}_l\}.$$

Definition 4.2.1. A classical modular form f without complex multiplication is said to have an extra twist if there is a $\tau \in Aut(\mathbb{C})$ along with a continuous character $\phi : G_{\mathbb{Q}} \to \mathbb{C}^\times$ which is unramified outside N such that $a_p = \tau(a_p)\phi(p)$ for all primes p in a set of primes of density 1.

We can now write down the main theorem of Ribet which we try to generalize here to Siegel modular forms.

Theorem 4.2.2. *We start with a newform f without complex multiplication. Let l be a prime. We get the strict inclusion of Lie algebras $\mathfrak{g}_l \subsetneq \mathfrak{a}_l$ if and only if the classical modular form f admits an extra twist.*

4.2.2 Extra twists for Siegel modular forms

Recall that we associate with every Siegel modular form F of genus $g = 2$, the compatible system of λ-adic Galois representations by the work of Laumon, Taylor, and Weissauer [40]:

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{K}_\lambda)$$

where $K = \mathbb{Q}(\{t_p\})$ is the number field obtained by adjoining the Hecke eigenvalues t_p of F for all primes p in a set of density 1. The images of these Galois representations are extensively studied by Dieulefait [11].

Note that if the Siegel modular forms are of genus two and non-lifts, these representations are expected to be irreducible but it is known only for large $k > 2$ and large λ ($\lambda \mid l$ with $l \geq 5$) by Ramakrishnan [25] and by Weiss [39] for $k = 2$. For $g > 2$, irreducibility is not known in general (cf. Patrikis-Taylor [22] for details) even for non-lifts.

Definition 4.2.3. [3, Definition 2.3] A Siegel modular form F is said to admit complex multiplication (or be CM) if there exists non trivial Dirichlet character ϵ such that $\rho_F \cong \rho_F \otimes \epsilon^{-1}$.

4.2.3 Lifted Siegel modular forms

Siegel modular forms F of genus 2 are of two types. Namely, either they are lifts (coming from modular forms of lower ranks) or nonlifts (those that are not lifted). Recall the possibility of the derived Lie algebra $\mathfrak{S} = [\mathfrak{g}, \mathfrak{g}]$ of the Lie algebra $\mathfrak{g}$ image G of Galois representation [13, Proposition 3.2].

Definition 4.2.4. A Siegel modular form F is said to be a lift if $\mathfrak{S}$ is one of the following forms:

1. $\mathfrak{S} \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$. This corresponds to the endoscopic lift.

2. $\mathfrak{S} \cong \mathfrak{sl}(2)$ with $\mathfrak{sl}(2)$ embedded into a Klingen parabolic subalgebra. This corresponds to CAP (cuspidal associated to Klingen parabolic).

3. $\mathfrak{S} \cong \mathfrak{sl}(2)$ with $\mathfrak{sl}(2)$ embedded into a Siegel parabolic subalgebra. This corresponds to CAP (cuspidal associated to Siegel parabolic).

4. $\mathfrak{S} \cong \mathfrak{sl}(2)$ via the symmetric cube representation of $\mathrm{SL}(2)$. This corresponds to the symmetric cube lifts of elliptic modular forms.

5. $\mathfrak{S} \cong 1$. In this case, $\rho_F \simeq Ind_L^{\mathbb{Q}}(\chi)$ for a degree 4 CM field L. This corresponds with the automorphic induction.

Hence a Siegel modular form F is a non lift if $\mathfrak{S}$ is not one of the above forms and hence $\mathfrak{S} \cong \mathfrak{sp}_4$.

Lemma 4.2.5. *For a non trivial character ϵ, the pair (c, ϵ^{-1}) is always an extra twist.*

Proof. Let $<,>$ be the Peterson inner product for the Siegel modular form. For any non-quadratic Dirichlet character ϵ, the pair (c, ϵ^{-1}) is always an extra twist. This follows from the fact that $t_p < F, F > = < t_pF, F > = < T_pF, F > = < F, T_p^* F > = < F, \epsilon(p)^{-1} t_p F > = \epsilon(p)\overline{t_p} < F, F >$. In this case, (c, ϵ^{-1}) is always an inner twist. □

Below, we will prove some important characteristics of the extra twists over a form.

Proposition 4.2.6. *[18, Proposition 2.4] For a Siegel modular form, if $\gamma \in \Gamma$ then $\gamma(K) \subset K$.*

Proof. Let (γ, χ_γ) be an inner twist. Comparing $\mathrm{Sim}(\gamma(\rho_l))$ and $\mathrm{Sim}(\rho_l \otimes \chi_\gamma)$, we get that $\chi_\gamma^2 = \gamma(\epsilon).\epsilon^{-1}$. Hence χ_γ takes values in the field $\mathbb{Q}(\epsilon) \subset K$. But also $\gamma(t_p) = t_p\chi_\gamma(p)$ for all $p \nmid lN$. Varying over the prime l, we get that $\gamma(t_p) \in E$ for all $p \nmid N$ and so $\gamma(K) \subset K$.

□

Lemma 4.2.7. 1. *The extra twists for ρ_F over $\mathbb{Q}$ form a group.*

2. *For a non CM modular form F, and an extra twist γ, the character χ_γ that satisfies the equivalence is uniquely determined.*

3. *The association $\gamma \to \chi^\gamma$ defines a cocycle on the group of self-twist with values in $K^\times$.*

Proof. 1. Let (γ_1, χ_1) and (γ_2, χ_2) be two extra twists associated with a modular form. This means that $\gamma_1(t_p) = t_p\chi_1(p)$ and $\gamma_2(t_p) = t_p\chi_2(p)$. Hence $\gamma_2(\gamma_1(t_p)) = \gamma_2(t_p\chi_1(p)) = t_p\chi_2(p).(\chi_1(p))^{\gamma_2}$. Hence the associated character to the composite embeddings is $\chi_2(p).(\chi_1(p))^{\gamma_2}$

2. Assume that we are starting with a non CM form F. Let there be two associated characters χ_1, χ_2 for the same embedding γ. This means that $\gamma(t_p) = t_p\chi_1(p)$ and $\gamma(t_p) = t_p\chi_2(p)$. Comparing the two, we get $t_p = \chi_1(p)\chi_2(p)^{-1}t_p$ which gives us a contradiction as we had assumed a form without complex multiplication.

3. If we start with $\gamma_1, \gamma_2 \in \Gamma$, then the identity $\chi_{\gamma_1\cdot\gamma_2} = \chi_{\gamma_1}\chi_{\gamma_2}^{\gamma_1}$, proves that the association as stated is a 1-cocycle.

□

4.2.4 A homomorphism associated to the group of extra twists

Restricting ourselves to $g \in G_{\mathbb{Q}}$, we observe that the association $\gamma \to \chi_\gamma(g)$ is a $1-$ cocyle again. By Hilbert 90, $\mathrm{H}^1(\Gamma, K^\times)$ is trivial, that is, there exists an element $\alpha(g) \in K^\times$ which satisfies :

$$\frac{\gamma(\alpha(g))}{\alpha(g)} = \chi_\gamma(g) \text{ for all } \gamma \in \Gamma.$$

As F is the fixed field upon the action by γ, the element $\alpha(g)$ is uniquely determined up to multiplication by elements of $F^\times$. Varying over $g \in G_{\mathbb{Q}}$, we get the well defined map:

$$\tilde{\alpha} : G_{\mathbb{Q}} \to K^\times/F^\times.$$

As each χ_γ is a character, $\tilde{\alpha}$ is a homomorphism. We may now verify all the relevant properties of α that hold in the case of classical modular forms as well.

Let $J = \begin{bmatrix} 0 & S \\ -S & 0 \end{bmatrix}$ with $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Lemma 4.2.8. *The homomorphism $\tilde{\alpha}$ satisfies :*

1. *The morphism $\tilde{\alpha}$ is unramified for all primes $p \nmid N$.*

2. *For each $g \in G_{\mathbb{Q}}$, we have $\alpha^2(g) \equiv \epsilon(g) \pmod{F^\times}$.*

3. *If the Hecke eigenvalue* $t_p \neq 0$*, then* $\alpha(\mathrm{Frob}_p) \equiv t_p (\mathrm{mod}\ F^\times)$.

Proof. 1. Both $\gamma(\rho_f)$ and ρ_f are unramified for $p \nmid N$ and so, $\chi_\gamma(g) = 1$ for all $g \in I_p$. Also, since $\alpha(g) \in F^\times$ for all $g \in I_p$, so $\tilde{\alpha}$ is also unramified at $p \nmid N$.

2. Since $\rho(g) \in \mathrm{GSp}_4(K)$, we have the following equality :

$$\rho^\sigma(g) J \rho^\sigma(g)^t = \mu(\rho^\sigma(g)) J.$$

This equates to $\chi_\sigma(g)\rho(g)J(\chi_\sigma(g)\rho(g))^t = \mu(\rho^\sigma(g))J$. Hence we get that

$$\frac{\sigma(\alpha^2(g))}{\alpha^2(g)} = \chi_\sigma(g)^2 = \frac{\mu(\rho^\sigma(g))}{\mu(\rho(g))} = \frac{(\mu(\rho(g)))^\sigma}{\mu(\rho(g))}.$$

We deduce, $\alpha^2(g) \equiv \mu(\rho(g)) \equiv \epsilon(g) \pmod{F^\times}$.

3. If we assume that the trace of $\rho_f(g)$ for $g \in G_\mathbb{Q}$ is not zero, then we can show that

$$\alpha(g) \equiv \mathrm{trace}(\rho_f(g))(\mathrm{mod} F^\times).$$

Specifically, if $p \nmid N$, with $t_p \neq 0$, we take $g = \mathrm{Frob}_p$ to get $\alpha(\mathrm{Frob}_p) \equiv t_p(\mathrm{mod} F^\times)$.

□

4.3 Examples of Siegel modular forms with extra twists

4.3.1 Yoshida lifts of classical modular forms with character

We recall the theory of Yoshida lifts [31]. This is basically coming from the embedding $\mathrm{GL}_2 \times \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4$. It is a specific case of Langlands functoriality that comes from the embedding of dual groups :

$$\{(g_1, g_2) \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) | \det(g_1) = \det(g_2)\} \to \mathrm{GSp}_4(\mathbb{C}),$$

given by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) \to \begin{bmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{bmatrix}.$$

If we start with two classical elliptic modular forms of weight 2, namely f and g of level N_1 and N_2, associated with the same primitive Dirichlet character χ, then the Yoshida lift F is the automorphic representation for GSp_4. We may recall that for the pair (f, g), the automorphic representation $F := Y(f \otimes g)$ is the Yoshida lift if the following conditions are satisfied :

1. The adelization of F generates an irreducible cuspidal representation π_F of $\mathrm{GSp}_4(\mathbb{A})$.
2. The local $L-$ parameter for $\pi_{F,v}$ at each place v is the direct sum of the $L-$ parameters for $\pi_{f,v}$ and $\pi_{g,v}$.

The determinant condition of the embedding is satisfied as in both the cases the determinant equals $\chi(p)p^{k-1}$ as the classical modular forms are associated with the same primitive Dirichlet character χ.

In line with Roberts [30], we mention below the necessary conditions for the existence of the Yoshida lifts. Starting with two classical newforms f and g, we say that the pair satisfies the conditions of a Yoshida lift it :

1. The modular form f is not a scalar multiple of g.
2. The characters of f and g come from the same primitive Dirichlet character.
3. One of the weights is 2 and the other weight has to be an even integer greater than or equal to 2.
4. For at least a finite prime p, $\pi_{f,p}$ and $\pi_{g,p}$ are both discrete series representations.

