Schrödinger Operators and Applications to Evolution Equations

A Thesis

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by

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CERTIFICATE

This is to certify that this dissertation entitled **Schrödinger Operators and Applications to Evolution Equations** towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Naren S Narayanan** at Indian Institute of Science Education and Research, Pune under the supervision of **Dr. Divyang Bhimani**, Assistant Professor, **Department of Mathematics**, during the academic year 2024-2025.

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DECLARATION

I hereby declare that the matter embodied in the report entitled "**Schrödinger Operators and Applications to Evolution Equations**" are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research (IISER) Pune, under the supervision of **Dr. Divyang Bhimani**, and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

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Contents

ABSTRACT

This thesis explores the spectral theory of Schrödinger operators and their applications to evolution equations. The primary focus is on the spectral decomposition of unbounded self-adjoint operators on Hilbert spaces, with particular emphasis on the Laplacian and Schrödinger operators of the form $-\Delta + V$, where V is a potential function. The work begins with a detailed exposition of the spectral theorem for unbounded self-adjoint operators, utilizing tools from functional analysis, including projection-valued measures, the Cayley transform, and the theory of Banach C^*-algebras. The spectral theorem is then applied to Schrödinger operators, where the potential V is assumed to satisfy various growth conditions, such as $V(x) \to \infty$ as $|x| \to \infty$ or $V(x) \to 0$ as $|x| \to \infty$.

The thesis also investigates the essential self-adjointness of these operators and their spectral properties, including the discrete and essential spectra. Specific examples, such as the quantum harmonic oscillator and the Coulomb potential, are analyzed in detail, with explicit calculations of eigenvalues and eigenfunctions provided. The work further extends to the study of evolution equations, where the semigroup theory is employed to solve initial value problems associated with PDEs involving Schrödinger operators.

In the final chapters, the thesis delves into the theory of spectral multipliers, which are operators of the form $m(T)$, where m is a bounded measurable function and T is a self-adjoint operator. The boundedness of these multipliers on L^p spaces is established using techniques from Fourier analysis, including the Mikhlin-Hörmander multiplier theorem and the Calderón-Zygmund decomposition. The results are applied to Schrödinger operators, providing sufficient conditions for the L^p-boundedness of spectral multipliers associated with these operators.

Overall, this thesis bridges the gap between abstract spectral theory and concrete applications in PDEs, offering a comprehensive treatment of the spectral properties of Schrödinger operators and their implications for solving evolution equations. The results presented have potential applications in quantum mechanics, mathematical physics, and the analysis of PDEs.

Keywords:

Spectral Theorem, Schrödinger Operators, Unbounded Self-Adjoint Operators, Hilbert Space, Laplacian, Potential Function (V), Projection-Valued Measures, Cayley Transform, Banach C^*-Algebras, Essential Self-Adjointness, Discrete Spectrum, Essential Spectrum, Quantum Harmonic Oscillator, Coulomb Potential, Eigenvalues and Eigenfunctions, Evolution Equations, Semigroup Theory, Initial Value Problems (IVP), Partial Differential Equations (PDEs), Spectral Multipliers, Fourier Analysis, Mikhlin-Hörmander Multiplier Theorem, Calderón-Zygmund Decomposition, L^p-Boundedness, Quantum Mechanics, Mathematical Physics, Functional Analysis, Spectral Decomposition, Heat Equation, Wave Equation.

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Notations

Notation	Description
$C(X)$	Continuous functions on the space X onto a scalar field
$B(\Omega)$	Bounded functions on the space X onto a scalar field
$\mathcal{B}(H)$	Bounded linear functionals on a Hilbert space
$C_c^\infty(\mathbb{R}^N)$	Space of complex-valued smooth functions on $\mathbb{R}^N$ with compact support
$L^2(X)$	Space of square integrable complex-valued functions on X
$\ell^2(X)$	Space of square summable sequences with entries in $\mathbb{R}$ indexed by X
$H^s(\mathbb{R}^N)$	Sobolev space with index s with norm given by $\|f\|s^2 = \int \mathbb{R}^N(1+\lvert\xi\rvert^2)^s \hat{f}(\xi)\ d\xi$
$H_0^2(\mathbb{R}^N)$	Space that is the closure of $C_c^\infty(\mathbb{R}^N)$ with the H^2-norm
$S(\mathbb{R}^N)$	Schwartz space in $\mathbb{R}^N$
$S'(\mathbb{R}^N)$	Space of Tempered distributions in $\mathbb{R}^N$
$L^p(\mathbb{R}^N)$	Space of complex-valued functions f on $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N} \lvert f\rvert^p < \infty$

Chapter 1

Preliminaries

1.1 Introduction

Spectral decomposition has been of interest to mathematicians since time immemorial. In the case of a university student, the first time we come across this concept is in the undergraduate linear algebra course, where we look at spectral decomposition of linear operators in finite dimensional spaces. This thesis wishes to explore this notion, progressing through compact and self-adjoint and normal bounded operators, upto unbounded self-adjoint operators. We then continue to explore the different applications of this spectral theorem.

We will venture into spectral theory of operators from the next chapter onwards. This chapter is devoted to develop some preliminary concepts that will be used repeatedly throughout this thesis.

1.2 Banach Algebras: Multiplicative Linear Functionals and Maximal Ideals

Definition 1.2.1. A *Banach Algebra* over $\mathbb{C}$ is a Banach space $\mathcal{A}$ with the norm $\|.\|$ that satisfies:

$$x(yz) = (xy)z \tag{1.1}$$

$$(x+y)z = xz + yz; \quad x(y+z) = xy + xz \tag{1.2}$$

$$\alpha(xy) = (\alpha x)y = x(\alpha y) \tag{1.3}$$

$$\|xy\| \leq \|x\|\|y\| \tag{1.4}$$

for all $x, y, z \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

In addition, if we have $xy = yx$ for all $x, y \in \mathcal{A}$, then $\mathcal{A}$ is said to be a commutative Banach algebra, and if $\mathcal{A}$ has an element e such that $xe = ex = x$ and $\|e\| = 1$ for all $x \in \mathcal{A}$, then $\mathcal{A}$ is said to be a Banach algebra with unity. If there exists an element $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = x^{-1}x = e$ where e is the unit element of $\mathcal{A}$, then x is said to be invertible, and it is easy to see that such an inverse is unique for each invertible x.

Definition 1.2.2. A *multiplicative linear functional* on a Banach algebra $\mathcal{A}$ is a map $\phi : \mathcal{A} \to \mathbb{C}$ such that

$$\phi(xy) = \phi(x)\phi(y) \tag{1.5}$$

for all $x, y \in \mathcal{A}$.

In this text, we usually assume ϕ is not identically 0 for convenience.

The set of all multiplicative linear functionals on a Banach algebra $\mathcal{A}$ shall be denoted by

$\Sigma_{\mathcal{A}}$.

Proposition 1.2.3 (Properties of multiplicative linear functionals). *Suppose $\phi \in \Sigma_{\mathcal{A}}$, where $\mathcal{A}$ is a commutative Banach algebra with unit. Then*

(a) $\phi(e) = 1$,

(b) if $x \in \mathcal{A}$ is invertible, then $\phi(x) \neq 0$ for every $\phi \in \Sigma_{\mathcal{A}}$,

(c) if $\|x\| \leq 1$, then $e - x$ is invertible,

(d) $|\phi(x)| \leq \|x\|$ for every $x \in \mathcal{A}$, i.e., $\|\phi\| = 1$.

Proof. Let $\phi \in \Sigma_{\mathcal{A}}$. Consider $x \in \mathcal{A}$ such that $\phi(x) \neq 0$. Then (a) immediately follows from

$$\phi(x) = \phi(ex) = \phi(e)\phi(x),$$

implying that $\phi(e) = 1$.

Note that

$$1 = \phi(e) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1})$$

which proves (b), as neither one of $\phi(x)$ nor $\phi(x^{-1})$ is zero.

By (1.4), we have $\|x^n\| \leq \|x\|^n$, and if $\|x\| < |\lambda|$ for $\lambda \in \mathbb{C}\ \{0\}$, then for $s_n = e + \lambda^{-1}x + (\lambda^{-1})^2x^2 + ... + (\lambda^{-1})^n x^n \in \mathcal{A}$, note that for $n > m$,

$$\|s_n - s_m\| = \|(\lambda^{-1})^{m+1}x^{m+1} + ... + (\lambda^{-1})^n x^n\|$$

can be made arbitrarily small for large n, m since $\|x\| < |\lambda|$. This means that the sequence $\{s_n\}$ forms a Cauchy sequence in $\mathcal{A}$. Due to the completeness of $\mathcal{A}$, there exists an element $s \in \mathcal{A}$ such that $s_n \to s$. Since $(\lambda^{-1})^n x^n \to 0$ as $n \to \infty$,

$$s_n\ .\ (e - \lambda^{-1}x) = e - (\lambda^{-1})^{n+1}x^{n+1} = (e - \lambda^{-1}x)\ .\ s_n$$

implies that s is the inverse of $e - \lambda^{-1}x$. Hence (c) is proved by taking $\lambda = 1$.
From (c), we have that if $\|x\| < |\lambda|$ for $\lambda \in \mathbb{C}$, then $\lambda e - x$ is invertible and hence by (b), $\phi(\lambda e - x) \neq 0$, which implies that $\phi(x) \neq \lambda$, for all $x \in \mathcal{A}$, which in turn implies the required result. $\square$

Remark 1.2.4. From the previous proposition, we can conclude that $\Sigma_{\mathcal{A}} \subset \mathcal{A}^*$, where $\mathcal{A}^*$ denotes the set of all continuous linear functionals on $\mathcal{A}$.

Corollary 1.2.5. *$\Sigma_{\mathcal{A}}$ is closed with respect to the weak* topology the closed unit ball in $\mathcal{A}^*$, and hence is weak* compact.*

Proof. Recall that if $f_n \in \mathcal{A}^*$, then $f_n \to f$ in the weak* topology if and only if $f_n(x) \to f(x)$ for all $x \in \mathcal{A}$.
Assume $\phi_n \to \phi$ in the weak* topology on $\mathcal{A}^*$, for $\phi_n \in \Sigma_{\mathcal{A}}$. We need to show that ϕ is a multiplicative linear functional, and that will prove our statement. However, that is immediate, as

$$\phi(xy) = \lim_{n\to\infty} \phi_n(xy) = \lim_{n\to\infty} \phi_n(x)\phi_n(y) = \phi(x)\phi(y).$$

By Banach-Alaoglu theorem [16, Ch. 3, Thm. 3.15], we know that the unit ball in $\mathcal{A}^*$ is weak* compact. Since $\Sigma_{\mathcal{A}}$ is a closed subset of the unit ball (Proposition 1.2.3), we have that $\Sigma_{\mathcal{A}}$ is (weak*) compact. $\square$

Remark 1.2.6. $\Sigma_{\mathcal{A}}$ is a compact Hausdorff space, and the Hausdorffness follows from the Hausdorffness of the weak* topology on $\mathcal{A}^*$ [16, Ch. 3, 3.14].

Definition 1.2.7. Let $\mathcal{A}$ be a Banach algebra with identity. $I \subset \mathcal{A}$ is called a *sub-algebra* if I is a subspace and $xy \in I$ for every $x, y \in I$.

(a) A sub-algebra $I \subset \mathcal{A}$ is called an *ideal* if for every $x \in \mathcal{A}$ and $y \in I$, we have $xy \in I$.

(b) Such an ideal is called a *proper ideal* if $I \neq \mathcal{A}$.

(c) An ideal I is called *maximal* if for I' ideal such that $I \subset I'$, then $I' = \mathcal{A}$.

Remark 1.2.8. Note that for non-commutative Banach algebras, the notions of left, right and two-sided ideals have to be considered. However, in this text, we will be needing ideals of commutative algebras, hence the above definition.

Remark 1.2.9. Also note that if $I \subset \mathcal{A}$ is a proper ideal, then $e \notin I$. If $e \in I$, then for $x \notin I$, $xe = x \in I$ since I is an ideal. This implies that $I = \mathcal{A}$, contradicting that I is a proper ideal. This is therefore a defining property of a proper ideal, i.e., I is a proper ideal if and only if $e \notin I$.
If I is an ideal, then so is $\bar{I}$, the closure of I in $\mathcal{A}$. This is because if $x \in \partial I$, then there is a sequence $\{x_n\}$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Thus for any $a \in \mathcal{A}$, $\|ax_n - ax\| \leq \|a\|\|x_n - x\| \to 0$ as $n \to \infty$.

Proposition 1.2.10. *Let $\mathcal{A}$ be a commutative Banach algebra with unit, and let $I \subset \mathcal{A}$ be a proper ideal. Then*

(a) I contains no invertible element,

(b) $\bar{I}$ is a proper ideal,

(c) I is contained in some maximal ideal.

(d) Maximal ideals are closed.

Proof. Straightaway, if $x \in I$ is invertible, we have $x^{-1} \in \mathcal{A}$. This implies that $xx^{-1} = e \in I$ which is a contradiction because I is a proper ideal. That proves (a).
For (b), since non-invertible elements in $\mathcal{A}$ is a closed set, both I and $\bar{I}$ are contained in this set. This shows that $\bar{I}$ is a proper ideal.
Consider $\mathcal{P} := \{J \subset \mathcal{A}, J$ is a proper ideal of $\mathcal{A} : I \subset J\}$. Using inclusion as the partial ordering, if $A_1 \subset A_2 \subset ...$, for $A_i \in \mathcal{P}$, we see that $B = \cup_{j=1}^{\infty}$ is an upper bound by Zorn's

lemma [13, Ch. 1, Sec. 11]. We note that $e \notin B$, which implies that B is a maximal ideal. We have thus produced a maximal ideal that contains I, proving (c).

Finally, suppose M is a maximal ideal in $\mathcal{A}$. Since M has no invertible elements and since the set of all invertible elements is open, $\bar{M}$ contains no invertible elements either. This implies that $\bar{M}$ is a proper ideal that contains M, which contradicts the maximality of M. Thus M has to be closed. $\square$

Definition 1.2.11. For each $x \in \mathcal{A}$, the *spectrum* of x, $\sigma(x)$ is the set of all complex numbers λ such that $\lambda e - x$ is not invertible. The complement of $\sigma(x)$ is called the *resolvent* set of x, denoted by $\rho(x)$.