Definition 4.3.1. Assuming that these conditions are satisfied, and $f, g \in S_k^1(N, \chi)$, there is a unique representation $\Pi_{F,p}$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$ which satisfies the following for all but finitely many primes p:

$$L(\Pi_{F,p}) = L(\pi_{f,p}) \oplus L(\pi_{g,p}). \tag{4.1}$$

In such a case, F is called a Yoshida lift of f and g.

If we have two classical modular forms f, g with coefficient fields K_1 and K_2 respectively, and Γ_1, Γ_2 be the relevant group of extra twists; then let us denote the Yoshida lift $Y(f \otimes g)$. We denote the group of extra twists for the lift as Γ_Y.

Lemma 4.3.2. *If two given classical modular forms f and g satisfy the conditions listed in 4.3.1, then $\Gamma_Y \supset \Gamma_1 \cap \Gamma_2$.*

Proof. We assume that $\gamma \in \Gamma_1 \cap \Gamma_2$. This means that the Hecke eigenvalue fields K_1 and K_2 have to be equal for a non zero γ to be in the intersection. Hence, their compositum will be $K = K_1 = K_2$. This implies that $\gamma(\rho_f) = \rho_f \otimes \chi_\gamma$ and $\gamma(\rho_g) = \rho_g \otimes \chi_\gamma$. Because of the equation 4.1 we have for the Yoshida lift $F = Y(f \otimes g)$, we get $\gamma(\rho_F) = \rho_F \times \chi_\gamma$. □

In the next example of Siegel modular form with extra twist, we show that it is not necessarily true that $\Gamma = \Gamma_1 \cap \Gamma_2$. In fact, for all of the examples we have $\Gamma_Y \supsetneq \Gamma_1 \cap \Gamma_2$.

Subsequently, we give examples of Siegel modular forms with extra twists. We find examples of Siegel modular forms with extra twists by considering the Yoshida lifts of two classical modular forms with a particular weight, character, and having extra twists themselves.

In this subsection, we show that the group of extra twists for Siegel modular forms can be quite large.

Proof. Let $f, g \in S_k(N, \chi)$ be two elliptic modular forms with extra twists different from complex conjugation. Let $K = K_f \cdot K_g$ be the compositum of the Hecke fields K_f, K_g of f and g. There exists at least one $\gamma \in Aut(K)$ such that

$$\gamma(\rho_f) = \rho_f \otimes \chi_\gamma, \gamma(\rho_g) = \rho_g \otimes \chi_\gamma. \tag{4.2}$$

Recall that we have assumed that the necessary condition for the existence of the Yoshida lift of (f, g) is satisfied and let $F = Y(f \otimes g)$ be the Yoshida lift for the pair (f, g) as defined in § 4.3.1. We claim that F is a Siegel modular form with an extra twist by the same γ. In other words, we need to show that there exists a character χ_γ such that

$$\gamma(\rho_F) = \rho_F \otimes \chi_\gamma. \tag{4.3}$$

This is an equality of two 4 dimensional Galois representations. By Brauer-Nesbitt theorem, this is only true if all the coefficients in the characteristic polynomial are equal. The characteristic polynomial of the left hand side in equation 4.3 is $(x^2 - \gamma(a_p)x + p^{k-1})(x^2 - \gamma(a'_p) + p^{k-1})$ and that of right hand side is $(x^2 - a_p\chi_\gamma(p)x + p^{k-1})(x^2 - a'_p\chi_\gamma(p)x + p^{k-1})$. The above equality follows from the equation 4.2.

Suppose (γ, χ_γ) be an extra twist for the form $Y(f \otimes g)$, for a modular form f of weight two and g of weight $k+2$ with coefficients in $\overline{\mathbb{Z}}$ (the ring of integers of $\overline{\mathbb{Q}}$). By [2, Cor. 5.8, p.2513], we deduce that $t_p = p^{\frac{k}{2}}a_p + b_p$. Assume that γ is a complex conjugation when restricted to $K_{Y(f\otimes g)}$, then we have the equation

$$\gamma(p^{\frac{k}{2}}a_p + b_p) = (p^{\frac{k}{2}}\overline{a}_p + b_p).$$

This gives us $\chi_\gamma(p)p^{\frac{k}{2}}a_p + \chi_\gamma(p)b_p = (p^{\frac{k}{2}}\overline{a}_p + \overline{b}_p)$.

$$p^{\frac{k}{2}}(\chi_\gamma(p)a_p - \overline{a}_p) = (-\chi_\gamma(p)b_p + \overline{b_p}).$$

Hence we have $(\chi_\gamma(p)a_p - \overline{a}_p) = \frac{1}{p^{\frac{k}{2}}}(-\chi_\gamma(p)b_p + \overline{b_p}) \notin \overline{\mathbb{Z}}$. This is a contradiction.

We expect that the endomorphism algebra of the conjectural motive associated to $Y(f \otimes g)$ is a direct sum of the endomorphism algebras for f and g. Note that we really need $k > 0$ for the validity of this theorem.

$\square$

4.3.2 Examples of the phenomenon described above

In this section, we find explicit examples of Siegel modular forms with extra twists that are different from complex conjugations. All these examples show that the Hecke field of Yoshida lift is smaller than the compositum of the individual elliptic modular forms.

1. Let $f, g \in S_2^1(30, \chi)$ be two newforms with χ is a Dirichlet character on $\chi : (\mathbb{Z}/30\mathbb{Z})^\times \to \mathbb{C}^\times$ such that χ is determined on its generators by $\chi(7) = -\zeta_8^2$ and $\chi(11) = -1$. Note we can do so because this is a four dimensional space. Since the conductor of the character is 15, condition (2) of Yoshida lifting is satisfied and we can consider the Yoshida lifting $Y(f \otimes g)$ and this is an example of an

explicit Siegel modular form with an extra twist. In this example, the classical modular forms could be chosen with the expansion $f = q + \zeta_8 q^2 + (\zeta_8^3 - \zeta_8^2 - 1)q^3 + \zeta_8^2 q^4 + ...$ and $g = q + \zeta_8^3 q^2 + (\zeta_8^9 - \zeta_8^6 - 1)q^3 + \zeta_8^6 q^4 + ...$ For a detailed study of this space, let $K_{f,g}$ be the compositum of the coefficient fields of f and g. It must be said that no matter what f and g we choose, in any case, $K_{Y(f \otimes g)} \subsetneq K_{f,g}$.

Let Γ_F be the group of extra twists of F. Then it is readily evident that $\Gamma_F \supseteq \Gamma_1 \cap \Gamma_2$. For the f and g we have chosen, $\Gamma_1 = \Gamma_2$. Note also that $K_{f,g} = \mathbb{Q}(\zeta_8)$. We see that in this expansion the coefficient of q^2 in the sum of $f + g$ is $\zeta_8 + \zeta_8^3$. We claim that all of the coefficients of the sum $f + g$ belong to the field $\mathbb{Q}(\zeta_8 + \zeta_8^3)$.

It is easy to check manually that for any power $1 \leq i \leq 8$, $\zeta_8^i + \zeta_8^{3i}$ can be written in terms of $\zeta_8 + \zeta_8^3$. Hence all the sum of Hecke eigenvalues of f and g are in $\mathbb{Q}(\zeta_8 + \zeta_8^3)$. From the definition of Yoshida lift 4.3.1, we get that the Hecke eigenvalues of $Y(f \otimes g) \in \mathbb{Q}(\zeta_8 + \zeta_8^3)$. Hence this is an explicit example of when the field of Hecke eigenvalues of the Yoshida lift is a proper subset of the compositum of the field of the Hecke eigenvalues of the concerned classical modular forms.

2. Consider the 8 dimensional complex vector space $S_2^1(100, \chi)$ where the character χ is defined by $\chi(51) = -1$ and $\chi(77) = \frac{\mu^6 - 3\mu^2}{4}$; here μ is a root of the polynomial $p(x) := x^8 - 7x^4 + 16 = 0$. Such a character χ has conductor 20 and the order of the group of inner twists is 8.

   Let $\mu_1 := \frac{7+\sqrt{15}i}{2}, \mu_2 := -(\frac{7+\sqrt{15}i}{2})$ be the root of $p(x)$. For $i = 1, 2$, we have the Fourier expansions

   $$f_i = q + \mu_i q^2 + \frac{3\mu_i^7 - 13\mu_i^3}{8} q^3 + ...$$

   The Hecke eigenvalue ring $K_{f_1 f_2}$ is of dimension 8 over $\mathbb{Q}$. The Hecke eigenvalue ring is determined by the coefficients of $1, q$ and q^2 itself. Hence by Definition 4.3.1, the coefficient ring, $K_{Y(f_1 \otimes f_2)}$ is determined by the sum of Hecke eigenvalues of q and q^2 under the two roots. By a small calculation, we see that this field happens to be $\mathbb{Q}(\sqrt{15}i)$ which is of extension degree 2 over $\mathbb{Q}$. Hence $K_{f_1 f_2}$ is of extension degree 4 over $K_{Y(f_1 \otimes f_2)}$. In these case also $K_{Y(f_1 \otimes f_2)} \subsetneq K_{f_1, f_2}$.

While we give just two examples, but from LFMFDB it is evident that the group of extra twists for Yoshida lifts can be very large, although it can be controlled (as expected) from the individual classical elliptic modular forms.

4.3.3 The Symmetric cube lift

Let $f = \sum_{n>1} a_n q^n$ be an elliptic cusp form that is a Hecke eigenform and has an extra twist different from complex conjugation.This implies that there exists as extra twist γ, different from complex conjugation such that $\gamma(a_p) = a_p \otimes \chi_\gamma(p)$. We show that $Sym^3(f)$, is a Siegel modular form with an extra twist as well. Let the Satake parameter weights of f at the prime p be $\{\alpha_p, \beta_p\}$. In such a case, the Satake parameters of $Sym^3(f)$ are $\{\alpha_p^3, \alpha_p^2\beta_p, \alpha_p\beta_p^2, \beta_p^3\}$. We calculate the trace $t_p := \mathrm{Trace}(\rho_{(Sym^3(f),\lambda)})$. The addition shows that

$$\begin{aligned} \alpha_p^3 + \alpha_p^2\beta_p + \alpha_p\beta_p^2 + \beta_p^3 = & \quad (\alpha_p + \beta_p)(\alpha_p^2 + \beta_p^2) \\ = & \quad (\alpha_p + \beta_p)((\alpha_p + \beta_p)^2 - 2\alpha_p\beta_p). \end{aligned}$$

Let $(\gamma, \chi_\gamma) \in \Gamma_f$ be an extra twist for the classical modular form f. Hence, we deduce that

$$\begin{aligned} \gamma(t_p) = & \quad (\alpha_p + \beta_p)^\gamma((\alpha_p + \beta_p)^2 - 2\alpha_p\beta_p)^\gamma \\ = & \quad a_p\chi_\gamma(p)(\chi_\gamma^2 a_p^2 - 2\chi_\gamma^2(p)\epsilon(p)p^{k-1}) \\ = & \quad \chi_\gamma^3(p)a_p(a_p^2 - 2\epsilon(p)p^{k-1}) \\ = & \quad \chi_\gamma^3(p)(\alpha_p + \beta_p)(\alpha_p^2 + \beta_p^2) \\ = & \quad \chi_\gamma^3(p)t_p. \end{aligned}$$

Note that traces are sufficient to determine if two irreducible Galois representations are isomorphic [34, p. 11]. Comparing the determinants we also get that $\frac{\gamma(\epsilon)}{\epsilon} = \chi_\gamma^2$ where ϵ is the nebentypus character. We show that $Sym^3(f)$ is a Siegel modular form with an extra twist as well. Assuming the form is of level N, we get that $\chi_\gamma^{\phi(N)} \equiv 1$.