The *spectral radius* of x is defined as follows:

$$\eta(x) = \sup_{\lambda \in \sigma(x)} |\lambda|.$$

Theorem 1.2.12 (Gelfand-Mazur). *If $\mathcal{A}$ is a Banach algebra in which every non-zero element is invertible, then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let $\lambda_1 \neq \lambda_2$. Then for $x \in \mathcal{A}$, at most one of $\lambda_1 e - x$ and $\lambda_2 e - x$ is zero, and hence the other is invertible. From here, and since $\sigma(x)$ is non-empty [16, Ch. 10, 10.13], there exists exactly one point λ_x in $\sigma(x)$. It also follows that $\lambda_x e - x$ is not invertible, and hence is zero. Thus $x = \lambda_x e$, and hence the map $x \to \lambda_x$ is an isomorphism of $\mathcal{A}$ onto $\mathbb{C}$. Also note $|\lambda_x| = \|\lambda_x e\| = \|x\|$ makes the map an isometry. $\square$

Proposition 1.2.13. *Let $\mathcal{A}$ be a commutative Banach algebra with unit. The map $\phi \to \ker \phi$ is a one-to-one correspondence between $\sigma_{\mathcal{A}}$ and the set of maximal ideals of $\mathcal{A}$.*

Proof. Let us first show that $\ker \phi$ is a maximal ideal of $\mathcal{A}$, for $\phi \in \Sigma_{\mathcal{A}}$. If $x \in \ker \phi$, $a \in \mathcal{A}$, we have $\phi(ax) = \phi(a)\phi(x) = 0$, i.e., $ax \in \ker \phi$. This shows that $\ker \phi$ is an ideal, which is maximal because the quotient, $\mathcal{A}/\ker \phi$ is one-dimensional.

The correspondence is one-to-one, as if $\ker\phi_1 = \ker\phi_2$, then $\phi_1 = \alpha\phi_2$ for $\alpha \in \mathbb{C}$. But $\phi_1(e) = \phi_2(e) = 1$ implies that $\alpha = 1$.

We now need to show that any maximal ideal of $\mathcal{A}$ is of the form $\ker\phi$ for some $\phi \in \Sigma_{\mathcal{A}}$. We have showed that M is closed (Proposition 1.2.10), and thus we can quotient $\mathcal{A}$ by M. Consider $\pi : \mathcal{A} \to \mathcal{A}/M$, the quotient map, i.e., $a \to a + M$. $\mathcal{A}/M$ is also a Banach space (*Ref 1., Thm 1.41*). We claim that $\mathcal{A}/M$ is a Banach algebra with the following multiplication:

$$(a + M)(b + M) = ab + M.$$

This multiplication is well-defined because of the following. If $a + M = a' + M, \quad b + M = b' + M$, then $a - a', b - b' \in M$. Clearly, since M is an ideal, $m = (a + a')(b - b') = ab + ab' - a'b - a'b' \in M, \quad m' = (a - a')(b + b') = ab + ab' - a'b - a'b' \in M$. This inturn implies that $ab - a'b' = \frac{1}{2}(m + m') \in M \Rightarrow ab + M = a'b' + M$. The only part we have to check in Definition 1.2.1 is (1.4), since the others just follow from $\mathcal{A}$ being a Banach algebra. This again follows from $\mathcal{A}$ being a Banach algebra, since for any $m, m' \in M$, $\|(a + m)(b + m')\| \leq \|a + m\|\|b + m'\|$.

We now claim that $\mathcal{A}/M$ has no proper ideals. Suppose $I \subset \mathcal{A}/M$ is a proper ideal. Consider $\pi^{-1}(I) = \{a \in \mathcal{A}, \pi(a) \in I\}$. $\pi^{-1}(I)$ is a proper ideal of $\mathcal{A}$ as $e \notin \mathcal{A}$ and $M \subset \pi^{-1}(I)$, which contradicts that M is a maximal ideal. Since $\mathcal{A}/M$ has no proper ideal, any non-zero $x \in \mathcal{A}/M$ is invertible as otherwise, the ideal generated by this element is proper. We can thus apply Theorem (1.2.12), which gives us $\mathcal{A}/M \cong \mathbb{C}$. Consider the isomorphism $\phi : x \to \lambda_x$ for all $x \in \mathcal{A}/M$. Define $\psi : \mathcal{A} \to \mathbb{C}$ by $\psi = \phi \circ \pi$, which is multiplicative since ϕ is. It is easy to see that $\ker\psi = M$. We have thus produced $\psi \in \Sigma_{\mathcal{A}}$ such that $\ker\psi = M$. $\square$

Example 1.2.14. Let $\mathcal{A} = C(X)$, where X is compact Hausdorff. Fix $x_0 \in X$, and define the multiplicative linear functional $\phi_{x_0} : f \to f(x_0)$ on $C(X)$. We claim that any multiplicative linear functional is the evaluation at a point.

Let $M_{x_0} = \ker(\phi(x_0)) = \{f \in C(X), f(x_0) = 0\}$. This is a maximal ideal because of

Proposition 1.2.13.

Suppose $M \subset C(X)$ is a maximal ideal. We try to show that there exists $x_0 \in X$ such that $M = M_{x_0}$. Suppose there exists no $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in M$. Then, for all $x \in X$, there exists $f_x \in M$ such that $f_x(x) \neq 0$, which implies there exists a neighborhood U_x of $x \in X$ such that $f_x(t) \neq 0$ for all $t \in U_x$. The collection of all U_x covers X. X being compact and Hausdorff implies that there exists a finite collection $\{x_1, ..., x_n\}$ such that $X = \cup_{j=1}^{n} U_{x_j}$. Define:

$$g = \sum_{j=1}^{n} f_{x_j} \bar{f_{x_j}} \in M$$

as $f_{x_j} \in M$. Thus if $y \in X$, there exists j_0 such that $y \in U_{x_{j_0}}$ which implies $g(y) = \sum_{j=1}^{n} |f_{x_j}|^2 > 0$. Hence, we have g strictly positive, which implies $\frac{1}{g}$ is also strictly positive making $g(\frac{1}{g}) = e \in M$, which contradicts the fact that M is a maximal ideal. Thus we have $M = M_{x_0}$ for some $x_0 \in X$, which proves our claim.

1.3 The Gelfand Transform and C^*- algebras

Definition 1.3.1. Let $\mathcal{A}$ be a commutative Banach algebra with unity. For $x \in \mathcal{A}$, define $\hat{x} : \Sigma_{\mathcal{A}} \to \mathbb{C}$ by

$$\hat{x}(\phi) = \phi(x).$$

The map $x \to \hat{x}$ is called the **Gelfand transform**.

Note that $\hat{x}$ is a continuous function on $\Sigma_{\mathcal{A}}$, since if $\phi_n \in \Sigma_{\mathcal{A}}$ such that $\phi_n \to \phi$ in the weak* topology, then $\hat{x}(\phi_\alpha) = \phi_\alpha(x) \to \phi(x) = \hat{x}(\phi)$.

Example 1.3.2. Let $\mathcal{A} = C(X)$ where X is a compact Hausdorff space. Since all multiplicative linear functionals are of the form ϕ_{x_0} as in Example 1.2.14. Thus we can identify

$\Sigma_{\mathcal{A}}$ with X. For each $f \in \mathcal{A}$, $\hat{f} : \Sigma_{\mathcal{A}} \to \mathbb{C}$, $\hat{f}(\phi_{x_0}) = \phi_{x_0}(f) = f(x_0)$. Upon the identification, the Gelfand transform is the identity map between $\mathcal{A}$ and $C(\Sigma_{\mathcal{A}})$.

The below theorem lists some important properties of the Gelfand transform, however the proof is left unattended in this document. The proof can be found in [16, Ch. 11, 11.19].

Theorem 1.3.3. *Suppose $\mathcal{A}$ is a commutative Banach algebra with unit.*

(a) The Gelfand transform is an algebra homomorphism from $\mathcal{A}$ onto $C(\Sigma_{\mathcal{A}})$, $\hat{e} \equiv 1$ where $e \in \mathcal{A}$ is the identity,

(b) $x \in \mathcal{A}$ is invertible if and only if $\hat{x}$ never vanishes, i.e., $\hat{x}$ is invertible in $C(\Sigma_{\mathcal{A}})$,

(c) Range $\hat{x} = \sigma(x)$,

(d) $\|\hat{x}\|_\infty = \eta(x) \leq \|x\|$, where $\eta(x)$ is the spectral radius.

Example 1.3.4. Let us consider the same example as in Example 1.2.14 and 1.3.2. The equality in Theorem 1.3.3 is attained in this case as the Gelfand transform is the identity map, $\|f\|_\infty = \|f\|$, for $f \in C(X)$, and we have $\sigma(f) =$ Range $\hat{f}=$ Range f.

Definition 1.3.5. An *involution* is a mapping $x \to x^*$ of an algebra that satisfies:

(a) $(x + y)^* = x^* + y^*$,

(b) $(xy)^* = y^*x^*$,

(c) $(\lambda x)^* = \lambda x^*$,

(d) $x^{**} = x$.

An Banach algebra with an involution is called a Banach* algebra.

Definition 1.3.6. A commutative Banach* algebra is said to be *symmetric* if for all $x \in \mathcal{A}$

$$\hat{x^*} = \bar{\hat{x}}.$$

The Gelfand transform has an interesting property, and this property will help us establish the spectral theorem for bounded normal operators. Before that however, we require one final tool in the form of the proposition below.

Proposition 1.3.7. *Let $\mathcal{A}$ be a commutative Banach* algebra with unit.*

(a) $\mathcal{A}$ is symmetric if and only if for self-adjoint $x \in \mathcal{A}$, i.e. when $x = x^*$, $\hat{x}$ is real-valued.

(b) If $\mathcal{A}$ is symmetric, then the range of the Gelfand map is dense in $C(\Sigma_{\mathcal{A}})$.

Proof. Let us prove (a) first. Assume that $\mathcal{A}$ is symmetric, then $\hat{x^*} = \bar{\hat{x}}$ for all $x \in \mathcal{A}$. Hence for $x = x*$, we have $\hat{x} = \hat{x*} = \bar{\hat{x}}$, which makes $\hat{x}$ real-valued. Conversely assume that for self-adjoint x, $\hat{x}$ is real valued. Let $y \in \mathcal{A}$, and define $u = \frac{y+y^*}{2}$ and $v = \frac{y-y^*}{2i}$. Note that both u and v are self-adjoint, and $y = u + iv$. This makes $y^* = u - iv$ and by taking tha Gelfand transform, we get $\hat{y^*} = \hat{u} - i\hat{v} = \bar{\hat{y}}$ (since the Gelfand transform is an algebra homomorphism, Theorem 1.3.3), which makes $\mathcal{A}$ symmetric.

For (b), we try to show that $\mathcal{G} = \{\hat{x} : \ x \in \mathcal{A}\}$ separates points in $\Sigma_{\mathcal{A}}$. This will mean that we can use the Stone-Weierstrass theorem [17, Ch. 7, 7.32] and obtain that $\mathcal{G}$ is dense in $C(\Sigma_{\mathcal{A}})$. Note that $\mathcal{G}$ is a sub-algebra of $C(\Sigma_{\mathcal{A}})$ closed under conjugation as $\mathcal{A}$ is symmetric. Also note that $1 \in \mathcal{G}$. To show $\mathcal{G}$ separates points in $\mathcal{A}$, we know that if $\phi_1 \neq \phi_2$, the $\phi_1(x) \neq \phi_2(x)$ for some $x \in \mathcal{A}$, $\phi_i \in \Sigma_{\mathcal{A}},\ i = 1, 2$. This implies that $\hat{x}(\phi_1) \neq \hat{x}(\phi_2)$. This implies that $\mathcal{G}$ separates points in $\Sigma_{\mathcal{A}}$, and hence we have the result. □

Definition 1.3.8. A Banach* algebra is called a C^*-algebra if $\|x^*x\| = \|x\|^2$.

Note that in C^* algebras, we have $\|x^*\| = \|x\|$, since $\|x * x\| = \|x\|^2 \leq \|x\|\|x^*\| \Rightarrow \|x\| \leq \|x^*\|$, and the other side follows from $x^{**} = x$.

C^*-algebras also have another important property, as shown by the following proposition.

Proposition 1.3.9. *If $\mathcal{A}$ is a C^*-algebra (with unit), then $\mathcal{A}$ is symmetric.*

Proof. Let $x = x^* \in \mathcal{A}$, $\phi \in \Sigma_{\mathcal{A}}$ and suppose $\phi(x) = \hat{x}(\phi) = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$. For $t \in \mathbb{R}$, consider $z = x + ite \in \mathcal{A}$. Then, $\phi(z) = \phi(x) + it\phi(e) = \alpha + i\beta + it = \alpha + i(\beta + t)$.

Note $z^* = x - ite$, which makes $z^*z = x^2 + t^2 e$. Hence

$$\alpha^2 + (\beta + t)^2 = |\phi(z)| \leq \|z\|^2 = \|z^*z\| \leq \|x^2\| + t^2 = \|x\|^2 + t^2,$$

where the inequality is from Proposition 1.2.3. This immediately implies that $\alpha^2 + \beta^2 + 2\beta t \leq \|x\|^2$ for all $t \in \mathbb{R}$, which forces $\beta = 0$. Thus we have that $\hat{x}$ is real-valued for self-adjoint x, which gives us that $\mathcal{A}$ is symmetric by Proposition 1.3.7. $\square$

Theorem 1.3.10 (Gelfand-Naimark). *Suppose $\mathcal{A}$ is a commutative C^*-algebra with unit, and maximal ideal space/ space of multiplicative linear functionals $\Sigma_{\mathcal{A}}$. Then the Gelfand transform is an isometric $^*-$isomorphism of $\mathcal{A}$ onto $C(\Sigma_{\mathcal{A}})$.*

Proof. It suffices to show that the Gelfand transform is an isometry. This will in turn prove that the range of the Gelfand transform is closed in $C(\Sigma_{\mathcal{A}})$, which will conclude the proof by Proposition 1.3.7, (b).

Fix $x \in \mathcal{A}$ and define $y = x^*x$. Then $y^* = x^*x = y$, and hence it is self-adjoint. For some $k \in \mathbb{N}$, $\|y^{2^k}\| = \|(y^{2^{k-1}})^* y^{2^{k-1}}\| = \|y^{2^{k-1}}\|^2$. Thus by induction, we get $\|y^{2^k}\| = \|y\|^{2^k}$.

From Theorem 1.3.3 we have that $\|\hat{y}\|_\infty = \eta(y)$. This gives us

$$\|\hat{y}\|_\infty = \eta(y) = \lim_{k\to\infty} \|y^{2^k}\|^{\frac{1}{2^k}} = \lim_{k\to\infty} \|y\| = \|y\|.$$

Thus for self-adjoint y, we have $\|\hat{y}\|_\infty = \|y\| = \|x^*x\| = \|x\|^2$.
Now, $y = x * x$ implies $\hat{y} = \bar{\hat{x}}\hat{x} = |\hat{x}|^2$. We thus have,

$$\|x\|^2 = \||\hat{x}|^2\| = \|\hat{x}\|_\infty^2,$$

giving us $\|x\| = \|\hat{x}\|_\infty$, proving that the Gelfand transform is an isometry. □

Suppose $\mathcal{B} \subseteq \mathcal{A}$ be commutative Banach algebras with unit, with $\mathcal{B}$ being a sub-algebra. For $x \in \mathcal{B}$, there exists two spectra:

$$\sigma_{\mathcal{B}}(x) = \{\lambda \in \mathbb{C} : \quad \lambda e - x \text{ is not invertible in } \mathcal{B}\},$$

$$\sigma_{\mathcal{A}}(x) = \{\lambda \in \mathbb{C} : \quad \lambda e - x \text{ is not invertible in } \mathcal{A}\},$$

and clearly, $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x)$.

The following proposition will apply rather beautifully to our treatment of bounded normal operators.

Proposition 1.3.11. *Suppose $\mathcal{A}$ is a commutative Banach algebra with unit and $\mathcal{B} \subseteq \mathcal{A}$ is a closed sub-algebra with unit. If $x \in \mathcal{B}$ and $\sigma_{\mathcal{B}}(x)$ is nowhere dense in $\mathbb{C}$, then $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$.*

Proof. Let $\lambda_0 \in \sigma_{\mathcal{B}}(x)$. Because of nowhere denseness, there exists a sequence $\lambda_n \in \mathbb{C} \setminus \sigma_{\mathcal{B}}(x)$ such that $\lambda_n \to \lambda_0$ in $\mathbb{C}$. Note that each $\lambda_n e - x$ is invertible in $\mathcal{B}$ and $\lambda_n e - x \to \lambda_0 e - x$. We claim that $\|(\lambda_n e - x)^{-1}\| \to \infty$ as $n \to \infty$. If not, by passing to a subsequence, assume $\|(\lambda_n e - x)^{-1}\| \leq c < \infty$ for all n, without loss of generality.
But then, for large n, we have

$$\|(\lambda_n e - x) - (\lambda_0 e - x)\| \leq \|\lambda_n - \lambda_0\| \leq c^{-1} \leq \|(\lambda_n e - x)^{-1}\|^{-1},$$

which implies that $\lambda_0 e - x$ is invertible in $\mathcal{B}$, which is a contradiction. Thus our claim is right.

This guarantees that $\lambda_0 e - x$ cannot be invertible in $\mathcal{A}$ either, because otherwise, $(\lambda_n e - x)^{-1} \to (\lambda_0 e - x)^{-1}$ by continuity. Thus, $\lambda_0 \in \mathcal{A}$. $\square$

Corollary 1.3.12. *Suppose $\mathcal{A}$ is a C^*-algebra with unit and $\mathcal{B} \subseteq \mathcal{A}$ be a sub-algebra with unit.*

(a) If $x \in \mathcal{B}$ and x is invertible in $\mathcal{A}$, then x is invertible in $\mathcal{B}$, and

(b) $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$.

Proof. Let $x \in \mathcal{B}$ and set $y = x^*x$. Let $\mathcal{C}$ be the C^*-sub-algebra generated by y and e. Then clearly, $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$.

Now if x is invertible in $\mathcal{A}$, then y is also invertible in $\mathcal{A}$. Since $\mathcal{C}$ is a commutative C^*-algebra with identity, $\sigma_{\mathcal{C}}(y) \subset \mathbb{R}$, which is nowhere dense in $\mathbb{C}$. Hence, by the previous proposition, we have $\sigma_{\mathcal{C}}(y) = \sigma_{\mathcal{A}}(y)$. This implies that $0 \notin \mathcal{C}$, which implies that y is invertible in $\mathcal{C} \subseteq \mathcal{B}$. Thus, $x^{-1} = y^{-1}x^* \in \mathcal{B}$ as $y^{-1} \in \mathcal{B}$. $\square$

This completes the chapter of required preliminaries to step into the main content of the thesis, which starts in the next chapter.

Chapter 2

Spectral Theorem for Unbounded Self-Adjoint Operators on a Hilbert Space

2.1 Introduction

The operators we consider in this thesis are linear, unless otherwise explicitly mentioned. Let us recall some definitions to start, and get them out of the way.

Definition 2.1.1. $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if $T - \lambda I$ is not injective.

Definition 2.1.2. A vector $0 \neq f \in \mathcal{H}$ is called an *eigenvector* of T associated with λ if $Tf = \lambda f$.

In the following definitions, the notion of adjoint is the usual one, for we consider at the most, bounded operators [16, Ch. 4, Thm. 4.10].

Definition 2.1.3. Let H be a Hilbert space. An operator T on H is *symmetric* if we have for $x, y \in \mathscr{D}(T)$,

$$\langle Tx, y\rangle = \langle x, Ty\rangle.$$

Clearly, $T \subset T^*$ whenever T is symmetric. If $T = T^*$, then T is self-adjoint. When H is finite-dimensional, the notions of symmetric and self-adjoint coincide. Lets look at a case where this breaks down when we are no longer in finite-dimensional Hilbert spaces.

Example 2.1.4. Consider the bounded linear operator on summable sequences with entries in $\mathbb{R}$ indexed by $\mathbb{Z}$, $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by:

$$(Tx)_j = x_{j+1} - x_j.$$

This operator is symmetric, because

$$\begin{aligned}\langle Tx, y\rangle &= \sum_{j\in\mathbb{Z}}(x_{j+1} - x_j)y_j \\ &= \langle x, Ty\rangle.\end{aligned}$$

However, computing the unique operator T^* such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, yields

$$(T^*y)_j = y_j - y_{j-1},$$

which for sure is not the same operator as T. Therefore the notions of symmetric and self-adjoint are distinct in the sense a self-adjoint operator is always symmetric, but a symmetric operator need not be self-adjoint.

Definition 2.1.5. An operator $T : H \to H$ is said to be *normal* if $TT^* = T^*T$. A normal operator is *unitary* if $TT^* = T^*T = I$.

Example 2.1.6. Consider $T : L^2([-\pi, \pi]) \to L^2([-\pi, \pi])$ given by

$$(Tf)(\varphi) = e^{i\varphi} f(\varphi).$$

The spectrum of this operator is the unit circle in the complex plane, $\sigma(T) = \mathbb{T}$ and $T^*T = TT^* = I$.

Definition 2.1.7. $T : H \to H$ is compact if for any bounded sequence $\{x_n\} \subset H$, the sequence $\{Tx_n\}$ has a convergent subsequence.

Let's state the finite dimensional version of the spectral theorem we all know well.

Theorem 2.1.8. *[1, Ch. 7, Thm. 7.29] Suppose $\mathbb{F} = \mathbb{R}/\mathbb{C}$ and T is a linear operator on $\mathbb{F}^N$. Then the following are equivalent:*

(a) T is normal. Moreover, T is self-adjoint if $\mathbb{F} = \mathbb{R}$.

(b) $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T.

We also note that in the finite-dimensional case,

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue}\}$$

since $T - \lambda I$ is not injective iff $T - \lambda I$ is not invertible. This is not transferrable to the infinite-dimensional case. Let us look at an example.

Example 2.1.9. Consider the right shift operator $T : \ell^2 \to \ell^2$ given by

$$T(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

Note, however T has no eigenvalues because:

$$(T - \lambda I)(a_1, a_2, \dots) = (0, a_1, a_2, \dots) - \lambda(a_1, a_2, \dots),$$

which shows that $T - \lambda I$ is injective. But the spectrum of T is:

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\},$$

which is an easy consequence of the fact that the above set consists of eigenvalues of the left shift operator, which is the adjoint of the right shift operator.

So we need a new theory for infinite-dimensional Hilbert spaces. Let us consider the special case of compact self-adjoint operators on a Hilbert space, H. Recall the spectral theorem in this case [2, Ch. 10, 10.106]:

Theorem 2.1.10. *Suppose T is a compact self-adjoint operator on $\mathcal{H}$. Then:*

(a) There exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of T.

(b) There exists a countable set Ω and an orthonormal family $\{e_k\}_{k\in\Omega}$ in $\mathcal{H}$, $\{\lambda_k\}_{k\in\Omega}$ in $\mathbb{R} \setminus \{0\}$, such that for every $f \in \mathcal{H}$,

$$Tf = \sum_{k\in\Omega} \lambda_k \langle f, e_k\rangle e_k.$$

If $\mathcal{H}$ is an $\mathbb{R}$ Hilbert space, then a compact operator T is self-adjoint if and only if (a) is true. This is however not true for $\mathbb{C}-$Hilbert spaces.

The class of compact self-adjoint operators is a restrictive one, even within the realm of bounded linear operators on a Hilbert space H, and hence it is natural to ask to extend the spectral theorem to more general settings. In this chapter, we will aim to prove the spectral theorem for unbounded self-adjoint operators by first using the theory of Banach C^*-Algebras to establish the spectral theorem for bounded normal operators, then establish the theory of the Cayley Transform, and finally apply the spectral theorem for bounded normal operators onto the Cayley Transform of an unbounded self-adjoint operator.

2.2 Projection-valued measures

Definition 2.2.1. Let Ω be a set and $\mathscr{F}$ be a $\sigma-$algebra of subsets of Ω, and let $(H, \|\cdot\|)$ be a Hilbert space. A *projection-valued measure* on $(\Omega, \mathscr{F})$ is a map $P : \mathscr{F} \to \mathcal{B}(H)$ such that:

(a) If $E \in \mathscr{F}$, then $P(E) = P(E)^* = P(E)^2$, i.e., each $P(E)$ is an orthogonal projection,

(b) $P(\emptyset) = 0$, $P(\Omega) = I$, the identity map,

(c) $P(E \cap F) = P(E)P(F)$,

(d) $\cup_{j=1}^{\infty} E_j = E$ such that E_js are disjoint, then $P(E) = \sum_{j=1}^{\infty} P(E_j)$ in the strong operator topology.

From here, we get that for each $u, v \in H$,

$$P_{u,v}(E) = \langle P(E)u, v\rangle$$

is a complex measure on Ω. All the properties are straightforward from the definition, except the countable additivity. To check this, suppose $\{E_j\}_{j=1}^{\infty}$ are disjoint. Then, $\langle P(\cup_{j=1}^{N} E_j)u, v\rangle = \sum_{j=1}^{N} \langle P(E_j)u, v\rangle \to \sum_{j=1}^{\infty} \langle P(E_j)u, v\rangle$ which is an absolute convergence due to the operator norm convergence of $\sum_{j=1}^{\infty} P(E_j)$. Also note that $P_{u,u}$ is a positive finite measure and $\|P_{u,u}\| = P_{u,u}(\Omega) = \langle P(\Omega)u, u\rangle = \|u\|^2$. Here is an example.

Example 2.2.2. Let $\mathcal{H} = L^2([0,1])$ and let $\mathscr{F}$ be the Borel σ-algebra. For $E \in \mathscr{F}$, define

$$P(E)f = \chi_E f$$

where χ_E is the characteristic function on E. By the definition of the characteristic function, all the properties of projection-valued measures are straightforward for this particular example.

Next, we try to define integration with respect to this measure. Let $B(\Omega)$ be the space of bounded measurable functions on Ω. If $f \in B(\Omega)$, $u \in H$, consider

$$\left|\int_\Omega f \quad dP_{u,u}\right| \leq \|f\|_\infty P_{u,u}(\Omega) = \|f\|_\infty \|u\|^2.$$

By polarizing, we can write $P_{u,v}$ for $u, v \in H$ in terms of $P_{u+v,u+v},\ P_{u-v,u-v},\ P_{u+iv,u+iv},\ P_{u-iv,u-iv}$ and obtain:

$$\left|\int_\Omega f \quad dP_{u,v}\right| \leq \frac{1}{4}\|f\|_\infty (\|u+v\|^2 + \|u-v\|^2 + \|u+iv\|^2 + \|u-iv\|^2).$$

Thus, if $\|u\| = \|v\| = 1$,

$$\left|\int_\Omega f \quad dP_{u,v}\right| \leq 4\|f\|_\infty,$$

and in general,

$$\left|\int_\Omega f \quad dP_{u,v}\right| \leq 4\|f\|_\infty \|u\|\|v\|,$$

making $(u, v) \to \int_\Omega f \quad dP_{u,v}$ a continuous sesquilinear form and by Riesz Representation theorem [2, Ch. 8, Sec. 8C], there exists a unique bounded operator, T_f, such that

$$\int_\Omega f \quad dP_{u,v} = \langle T_f u, v\rangle.$$

We will denote T_f by $\int_\Omega f \quad dP$.

The linearity of this operator is clear, by the definition. How does this operator look though? To see this, consider $g = \sum_{j=1}^N c_j \chi_{E_j}$, a simple function. Then

$$\int_\Omega g \quad dP_{u,v} = \int_\Omega (\sum_{j=1}^N c_j \chi_{E_j}) \quad dP_{u,v} = \sum_{j=1}^N \int_\Omega c_j \chi_{E_j} \quad dP_{u,v} = \left\langle \sum_{j=1}^N c_j P(E_j) u, v \right\rangle,$$

making $\int_\Omega g \quad dP = \sum_{j=1}^N c_j P(E_j)$. If $f \in B(\Omega)$, then there exists a sequence of simple functions $\{f_n\}$ such that $\|f_n - f\|_\infty \to 0$. Hence we have,

$$\left\| \int_\Omega f \quad dP - \int_\Omega f_n \quad dP \right\| = \left\| \int_\Omega (f - f_n) \quad dP \right\| \leq 4\|f - f_n\|_\infty \to 0,$$

where the first expression is in the operator norm.

Theorem 2.2.3. *If P is a projection-valued measure on $(\Omega, \mathscr{F})$, then the map $f \to \int_\Omega f \quad dP$ is a *-homomorphism from $B(\Omega)$ onto $\mathcal{B}(H)$.*

Proof. Let us work with simple functions, and prove the $*-$homomorphism property. Suppose f and g are simple functions:

$$f = \sum_{j=1}^{n} c_j \chi_{E_j}; \qquad g = \sum_{k=1}^{m} d_k \chi_{F_k},$$

for $E_j, \ F_k \in \mathscr{F}$.

Then, $fg = \sum_{j=1}^{n} \sum_{k=1}^{m} c_j d_k \chi_{E_j \cap F_k}$, and

$$\begin{aligned} \int_\Omega fg \quad dP &= \sum_{j=1}^{n} \sum_{k=1}^{m} c_j d_k \int_\Omega \chi_{E_j \cap F_k} \quad dP \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} c_j d_k P(E_j \cap F_k) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} c_j d_k P(E_j) P(F_k) \\ &= \sum_{j=1}^{n} c_j P(E_j) \sum_{k=1}^{m} d_k P(E_k) \\ &= \int_\Omega f \quad dP \int_\Omega g \quad dP, \end{aligned}$$

and that shows that the map $f \to \int_\Omega f \quad dP$ is a homomorphism.