Since $\rho_{Sym^3(f)}$ is an irreducible Galois representation, we get the representations are similar $\rho^\gamma_{Sym^3(f),\lambda} \sim \rho_{Sym^3(f)} \otimes \chi_\gamma^3$. By assumption, we have $3 \nmid \phi(N)$, so $\chi_\gamma^3 \neq 1$. As a consequence, the Siegel modular form $Sym^3(f)$ has an extra twist different from complex conjugation.

4.3.4 Lifted modular forms coming from CAP representations

In this subsection, we prove part (3) of Theorem 4.1.2.

Let $f \in S_2(M, \epsilon^2)$ be a classical elliptic modular form. We can associate a lift (Mass specialization) $F_f \in S_2^2(M, \epsilon)$ to this f. The Hecke eigenvalues are given by [8, p. 1496]

$$\lambda_{F_f}(p) = \lambda_f(p) + \epsilon(p)p^{k-2}(p+1).$$

Suppose f is a modular form with an extra twist (γ, χ_γ) different from the complex conjugation. Hence, we have $\chi_\gamma^2 = (\epsilon^2)^{\gamma-1}$. Hence, we have $\chi_\gamma = (\epsilon)^{\gamma-1}\psi$ for a quadratic character ψ. Let p be a prime that is not in the kernel of ψ. Then F_f will not satisfy the condition $\lambda_{F_f}(p)^\gamma = \lambda_{F_f}(p)\chi_\gamma(p)$.

5

The image of the associated representation

In this section, we prove our main theorem, which gives a necessary and sufficient condition for the existence of extra twists analogous to the classical case. Let π be an automorphic representation with respect to a reductive group $\underline{G}$, with the Langlands dual group $\underline{\hat{G}}$. The content of the section also carries forward for any compatible system of Galois representation :

$$\rho : G_{\mathbb{Q}} \to \underline{\hat{G}}(\overline{\mathbb{Q}_l})$$

with $\underline{G}$ being either the general or symplectic group as long as the corresponding Galois representation is also irreducible. Now, since the irreducibility of this Galois representation is not known in generality, we may assume that we have a Galois representation to GSp_4.

5.1 The image of the representation

Recall the following setup [13, §3]. We start by considering a Siegel modular cusp form F of level N, nebentypus ϵ and weight (k_1, k_2) with $k_1 \geq k_2 \geq 3$. We have a Galois representation for all primes l with $\lambda \mid l$:

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{V_\lambda}).$$

By a theorem due to Kumar et all [18], for most primes l with $\lambda|l$, we have

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(K_\lambda);$$

with K_λ a finite extension of $\mathbb{Q}_l$. For all primes p co-prime to N and l, the matrix $\rho_{F,\lambda}(\mathrm{Frob}_p)$ satisfies the polynomial:

$$P_{F,p}(X) = (X-\alpha)(X-\beta)(X-\gamma)(X-\delta).$$

Denote by $K_l = \prod_{\lambda|l} K_\lambda$ be a product of fields. Every module V_l over K_l breakes up into product of vector spaces $V_l = \prod_{\lambda|l} V_\lambda$. Fix a prime l with $\lambda \mid l$ for which the representation takes values in the finite extension K_λ of $\mathbb{Q}_l$. Let $\mathcal{O}_{K_\lambda}$ be the ring of integers of K_λ. Then $\mathcal{O}_{K_\lambda}$ is free of rank 4.

Let $\mathfrak{g}_l$ be the Lie algebra of the image of the representation and

$$\mathfrak{a}_l = \{u \in \mathfrak{gsp}_4(K \otimes_{\mathbb{Q}} \mathbb{Q}_l) = \prod_{\lambda|l} \mathfrak{gsp}_4(K_\lambda) \mid \Lambda(u) \in \mathbb{Q}_l\}$$

where Λ denotes the symplectic similitude.

Now $G_{\mathbb{Q}}$ is compact and $\rho_{F,\lambda}$ is continuous, the image is a *closed* subgroup of $\mathrm{GSp}_4(K_\lambda)$. Denote this by G. It is a l-adic Lie group with Lie algebra $\mathfrak{g}_\lambda$. We have $\exp : \mathfrak{g}_\lambda \to G$ with the inverse $\log : G \to \mathfrak{g}_\lambda$. These are analytic maps for the l adic topology that are inverse to each other. For any subgroup U of G that is of finite index we have

$$K_\lambda \log(U) = \mathfrak{g}_\lambda.$$

This is because for $x \in \mathfrak{g}_\lambda$, $\exp(x) \in G$ and since U is of finite index we have $\exp(nx) \in U$. Taking the inverse we get $x \in K_\lambda \log(U)$. The other containment is obvious. For a detailed look on exp and log maps, one can refer to [32, Section III and Section VII].

We have a Galois representation:

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}_l}).$$

At the level of the Lie algebra, this produces a representation:

$$Lie(\rho_{F,\lambda}) : \mathfrak{g}_\lambda \to \mathfrak{gsp}_4(\overline{\mathbb{Q}}_l).$$

Clearly, we have $\mathrm{GSp}_4(K \otimes \overline{\mathbb{Q}_l}) = \prod_{\lambda|l} \mathrm{GSp}_4(\overline{K_\lambda})$ with each $\overline{K_\lambda} = \overline{\mathbb{Q}_l}$.

The representation $\rho_{F,\lambda}$ is irreducible even if we restrict it to any finite index subgroup H of $G_{\mathbb{Q}}$ (cf. [39, Lemma 5.9, p. 379]). We denote by $G_\lambda := \mathrm{GSp}_4(\overline{V_\lambda})$ with $\overline{V_\lambda} = V_\lambda \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l}$. The group GSp_4 is equipped with a bilinear form Ψ.

For a subgroup $H \subset G_{\mathbb{Q}}$, let us denote by

$$I_{L[H]}(V) := \{\Phi : (V, \Psi) \to (V, \Psi), \Phi \circ \rho_{F,\lambda}(h) = \rho_{F,\lambda}(h) \circ \Phi \text{ for all } h \in H\};$$

the set of all intertwining operators from (V, Ψ) to itself. If $H = G_{\mathbb{Q}}$, we denote $I^H_\lambda = I_\lambda$.

Denote by $\rho_l := \prod_{\lambda|l} \rho_{F,\lambda}$ and $\mathfrak{g}_l := \prod_{\lambda|l} \mathfrak{g}_\lambda$. For any field K, let $\mathfrak{gsp}_4(K) = \mathrm{Lie}(\mathrm{GSp}_4(K))$ and

$$\mathfrak{a}_l = \{u \in \mathfrak{gsp}_4(K \otimes_{\mathbb{Q}} \mathbb{Q}_l) = \prod_{\lambda|l} \mathfrak{gsp}_4(K_\lambda) | \Lambda(u) \in \mathbb{Q}_l\};$$

where $\Lambda(u)$ denotes the similitude of the matrix u for $\mathfrak{gsp}_4(K)$(cf. [21, p. 238]).

For a subgroup $U \subset G_{\mathbb{Q}}$ and a vector space V over K_λ, let us denote by

$$\iota^U_\lambda(V) := \{\Phi : (V, \Psi) \to (V, \Psi), \Phi \circ \mathfrak{g}_\lambda(h) = \mathfrak{g}_\lambda(h) \circ \Phi \text{ for all } h \in \log(U)\}.$$

We recall the following results from [26, p. 789-791] that carry forward from GL_2 to GSp_4. Following [4, Lemma 4.12(iii), p. 50], observe that

$$\bigcup_U \iota^U_\lambda(V_\lambda) = \iota^{\mathfrak{g}_\lambda}_\lambda(V_\lambda). \tag{5.1}$$

On the other hand, for a module V over $K_l = \prod_{\lambda|l} K_\lambda$ and a subgroup U of $G_{\mathbb{Q}}$, $\iota^U_l(V) := \prod_{\lambda|l} \iota^U_\lambda(V)$. If $H = G_{\mathbb{Q}}$, we denote $\iota^H(V) = \iota(V_\lambda)$ and $\iota(V_l) := \prod_{\lambda|l} \iota(V_\lambda)$. Similarly, we define

$$\iota^{\mathfrak{g}_l}_l := \prod_{\lambda|l} \iota^{\mathfrak{g}_\lambda}_\lambda(V_\lambda); \iota^{\overline{\mathfrak{g}_l}}_l((\overline{V_l}) := \prod_{\lambda|l} \iota^{\mathfrak{g}_\lambda}_\lambda(\overline{V_\lambda});$$

with $\overline{V_\lambda}$ vector space over $\overline{\mathbb{Q}_l}$ with embedding induced by λ.

5.1.1 Lie algebra of images of Galois representations

We start with a lemma that perhaps holds for much more general reductive groups.

Lemma 5.1.1. *For an algebraically closed field E, we have the equality*

$$\mathfrak{gsp}_{2g}(E) = \mathfrak{sp}_{2g}(E) \bigoplus EI_{2g}.$$

Proof. The algebra $\mathfrak{gsp}_{2g}(E) = \{A \in M_{2g\times 2g}(E) | JA + A^tJ = \lambda(A)J\}$. The subalgebra $\mathfrak{sp}_{2g}(E) = \{A \in \mathfrak{gsp}_{2g}(E) | \lambda(A) = 0\}$. The map of the equality is $\phi : M \to (M - \frac{1}{2}\lambda(M)I_{2g}, \frac{1}{2}\lambda(M)I_{2g})$. We need to show that $(M - \frac{1}{2}\lambda(M)I_{2g}) \in \mathfrak{sp}_{2g}(E)$ given that $M \in \mathfrak{gsp}_{2g}(E)$. So, $(M - \frac{1}{2}\lambda(M)I_{2g})^tJ + J(M - \frac{1}{2}\lambda(M)I_{2g}) = M^tJ + JM - \lambda(M)J = 0$. Hence $(M - \frac{1}{2}\lambda(M)I_{2g}) \in \mathfrak{sp}_{2g}(E)$. We also show that the map is a Lie algebra homomorphism. We need to show that $\phi([M_1, M_2]) = [\phi(M_1), \phi(M_2)]$. Let $M_1 = M_1' \oplus \lambda(M_1)$ and $M_2 = M_2' \oplus \lambda(M_2)$. Then $[\phi(M_1), \phi(M_2)] = (M_1' + \lambda(M_1))(M_2' + \lambda(M_2)) - (M_2' + \lambda(M_2))(M_1' + \lambda(M_1)) = M_1'M_2' - M_2'M_1'$. Also $\phi([M_1, M_2]) = \phi(M_1'M_2' - M_2'M_1')$. We will show that $M_1'M_2' - M_2'M_1' \in \mathfrak{sp}_{2g}(E)$ which would complete the proof. We have $(JM_1' + M_1'^tJ)M_2' = 0$ and $(JM_2' + M_2'^tJ)M_1' = 0$. Subtracting we get $J(M_1'M_2' - M_2'M_1') + (M_1'^tJM_2' - M_2'^tJM_1') = 0$. Now $(M_1'M_2' - M_2'M_1')^tJ = (M_2'^tM_1'^tJ - M_1'^tM_2'^tJ) = M_2'^t(-JM_1') + M_1'^tJM_2'$ and this completes the proof as it shows that $\phi(M_1'M_2' - M_2'M_1') = (M_1'M_2' - M_2'M_1')$.

□

Lemma 5.1.2. *[Goursat's lemma] Let G_1 and G_2 be two groups and H be a subgroup of $G_1 \times G_2$ such that the two projections $p_1 : H \to G_1$ and $p_2 : H \to G_2$ are surjective. If N_1 is the kernel of p_2 and N_2, the kernel of p_1, then H is the graph of an isomorphism from $G_1/N_1 \cong G_2/N_2$.*

For an embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}}_l$, let ρ_σ be the map

$$\rho_\sigma : G_{\mathbb{Q}} \to \mathrm{GSp}_4(K \otimes \overline{\mathbb{Q}_l}) \to \mathrm{GSp}_4(\overline{\mathbb{Q}_l});$$

where the first map is given by ρ_l and the latter map is induced by σ entry wise by $\sigma(k, t) := \sigma(k)t$. For a field $K \subset \overline{\mathbb{Q}}$ and a vector space V over K, let us denote by $\overline{V}$ be the vector space $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l}$. Let $\mathfrak{g}_\sigma \subset \mathfrak{gsp}_4(\overline{\mathbb{Q}_l})$ be the Lie algebra of the image of $\rho_\sigma(G_{\mathbb{Q}})$. Consider the set $\mathfrak{o} := \mathfrak{g}_\sigma \times \mathfrak{g}_\tau$.

Lemma 5.1.3. *Let F be a non-CM (cf. Definition 4.2.3) Siegel modular form of genus two, weight (k_1, k_2) with $(k_1, k_2) \neq (2k-1, k+1)$ for some $k \geq 2$ and $k_1, k_2 \geq 2$, level N. Let $\mathfrak{g}_\sigma, \mathfrak{g}_\tau$ and $\mathfrak{o}$ be as above, and $\overline{\mathfrak{g}_l} \subsetneq \overline{\mathfrak{a}_l}$. The set $\mathfrak{o}$ is the graph of an isomorphism $\mathfrak{g}_\sigma \simeq \mathfrak{g}_\tau$ for some choice of σ and τ.*

Proof. The idea of this proof borrows heavily from [26, Lemma 4.4.10]. By Lemma 5.1.1, we deduce that

$$\mathfrak{gsp}_4(V_\sigma) = \mathfrak{sp}_4(V_\sigma) \bigoplus \overline{\mathbb{Q}_l}, \mathfrak{gsp}_4(V_\tau) = \mathfrak{sp}_4(V_\tau) \bigoplus \overline{\mathbb{Q}_l}.$$

By assumption $\mathfrak{g}_l \subsetneq \mathfrak{a}_l$. Hence, for some σ we have $\mathfrak{g}_\sigma \subsetneq \mathfrak{a}_\sigma$. The kernel is an normal subgroup of $\mathfrak{sp}_4(V_\sigma) \bigoplus \overline{\mathbb{Q}_l}$. Let $N_\sigma = ker(\pi_\tau)$ and $N_\tau = ker(\pi_\sigma)$. We use Goursat's lemma 5.1.2 with $G_1 = \mathfrak{gsp}_4(V_\sigma), G_2 = \mathfrak{gsp}_4(V_\tau)$. We show that for each embedding σ, the equality $\mathfrak{g}_\sigma = \mathfrak{gsp}_4(V_\sigma)$ holds true. The proof follows from [39, Proposition 5.4] as our weights are not of the form $(2k-1, k+1)$.

We have the following subgroup $\mathfrak{o} = \mathfrak{g}_\sigma \times \mathfrak{g}_\tau \subset G_1 \times G_2$ on which we can apply Goursat's lemma. By this lemma, we have $\mathfrak{g}_\sigma \times \mathfrak{g}_\tau$ is the graph of an isomorphism from $\frac{\mathfrak{gsp}_4(V_\sigma)}{N_\sigma} \to \frac{\mathfrak{gsp}_4(V_\tau)}{N_\tau}$.

Now suppose $N_\sigma = \mathfrak{sp}(V_\sigma)$ and as a consequence $N_\tau = \mathfrak{sp}(V_\tau)$. In this case, we show that $\mathfrak{g}_\sigma = \mathfrak{a}_\sigma$. Now, $\mathfrak{g}_\sigma = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \bigoplus \overline{\mathbb{Q}}_l$ (respectively $\mathfrak{a}_\sigma = [\mathfrak{a}_\sigma, \mathfrak{a}_\sigma] \bigoplus \overline{\mathbb{Q}}_l$ is the Cartan decomposition of the Lie algebra of the image of the Galois representation (respectively the other algebra). Observe that by definition $\mathfrak{sp}(V_\sigma) = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \subset [\mathfrak{a}_\sigma, \mathfrak{a}_\sigma] \subset \mathfrak{sp}(V_\sigma)$.

By assumption, $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] = [\mathfrak{a}_\sigma, \mathfrak{a}_\sigma] = \mathfrak{sp}_4(V_\sigma) = N_\sigma$. As a consequence, we deduce that $\mathfrak{g}_\sigma = \mathfrak{a}_\sigma$. This is a contradiction. Hence, we deduce that N_σ and hence N_τ are trivial. Hence, we have an isomorphism $\mathfrak{g}_\sigma \simeq \mathfrak{g}_\tau$ for some choice of σ and τ in Γ. □

We now state our theorem that gives a necessary and sufficient condition for the existence of extra twists for non-CM Siegel modular forms:

Theorem 5.1.4. *Let F be a non-CM (cf. Definition 4.2.3) Siegel modular form of genus two, weight (k_1, k_2) with $(k_1, k_2) \neq (2k-1, k+1)$ for some $k \geq 2$ and $k_1, k_2 \geq 2$, level N such that F is a non lift (cf. Definition 4.2.4). Then F admits an extra twist if and only if we have a strict inclusion $\iota_l^{\mathfrak{g}_l} \subsetneq \mathfrak{a}_l$.*

Note that $\iota_l^{\mathfrak{g}_l} \subset \mathfrak{a}_l$ and hence $\iota_l^{\mathfrak{g}_l} \subsetneq \mathfrak{a}_l$ if and only if $\iota_l^{\mathfrak{g}_l} \neq \mathfrak{a}_l$. As a consequence, we deduce that $\iota_l^{\mathfrak{g}_l} = \mathfrak{a}_l$ if and only if $\overline{\iota_l^{\mathfrak{g}_l}} = \overline{\mathfrak{a}_l}$.

The proof of the next lemma copies several arguments from elliptic modular forms [28].

Lemma 5.1.5. *Let V and W be two four dimensional vector spaces equipped with a symplectic inner product given by Ψ. Let $\rho_V : G_{\mathbb{Q}} \to \mathrm{GSp}(V)$ and $\rho_W : G_{\mathbb{Q}} \to \mathrm{GSp}(W)$ be two representations where V and W are vector spaces over $\overline{\mathbb{Q}_l}$. Let H be an open, normal subgroup of $G_{\mathbb{Q}}$ such that ρ_V and ρ_W are irreducible when restricted to H. Then $\rho_V \simeq \rho_W$ as H modules if and only if $\rho_V \simeq \rho_W \otimes \phi$ as $G_{\mathbb{Q}}$-modules, for some finite order character $\phi : G_{\mathbb{Q}} \to \overline{\mathbb{Q}_l}^*$ which is trivial on H.*

Proof. Let $T : (V, \Psi) \to (W, \Psi)$ be a H-isomorphism with inverse $S : (W, \Psi) \to (V, \Psi)$. For an$g \in G_{\mathbb{Q}}$, define an element of $\mathrm{GSp}_4(W)$ by

$$\phi(g) = T \cdot \rho_V(g) \cdot S \cdot \rho_W(g^{-1}) \tag{5.2}$$

Since H is a normal subgroup of $G_{\mathbb{Q}}$, $\phi(g)$ is H-equivariant. Hence $\phi(g) \in \iota_l^H(W)$. Evidently ϕ has an inverse as well, so $\phi(g) \in (\iota_l^H(W))^\star$. Since H acts irreducibly, a version of Schur's Lemma (cf. [3, Lemma 2.4]) implies that $\phi(g) \in \overline{\mathbb{Q}_l}^*$.

To help us with the proof, we consider the following commutative diagram :

$$\begin{tikzcd} V \arrow[r, "\rho_v(g)"] & V \arrow[d, "T"] \\ W \arrow[u, "S"] & W \arrow[l, "\rho_w(g^{-1})"] \end{tikzcd}$$

Now ϕ is a homomorphism because

$$\begin{aligned} \phi(g_1 g_2) &= T \cdot \rho_V(g_1 g_2) \cdot S \cdot \rho_W(g_2^{-1} g_1^{-1}) \\ &= T \cdot \rho_V(g_1) \cdot S \cdot T \cdot \rho_V(g_2) \cdot S \cdot \rho_W(g_2^{-1}) \cdot \rho_W(g_1^{-1}) \\ &= T \cdot \rho_V(g_1) \cdot S \cdot \phi(g_2) \cdot \rho_W(g_1^{-1}) \\ &= \phi(g_1)\phi(g_2). \end{aligned}$$

The last step uses the assumption that $\phi(g_2)$ is a scalar matrix. Since ϕ is trivial on H, and H is a subgroup of finite index of $G_{\mathbb{Q}}$, ϕ has finite order. The definition of ϕ in (5.2) shows $V \simeq W \otimes \phi$ as $G_{\mathbb{Q}}$-modules, proving one direction of the lemma. The other direction is clear. □

We return to the proof of the main result of this section. We recall some notations which are used for classical modular forms as well. If H is a subgroup of $G_{\mathbb{Q}}$, then let

$$\Gamma_H := \{\gamma \in \Gamma \mid \chi_\gamma \text{ is trivial on } H\}$$

and let F^H be the fixed field of Γ_H in E. We view E as a vector space over F^H. The following theorem (compare with [28, Theorem 4.4]) is important for dimension counting purposes.

Theorem 5.1.6. *If H is an open and normal subgroup of $G_{\mathbb{Q}}$, then*

$$\iota_l^H(\overline{V_l}) \simeq M(a, \overline{\mathbb{Q}_l})^b;$$

with $a = \#\Gamma_H$ and $b = \#\Sigma/\Gamma_H$.