Now if $f, g \in B(\Omega)$, then there exist sequences $\{f_n\}, \ \{g_n\}$ of simple functions such that

$f_n \to f$, $g_n \to g$ in the L^∞ norm. This does also mean that $f_n g_n \to fg$ in the L^∞ norm, which implies

$$\left\| \int_\Omega f_n g_n \quad dP - \int_\Omega fg \quad dP \right\| = \left\| \int_\Omega (f_n g_n - fg) \quad dP \right\| \leq 4\|f_n g_n - fg\|_\infty \to 0.$$

Since f_n, g_n are simple, we have

$$\int_\Omega f_n g_n \quad dP = \int_\Omega f_n \quad dP \int_\Omega g_n \quad dP. \tag{2.1}$$

An observation here will help us conclude the result: if $\|T_n - T\| \to 0$, $\|S_n - S\| \to 0$, then $T_n S_n \to TS$ in the operator norm. This follows straight from the triangle inequality: $\|T_n S_n - TS\| = \|T_n S_n - T_n S + T_n S - TS\| \leq \|T_n\|\|S_n - S\| + \|T_n - T\|\|S\| \to 0$. This concludes our proof of the homomorphism as we just take $\lim_{n\to\infty}$ on both sides in (2.1).
We now aim to show that $(\int_\Omega f \quad dP)^* = \int_\Omega \bar{f} \quad dP$. Again, consider a simple function $f = \sum_{j=1}^n c_j \chi_{E_j}$, which makes $\bar{f} = \sum_{j=1}^n \bar{c_j} \chi_{E_j}$.
Now, $\int_\Omega f \ dP = \sum_{j=1}^n c_j P(E_j)$, which makes $(\int_\Omega f \ dP)^* = \sum_{j=1}^n \bar{c_j} P(E_j)^* = \sum_{j=1}^n \bar{c_j} P(E_j) = \int_\Omega \bar{f} \quad dP$, proving the assertion for simple functions. The approximation of bounded measurable functions by simple functions gives us the result. □

We now set $\Omega = \Sigma_{\mathcal{A}}$, and write down the first version of the spectral theorem that we will prove.

2.3 Spectral Theorem for Bounded Normal Operators

Theorem 2.3.1 (**Spectral Theorem I**). *Let $\mathcal{A}$ be a commutative C^*-sub-algebra of $\mathcal{B}(H)$ with identity. Let Σ be the set of multiplicative linear functionals on $\mathcal{A}$ (or) the* *maximal ideal space, and let $T \to \hat{T}$ be the Gelfand Transform. Then there exists a unique regular projection-valued measure P on Σ such that*

$$T = \int_{\Sigma} \hat{T} \quad dP,$$

for all $T \in \mathcal{A}$. Moreover, there exists $T_f \in \mathcal{B}(H)$ such that for all $f \in B(\Sigma)$, such that

$$T_f = \int_{\Sigma} f \quad dP.$$

Furthermore, if $S \in \mathcal{B}(H)$ then the following are equivalent:

(a) S commutes with every operator $T \in \mathcal{A}$,

(b) S commutes with every $P(E)$ for all $E \in \mathscr{F}$, where $\mathscr{F}$ is the Borel sigma-algebra on Σ,

(c) S commutes with every $\int_{\Sigma} f \quad dP$ for all $f \in B(\Sigma)$.

Proof. The proof of the first two statements are simple consequences of Theorems 1.3.10 and 2.2.3. This means that we just have to prove the equivalence statements. (c) implies (b) follows from letting $f = \chi_E$ for $E \in \mathscr{F}$. (c) implies (a) follows from Theorems 1.3.10 and 2.2.3, by the identification of $\mathcal{A}$ and $C(\Sigma)$ and the extension of the representation of T_f as $\int_{\Sigma} f \quad dP$ to all bounded measurable functions on Σ. The operator $\int_{\Sigma} f \quad dP = T_f$ is bounded for all $f \in B(\Sigma)$, and hence (a) implies (c). (b) implies (c) can be concluded by approximation with simple functions. $\square$

Let us now restrict ourselves to a single operator.

Theorem 2.3.2 (**Spectral Theorem for Bounded Normal Operators**). *Let $T \in \mathcal{B}(H)$ be a normal operator. Then there exists a unique projection-valued measure P on* *the Borel subsets of $\sigma(T)$ satisfying*

$$T = \int_{\sigma(T)} \lambda \;\; dP(\lambda). \tag{2.2}$$

Proof. Let $\mathcal{A}$ be the commutative C^*-algebra with identity generated by T, T^* and I. Then $\Sigma_{\mathcal{A}}$ can be identified with $\sigma(T) \subset \mathbb{C}$. This is because of theorem 1.3.3 and the Gelfand-Naimark Theorem (1.3.10), for we have $\phi \in \Sigma_{\mathcal{A}}$, $\phi(T) = \hat{T}(\phi) = \lambda \in \sigma(T)$. Since $\Sigma_{\mathcal{A}}$ is compact Hausdorff, by Example 1.3.2, we have that the Gelfand Transform is the identity map after the identification, i.e $\hat{T}(\lambda) = \lambda$. For clarity, the identification from the Gelfand-Naimark theorem is as follows:

$$\begin{array}{c|c} \mathcal{A} & C(\Sigma_{\mathcal{A}}) \\ T & \lambda \to \lambda \\ T^2 & \lambda \to \lambda^2 \\ T^* & \lambda \to \bar{\lambda} \\ p(T, T^*) & \lambda \to p(\lambda, \bar{\lambda}) \end{array}$$

And the right column becomes the whole of all continuous functions on $C(\Sigma_{\mathcal{A}})$ by the Stone-Weierstrass theorem.

We now have the existence of the unique projection-valued measure by Theorem 2.3.1, call it P, such that

$$T_f = \int_{\sigma(T)} f \;\; dP = \int_{\sigma(T)} f(\lambda) \;\; dP(\lambda),$$

for all $f \in C(\Sigma_{\mathcal{A}})$. In particular, $T = \int_{\sigma(T)} \lambda \;\; dP(\lambda)$. This proves our theorem. □

We finally prove one small but useful result, before we venture into the realm of unbounded operators.

Theorem 2.3.3. *A normal operator $T \in \mathcal{B}(H)$ is* *(a) self-adjoint if and only if* $\sigma(T) \subseteq \mathbb{R}$,

(b) unitary, i.e. $T^*T = I = TT^*$, *if and only if* $\sigma(T)$ *lies on the unit circle in the complex plane.*

Proof. Let $\mathcal{A}$ be the smallest C^*-sub-algebra of $\mathcal{B}(H)$ generated by T, T^* and I. Then $\hat{T}(\lambda) = \lambda$ (Theorem 2.3.2) and $(T^*)\hat{}\,(\lambda) = \bar{\lambda}$. Due to the correspondence between T and $\hat{T}$, we have that $T = T^*$ if and only if $\lambda = \bar{\lambda}$ and $T^*T = I = TT^*$ if and only if $\lambda\bar{\lambda} = 1$, i.e., $|\lambda| = 1$ for $\lambda \in \sigma(T)$. $\square$

2.4 The Cayley Transform

The operators (by default linear) in this section are necessarily not bounded, and hence the theories we know for bounded operators need not necessarily apply to this setting. One particular definition we need before we step into the theory of unbounded operators is stated below.

Definition 2.4.1. A linear operator $T : H \to H$ is said to be *densely defined* if its domain $\mathscr{D}(T)$ is dense in H.

This naturally means that we can think of extensions of such an operator T into larger domains than $\mathscr{D}(T)$ (even when T is not densely defined). We use the notation $T \subset S$ if S is an extension of T, i.e. $\mathscr{D}(T) \subset \mathscr{D}(S)$ and $Sx = Tx$ for all $x \in \mathscr{D}(T)$.

Remark 2.4.2. Adjoints: If we want to associate an adjoint T^* to an operator T, we want its domain $\mathscr{D}(T^*)$ to consist of all those $y \in H$ such that the linear functional,

$$x \to \langle Tx, y\rangle$$

is continuous on $\mathscr{D}(T)$. We now try to apply the Hahn-Banach extension theorem [10, Ch. 3, Thm. 3.1.2] and for $y \in \mathscr{D}(T^*)$ we can extend the above functional to a continuous linear functional on H, i.e. there exists $T^*y \in H$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in \mathscr{D}(T)$. T^*y is uniquely determined if and only if $\mathscr{D}(T)$ is dense in H. It is seen that T^* is linear as if $z \in \mathscr{D}(T)$ and $x, y \in \mathscr{D}(T^*)$,

$$\langle z, T^*x + T^*y\rangle = \langle z, T^*x\rangle + \langle z, T^*y\rangle = \langle Tz, x\rangle + \langle Tz, y\rangle = \langle Tz, x + y\rangle = \langle z, T^*(x + y)\rangle.$$

The verification of scalar multiplication is much easier, as

$$\langle z, T^*(\alpha x)\rangle = \langle Tz, \alpha x\rangle = \bar{\alpha}\langle Tz, x\rangle = \langle T(\bar{\alpha}z), x\rangle = \langle \bar{\alpha}z, T^*x\rangle = \langle z, \alpha T^*x\rangle,$$

for $\alpha \in \mathbb{C}$. This shows that $\mathscr{D}(T^*)$ is a subspace of H.

Definition 2.4.3. A symmetric operator T is called *maximally symmetric* if there exists no proper symmetric extension of T, i.e if $T \subset S$ and S is symmetric, then $S = T$.

Theorem 2.4.4. *Let T be a symmetric operator in H not necessarily densely defined. Then,*

(a) $\|Tx + ix\|^2 = \|x\|^2 + \|Tx\|^2$,

(b) T is a closed operator if and only if $\mathscr{R}(T + iI)$ is closed,

(c) $T + iT$ is one-one,

(d) If $\mathscr{R}(T + iI) = H$, then T is maximally symmetric,

(e) (a), (b), (c) and (d) are true if i is replaced with $-i$.

Proof. The proof of (a) is quite straightforward:

$$\begin{aligned}\|Tx+ix\|^2 &= \|Tx\|^2+\|x\|^2+\langle Tx, ix\rangle+\langle ix, Tx\rangle \\ &= \|Tx\|^2+\|x\|^2-i\langle Tx, x\rangle+i\langle Tx, x\rangle \\ &= \|Tx\|^2+\|x\|^2.\end{aligned}$$

For (b), consider the map $\phi : \mathscr{R}(T+iI) \to \mathscr{G}(T)$ given by $(T+iI)x \to [x, Tx]$ (ordered pair), where the graph of T, $\mathscr{G}(T) = \{[x, Tx] : \; x \in \mathscr{D}(T)\}$. $\mathscr{G}(T)$ is a subspace of $H \times H$ where the norm is defined as follows: for $[a, b] \in H \times H$, define $\|[a, b]\|^2 = \|a\|^2 + \|b\|^2$. Clearly, by (a), ϕ is an isometry. If we show that ϕ is a one-to-one correspondence we're done. Note that for $x \in \mathscr{D}(T) = \mathscr{D}(T+iI)$, we have $[x, Tx] \in \mathscr{G}(T)$ and that implies $(T+iI)x$ exists, proving (b).

(c) follows readily from (a) because when $(T+iI)x = 0$, we have $\|Tx+ix\|^2 = 0$ implying $\|x\|^2 + \|Tx\|^2 = 0$ which forces $x = 0$.

To show that T is maximally symmetric, we show this by contradiction. Suppose $\mathscr{R}(T+iI) = H$ and S is a proper symmetric extension of T, then $S+iI$ is a proper extension of $T+iI$. For any symmetric operator T, if $\mathscr{R}(T)$ is dense in H, for $Tx = 0$, we have $0 = \langle Tx, y\rangle = \langle x, Ty\rangle$, which makes $x \perp \mathscr{R}(T)$ and $x = 0$, making T one-one. Thus in our situation, we have that $S+iI$ is one-one. This makes $T+iI$ not one-one too, and not symmetric. Hence, $T+iI$ has to be maximally symmetric.

Since (a) is valid if $i \to -i$, all other statements are valid too. $\square$

Definition 2.4.5. The *Cayley transform* of an operator T on H is defined as follows:

$$U = (T-iI)(T+iI)^{-1}.$$

Let us compute the Cayley transform for a bounded linear operator, to illustrate the power of Theorem 2.4.8, which comes later.

Example 2.4.6. For $k \in \mathbb{N}$, define $T_k : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by:

$$T_k(x_1, x_2, \dots) = (0, 0, \dots, kx_k, 0, \dots).$$

Note that T_k is a self-adjoint bounded operator. We then compute

$$(T_k + iI)^{-1}(x_1, x_2, \dots) = \left(\frac{1}{i}x_1, \frac{1}{i}x_2, \dots, \frac{1}{i+k}x_k, \frac{1}{i}x_{k+1}, \dots\right).$$

Then, the Cayley transform of T is given by $U : \mathscr{R}(T_k + iI) \to \ell^2(\mathbb{N})$, a unitary operator, is given by:

$$U(x_1, x_2, \dots) = \left(x_1, x_2, \dots, \frac{k-i}{k+i}x_k, x_{k+1}, \dots\right).$$

Theorem 2.4.7. *Suppose that U is an operator in H and an isometry, i.e. $\|Ux\| = \|x\|$ for $x \in \mathscr{D}(U)$.*

(a) If $x \in \mathscr{D}(U)$, $y \in \mathscr{D}(U)$ then $\langle Ux, Uy\rangle = \langle x, y\rangle$,

(b) If $\mathscr{R}(U)$ is dense in H, then $I - U$ is one-one,

(c) If any one of $\mathscr{D}(U)$, $\mathscr{R}(U)$ or $\mathscr{G}(U)$ is closed, so are the other two.

Proof. (a) immediately follows from [16, Ch. 12, Ex. 2], which says that any Hilbert space inner product satisfies

$$\langle x, y\rangle = \frac{1}{N}\sum_{n=1}^{N} \|x + \alpha^n y\|^2 \alpha^n,$$

for $\alpha \in \mathbb{C}$, $\alpha^n = 1$ and $\alpha^2 \neq 1$. Therefore, using the fact that U is linear and an isometry, we conclude.

For (b), assume there exists $x \in H$ such that $(I - U)x = 0$. If we show $x = 0$, we are done.

For $z = (I - U)y \in \mathscr{R}(I - U)$, we have

$$\begin{aligned}\langle x, (I - U)y \rangle &= \langle x, y \rangle - \langle x, Uy \rangle \\ &= \langle Ux, Uy \rangle - \langle x, Uy \rangle \\ &= \langle Ux - x, Uy \rangle = 0,\end{aligned}$$

which gives us that $x \in [\mathscr{R}(I - U)]^{\perp}$ forcing x to be 0.