Proof. By definition $\overline{V_l}$ is a sum of modules V_σ, each of which is simple as an H-module, and hence satisfies

$$\iota_\sigma^H(V_\sigma) = \overline{\mathbb{Q}_l}.$$

To compute $\iota_l^H(\overline{V_l})$, we only have to determine when ρ_{V_σ} and ρ_{V_τ} are isomorphic as H-modules, for any two embeddings σ, τ of K into L. By Lemma 5.1.5 this occurs if and only if there is a character of finite order

$$\phi : G_{\mathbb{Q}} \to \overline{\mathbb{Q}_l}^*;$$

trivial on H, such that $\rho_{V_\sigma} \sim \rho_{V_\tau} \otimes \phi$ as $G_{\mathbb{Q}}$-modules. Taking traces on both sides we get

$$\sigma(a_p) = \tau(a_p) \cdot \phi(p)$$

for almost all primes p. This happens if and only if $\sigma = \tau \cdot \gamma$ for some $\gamma \in \Gamma_H$. Recall Σ is the set of all embeddings. Let ψ be an inner form on four dimensional vector space V and W. For a subgroup H of $G_{\mathbb{Q}}$ and two symplectic vector spaces V ad W , let us denote

$$I_\lambda^H(V,W) := \{\Phi : (V,\Psi) \to (W,\Psi), \Phi \circ \rho_\lambda(h) = \rho_\lambda(h) \circ \Phi \text{ for all } h \in H\}.$$

Note that $\iota_\lambda^H(V,W) = 0$ or L depending on $V \nsim W$ or $V \sim W$. This is a version of Schur's Lemma (cf. [3, Lemma 2.4]). We have

$$\begin{aligned}\iota_l^H(\overline{V_l}) &= \prod\nolimits_{\sigma\in\Sigma/\Gamma_H}\left(\iota_\sigma^H\left(\prod\nolimits_{\gamma\in\Gamma_H} V_{\sigma\cdot\gamma}\right)\right)\\ &= \prod\nolimits_{\sigma\in\Sigma/\Gamma_H}\left(\prod\nolimits_{\gamma\in\Gamma_H} M(a,\overline{\mathbb{Q}_l})\right)\\ &\simeq M(a,\overline{\mathbb{Q}_l})^b,\end{aligned}$$

where

$$a = \#\Gamma_H, \quad \text{and} \quad b = \#\Sigma/\Gamma_H.$$

□

Lemma 5.1.7. *We have an equality $\iota_l^{\mathfrak{g}_l} = K_l$ if and only if $K_l^\Gamma = K_l$.*

Proof. From Equation 5.1, we get that $\iota_l^{\mathfrak{g}_l} = K_l$ if for any open subgroup $U \subset G_{\mathbb{Q}}$, $\iota_l^U(V) = K_l$. Recall that open subgroups in the Krull topology correspond to the finite extensions of $\mathbb{Q}$. Take U to be a sufficiently large open subgroup such that $\Gamma_U = \{e\}$. Then, $\iota_l^U = K_l$ since $a = 1$. Now if $\iota_l^{\mathfrak{g}}(V) = K_l$ then $\iota_l^U(V) = K_l$.

We have $K_l^\Gamma = K_l$ if and only if $\Gamma = \{e\}$ if and only if $\Gamma_U = \{e\}$ if and only if $\iota_l^U = K_l$ if and only if $\iota_l^{\mathfrak{g}_l} = K_l$.

□

Theorem 5.1.8. *The algebra $\iota_l^{\mathfrak{g}_l} = K_l$ if and only if $\mathfrak{g}_l = \mathfrak{a}_l$.*

Proof. Assume $\iota_l^{\mathfrak{g}_l} = K_l$, we prove that $\mathfrak{g}_l = \mathfrak{a}_l$.

- Step 1 : We show that for each embedding σ, the equality $\mathfrak{g}_\sigma = \mathfrak{gsp}_4(V_\sigma)$ holds true. The proof follows from [39, Proposition 5.4] as our weights are scalars and hence not of the form $(2k-1, 2k+1)$.

- Step 2 : Recall that

$$\overline{\mathfrak{g}_l} = \prod_{\sigma\in\Gamma} \mathfrak{g}_\sigma \subsetneq \overline{\mathfrak{a}_l} = \prod_{\sigma\in\Gamma} \mathfrak{a}_\sigma \subset \prod_{\sigma\in\Gamma} \mathfrak{gsp}_4(V_\sigma).$$

For two distinct embeddings σ, τ the projection of $\overline{\mathfrak{g}_l}$ into $\mathfrak{gsp}_4(V_\sigma) \times \mathfrak{gsp}_4(V_\tau)$ is the same as the projection of $\overline{\mathfrak{a}_l}$ into the same. We use the same argument as in [33, Lemma 7] and modify it for our case. Recall that $\mathfrak{g}'_{\sigma,\tau} = \{(u, u') \in \mathfrak{g}_\sigma \times \mathfrak{g}_\tau\}$. At the Lie algebra level, $\mathfrak{g}'_{\sigma,\tau} \subset \mathfrak{gsp}_4(V_\sigma) \times \mathfrak{gsp}_4(V_\tau)$. Suppose $\mathfrak{g}_l \subsetneq \mathfrak{a}_l$, as the projections are surjective by Step 1. By applying Goursat's Lemma [17, Lemma 3.8, p. 12], we deduce that $\mathfrak{g}'_{\sigma,\tau}$ is the graph of an isomorphism by Lemma 5.1.3 $\alpha_{\sigma,\tau} : \mathfrak{gsp}_4(V_\sigma) \to \mathfrak{gsp}_4(V_\tau)$ which takes 1 to 1. Note that as the projections are surjective, the graph always exists. However only when the inclusion is proper, are the kernels trivial.

Consider the vector space $\overline{V_l} = \bigoplus_\sigma V_\sigma$. Recall that for a vector space V over $\overline{\mathbb{Q}_l}$, we have $\mathfrak{gsp}_4(V) = \mathfrak{sp}_4(V) \oplus \overline{\mathbb{Q}_l}.I_4$. Hence, any automorphism $\sigma : \overline{\mathbb{Q}_l} \to \overline{\mathbb{Q}_l}$ produces an automorphism of $\mathfrak{gsp}_4(V)$. This is determined by what it does to $\mathfrak{sp}_4(V)$. Hence our automorphism is determined by its restriction to $\mathfrak{sp}_4(V)$. By [14], we conclude that $\alpha_{\sigma,\tau}(u) = f_{\sigma,\tau} \circ u \circ f^{-1}_{\sigma,\tau}$ where $u \in \mathfrak{gsp}_4(V_\sigma)$ and $f_{\sigma,\tau} : V_\sigma \to V_\tau$. Here $f_{\sigma,\tau}$ is an isomorphism of $\mathfrak{g}'_l$ modules. As a consequence, we have an isomorphism of vector spaces $V_\sigma {\simeq}^{f_{\sigma,\tau}} V_\tau$. This is a contradiction because σ and τ are not isomorphic.

- Step 3 : We prove that $\overline{\mathfrak{g}_l} = \overline{\mathfrak{a}_l}$. Let m be the intersection of $\overline{\mathfrak{g}_l}$ with $\prod_{\sigma\in\Gamma} \overline{\mathfrak{sp}_4}(V_\sigma)$. The statement follows from [26, p. 790, Lemma in Step 3]. Hence, we have an isomorphism $\overline{\mathfrak{g}_l} = \overline{\mathfrak{a}_l}$. This implies $\mathfrak{g}_l = \mathfrak{a}_l$ because otherwise we can choose an $a_l \in \mathfrak{a}_l - \mathfrak{g}_l$. This produces an element $\overline{a}_l := a_l \otimes \overline{\mathbb{Q}_l} \in \overline{\mathfrak{a}_l} - \overline{\mathfrak{g}_l}$.

We now prove the containment of the other side. Assume that $\mathfrak{g}_l = \mathfrak{a}_l$. Let J be the matrix $\begin{bmatrix} 0 & S \\ -S & 0 \end{bmatrix}$ with $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consider the matrix $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \iota^{\mathfrak{a}_l}$. Consider the matrices $X_1 := \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}$, $X_2 := \begin{bmatrix} 0 & 0 \\ I_2 & 0 \end{bmatrix}$. These are matrices in $\mathfrak{a}_l$ since $X_i^t J + J X_i = 0$ for $i \in \{1, 2\}$. Solving for $XX_1 = X_1 X$ and $XX_2 = X_2 X$, we get that $A = D$ and

$C = B = 0$. Now, X commuting with these matrices will ensure off diagonal entries of X are zero and the matrix is block diagonal with the same blocks. Consider the matrix $M = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The matrix satisfies $M^t J + JM = 0$. Writing $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we see that $XM = MX$ gives the relations $a_{11} = a_{22}$ and $a_{12} = a_{21} = 0$. Hence, the commuting matrices are only scalars. We deduce that $\iota_l^{\mathfrak{g}_l} = K_l$. □

We now prove one of our main Theorem 5.1.4 that gives a necessary and sufficient condition for the existence of extra twists for non-lifted Siegel modular forms.

Proof. The Siegel modular form F has an extra twist if and only if $\Gamma \neq \varnothing$. From the above theorem, this is tantamount to $\iota_l^{\mathfrak{g}_l} \subsetneq \mathfrak{a}_l$. □

5.2 The central simple algebra in the image of the Galois representation

Consider the vector space $\overline{V_l} = \prod\limits_{\lambda | l} V_\lambda$. We have a Galois representation:

$$\rho_{F,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}_l}).$$

Let us consider the algebraic group GSp_4 defined over $\mathbb{Q}$. Write elements of the set $\overline{V_l}$ in the form $v = (v_\sigma)_{\sigma \in \Sigma}$. Then $\mathrm{GSp}_4(K \otimes \overline{\mathbb{Q}_l})$ can be thought of as an E-algebra using the following map:

$$\iota : K \to \mathrm{GSp}_4(K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l}) = \prod_{\lambda | l} \mathrm{GSp}_4(\overline{K_\lambda})$$

$$k \mapsto \{(v_\sigma)_{\sigma \in \Sigma} \mapsto (\sigma(k) v_\sigma)_{\sigma \in \Sigma}\}.$$

Here, we have the equality of fields $\overline{K_\lambda} = \overline{\mathbb{Q}_l}$.

Lemma 5.2.1. *The action of K as defined above commutes with the action of $G_{\mathbb{Q}}$.*

Proof. Let $g \in G_{\mathbb{Q}}$ and $k \in K$. We compute

$$\rho(g)k \cdot \prod_{\sigma\in\Sigma}(v_\sigma) = \prod_{\sigma\in\Sigma}\rho_\sigma(g)\prod_{\sigma\in\Sigma}(\sigma(k)v_\sigma) = \prod_{\sigma\in\Sigma}(\sigma(k)\rho_\sigma(g)v_\sigma),$$

while

$$k \cdot \rho(g)(v_\sigma)_{\sigma\in\Sigma} = k \cdot (\rho_\sigma(g)v_\sigma)_{\sigma\in\Sigma} = (\sigma(k)\rho_\sigma(g)v_\sigma)_{\sigma\in\Sigma}.$$

□

In particular, the elements of E are defined over $\mathbb{Q}$. Let L be a finite extension of $\mathbb{Q}_l$ that contains all the Gauss sums $G(\chi_\gamma^{-\sigma})$, for $\gamma \in \Gamma$ and $\sigma \in \Sigma$.

Definition 5.2.2. For each $\gamma \in \Gamma$, we define an intertwining operator η_γ by

$$\eta_\gamma((v_\sigma)_{\sigma\in\Sigma}) = (G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma}.$$

Lemma 5.2.3. *If H is a subgroup of $G_{\mathbb{Q}}$ such that $H \subset \ker(\chi_\gamma)$, then η_γ is H-equivariant. In particular, η_γ is defined over the fixed field cut out by χ_γ, an abelian extension of $\mathbb{Q}$.*

Proof. If $h \in H$, we have

$$\eta_\gamma\rho(h)(v_\sigma)_{\sigma\in\Sigma} = \eta_\gamma(\rho_\sigma(h)v_\sigma)_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})\rho_{\sigma\gamma}(h)v_{\sigma\gamma})_{\sigma\in\Sigma}.$$

On the other hand

$$\rho(h)\eta_\gamma(v_\sigma)_{\sigma\in\Sigma} = \rho(h)(G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})\rho_\sigma(h)v_{\sigma\gamma})_{\sigma\in\Sigma}.$$

By applying σ on the definition extra twists we get that

$$\rho_{\sigma\gamma}(g) = \rho_\sigma(g)\chi_\gamma^\sigma(g),$$

for all $g \in G_{\mathbb{Q}}$. In particular, if $h \in H \subset \ker(\chi_\gamma^\sigma)$, then $\rho_{\sigma\gamma}(h) = \rho_\sigma(h)$, and the two expressions above are coincide. □

Define the K-valued 2-cocycle c on Γ by [36, Lemma 8]

$$c(\gamma,\delta) = \frac{G(\chi_\delta^{-\gamma})G(\chi_\gamma^{-1})}{G(\chi_{\gamma\cdot\delta}^{-1})},$$

for γ, $\delta \in \Gamma$.

Recall that we have fixed an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_l}$ and we consider the Gauss sums $G(\chi) \in \overline{\mathbb{Q}_l}$ using this isomorphism. We have the following.

Lemma 5.2.4. *The endomorphisms η_γ defined as above satisfies the relations*

$$\eta_\gamma \cdot k = \gamma(k) \cdot \eta_\gamma,$$

$$\eta_\gamma \cdot \eta_\delta = c(\gamma, \delta) \cdot \eta_{\gamma \cdot \delta}.$$

Proof. Again, we compute:

$$\eta_\gamma(k \cdot (v_\sigma)_{\sigma\in\Sigma}) = \eta_\gamma(\sigma(k)v_\sigma)_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})(\sigma\gamma)(k)v_{\sigma\gamma})_{\sigma\in\Sigma},$$

whereas

$$\gamma(k) \cdot (\eta_\gamma(v_\sigma)_{\sigma\in\Sigma}) = \gamma(k) \cdot (G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma} = ((\sigma\gamma)(k)G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma},$$

proving the first relation. For the second, we have:

$$\eta_\gamma\eta_\delta(v_\sigma)_{\sigma\in\Sigma} = \eta_\gamma(G(\chi_\delta^{-\sigma})v_{\sigma\delta})_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})G(\chi_\delta^{-\gamma\cdot\sigma})v_{\sigma\gamma\delta})_{\sigma\in\Sigma}$$

On the other hand

$$c(\gamma, \delta) \cdot \eta_{\gamma\cdot\delta}(v_\sigma)_{\sigma\in\Sigma} = c(\gamma, \delta) \cdot (G(\chi_{\gamma\cdot\delta}^{-\sigma})(v_{\sigma\gamma\delta}))_{\sigma\in\Sigma} = (\sigma(c(\gamma, \delta))G(\chi_{\gamma\cdot\delta}^{-\sigma})v_{\sigma\gamma\delta})_{\sigma\in\Sigma}.$$

So $\eta_\gamma \cdot \eta_\delta = c(\gamma, \delta) \cdot \eta_{\gamma\cdot\delta}$, if

$$\sigma(c(\gamma, \delta)) = \frac{G(\chi_\gamma^{-\sigma})G(\chi_\delta^{-\gamma\cdot\sigma})}{G(\chi_{\gamma\cdot\delta}^{-\sigma})}.$$

But this is proved in [36, Lemma 8, p. 797] □

Let X be the cross product algebra associated to the cocycle $c(\gamma, \delta)$ defined by

$$X = \bigoplus\nolimits_{\gamma\in\Gamma} K \cdot x_\gamma \tag{5.3}$$

with relations

$$
\begin{aligned}
x_\gamma \cdot x_\delta &= c(\gamma, \delta) \cdot x_{\gamma \cdot \delta} \\
x_\gamma \cdot k &= \gamma(k) \cdot x_\gamma
\end{aligned}
$$

for γ, $\delta \in \Gamma$, $k \in K$. Then X is a central simple algebra over $F := E^\Gamma$, the fixed field of Γ in E. We note that F is the number field generated by $\frac{t_p^2}{\epsilon(p)}$ for $p \nmid N$.

Let K_0 be the fixed field of $H = \cap \ker \chi_\gamma$. Note K_0 is an abelian number field. By Lemma 5.2.3 each η_γ is defined over K_0, and so there is a natural map:

$$X \otimes_F \overline{\mathbb{Q}_l} \to \iota_l^{\overline{\mathfrak{g}_l}}((\overline{V_l}) \tag{5.4}$$

induced by

$$
\begin{aligned}
x_\gamma &\mapsto \eta_\gamma \\
k &\mapsto \iota(k).
\end{aligned}
$$

Theorem 5.2.5. *We have an equality:*

$$\iota_l^{\overline{\mathfrak{g}_l}}((\overline{V_l}) \simeq X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l}.$$

Proof. Let K_0 be a sufficiently large abelian extension of $\mathbb{Q}$. Here 'sufficiently large' means that K_0 contains the fixed field of $\cap_{\gamma \in \Gamma} \ker(\chi_\gamma)$, or equivalently if $H = \mathrm{Gal}(\bar{\mathbb{Q}}/K_0)$, then

$$H \subset \cap_{\gamma \in \Gamma} \ker(\chi_\gamma). \tag{5.5}$$

Consider the natural map defined in (5.4):

$$X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l} \to \iota_l^{\overline{\mathfrak{g}_l}}((\overline{V_l}).$$

This map is injective since $X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l} = \prod_{\sigma : F \hookrightarrow \overline{\mathbb{Q}_l}} (X \otimes_{F,\sigma} \overline{\mathbb{Q}_l})$ and the restriction of the above map to each factor is injective (each factor is a central simple algebra over L, so has no two-sided ideals).

Now X has dimension $[K : F]$ over K, and hence has dimension $[K : F][K : \mathbb{Q}]$

over $\mathbb{Q}$. Hence $X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l}$ has dimension $[K:F][K:\mathbb{Q}]$ over $\overline{\mathbb{Q}_l}$. If we can prove $X_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_l}$ also has dimension $[K:F][K:\mathbb{Q}]$ over $\overline{\mathbb{Q}_l}$ then the above inclusion is an isomorphism. By Theorem 5.1.6, we have

$$\dim_{\overline{\mathbb{Q}_l}} \iota_l^H(\overline{V_l}) = [K:F_H]^2[F_H:\mathbb{Q}].$$

Since $F = F_H$ by Equation 5.5, this dimension of $\iota_l^{\mathfrak{g}_l}((\overline{V_l})$ is $[K:F][K:\mathbb{Q}]$, as desired. □

5.2.1 Local Brauer classes of the central simple algebra X

Recall that the invariant maps are explicitly computed in this situation [5, §3.3]. By definition, the invariant map at v is given by

$$\mathrm{inv}_v = \mathrm{Ev} \circ \delta^{-1} \circ v \circ \mathrm{Inf}^{-1} : \mathrm{Br}(F_v) \to \mathbb{Q}/\mathbb{Z}.$$

Let $K_1 : G_v \to \bar{F}_v^*$ be any map. Then

$$c_{K_1}(g,h) = \frac{K_1(g)K_1(h)}{K_1(gh)};$$

defines a local 2-cocycle on G_v. Note $c_{K_1}(g,h) \in F_v$, for all g, $h \in G_v$. We call it the local 2-cocycle defined by the function K_1. The following general lemma regarding the Brauer class of this local 2-cocycle will be very useful in our computations [5, Lemma 9]:

Lemma 5.2.6. *Let $K_1 : G_v \to \bar{F}_v^*$ be a map and let $t : G_v \to \bar{F}_v^*$ be an unramified homomorphism such that*

1. *$K_1(i) \in F_v^*$, for all $i \in I_v$,*
2. *$K_1(g)^2/t(g) \in F_v^*$, for all $g \in G_v$.*

For any arithmetic Frobenius Frob_v at v, we have

$$\mathrm{inv}_v(c_{K_1}) = \frac{1}{2} \cdot v\left(\frac{K_1(\mathrm{Frob}_v)^2}{t(\mathrm{Frob}_v)}\right) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z},$$

where $v : F_v^* \to \mathbb{Z}$ is the surjective valuation.

5.2.2 Good primes

Let t_p be as in the introduction.

Lemma 5.2.7. *Assume $\gcd(p, N) = 1$ and assume $t_p \neq 0$. Let v be a place of F lying over p. Then X_v is a matrix algebra over F_v, if and only if the slope*

$$m_v := [F_v : \mathbb{Q}_p] \cdot v(t_p^2/\epsilon(p)) \in \mathbb{Z}$$

is even, where v is normalized such that $v(p) = 1$.

Proof. This follows immediately from the lemma above taking $K_1 = \alpha$ and $t = \epsilon$ from § 4.2.8. Similar to the classical case, we have

$$\mathrm{inv}_v(c_\alpha) = \frac{1}{2} v \left(\frac{\alpha^2(\mathrm{Frob}_v)}{\epsilon(\mathrm{Frob}_v)} \right)$$

mod $\mathbb{Z}$, and $\alpha(\mathrm{Frob}_v) \equiv t_p^{f_v} \pmod{F^*}$, by part (3) of Proposition 3. □

We use the valuation v such that $v(p) = 1$ introducing the ramification index e. We also have the inertia degree f since $\mathrm{Frob}_v = \mathrm{Frob}_p^f$. Since $[F_v : \mathbb{Q}_p] = ef$, we get the extra factor $[F_v : \mathbb{Q}_p]$.

Recall that $p^\dagger \nmid N$ is a prime such that $p^\dagger \equiv p \bmod N$ and $t_p^\dagger \neq 0$.

Lemma 5.2.8. *Let $\gcd(p, N) = 1$ and suppose $t_p = 0$. Let v be a place of F lying over p. Then X_v is a matrix algebra over F_v if and only if*

$$m_v^\dagger := [F_v : \mathbb{Q}_p] \cdot v(t_p^2/\epsilon(p^\dagger)) \in \mathbb{Z}$$

is even.

Proof. The proof is similar to that of the previous theorem with minor changes. Note that $p^\dagger \equiv p \bmod N$ implies $\chi_\gamma(p) = \chi_\gamma(p^\dagger)$, for all $\gamma \in \Gamma$. If Frob_p and $\mathrm{Frob}_{p^\dagger}$ denote the Frobenii at the prime p and $p^\dagger$, then by 4.2.8, we have $\alpha(\mathrm{Frob}_p) \equiv \alpha(\mathrm{Frob}_{p^\dagger}) \equiv a_{p^\dagger}$ mod

F^*. Hence

$$\mathrm{inv}_v(c_\alpha) = \frac{1}{2}v\left(\frac{\alpha^2(\mathrm{Frob}_v)}{\epsilon(\mathrm{Frob}_v)}\right) = \frac{1}{2}\cdot f_v \cdot v\left(\frac{\alpha^2(\mathrm{Frob}_p)}{\epsilon(p)}\right) = \frac{1}{2}\cdot f_v\cdot v\left(\frac{a^2_{p^\dagger}}{\epsilon(p^\dagger)}\right) \mod \mathbb{Z}.$$

□

5.2.3 Infinite places

Lemma 5.2.9. *For all infinite places v, X_v is a split central simple algebra.*

Proof. Note that F is contained in $\mathbb{R}$ since complex conjugation is always an extra twist. Hence, for all infinite places v, we have $F_v = \mathbb{R}$ and hence $\overline{F_v} = \mathbb{C}$. As a consequence, we have $\mathrm{Gal}(\overline{F_v}/F_v) = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We have a restriction map

$$\mathrm{H}^2(G_{\mathbb{Q}}, \overline{\mathbb{Q}}^\times) \to^{res} \mathrm{H}^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \simeq \mathbb{R}^\times/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) \simeq \mathbb{R}^\times/\mathbb{R}^{\times 2}.$$

Here, $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$ is the image of the norm map. This is a set consisting of set of positive real numbers. The restriction map

$$\mathrm{H}^2(G_{\mathbb{Q}}.\overline{\mathbb{Q}}^\times) \to \mathrm{H}^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times);$$

is give by $[c_\alpha] \to res[c_\alpha]$. By [19, p. 139], the nontrival element of the local group cohomology $\mathrm{H}^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$ is given by

$$\tilde{\alpha}(g,h) = \begin{cases} -1 & \text{if } g = h = c, \\ 1 & \text{otherwise.} \end{cases}$$

Hence, $res[c_\alpha]$ is completely determined by $\alpha^2 : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathbb{R}^\times/\mathbb{R}^{\times 2}$.