We have that

$$\begin{aligned}\left[\frac{1}{\sqrt{2}}\|[x, Ux] - [y, Uy]\|\right]^2 &= \frac{1}{2}\|[x - y, Ux - Uy]\|^2 \\ &= \frac{1}{2}\left(\|x - y\|^2 + \|U(x - y)\|^2\right) \\ &= \|x - y\|^2 \\ &= \|U(x - y)\|^2,\end{aligned}$$

establishing a one-to-one correspondence between the three spaces, giving us (c). $\square$

This next theorem carries the major result of the section, i.e. the Cayley transform of a self-adjoint operator is unitary. This will help us establish the spectral theorem for unbounded self-adjoint operators in the next section. Before the proof, however, the following fact is imperative: for a self-adjoint closed densely defined operator, $\mathscr{R}(I + T^2) = H$. This fact is a ready conclusion from [16, Ch. 13, 13.13].

Theorem 2.4.8. *Suppose U is the Cayley transform of a symmetric operator T in H. Then*

(a) U is closed if and only if T is closed,

(b) $\mathscr{R}(I - U) = \mathscr{D}(T)$, $I - U$ is one-one and T can be reconstructed from U by the

formula:

$$T = i(I + U)(I - U)^{-1},$$

(c) U is unitary if and only if T is self-adjoint.

Conversely, if V is an operator on H that is an isometry and $I - V$ is one-one, then V is the Cayley transform of a symmetric operator in H.

Proof. We have that $U = (T - iI)(T + iI)^{-1}$, making $\mathscr{D}(U) = \mathscr{R}(T + iI)$. From Theorem 2.4.4 (b), we get that the latter is closed, and this makes U a closed operator as $\mathscr{D}(U)$ is closed, from Theorem 2.4.7. This proves (a).

The one-one correspondence between $\mathscr{R}(T + iI)$ and $\mathscr{G}(T)$ in Theorem 2.4.4 gives us a one-one correspondence between $\mathscr{D}(T)$ and $\mathscr{R}(T + iI)$, i.e. $x \leftrightarrow z$ such that $z = Tx + ix$. From Theorem 2.4.7 we have that $T + iI$ is one-one, which makes sure that its inverse exists. Therefore,

$$Uz = (T - iI)(T + iI)^{-1}(T + iI)x = Tx - ix,$$

for $x \in \mathscr{D}(T)$.

Now, to establish the reconstruction, we need to first show that $D(T) = \mathscr{R}(I - U)$, which is straightforward as:

$$(I - U)z = z - Uz = (Tx + ix) - (Tx - ix) = 2ix.$$

We also note that $(I + U)z = 2Tx$. Note that the injectivity of $I - U$ is clear, as if for $(T + iI)x_1 = z_1 \neq z_2 = (T + iI)x_2 \in \mathscr{R}(T + iI)$, implying $x_1 \neq x_1$, which will give us $(I - U)z_1 \neq (I - U)z_2$. This ensures the existence of $(I - U)^{-1}$. Therefore,

$$2Tx = (I + U)z = (I + U)(I - U)^{-1}(2ix),$$

for all $x \in \mathscr{D}(T)$ which gives us $T = i(I+U)(I-U)^{-1}$.

Now for (c), assume first that T is self-adjoint. Then from the fact stated just before this theorem, we have $\mathscr{R}(I+T^2) = H$. Now, we have

$$I + T^2 = (T+iI)(T-iI) = (T-iI)(T+iI),$$

making $\mathscr{D}(U) = \mathscr{R}(T+iI) = H = \mathscr{R}(T-iI) = \mathscr{R}(U)$. If we just show that U is an isometry, we have that U is unitary, since $\mathscr{R}(U) = H$. *(Ref 1., Thm. 12.13)*.

This can be shown readily using Theorem 2.4.4 (a). For $x \in \mathscr{D}(U) = \mathscr{R}(T+iI)$, we have $x = (T+iI)y$, $y \in \mathscr{D}(T)$. Thus,

$$\|Ux\|^2 = \|(T-iI)(T+iI)^{-1}(T+iI)y\|^2 = \|(T-iI)y\|^2 = \|(T+iI)y\|^2 = \|x\|^2.$$

Now assume that U is unitary. Suppose we have reconstructed T as in (b). To talk about its adjoint, we need to ensure that $\mathscr{D}(T) = \mathscr{R}(I-U)$ is dense in H. We have shown that $I-U$ is one-one, in (b). Note that $I-U$ is a normal operator since U is unitary: $(I-U)^*(I-U) = 2I - U - U^* = (I-U)(I-U)^*$. For a normal operator S, $\mathscr{R}(S)$ is dense in H if and only if S is one-one because for normal operators we have the relation $\|Sx\| = \|S^*x\|$ for all $x \in \mathscr{D}(S)$ if and only if $\ker(S) = \ker(S^*)$ if and only if $\mathscr{R}(S)$ is dense in H since $\ker(S^*) = [\mathscr{R}(S)]^\perp = \ker(S) = \{0\}$. This makes $\mathscr{D}(T) = \mathscr{R}(I-U)$ dense in H, and hence we can talk about T^*. And since T is symmetric, we have $T \subset T^*$. If we show the other side of the inclusion, i.e. $T^* \subset T$, we are done.

Now we know that U is unitary if and only if $\mathscr{R}(U) = H$ and U is an isometry. Since by definition $\mathscr{R}(T-iI) = \mathscr{R}(U) = H$, by Theorem 2.4.4 (d) and (e) we conclude that T is maximally symmetric, and hence $T = T^*$. This shows that T is self-adjoint.

Finally to prove the converse, let V be as in the statement. Since V is an isometry and $I - V$ is one-one, we have a one-one correspondence $z \leftrightarrow x$ between $\mathscr{D}(V)$ and $\mathscr{R}(I-V)$ given by $x = z - Vz$. We aim to reconstruct S as in (b) and show that it is symmetric and verify that V is indeed the Cayley Transform of S.

Define S on $\mathscr{R}(I-V)$ by

$$Sx = i(z + Vz), \tag{2.3}$$

for $x = z - Vz$. Let us now show that S is a symmetric operator. Let $x = z - Vz$ and $y = u - Vu$ for $x, y \in \mathscr{D}(S)$ and $u, z \in \mathscr{D}(V)$. In the calculations below, we use Theorem 2.4.7 (a).

$$\begin{aligned}
\langle Sx, y\rangle &= i\langle z + Vz, u - Vu\rangle \\
&= i[\langle z, u\rangle + \langle z, -Vu\rangle + \langle Vz, u\rangle + \langle Vz, Vu\rangle] \\
&= i[\langle Vz, u\rangle - \langle z, Vu\rangle] \\
\langle x, Sy\rangle &= -i\langle z - Vz, u + Vu\rangle \\
&= -i[\langle z, u\rangle + \langle z, Vu\rangle + \langle -Vz, u\rangle + \langle -Vz, Vu\rangle] \\
&= i[\langle Vz, u\rangle - \langle z, Vu\rangle],
\end{aligned}$$

giving us that $\langle Sx, y\rangle = \langle x, Sy\rangle$, making S symmetric.

To verify that $V = (S - iI)(S + iI)^{-1}$, we show that $V(Sx + ix) = Sx - ix$ for all $x \in \mathscr{D}(S)$, which is an equivalent expression.

$$\begin{aligned}
Sx - ix &= i(z + Vz) - i(z - Vz) \\
&= 2i\ Vz \\
Sx + ix &= i(z + Vz) + i(z - Vz) \\
&= 2iz \\
V(Sx + ix) &= V(2iz) \\
&= 2i\ Vz \\
&= Sx - ix,
\end{aligned}$$

proving that V is the Cayley Transform of S. □

2.5 The Spectral Theorem

As explained before, we look to apply the results from section 1.4 to the Cayley Transform of an unbounded self-adjoint operator T, which is unitary by Theorem 2.4.8. However integral with respect to the projection-valued measure of the map $f : \sigma(T) \to \mathbb{C}$ given by $f(\lambda) = \lambda$ may not exist anymore due to the unboundedness of T. Hence, we will have to check this apriori to writing the main theorem.

In this section, consider $P : \mathscr{F} \to \mathcal{B}(H)$ to be a projection-valued measure defined as in Definition **??**. If we recall, we had defined T_f for each $f \in B(\Omega)$ to be as follows:

$$\langle T_f x, y\rangle = \int_\Omega f \quad dP_{x,y},$$

for every $x, y \in H$. We extend this to unbounded measurable functions f in the following results.

Lemma 2.5.1. *Let $f : \Omega \to \mathbb{C}$ be a measurable function. Define*

$$\mathscr{W}_f = \left\{ x \in H, \ \int_\Omega |f|^2 \ \ dP_{x,x} < \infty \right\}.$$

Then $\mathscr{W}_f$ is a dense subspace of H. Moreover, if $x, y \in H$, then

$$\int_\Omega |f| \ \ d|P_{x,y}| \leq \|y\| \left\{ \int_\Omega |f|^2 \ \ dP_{x,x} \right\}^{\frac{1}{2}}. \tag{2.4}$$

Furthermore, if f is bounded and $v = T_f z$ then

$$dP_{x,v} = \bar{f} \ \ dP_{x,z} \tag{2.5}$$

for $x, z \in H$.

Proof. We first check if $\mathscr{W}_f$ is a subspace of H. Since by the property of projection-valued measures, $P(E)$ is self-adjoint for all $E \in \mathscr{F}$ and hence for $x \in H$:

$$\|P(E)x\|^2 = \langle P(E)x, P(E)x \rangle = \langle x, P(E)^2 x \rangle = \langle x, P(E)x \rangle = \langle P(E)x, x \rangle = P_{x,x}(E).$$

Now suppose that $x, y \in \mathscr{W}_f$ and $x + y = z$. Then

$$P_{z,z}(E) = \|P(E)z\|^2 \leq \|P(E)x + P(E)y\|^2 \leq 2\|P(E)x\|^2 + 2\|P(E)y\|^2 = 2[P_{x,x}(E) + P_{y,y}(E)],$$

implying $z \in \mathscr{W}_f$. For scalar multiplication, assume $u = \alpha x$ for $\alpha \in \mathbb{C}$. Then

$$P_{z,z}(E) = \|P(E)(\alpha x)\|^2 \leq |\alpha|^2 \|P(E)x\|^2 = |\alpha|^2 P_{x,x}(E),$$

which makes $\mathscr{W}_f$ a subspace of H.

To show $\mathscr{W}_f$ is dense in H, consider, for $n = 1, 2, 3, ...,$ $E_n = \{x \in \Omega, \ |f(x)| < n\}$. Then if $x \in \mathscr{R}(P(E_n))$, then $P(E)x = P(E)P(E_n)x = P(E \cap E_n)x$ for $E \in \mathscr{F}$. We have for any

$E \in \mathscr{F}$, $|P_{x,x}(E)| = \|P(E)x\|^2 \leq \|x\|^2$ since orthogonal projection does not increase norm [10, Ch. 7, Prop. 7.1.2]. This gives us that

$$\int_\Omega |f|^2 \ \ dP_{x,x} = \int_{E_n} |f|^2 \ \ dP_{x,x} \leq n^2 \|x\|^2 < \infty,$$

which concludes that $\mathscr{R}(E_n) \subset \mathscr{W}_f$. Observing that $\Omega = \cup_{n=1}^{\infty} E_n$ and the fact that the map $E \to P(E)y$ is countably additive gives us that $y = \lim_{n\to\infty} P(E_n)y$ for all $y \in H$. This gives us that $y \in \bar{\mathscr{W}}_f$ for all $y \in H$, implying that $\mathscr{W}_f$ is dense in H.

Now, we need to prove the next statement. If $x, y \in H$ and f is a bounded measurable function on Ω, then by the Radon-Nikodym theorem [2, Ch. 9, 9.36] there exists a measurable function u on Ω with $|u| = 1$ a.e. such that

$$uf \ \ dP_{x,y} = |f| \ \ d|P_{x,y}|.$$

This gives us,

$$\int_\Omega |f| \ \ d|P_{x,y}| = \int_\Omega uf \ \ dP_{x,y} = \langle T_{uf}x, y\rangle \leq \|T_{uf}x\|\|y\|,$$

and by Theorem 2.2.3, we have that $\|T_s x\|^2 = \langle T_s^* T_s x, x\rangle = \langle T_{\bar{s}} T_s x, x\rangle = \langle T_{|s|^2} x, x\rangle = \int_\Omega |s|^2 \ \ dP_{x,x}$ for any simple function s, and hence this extends to bounded measurable functions in a natural way. Thus,

$$\|\psi(uf)x\|^2 = \int_\Omega |uf|^2 \ \ dP_{x,x} = \int_\Omega |f|^2 \ \ dP_{x,x},$$

giving us the required result for bounded f. For general measurable f, we again approximate by simple functions.

The last statement is obtained by the following calculation using Theorem 2.2.3: for any measurable g,

$$\int_{\Omega} g \quad dP_{x,v} = \langle T_g x, v\rangle = \langle T_g x, T_f z\rangle = \langle T_{\bar{f}} T_g x, z\rangle = \langle T_{\bar{f}g} x\rangle = \int_{\Omega} g\bar{f} \quad dP_{x,z},$$

concluding the proof. $\square$

The following theorem establishes T_f for any measurable f and basically extends Theorem 2.2.3 to general measurable functions. The proof consists of considering truncations of f, $f_n = f\phi_n$, where each ϕ_n is defined as $\phi_n(x) = 1$ if $|f(x)| \leq n$ and 0 otherwise and deriving the result in the limiting sense. We will omit the proof in this text, as there is little to elaborate from W. Rudin's treatment of [16, Ch. 13, 13.24], and just state the theorem.

Theorem 2.5.2. *Let P be a projection-valued measure on a set Ω.*

(a) To every measurable function $f : \Omega \to \mathbb{C}$ there exists a densely defined closed operator T_f in H with $\mathscr{D}(T_f) = \mathscr{W}_f$ characterized by

$$\langle T_f x, y\rangle = \int_{\Omega} f \quad dP_{x,y}, \tag{2.6}$$

for $x \in \mathscr{W}_f$ and $y \in H$ also satisfying

$$\|T_f x\|^2 = \int_{\Omega} |f|^2 \quad dP_{x,x}, \tag{2.7}$$

for $x \in \mathscr{W}_f$.

(b) The multiplication theorem (the map $f \to \int_{\Omega} f \quad dP = T_f$ was a *-homomorphism, Theorem 2.2.3) holds in the following way: if f and g are measurable, then

$$T_f T_g \subset T_{fg} \text{ and } \mathscr{D}(T_f T_g) = \mathscr{W}_g \cap \mathscr{W}_{fg}, \tag{2.8}$$

and hence $T_f T_g = T_{fg}$ if and only if $\mathscr{W}_{fg} \subset \mathscr{W}_g$.

(c) For each measurable $f : \Omega \to \mathbb{C}$ we have

$$T_f^* = T_{\bar{f}} \tag{2.9}$$

and

$$T_f T_f^* = T_{|f|^2} = T_f^* T_f, \tag{2.10}$$

i.e. T_f is normal.

Definition 2.5.3. The *essential range of f*, denoted by $\mathscr{E}\mathscr{R}(f)$, is the smallest closed subset of $\mathbb{C}$ that contains $f(x)$ for almost all $x \in \Omega$. In other words, it is the set of all $x \in \Omega$ excluding those x that lie in some set $E \in \mathscr{F}$ with $P(E) = 0$.

This next theorem relates the spectrum of T_f and the essential range of f, which will be instrumental in our proof of the spectral theorem. The theorem is basically the definition of the point spectrum and continuous spectrum for the operator T_f.

Theorem 2.5.4. *Suppose P is a projection-valued measure on a set Ω and f be an $\mathscr{F}-$measurable function, $f : \Omega \to \mathbb{C}$. Define*

$$E_\alpha = \{x \in \Omega,\ f(x) = \alpha\}$$

for $\alpha \in \mathbb{C}$.

(a) If $\alpha \in \mathscr{E}\mathscr{R}(f)$ and $P(E_\alpha) \neq 0$, then $T_f - \alpha I$ is not one-one, and hence not invertible (point spectrum).

(b) If $\alpha \in (ER)(f)$ but $P(E_\alpha) \neq 0$, then $T_f - \alpha I$ is a one-one map of $\mathscr{W}_f$ onto a dense proper subspace of H and there exist vectors $x_n \in H$ with $\|x_n\| = 1$ such that $\lim_{n\to\infty} T_f x_n - \alpha x_n = 0$ (continuous spectrum).

(c) $\sigma(T_f) = \mathscr{E}\mathscr{R}(f)$.

Proof. Let us assume without loss of generality, $\alpha = 0$.

(a) If $P(E_0) \neq 0$, then there exists an $x_0 \in \mathscr{R}(P(E_0))$ with $\|x_0\| = 1$. Let $\phi_0 = \chi_{E_0}$. By the definition of E_0, we have $f\phi_0 = 0$, which also makes it a bounded function. By Theorem 2.2.3 we have $T_f T_{\phi_0} = 0$. Note that $T_{\phi_0} = \int_\Omega \chi_{E_0} dP = P(E_0)$. This implies:

$$T_f x_0 = T_f P(E_0) x_0 = T_f T_{\phi_0} x_0 = 0,$$

which shows that there exists a non-zero x_0 such that $T_f x_0 = 0$, making T_f not one-one.

(b) Let $P(E_0) = 0$ but $P(E_n) \neq 0$ for $n = 1, 2, ...$ where

$$E_n = \{x \in \Omega,\ |f(x)| < \frac{1}{n}\}.$$

Pick $x_n \in \mathscr{R}(P(E_n))$ with $\|x_n\| = 1$ and let $\phi_n = \chi_{E_n}$. Note that we have

$$\|T_f x_n\| = \|T_{f\phi_n} x_n\| \leq \|T_{f\phi_n}\| = \|f\phi_n\|_\infty < \frac{1}{n},$$

upon using (2.7). This makes $T_f x_n \to 0$ even when $\|x_n\| = 1$, which clearly shows that $T_f \notin \mathcal{B}(H)$.

Now, if $T_f x = 0$ for some $x \in \mathscr{W}_f$, then $\|T_f x\|^2 = 0 = \int_\Omega |f|^2\, dP_{x,x}$. Since $|f| > 0$ almost everywhere, it must be that $P_{x,x}(\Omega) = \|x\|^2 = 0$ which makes $x = 0$ and hence, T_f is one-one. This also makes $T_{\bar{f}} = T_f^*$ one-one.

If $\mathscr{R}(T_f)$ filled H, then T_f is bijective and the closed graph theorem [10, Ch. 4, Thm. 4.42] will help us conclude that T_f^{-1} is a bounded operator, but the sequence we produced above says otherwise, as $T_f x_n \to 0$, but $T_f^{-1} T_f x_n = x_n$ does not converge to zero. Thus the range is a proper subspace of H.

If $y \in [\mathscr{R}(T_f)]^{\perp}$ and we show $y = 0$ we show that T_f maps into a dense subspace of H, and we're done with (b). The map $x \to \langle T_f x, y\rangle = 0$ is continuous on $\mathscr{W}_f$ and hence $y \in \mathscr{D}(T_f^*)$ by the definition of the adjoint (Remark 2.2.3). This in turn gives us:

$$\langle x, T_{\bar{f}} y\rangle = \langle T_f x, y\rangle = 0,$$

for all $x \in \mathscr{W}_f$ which makes $T_{\bar{f}} y = 0$ and as $T_{\bar{f}}$ is one-one, $y = 0$. This concludes that $\mathscr{R}(T_f)$ is a dense proper subspace of H.

(c) From the proof of (a) and (b), we see that $\mathscr{E}\mathscr{R} \subset \sigma(T_f)$. Let us assume $0 \notin \mathscr{E}\mathscr{R}(f)$. Then $g = \frac{1}{f}$, $fg = 1$ is bounded and $T_f T_g = T_{fg} = I$ which makes $\mathscr{R}(T_f) = H$ contradicted by (b) since T_f maps into a dense proper subspace of H.

□

We now have almost all the tools to prove the spectral theorem for unbounded self-adjoint operators. This last final theorem before the finale is called the Change of Measure Principle.

Theorem 2.5.5. *Suppose that*

(a) $\mathscr{F}$ *and* $\mathscr{F}'$ *are* $\sigma-$*algebras in the sets* Ω *and* Ω'*,*

(b) $P : (F) \to \mathcal{B}(H)$ *be a projection-valued measure,*

(c) $\phi : \Omega \to \Omega'$ *is such that* $\phi^{-1}(E') \in \mathscr{F}$ *for all* $E' \in \mathscr{F}'$*.*

Then, $P' : \mathscr{F}' \to \mathcal{B}(H)$ *defined by* $P'(E') = P(\phi^{-1}(E'))$ *is also a projection-valued measure on* Ω' *and*

$$\int_{\Omega'} f \quad dP'_{x,y} = \int_{\Omega} f \circ \phi \quad dP_{x,y} \tag{2.11}$$

for f $\mathscr{F}'-$measurable such that either of these integrals exists.

Proof. The proof of (2.11) is straightforward, since it is just the definition of P' for characteristic functions. This then translates to simple functions an hence all measurable f. The projection-valued measure part follows from the fact that P is a projection-valued measure. $\square$

Theorem 2.5.6 (**Spectral Theorem for Unbounded Self-Adjoint Operators**). *To every self-adjoint operator T in H corresponds a unique projection-valued measure P on the Borel subsets of the real line such that*

$$\langle Tx, y\rangle = \int_{-\infty}^{\infty} t \;\; dP_{x,y}(t), \tag{2.12}$$

where $x \in \mathscr{D}(T)$ and $y \in H$. Moreover, P is concentrated on $\sigma(T) \subset \mathbb{R}$, i.e. $P(\sigma(T)) = I$. Such a P is called the spectral decomposition of T.

Proof. Let U be the Cayley Transform of T, $U = (T - iI)(T + iI)^{-1}$, and let Ω be the unit circle in $\mathbb{C}$ with the point 1 removed. Hence U is a bounded normal (unitary) operator from Theorem 2.4.8. Let P' be the spectral decomposition of U, as in Theorem 2.3.2. Theorem 2.4.8 also tells us that $I - U$ is one-one, which gives us $P'(\{1\}) = 0$, since 1 is not an eigenvalue of U. This is because any λ_0 is an eigenvalue if and only if $P_0 = P(\{\lambda_0\}) \neq 0$. (Theorem 2.5.4).

From the spectral decomposition of U we have that

$$\langle Ux, y\rangle = \int_{\Omega} \lambda \;\; dP'_{x,y}(\lambda), \tag{2.13}$$

for $x, y \in H$. We then define $f : \Omega \to \mathbb{R}$ by

$$f(\lambda) = \frac{i(1+\lambda)}{1-\lambda},$$

from which we can define T_f (Theorem 2.5.2) as follows:

$$\langle T_f x, y\rangle = \int_\Omega f \quad dP'_{x,y} \tag{2.14}$$

for $x \in \mathscr{W}_f$ and $y \in H$. Note that f is real valued since $\lambda \in \Omega$, which implies T_f is self-adjoint by Theorem 2.5.2 and we have by (2.8) (the multiplication theorem),

$$T_f(I - U) = i(I + U). \tag{2.15}$$

From (2.15) we see that $\mathscr{D}(T) = \mathscr{R}(I - U) \subset \mathscr{W}_f$, and by Theorem 2.4.8 we have that

$$T(I - U) = i(I + U),$$

making T_f a self-adjoint extension of T. However, when an operator S is self-adjoint, it is maximally symmetric, because if not and if there exists a symmetric extension to S, say S_1, then $S_1^* \subset S^*$, and we have $S_1 \subset S_1^* \subset S^* = S \subset S$, concluding that $S_1 = S$. Since T_f is a self-adjoint extension of T, which is also self-adjoint, we have that $T_f = T$. Thus,

$$\langle Tx, y\rangle = \int_\Omega f \quad dP'_{x,y} \tag{2.16}$$

for $x \in \mathscr{D}(T)$ and $y \in H$. We also observe from Theorem 2.5.4 that $\sigma(T) = \sigma(T_f) = \mathscr{E}\mathscr{R}(f) \subset (-\infty, \infty)$. f being one-one on Ω guarantees the application of Theorem 2.5.5 by defining

$$P(f(E)) = P'(E) \tag{2.17}$$

for E Borel subset of Ω. This provides us with the required projection-valued measure that converts (2.16) to (2.12). The uniqueness of this projection-valued measure follows from the uniqueness of P'. We have thus established the spectral decomposition of an unbounded self-adjoint operator on a Hilbert space. $\square$

Chapter 3

Schrödinger Operators and Examples

3.1 Introduction

The previous chapter developed the necessary abstract theory required for this chapter. In here, we will explore operators of the kind $-\Delta + V$, or Schrödinger Operators, and focus on applying the results of Chapter 1 and 2 to these operators. For that, we will have to prove that operators of this kind, for specific V, are self-adjoint. For this purpose, we will look at some preliminary definitions and notions before we venture into the spectral theory of such operators.

3.2 The Laplacian as a Multiplication Operator and its Spectrum

Lemma 3.2.1. *Let $f : \mathbb{R}^N \to \mathbb{C}$ be a measurable function. Define $M_f : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, the multiplication operator, mapping $v \mapsto fv$ with domain*

$$\mathscr{D}(M_f) = \left\{ v \in L^2(\mathbb{R}^N);\ fv \in L^2(\mathbb{R}^N) \right\}.$$

Then,

(a) M_f is bounded if and only if $f \in L^\infty(\mathbb{R}^N)$,

(b) M_f is densely-defined,

(c) M_f is conjugate-adjoint, i.e. $M_f = M_{\bar{f}}$,

(d) M_f is self-adjoint if and only if f is real-valued, and

(e) M_f is closed.

Proof. The proof of this lemma is not difficult, but will be done in detail as this is an important result.

(a) Assume that $f \in L^\infty(\mathbb{R}^N)$. Then

$$\|M_f v\|_2^2 = \int_{\mathbb{R}^N} |fv|^2 \, dx \leq \|f\|_\infty^2 \, \|v\|_2^2,$$

making M_f bounded.

For the converse, suppose that $\|M_f v\|^2 \leq c\|v\|^2$, for some constant $c > 0$ independent of v. Set $A_n = \{x \in \mathbb{R}^N;\ |f| \geq n\}$. Let χ_n be the characteristic function of A_n. Then clearly,

$$\int_{\mathbb{R}^N} |fv|^2 \chi_n = \int_{A_n} |fv|^2 \leq \int_{\mathbb{R}^N} |M_f v|^2 \leq c\|v\|^2.$$

This then implies for each $n \in \mathbb{N}$,

$$n^2 \int_{A_n} |v|^2 \leq \int_{A_n} |fv|^2 \leq c \int_{\mathbb{R}^N} |v|^2.$$

If we take $n \to \infty$ everywhere, we get a contradiction unless f is bounded.

(b) To show that $\mathscr{D}(M_f)$ is dense in $L^2(\mathbb{R}^N)$, we pick an element $v \in L^2(\mathbb{R}^N)$ and show that there exists a sequence in $\mathscr{D}(M_f)$ that converges to v in $L^2(\mathbb{R}^N)$.

Define $E_n = \{x \in \mathbb{R}^N;\ |f(x)| < n\}$, and let χ_n be the characteristic function of E_n. Define a sequence $\phi_k; = v\chi_k$, $k \in \mathbb{N}$. Then each $\phi_k \in \mathscr{D}(M_f)$, since

$$\int_{\mathbb{R}^N} |f\phi_k|^2 = \int_{\mathbb{R}^N} |fv\chi_k|^2 = \int_{E_k} |fv|^2 < k^2\|v\|^2.$$

Now,

$$\int_{\mathbb{R}^N} |\phi_k - v|^2 = \int_{\mathbb{R}^N} |v\chi_k - v|^2 = \int_{\mathbb{R}^N} |v|^2|\chi_k - 1|^2 \to 0,$$

as $k \to \infty$.

(c) Since M_f is densely-defined, M_f^* is uniquely defined and characterized by

$$\langle fu, v\rangle = \langle M_f u, v\rangle = \langle u, M_f^* v\rangle,$$

for $u \in \mathscr{D}(M_f)$ and $v \in \mathscr{D}(M_f^*)$. Since $\mathscr{D}(M_f)$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $\{\phi_k\} \subset \mathscr{D}(M_f)$ such that $\phi_k \to v$ in L^2, i.e. there exists a subsequence $\{\phi_{k_l}\}$ such that $\phi_{k_l}(x) \to v(x)$ pointwise almost everywhere (a.e.). For brevity, call this subsequence ϕ_k. Then,

$$\langle fu, \phi_k \rangle \to \langle fu, v \rangle = \langle M_f u, v \rangle,$$

due to the continuity of inner product. Similarly,

$$\langle u, \bar{f}\phi_k \rangle \to \langle u, \bar{f}v \rangle = \langle u, M_{\bar{f}} v \rangle,$$

but

$$\langle fu, \phi_k \rangle = \int_{\mathbb{R}^N} fu\bar{\phi_k} = \int_{\mathbb{R}^N} u\overline{\bar{f}\phi_k} = \langle u, \bar{f}\phi_k \rangle.$$

Therefore, since the adjoint is unique, $M_f^* = M_{\bar{f}}$. This also makes $\mathscr{D}(M_f^*) = \mathscr{D}(M_f)$.