Note that $S_k^g(N,\chi) = 0$ unless $\epsilon(c) = (-1)^{gk}$. Since g is even, we have $\epsilon(c) = 1$ and hence $\alpha^2(c) \equiv \epsilon(c) \pmod{(\mathbb{R}^\times)^2}$. Since $\epsilon(c) = 1$, we have $Res([c_\alpha])(c,c) = 1$. Hence, we get the trivial element of the local Brauer group $\mathrm{H}^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$. We deduce that $X_v \sim 1$ and it is split algebra. □

Our Theorem 4.1.3 follows from Lemma 5.2.7, Lemma 5.2.8 and Lemma 5.2.9. Our

Lemma also gives alternate proof of the ramification of local Brauer classes for central simple associated to elliptic modular forms as given by Momose [20]. This also gives us a recipe to find the local Brauer classes for higher genus.

6

Fourier Expansions of Siegel modular forms

6.1 Some basic notations

Below we consider two natural numbers $n, k \in \mathbb{N}$, and list the following notations which will be important for us in this and the following chapter.

- $P_n(\mathbb{R}) = \text{The set of real positive definite } n \times n \text{ matrices.}$
- $P_n(\mathbb{R})^{\text{semi}} = \text{The set of real positive semi} - \text{definite } n \times n \text{ matrices}$
- $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ is the full Siegel modular group.
- $\Gamma_0(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0 \pmod N\}$
- $\Delta_n(\mathbb{Z}) = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n\}$
- $\Gamma_1(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0 \text{ and } A, D \equiv I_n \pmod N\}$
- $V_n(\mathbb{Z})$ be the set of symmetric $n \times n$ matrices over $\mathbb{Z}$.

- For matrices $T, u \in \mathrm{GL}_n\mathbb{R}$, we define the action $T[u] = u^t T u$.
- For two matrices $< t, \Omega >= \mathrm{tr}(t\Omega)$.
- For a matrix $s \in P_n^{\mathrm{semi}}(\mathbb{R})$, define
  1. $m(s) = \inf_{u \in \mathbb{Z}_n - \{0\}} u^t s u$, is called the Minimum function
  2. $\tilde{\mathrm{tr}}(s) = \inf_{u \in \mathrm{GL}_n(\mathbb{Z}) \mathrm{tr}({}^t u s u)}$, is called the trace function.
  3. $w(s) = \inf_{u \in P_n(\mathbb{R}) \frac{<u,s>}{m(u)}}$, which we call the dyadic trace function.

In the following subsection we will show how we get Fourier expansions for the restriction map for an arbitrary prime level p.

6.2 The Restriction map

For any natural number $N \in \mathbb{N}$, let $S_1^k(\Gamma_0(N)$ be the space of all the classical elliptic cusp forms [10] of weight k for the congruence subgroup $\Gamma_0(N)$. Denote the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ as $[a^b c]$. For a matrix $[a^b c]$, we denote the dyadic trace $w([a^b c]) = \frac{1}{2}(a + c - |b|)$. For $F \in S_2^k(\Gamma_0^{(2)}(p))$, consider the set of matrices $[a^b c]$ with $w([a^b c]) < \frac{1}{6}(1+p)k$. Let X_2 be integral valued half integral positive definite 2×2 matrices. Consider also the set of reduced matrices

$$X_2^{red} := \{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in X_2 \text{ such that } 0 \leq 2b \leq a \leq c \}.$$

We say that $v \in [t]$ if there exists a matrix $m \in \mathrm{SL}_2(\mathbb{Z})$ such that $v = mtm^{-1}$. Following [24], consider the quantity

$$v(j, s, t) = \#\{ v \in [t] \mid \; < v, s >= j \}.$$

Consider the map $\phi_s : \mathbb{H}_1 \to \mathbb{H}_2$ by $\phi_s(\tau) = s\tau$ where s is a symmetric positive definite 2×2 matrix. For $f \in S_2^k(\Gamma_0^{(2)}(p))$, the restriction map $\phi_s^*(f) \in S_1^{2k}(\Gamma_0(pl))$ have the

Fourier series expansion given by:

$$(\phi_s^* f)(\tau) := \sum_{j\in\mathbb{N}} \left(\sum_{t\in X_2 : <t,s>=j} a_0(t) \right) q^j = \sum_{j\in\mathbb{N}} \left(\sum_{t\in X_2^{red}} v(j,s,t) a_0(t) \right) q^j.$$

For $c \equiv 0 \pmod{Nl}$ and $l = det(s)$, we have

$$(\phi_s^*(f))|_{2k} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = f|_k(\begin{pmatrix} aI_2 & bs \\ cs^{-1} & dI_2 \end{pmatrix})(s\tau) = f|_k(s\tau) = \phi_s^*(f)(\tau).$$

Note that this makes sense as for $l = det(s)$ and hence as $Nl|c$, so also $cs^{-1} \equiv 0 \pmod{N}$.

6.3 Upper bound of the dyadic traces of the set of Determining coefficients

We study the restriction map

$$\phi_s^\star : S_2^2(\Gamma_0^2((p))) \to S_1^4(\Gamma_0(p.l))$$

for $p \neq l$. If $p = 43$, we take $l \in \{1,2,3,5\}$. For every choice of l, we choose a 2×2 matrix s with $l = det(s)$. For instance, we choose $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $l = 1$. Similarly, we choose $s = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ for $l = 2$, $s = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ for $l = 3$ and $s = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ for $l = 5$. Let $w([t])$ be the dyadic traces of the matrix $[t]$.

Definition 6.3.1. (Set of determining coefficients)

From [24, Theorem 3.1], we get that for a Siegel modular form $F \in S_2^k(\Gamma_0^{(2)}(p))$, the set of determining coefficients is the set of matrices of the form $[t]$ with dyadic traces $w([t]) < \frac{1}{6}(1+p)k$. In the next proposition, we show that this is dependent on the power of prime that appear in the level of the Siegel modular forms.

Lemma 6.3.2. *The set of determining coefficients for the space $S_2^k(\Gamma_0^{(2)}(p^i))$ is the set of matrices of the form $[t]$ with dyadic traces*

$$w([t]) < \frac{3}{2} + p^i(\frac{2k}{2\sqrt{3}\pi} - \frac{3}{2p^i})\prod_{j=1}^{i}(1+\frac{1}{p^j}).$$

Proof. The proof follows from [16, Page 103] and putting p^i as the level in the general formula.

□

Lemma 6.3.3. *The upper bound of the dyadic traces of the set of determining coefficients for the space $S_2^k(\Gamma_0^{(2)}(N),\chi)$, is the same as that of the space $S_2^k(\Gamma_0^{(2)}(N))$.*

Proof. Let $F, G \in S_2^k(\Gamma_0^{(2)}(N),\chi)$ and consider the Siegel modular form $H = F - G$. The Fourier coefficients of H are zero for all matrices t with dyadic traces $w(t) < M$ for some natural number M. If χ is a Dirichlet character of order r, then $H^r \in S_2^{kr}(\Gamma_0^{(2)}(p))$ and $a_0(t) \equiv 0$ for all t with $w(t) < Mr$. Hence $H^r \equiv 0$ as $a_0(t) \equiv 0$ for the relevant bound and so $H \equiv 0$. So with these conditions, $F = G$. □

6.4 Program for the expansion

We first write down a program using the language `Java` to find the index matrices of the restriction of the Fourier expansions:

```
public class matrices{
    public static void main(String args[]) {
      int a,b,c;
      int count=0;
     for(a=2;a<=60;a+=2){
         for(b=0;b<=60;b++){
             for(c=2;c<=60;c+=2){
                 if((2*b<=a)&&(a<=c)&&(a+c-b<=28)){
                 System.out.println(a+" "+b+" "+c);
                 count++;
```

```
                    }
                }
            }
        }
        System.out.println(count);


    }
}
```

We are interested in the prime $p = 43$ since the index matrices are already constructed for $p \leq 41$ by Poor-Yuen [24]. Running the program, we get the set of determining coefficients for $p = 43$ and $k = 2$ is given by the following list of 234 positive definite matrices:

$$[2^0 2], [2^0 4], [2^0 6], [2^0 8], [2^0 10], [2^0 22, [2^0 14, [2^0 26, [2^0 28, [2^0 2].$$

$$[2^1 6], [2^1 8], [2^1 10], [2^1 12], [2^1 14], [2^1 16], [2^1 18], [2^1 20], [2^1 22], [2^1 24],$$

$$[2^1 26], [4^0 4], [4^0 6], [4^0 8], [4^0 10], [4^0 12], [4^0 14], [4^0 16], [4^0 18], [4^0 20],$$

$$[4^0 22], [4^0 24], [4^1 4], [4^1 6], [4^1 8], [4^1 10], [4^1 12], [4^1 14], [4^1 16], [4^1 18],$$

$$[4^1 20], [4^1 22], [4^1 24], [4^2 4], [4^2 6], [4^2 8], [4^2 10], [4^2 12], [4^2 14], [4^2 16],$$

$$[4^2 18], [4^2 20], [4^2 22], [4^2 24], [4^2 26], [6^0 6], [6^0 8], [6^0 10], [6^0 12], [6^0 14],,$$

$$[6^0 16], [6^0 18], [6^0 20], [6^0 22], [6^1 6], [6^1 8], [6^1 10], [6^1 12], [6^1 14], [6^1 16],$$

$$[6^1 18], [6^1 20], [6^1 22], [6^2 6], [6^2 8], [6^2 10], [6^2 12], [6^2 14], [6^2 16], [6^2 18],$$

$$[6^2 20], [6^2 22], [6^2 24], [6^3 6], [6^3 8], [6^3 10], [6^3 12], [6^3 14], [6^3 16], [6^3 18],$$

$$[6^3 20], [6^3 22], [6^3 24], [8^0 8][8^0 10], [8^0 12], [8^0 14], [8^0 16], [8^0 18], [8^0 20],$$

$$[8^1 8], [8^1 10], [8^1 12], [8^1 14], [8^1 16], [8^1 18], [8^1 20], [8^2 8], [8^2 10], [8^2 12],$$

$$[8^2 14], [8^2 16][8^2 18], [8^2 20], [8^2 22], [8^3 8], [8^3 10], [8^3 12], [8^3 14], [8^3 16],$$

$$[8^3 18], [8^3 20], [8^3 22], [8^4 8], [8^4 10], [8^4 12], [8^4 14], [8^4 16], [8^4 18], [8^4 20],$$

$$[8^4 22], [8^4 24], [10^0 10], [10^0 12], [10^0 14], [10^0 16][10^0 18], [10^1 10], [10^1 12],$$

$$[10^1 14], [10^1 16], [10^1 18], [10^2 10], [10^2 12], [10^2 14], [10^2 16], [10^2 18],$$

$$[10^2 20], [10^3 10], [10^3 12], [10^3 14], [10^3 16], [10^3 18], [10^3 20], [10^4 10], [10^4 12],$$

$$[10^4 14], [10^4 16], [10^4 18], [10^4 20], [10^4 22], [10^5 10], [10^5 12], [10^5 14], [10^5 16],$$

$$, [10^5 18], [10^5 20], [10^5 22], [12^0 12], [12^0 14], [12^0 16], [12^1 12], [12^1 14],$$

$$[12^1 16], [12^2 12], [12^2 14], [12^2 16], [12^2 18], [12^3 12], [12^3 14]$$

$$, [12^3 16], [12^3 18], [12^4 12], [12^4 14], [12^4 16], [12^4 18], [12^4 20], [12^5 12], [12^5 14],$$

$$[12^5 16], [12^5 18], [12^5 20], [12^6 12], [12^6 14], [12^6 16], [12^6 18], [12^6 20], [12^6 22],$$

$$[14^0 14], [14^1 14], [14^2 14], [14^2 16], [14^3 14], [14^3 16], [14^4 14], [14^4 16],$$

$$, , [14^4 18], [14^5 14], [14^5 16], [14^5 18], [14^6 14], [14^6 16], [14^6 18], [14^6 20], [14^7 14],$$

$$[14^7 16], [14^7 18], [14^7 20], [16^4 16], [16^5 16], [16^6 16], [16^6 18], [16^7 16], [16^7 18],$$

$$[16^8 16], [16^8 16], [16^8 18], [16^8 20], [18^8 18], [18^9 18].$$

We now compute the image of the restriction map for our choice of s, power j and p:

```
public class participating {
 public static void main(String args[]) {
  double arr[]={2.0,1.0,2.0,2.0,0.0,2.0,2.0,1.0,4.0,2.0,0.0,4.0,4.0,
  2.0,4.0,2.0,1.0,6.0,4.0,1.0,4.0,4.0,0.0,4.0,4.0,2.0,6.0,2.0,0.0,6
  double s1=1;
  double s2=0;
  double s4=1;
  int i,j,a,b,d;
  int m,n,o,p;
   for(j=1;j<=8;j++){
  for(i=0;i<=29;i+=3){
      int coeff=0;
       for(a=2;a<=10;a+=2){
      for(b=-10;b<=10;b++){
          for(d=-10;d<=10;d+=2){
              int count=0;
  for(m=-8;m<=8;m++){
```

```
    for(n=-8;n<=8;n++){
    for(o=-8;o<=8;o++){
    for(p=-8;p<=8;p++){
        if((m*p-n*o==1)||(m*p-n*o==-1)){
        if  (a*d-b*b>0){
        if((a*s1*0.5+b*s2+s4*d*0.5==j)&&(m*m*a+2*m*n*b+n*n*d==arr[i])&
        count+=1;
        coeff=coeff+count;
        break;
        }
    }
    }
    if(count>0)
    break;
   }
   if(count>0)
   break;
}
if(count>0)
break;
}
if(count>0)
break;
}
}
}
}
if(coeff>0)
System.out.println(coeff+"  "+arr[i]+" "+arr[i+1]+" "+arr[i+2]+"  q^"+j
}
}
}
}
```

We run the same in terminal. We prove here that the set of determining coefficients for Siegel modular forms with character is the same as for Siegel modular forms for the subgroup $\Gamma_0^{(2)}(p)$. We will prove below how such a bound is sufficient for the space of cusp forms with character namely $S_2^k(p)$. In this paper. we use mathematical software [38] to compute the image of the restriction map. Putting $p = 43$, it is evident that we are looking at t for which $w(t) \leq 14$. For instance, if $p = 43$ and $s = 1$ we choose the matrix $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and we get the following expansion:

$$\begin{aligned}
\phi_s^\star(f) = a_0(2^02)q^2 + 2a_0(2^12)q^2 + 4a_0(2^02)q^3 + 2a_0(2^04)q^3 + 4a_0(2^14)q^3 + 4a_0(2^04)q^4 + \\
2a_0(2^06)q^4 + 4a_0(2^12)q^4 + 2a_0(2^14)q^4 + 4a_0(2^16)q^4 + 1a_0(4^04)q^4 + \\
2a_0(4^14)q^4 + 2a_0(4^24)q^4 + 4a_0(2^04)q^5 + 4a_0(2^06)q^5 + 2a_0(2^08)q^5 + 4a_0(2^14)q^5 + \\
4a_0(2^18)q^5 + 2a_0(4^06)q^5 + 4a_0(4^14)q^5 + 4a_0(4^16)q^5 + 4a_0(4^26)q^5 + \\
4a_0(2^02)q^6 + 4a_0(2^08)q^6 + 2a_0(2^010)q^6 + 4a_0(2^14)q^6 + 6a_0(2^16)q^6 + 4a_0(2^110)q^6 + \\
4a_0(4^04)q^6 + 2a_0(4^08)q^6 + 4a_0(4^16)q^6 + 4a_0(4^18)q^6 + 2a_0(4^26)q^6 + \\
4a_0(4^28)q^6 + 1a_0(6^06)q^6 + 2a_0(6^16)q^6 + 2a_0(6^26)q^6 + \\
2a_0(6^36)q^6 + 4a_0(2^02)q^7 + 4a_0(2^06)q^7 + 4a_0(4^06)q^7 + \\
2a_0(4^010)q^7 + 4a_0(4^14)q^7 + 4a_0(4^16)q^7 + 4a_0(4^18)q^7 + \\
4a_0(4^110)q^7 + 4a_00(4^210)q^7 + 2a_0(6^08)q^7 + 4a_0(6^18)q^7 + \\
4a_0(6^26)q^7 + 4a_0(6^28)q^7 + 4a_0(6^38)q^7
\end{aligned}$$

Also for $s = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ we get the expansion :

$$\begin{aligned}\phi_s^\star(f) = a_0(2^0 2)q^3 + 2a_0(2^1 2)q^3 + 2a_0(2^0 2)q^4 + a_0(2^0 4)q^4 + \\ 2a_0(2^1 4)q^4 + 2a_0(2^0 2)q^5 + 3a_0(2^0 4)q^5 + 1a_0(2^0 6)q^5 + 2a_0(2^1 2)q^5 + \\ 2a_0(2^1 4)q^5 + 2a_0(2^1 6)q^5 + 2a_0(2^0 6)q^6 + a_0(2^0 8)q^6 + \\ 4a_0(2^1 4)q^6 + 2a_0(2^1 8)q^6 + 1a_0(4^0 4)q^6 + 2a_0(4^1 4)q^6 + \\ 2a_0(4^2 4)q^6 + 2a_0(2^0 2)q^7 + 4a_0(2^0 4)q^7 + a_0(2^0 6)q^7 + 2a_0(2^0 8)q^7 + \\ a_0(2^0 10)q^7 + 2a_0(2^1 2)q^7 + 4a_0(2^1 6)q^7 + 2a_0(2^1 10)q^7 + \\ a_0(4^0 6)q^7 + 2a_0(4^1 4)q^7 + 2a_0(4^1 6)q^7 + 2a_0(4^2 6)q^7 + \dots\end{aligned}$$

For $s = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ we get the expansion :

$$\begin{aligned}\phi_s^\star(f) = a_0(2^1 2)q^3 + 3a_0(2^0 2)q^4 + 3a_0(2^1 2)q^5 + 3a_0(2^1 4)q^5 + \\ 6a_0(2^0 4)q^6 + a_0(4^2 4)q^6 + 6a_0(2^1 4)q^7 + 3a_0(2^1 6)q^7 + 3a_0(4^1 4)q^7 + \dots\end{aligned}$$

Finally for $s = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ we get the expansion :

$$\begin{aligned}\phi_s^\star(f) = a_0(2^1 2)q^4 + 2a_0(2^0 2)q^5 + 1a_0(2^0 2)q^6 + 2a_0(2^1 2)q^6 + \\ a_0(2^1 4)q^6 + 2a_0(2^0 4)q^7 + 2a_0(2^1 4)q^7 + \dots\end{aligned}$$

6.5 Upper bound for the space of Siegel modular forms using the restriction map

In this section, we state an upper bound for the Siegel cusp space. We investigate the space $S_2^2(\Gamma_0^{(2)}(9))$. From Klien Thesis [16, Page 103], we need to look at $t \in X_2$ such that

$$w(t) < \frac{3}{2} + 9\left(\frac{4}{2.\sqrt{3}\pi} - \frac{3}{2.9}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{3^2}\right).$$

In other words, we are interested in the set of t such that $w(t) < 8.5$. This bound may not be optimal. In the context of this level, the corresponding set X_2 is:

$$[a_0(2^02)] \cup [a_0(2^04)] \cup [a_0(2^06)] \cup [a_0(2^12)] \cup [a_0(2^14)] \cup [a_0(2^16)] \cup [a_0(4^04)] \\ \cup [a_0(4^14)] \cup [a_0(4^24)] \cup [a_0(4^26)].$$

We choose $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, with $l = 1, 2$ respectively.

For the matrix $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get the expansion :

$$(2a_0(2^12) + a_0(2^02))q^2 + \\ (4a_0(2^02) + 4a_0(2^14) + 2a_0(2^04))q^3 + \\ (a_0(2^12) + 2a_0(2^14) + 4a_0(2^04) + 2a_0(4^24) + 4a_0(2^16) + 2a_0(4^14) + a_0(4^04) + 2a_0(2^06))q^4 + \\ (4a_0(2^14) + 4a_0(2^04) + 4a_0(4^14) + 4a_0(4^26) + 4a_0(2^06))q^5 + ...$$

Since there is only one cusp form in $S_1^4(\Gamma_0(9))$ and that too starts with q^3, the only option is for $\phi_s^*(f) \equiv 0$ and this gives us the following set of equations.

$$(2a_0(2^12) + a_0(2^02)) = 0 \\ (4a_0(2^02) + 4a_0(2^14) + 2a_0(2^04)) = 0 \\ (a_0(2^12) + 2a_0(2^14) + 4a_0(2^04) + 2a_0(4^24) + 4a_0(2^16) + 2a_0(4^14) + a_0(4^04) + 2a_0(2^06)) = 0 \\ (4a_0(2^14) + 4a_0(2^04) + 4a_0(4^14) + 4a_0(4^26) + 4a_0(2^06)) = 0$$

Next we choose $s = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The corresponding expansion we get in this case is ;

$$(2a_0(2^12)+a_0(2^02))q^3+(2a_0(2^02)+2a_0(2^14)+a_0(2^04))q^4+$$
$$(2a_0(2^12)+2a_0(2^02)+2a_0(2^14)+3a_0(2^04)+2a_0(2^16)+a_0(2^06))q^5+...$$

Using the equality $\phi_s^*(f)=W_2(\phi_s^*(f))$, we get the relations :

$$2a_0(2^12)+a_0(2^02)=0$$
$$2a_0(2^02)+2a_0(2^14)+a_0(2^04)=0$$
$$2a_0(2^12)+2a_0(2^02)+2a_0(2^14)+3a_0(2^04)+2a_0(2^16)+a_0(2^06)=0.$$

Solving for this gives that the dimension of the space $S_2^2(\Gamma_0^2(9))\leq 6$.

7

Future problems on extra twists of Siegel modular forms

In this chapter, we list some future problems which we might pursue. While there are examples of Siegel modular forms, all of them, to our understanding, are lifted Siegel modular forms.

Problem 7.0.1. *Constructing an explicit non lifted Siegel modular form with an extra twist.*

We have shown examples of Siegel modular forms with extra twists but the discussion was about lifted Siegel modular forms in terms of the examples. An explicit example of a non lifted Siegel modular form with extra twists could be intriguing.

Problem 7.0.2. *The modularity of Siegel modular forms is not known for now. Let A be a principally polarized abelian scheme of dimension two over $\mathbb{Q}$ with $End_{\overline{\mathbb{Q}}}(A)$ bigger than $\mathbb{Z}$. In a joint work with my advisor and Professor Krishna Kishore of IIT, Tirupati, we are investigating Ribet type big image results for the image of Galois representations associated with these abelian surfaces. We expect that there are close connections with suitably defined extra twists for the Galois representations.*

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