(d) This immediately follows from (c).

(e) Let $\{u_n\} \subset \mathscr{D}(M_f)$ such that $u_n \to u$ in $L^2(\mathbb{R}^N)$ and $M_f u_n = fu_n \to w$ in $L^2(\mathbb{R}^N)$. To show that M_f is closed, we show that $u \in \mathscr{D}(M_f)$ and $M_f u = w$, we're done. But the assumptions imply that $u_n \to u$ and $fu_n \to w$ pointwise a.e., making $w = fu$ a.e., which proves our claim.

$\square$

Before we address the question so as to what important applications these multiplication operators can have, a rudimentary tool used in a majority of proofs henceforth, is integration by parts. The following theorem establishes this.

Theorem 3.2.2 (Integration by Parts). *[5, App C.2] Let $\Omega \subseteq \mathbb{R}^N$ be an open, bounded set. Let $u, v \in C^1(\bar{\Omega})$. Then,*

$$\int_{\Omega} u_{x_i} v \; dx = - \int_{\Omega} u_{x_i} \; dx + \int_{\partial\Omega} uv \; \nu^i \; dS(x), \tag{3.1}$$

for $i = 1, 2, ..., N$, where $u_{x_i} = \frac{\partial u}{\partial x_i}$, $\partial\Omega$ is the boundary of Ω and ν^i is the unit outward normal in the $i-th$ direction.

Remark 3.2.3. Consider a function $\psi \in C_c^\infty(\mathbb{R}^N)$. Let $\mathcal{F} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be the Fourier Transform. Then,

$$\begin{aligned}
\mathcal{F}(-\Delta\psi)(\xi) &= \int_{\mathbb{R}^N} e^{-ix.\xi} \; ((-\Delta)(\psi)) \; dx, \\
&= - \int_{\mathbb{R}^N} \nabla(e^{-ix.\xi}).\nabla\psi(x) \; dx, \\
&= \int_{\mathbb{R}^N} \psi(x) \; \Delta(e^{-ix.\xi}) \; dx, \\
&= \int_{\mathbb{R}^N} |\xi|^2 \; \psi(x) \; e^{-ix.\xi} \; dx, \\
&= |\xi|^2 \; \hat{\psi}(\xi),
\end{aligned} \tag{3.2}$$

where the first two steps are obtained by repeated integration by parts (3.1). We can perform integration by parts since the integration is actually over a bounded domain, the support of the function ψ. Since this support is compact, the boundary terms vanish in (3.1).

Since $\mathcal{F}$ is a unitary map and a bijection on $L^2(\mathbb{R}^N)$ by Plancherel's theorem, we have

$$\mathcal{F}(-\Delta) = |\xi|^2 \hat{\psi},$$

for $\psi \in C_c^\infty(\mathbb{R}^N)$. This gives us that the operator $-\Delta$ is conjugate to the multiplication operator $M_{|\xi|^2}$. Define the domain of $M_{|\xi|^2}$ as in Lemma 3.2.1. This gives us that

$$\mathscr{D}(-\Delta) = \{f \in L^2(\mathbb{R}^N);\ |\xi|^2 \ \ (\mathcal{F}f) \in L^2(\mathbb{R}^N)\}.$$

Clearly, by Lemma 3.2.1 (d), $-\Delta$ is self-adjoint on $\mathscr{D}(-\Delta)$.

Let us now look at a couple definitions.

Definition 3.2.4. Let $\mathcal{H}$ be a Hilbert space. An operator T on $\mathcal{H}$ is **essentially self-adjoint** if $\bar{T} = T^*$, where the closure of T, $\bar{T}$, is the operator whose graph is $\overline{\mathscr{G}(T)}$.

Definition 3.2.5. A domain on which a symmetric operator is essentially self-adjoint is called a **core domain**.

Remark 3.2.6. Consider $T = -\Delta$ on $L^2(\mathbb{R}^N)$ with $\mathscr{D}(T) = C_c^\infty(\mathbb{R}^N)$. Therefore, $\mathscr{D}(T^*)$ consists of $f \in L^2(\mathbb{R}^N)$ such that $\phi \to \langle f, -\Delta\phi\rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^N)$. By Plancherel's theorem,

$$\langle f, -\Delta\phi\rangle = \langle \mathcal{F}f, |\xi|^2\hat{\phi}\rangle.$$

This implies that $f \in \mathscr{D}(-\Delta^*)$ if and only if there exists a function $g \in L^2(\mathbb{R}^N)$ such that $g - |\xi|^2\ \mathcal{F}f$ is orthogonal to $\mathcal{F}(C_c^\infty(\mathbb{R}^N))$, i.e.

$$\langle g - |\xi|^2\ \mathcal{F}f, \hat{\phi}\rangle = 0,$$

for all $\phi \in C_c^\infty(\mathbb{R}^N)$. But since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, $g = |\xi|^2\ \ \mathcal{F}f$. This implies that $f \in \mathscr{D}(-\Delta^*)$ if and only if $|\xi|^2\mathcal{F}f \in L^2(\mathbb{R}^N)$, which makes the operator $-\Delta$ essentially self-adjoint on the core $C_c^\infty(\mathbb{R}^N)$.

Remark 3.2.7. Note that the closure of an operator is the unique self-adjoint extension for an essentially self-adjoint operator. This is because if A is essentially self-adjoint on a core domain and if B is a self-adjoint extension, since $\bar{A}$ is the smallest such extension, we have $\bar{A} \subset B$. Taking adjoint will give us the reverse inclusion, making $B = \bar{A}$.

Hence, the Laplacian $-\Delta$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^N)$ with domain of self-adjoint extension $\mathscr{D}(-\Delta)$. We call the unique self-adjoint extension of $-\Delta$ to $\mathscr{D}(-\Delta)$ the **Free Hamiltonian**, denoted by H_0.

Even though we know the only unique self-adjoint extension of an essentially self-adjoint operator is the closure of the operator itself, we will be using the Friedrichs extension, elaborated below. Note that the Friedrichs extension method is applicable for positive, densely-defined symmetric operators, and yields a positive self-adjoint extension. However, in this text, we are solely concerned with this process in the context of Schrödinger Operators.

Definition 3.2.8. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We define the space $H_0^1(\Omega)$ to be the closure of the space $C_c^\infty(\Omega)$ with respect to the norm:

$$\|\phi\|_1^2 = \|\phi\|^2 + \|\nabla\phi\|^2,$$

where $\|.\|$ denotes the L^2-norm.

The following theorem characterizes the Friedrichs extension. For a proof, refer [3, Ch. 7, Thm. 7.2].

Theorem 3.2.9. *Let $V \in L^1_{loc}(\mathbb{R}^N)$ be real-valued with $V \geq 0$. Then define the Hilbert space*

$$\mathcal{H}_q := \{f \in H_0^1(\mathbb{R}^N); \quad f\sqrt{V} \in L^2(\mathbb{R}^N)\} \tag{3.3}$$

with respect to the inner product

$$\langle f, g\rangle_q = \langle f, g\rangle_1 + \langle Vf, g\rangle.$$

There exists a self-adjoint extension of $H := -\Delta + V$ with the domain $\mathscr{D}(H) = \{u \in \mathcal{H}_q\ ;\ f \mapsto \langle f, u\rangle_q$ extends to $L^2(\mathbb{R}^N)$ continuously$\}$, with

$$\langle f, g\rangle_q = \langle(-\Delta + V + 1)f, g\rangle,$$

for all $f, g \in \mathscr{D}(H)$.

Remark 3.2.10. The Free Hamiltonian has no eigenvalues and its spectrum is purely continuous. This property is illustrated readily in the following calculation:

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $u \in C_c^\infty(\Omega)$ be such that $\|u\| = 1$ and u satisfies the eigenvalue equation: $\Delta u = \lambda u$. Then taking the Fourier Transform will give us $\hat{u}(|\xi|^2 - \lambda) = 0$. This violates Plancherel's theorem as it forces $\hat{u} = 0$ a.e..

We will require a couple definitions before we set foot into determining the spectra of various operators, starting with the Free Hamiltonian.

Definition 3.2.11. Let $T : \mathscr{D}(T) \to \mathcal{H}$ be a self-adjoint operator and $\lambda \in \mathbb{R}$. A sequence $\{u_n\}_{n\in\mathbb{N}}$ is a **singular sequence for T and λ** if:

(a) $\|u_n\| = 1$ for all $n \in \mathbb{N}$ or $\liminf_{n\to\infty} \|u_n\| > 0$,

(b) $u_n \xrightarrow{w} 0$, and

(c) $\|(T - \lambda I)u_n\| \to 0$.

Definition 3.2.12. $\lambda \in \mathbb{R}$ is said to be in the **essential spectrum** of T if there exists a singular sequence for T and λ. We denote the essential spectrum by $\sigma_{ess}(T)$.
The **discrete spectrum** of T is defined to be $\sigma_{disc}(T) = \sigma(T) \setminus \sigma_{ess}(T)$. This consists of isolated points in the spectrum having at most finite multiplicity.

The following theorem characterizes the spectrum of the Free Hamiltonian.

Theorem 3.2.13. $\sigma(H_0) = \sigma_{ess}(H_0) = [0, \infty)$.

Proof. We first show that H_0 is a positive operator. For any $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
\langle H_0\varphi, \varphi\rangle &= -\int_{\mathbb{R}^N} \Delta\varphi(x)\overline{\varphi(x)}\,dx \\
&= -\int_{\operatorname{supp}\varphi} \varphi(x)\overline{\varphi(x)}\,dx \\
&= \int \nabla\varphi\cdot\overline{\nabla\varphi}\,dx + \text{boundary term} \quad \text{(Integration by parts)} \\
&= \langle\nabla\varphi, \nabla\varphi\rangle = \int_{\mathbb{R}^N} |\nabla\varphi(x)|^2\,dx \geq 0.
\end{aligned}$$

Note: The boundary term becomes zero as $\operatorname{supp}\varphi$ is compact.

Now, for any $f \in \mathscr{D}(H_0) = H_0^2(\mathbb{R}^N)$, since H_0 is essentially self-adjoint with core $C_c^\infty(\mathbb{R}^N)$, we have that $\exists$ a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)$ such that $\varphi_n \to f$ and $-\Delta\varphi_n \to H_0 f$. Therefore,

$$\langle H_0 f, f\rangle \geq 0 \quad \forall f \in \mathscr{D}(H_0)$$

This implies H_0 is a positive operator.

The positivity forces that $P(\lambda) \geq 0$ whenever $\lambda < 0$, where $P(\lambda)$ is the associated projection valued measure.

$$\implies \qquad \sigma\left(H_0\right) \subset [0, \infty).$$

From Remark 3.2.10, we have that $\sigma\left(H_0\right) = \sigma_{ess}\left(H_0\right)$. Therefore we only need to show $\sigma_{ess}\left(H_0\right) \supset [0, \infty)$. We shall use the Definition 3.2.11 and construct a singular sequence for H_0 and any $\lambda \in [0, \infty)$.

Note that the function $\omega(x) := e^{i\xi\cdot x}$, where $\xi \in \mathbb{R}^N$ fixed with $\xi\cdot\xi = \lambda$, behaves like an eigenfunction for $-\Delta$. However in lieu of Remark 3.2.10, we see that $e^{i\xi\cdot x} \notin L^2\left(\mathbb{R}^N\right)$, and hence is not an eigenfunction.

Thus point-wise we have,

$$(-\Delta\omega)(x) = \lambda\omega(x), \quad x \in \mathbb{R}^N.$$

Choose a cutoff function $\psi \in C_c^\infty(B(0,2))$ with $0 \leq \psi \leq 1$, where $B(0,2)$ is the ball in R^N of radius 2 centred at origin, such that $\psi|_{B(0,1)} = 1$ Define now the truncations $\psi_k(x) := \psi\left(\frac{x}{k}\right)$, $x \in \mathbb{R}^N, k \in \mathbb{N}$.
These cutoff functions help us design our singular sequence $\{u_k\}$ with $\|u_k\| = 1$:

$$u_k := \frac{\psi_k\omega}{\|\psi_k\omega\|}$$

We just need to show the remaining two properties. Choose $f \in C_c^\infty\left(\mathbb{R}^N\right)$ and observe that

$$\begin{aligned}
|\langle f, u_k\rangle| &= \left|\int_{\mathbb{R}^N} f u_k dx\right| \\
&= \left|\int_{B(0,2k)} f u_k dx\right| \\
&\leq \left|\int_{B(0,k)} \frac{f}{\|\psi_k\omega\|} dx\right| + \left|\int_{B(0,2k)\setminus B(0,k)} f\frac{\psi_k\omega}{\|\psi_k\omega\|} dx\right| \\
&\leq \frac{\|f\|_{L^1}}{\|\psi_k\omega\|} + \left|\int_{B(0,2k)\setminus B(0,k)} f\frac{\psi_k\omega}{\|\psi_k\omega\|} dx\right| \qquad \to 0 \text{ as } k \to \infty,
\end{aligned}$$

since

$$\begin{aligned}
\|\psi_k\omega\|^2 &= \int_{\mathbb{R}^N} |\psi_k(x)\omega(x)|^2\, dx \\
&\geqslant \int_{B(0,k)} |\omega(x)|^2 dx = |B_k| \to \infty \text{ as } k \to \infty,
\end{aligned}$$

and for $k > N_0$ the term inside the second integral vanishes for large N_0 as supp $f \subset B_{N_0}$.

Thus $u_k \xrightarrow{w} 0$ as $k \to \infty$.

To show that $\|(H_0 - \lambda I)\, u_k\| \to 0, \quad k \to \infty$, for $f \in C^\infty\left(\mathbb{R}^N\right)$ note that

$$\begin{aligned}
-\Delta\left(\psi_k f\right) &= -\sum_{j=1}^{N} \frac{\partial^2}{\partial j_j^2}\left(\psi_k f\right) \\
&= -\sum_{j=1}^{N}\left[\left(\frac{\partial^2 \psi_k}{\partial x_j^2}\right) f + 2\frac{\partial \psi_k}{\partial x_j}\frac{\partial f}{\partial x_j} + \psi_k \frac{\partial^2 f}{\partial x_j^2}\right] \\
&= -\left(\Delta \psi_k\right) f - 2\left\langle \nabla \psi, \nabla f\right) - \psi_k \Delta f.
\end{aligned}$$

Again using the essential self-adjointness of H_0,

$$\begin{aligned}
\|(H_0 - \lambda I)\, u_k\|^2 &= \int_{\mathbb{R}^N} \left|(-\Delta - \lambda I)\frac{\psi_k \omega}{\|\psi_k \omega\ \|}\right|^2 dx \\
&= \frac{1}{\|\psi_k \omega\|^2}\left[\int_{B(0,k)} \left|(-\Delta - \lambda I)e^{i\xi\cdot x}\right|^2 dx\right. \\
&\quad \left. + \int \left|(-\Delta - \lambda I)\left(\psi_k \omega\right)\right|^2 dx\right]
\end{aligned}$$

The first term vanishes as $-\Delta\left(e^{i\xi\cdot x}\right) = \lambda e^{i\xi\cdot x}$.

Using the identity above, we have

$$\begin{aligned}
&= \frac{1}{\|\psi_k \omega\|^2}\int_{B(0,2k)\setminus B(0,k)} \left|2\left\langle \nabla \psi_k, \nabla \omega\right\rangle + \Delta \psi_k \omega\right|^2 dx \\
&\leqslant \frac{2}{\|\psi_k \omega\|^2}\left[\int_{B(0,2k)\setminus B(0,k)} \left|2\left\langle \xi, \nabla \psi_k\right\rangle\right|^2 dx + \int \left|\Delta \psi_k\right|^2 dx\right] \\
&\leqslant \frac{2}{\|\psi_k \omega\|^2}\left[\int_{B(0,2k)\setminus B(0,k)} 4\lambda \left|\nabla \psi_k\right|^2 dx + \int_{B(0,2k)\setminus B(0,k)} \left|\Delta \psi_k\right|^2 dx\right]
\end{aligned}$$

From our construction of the cutoff functions $\exists \quad \alpha, \beta$ such that

$$
\begin{aligned}
&\|(H_0 - \lambda I)\, u_k\|^2 \leq \frac{|B(0,2k)\backslash B(0,k)|}{|B(0,k)|} \left(\frac{\alpha}{k^2} + \frac{\beta}{k^4} \right) \\
&= \left(2^N - 1\right) \left(\frac{\alpha}{k^2} + \frac{\beta}{k^4} \right) \\
&\longrightarrow 0 \text{ as } k \to \infty.
\end{aligned}
$$

Hence $\sigma_{ess}(H_0) \supset [0, \infty)$, proving our theorem. $\square$

Let us now look at examples of Schrödinger Operators $-\Delta + V$ with different V. We will mostly be concerned with the spectral data associated with such operators and largely take the self-adjointness and essential self-adjointness on $C_c^\infty(\mathbb{R}^N)$ for granted. This is because these properties need to be verified for each example separately because the operators are vastly different by just changing the kind of V we consider. This thesis is more concerned with spectral decomposition and solutions to partial differential equations comprising of such differential operators. If one should require the proofs to self-adjointness and essential self-adjointness, then refer to [3, Ch. 7].

Let us straightaway jump into the first example.

3.3 $V(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$

In addition to the assumptions above, we will also impose that $V(x) \geq c_0$ for some constant $c_0 \in \mathbb{R}$ and for all $x \in \mathbb{R}^N$. Then $(-\Delta + V) : C_c^\infty(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is bounded below, i.e., there exists a constant $\alpha > 0$ such that

$$\|(-\Delta + V)u\| \geq \alpha \|u\|, \quad \forall u \in C_c^\infty(\mathbb{R}^N).$$

This arises from the property that $-\Delta$ is a positive definite operator.

Then, since $-\Delta + V$ is essentially self-adjoint on the core $C_c^\infty(\mathbb{R}^N)$, there exists a unique self-adjoint extension. Let this extension be the Friedrichs extension, which is also the same as

$$\overline{-\Delta + V\big|_{C_c^\infty(\mathbb{R}^N)}} =: H,$$

defined on $\mathscr{D}(H) = H_0^1(\mathbb{R}^N)$, defined by Definition 3.2.8.

The following result will prove instrumental in our characterization of the spectrum of the operator H. The proof is based on standard functional analytic techniques such as the Riesz Representation Theorem [2, Ch. 8, Sec. 8C] and is hence omitted from this thesis.

Lemma 3.3.1. *Let $Q := (0, 2\pi)^N \subset \mathbb{R}^N$ be a cube and $\{u_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)$ be a sequence such that*

$$\|u_n\|_{H_0^1} \leq c_0, \quad \forall n \in \mathbb{N}.$$

Then there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ and $u \in H_0^1(\mathbb{R}^N)$ such that

$$u_{n_k} \to u \text{ in } L^2(Q) \quad \text{and} \quad u_{n_k} \xrightarrow{w} u \text{ in } H_0^1(Q) \text{ as } k \to \infty.$$

This lemma is an important tool in proving the following theorem due to Rellich.

Theorem 3.3.2 ((Rellich's Compactness Theorem)). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Then for any bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H_0^1(\mathbb{R}^N)$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ such that $\{u_{n_k}\}$ converges strongly in $L^2(\Omega)$. Let this strong limit be u. Then, $u \in H_0^1(\Omega)$ and $u_{n_k} \xrightarrow{w} u$ in $H_0^1(\mathbb{R}^N)$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ be given and bounded, i.e., there exists K_0 such that $K_0 > 0$ and $\|u_n\|_1 \leq K_0$.

Consider the open cube $W \subset \mathbb{R}^N$ such that $\overline{\Omega} \subset W$ (always possible since Ω is bounded). Since $H_0^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$, for any u_n, we can find a $\varphi_n \in C_c^\infty(\Omega) \subset C_c^\infty(W)$ such that

$$\|u_n - \varphi_n\|_1 \leq \frac{1}{n}, \quad \text{and} \quad \|\varphi_n\|_1 \leq K_1. \tag{3.4}$$

Then, by Lemma 3.3.1, there exists a subsequence $\{\varphi_{n_j}\}_{j\in\mathbb{N}} \subset \{\varphi_n\}_{n\in\mathbb{N}}$ and $u \in H_0^1(W)$ such that

$$\varphi_{n_j} \to u \text{ in } L^2(W), \quad \varphi_{n_j} \xrightarrow{w} u \text{ in } H_0^1(W) \text{ as } j \to \infty.$$

Define u' to be the restriction of u to Ω. Then clearly, $u' \in L^2(\Omega)$ and $\varphi_{n_j} \to u'$ in $L^2(\Omega)$. By 3.4 and using the fact that $H_0^1(\Omega) \subset H_0^1(W)$, we get

$$\langle \varphi_{n_j}, \psi \rangle_1 \longrightarrow \langle u, \psi \rangle_1 \quad \forall \psi \in H_0^1(\Omega),$$

which implies

$$\varphi_{n_j} \xrightarrow{w} u' \text{ in } H_0^1(\Omega).$$

Since Hilbert spaces such as $H_0^1(\Omega)$ are weakly closed, $u' \in H_0^1(\Omega)$. But by the uniqueness of the weak limit, $v = u'$ and we have $u' \in H_0^1(\Omega)$.

Thus, $u_{k_j} \to u'$ in $L^2(\Omega)$ and $u_{k_j} \xrightarrow{w} u'$ in $H_0^1(\Omega)$, completing the proof. □

We will use Rellich's compactness theorem to show that for an operator of the form

$$H = -\Delta + V$$

with $V(x) \to \infty$ as $|x| \to \infty$ and $V(x) \geq 0$, $\forall x \in \mathbb{R}^N$, the operator $(H+1)^{-1}$ is compact. However, before we can do this, we observe that the continuous embedding $H_0^1(\Omega) \subset L^2(\Omega)$ gives rise to a self-adjoint positive operator. To wit, $\mathscr{D}(T_0) \subset H_0^1(\Omega)$ and such that

$$\langle T_0 u, v \rangle = \langle u, v \rangle_1 \quad \forall u, v \in \mathscr{D}(T_0),$$

T_0 turns out to be the Friedrichs extension of $-\Delta|_{C_c^\infty(\Omega)}$.

We designate T_0 to be called the Laplace operator with Dirichlet boundary conditions.

Define the inner product

$$\langle \varphi, \psi \rangle_{1,V} = \langle \varphi, \psi \rangle_1 + \int_{\mathbb{R}^N} V(x)\varphi(x)\overline{\psi(x)}\, dx,$$

and complete $C_c^\infty(\mathbb{R}^N)$ with respect to this inner product. Call the space so obtained $H_{0,V}^1(\mathbb{R}^N)$.

Note that whenever $u \in H_0^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx < \infty$,

$$\|u\|_{1,V}^2 = \|u\|_1^2 + \int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx < \infty,$$

we have $u \in H_{0,V}^1(\mathbb{R}^N)$.

The reverse inclusion is straightforward, and hence

$$H_{0,V}^1(\mathbb{R}^N) = H_0^1(\mathbb{R}^N) \cap \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx < \infty \right\}.$$

Because of this above equality, the Friedrichs extension $H = H_0 + V$ of $(-\Delta + V)|_{C_c^\infty(\mathbb{R}^N)}$

is characterized by:

$$C_c^\infty(\mathbb{R}^N) \subset \mathscr{D}(H) \subset H^1_{0,V}(\mathbb{R}^N)$$

and

$$\langle Hu, v\rangle = \langle u, v\rangle_{1,V}, \quad u \in \mathscr{D}(H),\, v \in H^1_{0,V}(\mathbb{R}^N).$$

Furthermore, $\mathscr{D}(H) \subset H^1_{0,V}(\mathbb{R}^N)$ is dense with respect to $\|\cdot\|_{1,V}$.

Theorem 3.3.3. *Let $V : \mathbb{R}^N \to \mathbb{R}$ be continuous with $V(x) \to \infty$ for $|x| \to \infty$, and let $H = H_0 + V$ be the Friedrichs extension of $(-\Delta + V)|_{C_c^\infty(\mathbb{R}^N)}$. Then if $V(x) \geq 0$, H is a positive operator and $(H+1)^{-1}$ is compact.*

Proof. Since $H = H_0 + V$ is the Friedrichs extension of $(-\Delta + V)|_{C_c^\infty(\mathbb{R}^N)}$, choosing $\varphi \in C_c^\infty(\mathbb{R}^N)$ we have,

$$\langle(-\Delta + V)\varphi, \varphi\rangle = \langle -\Delta\varphi, \varphi\rangle + \langle V\varphi, \varphi\rangle \geq 0,$$

since $-\Delta|_{C_c^\infty(\mathbb{R}^N)}$ is a positive operator and $V \geq 0$. Again by the density argument used in Theorem 3.2.13, we have that $H \geq 0$.

To show that $(H+1)^{-1}$ is compact, we take a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^N)$ such that $f_n \xrightarrow{w} 0 \quad \text{in } L^2(\mathbb{R}^N)$.

We then consider a sequence

$$u_n = (H+1)^{-1} f_n$$

and produce a subsequence $\{u_{n_j}\}$ such that $\{u_{n_j}\}$ converges in $L^2(\mathbb{R}^N)$ using Rellich's compactness theorem.

The fact that $\{f_n\}$ converges weakly implies that $\{f_n\}$ is weakly bounded in $L^2(\mathbb{R}^N)$.

$(H+1)^{-1}$ is bounded, which implies that $\{u_n\}$ is bounded and is a weak null sequence in $L^2(\mathbb{R}^N)$.

Note that $u_n \in \mathscr{D}(H)$ and

$$\|Hu_n\|_{1,V} \leq \|(H+1)(H+1)^{-1}f_n\| + \|u_n\| \leq K_1.$$

But then by the definition of H,

$$\|u_n\|_{1,V}^2 = \langle u_n, u_n\rangle_{1,V} = \langle Hu_n, u_n\rangle \leq \|Hu_n\|\|u_n\| \leq K_2,$$

where K_1, K_2 are positive constants.

Since $V(x)$ is continuous, we can choose $R \geq 0$ such that, given $\epsilon > 0$,

$$V(x) \geq \frac{1}{\epsilon}, \quad |x| \geq R.$$

Note that

$$\int_{\mathbb{R}^N} V(x)|u_n(x)|^2\, dx \leq K_2$$

by the definition of $\langle\cdot,\cdot\rangle_{1,V}$, and this gives us

$$\int_{|x|\geq R} |u_n(x)|^2\, dx \leq K_2\epsilon.$$

Now define the cutoff function $\psi_R \in C_c^\infty(B(0,2R))$ with $\psi_R \equiv 1$ on $B(0,R)$ and $0 \leq \psi_R \leq 1$ as in Theorem 3.2.13.

See that

$$\begin{aligned}
\|\psi_R u_n\|_1^2 &\leq 2\int_{\mathbb{R}^N} \psi_R^2|\nabla u_n|^2\, dx + 2\int_{\mathbb{R}^N} |\nabla\psi_R|^2|u_n|^2\, dx + \|u_n\|^2, \\
&\leq 2\int_{B(0,2R)} |\nabla u_n|^2\, dx + 2\int_{B(0,2R)\setminus B(0,R)} |\nabla\psi_R|^2|u_n|^2\, dx + \|u_n\|^2, \\
&\leq 2\int_{B(0,2R)} |\nabla u_n|^2\, dx + 2\int_{B(0,2R)\setminus B(0,R)} |\nabla\psi_R|^2|u_n|^2\, dx + \|u_n\|^2, \\
&\leq K_3,
\end{aligned}$$

for some positive constant $K_3 > 0$, since all the terms are bounded.

Thus, $\{\psi_R u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H_0^1(B(0,2R))$.

Now we can apply Rellich's compactness theorem. There exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$ such that

$$\|\psi_R u_{n_j}\| \to 0 \quad \text{in } L^2(B(0,2R)), \text{ as } j \to \infty,$$

since $u_n \xrightarrow{w} 0$.

Now choose $j_0 \in \mathbb{N}$ such that

$$\|\psi_R u_{n_j}\|^2 < \epsilon, \quad \forall j \geq j_0.$$

Thus,

$$\|u_{n_j}\|^2 < K_2\epsilon + \epsilon, \quad \forall j \geq j_0.$$

This implies $u_{n_j} \to 0$ as $j \to \infty$. Therefore, $(H+1)^{-1}$ is a compact operator. $\square$

Corollary 3.3.4. *$(H+1) : \mathscr{D}(H) \to L^2(\mathbb{R}^N)$ is a bijective map with a compact inverse $(H+1)^{-1}$.*
By the spectral theorem for compact self-adjoint operators, the spectrum of $(H+1)$ and, in turn, H, is discrete, and the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ form an increasing sequence diverging to infinity. The eigenfunctions φ_k corresponding to λ_k form an orthonormal basis for $L^2(\mathbb{R}^N)$.

An example of an operator of the form $H = -\Delta + V$ with $V(x) \to \infty$ as $|x| \to \infty$ is the quantum harmonic oscillator with $V(x) = x^2$, $x \in \mathbb{R}$. The operator

$$-\frac{d^2}{dx^2} + x^2 : C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$$

is essentially self-adjoint with the closure being the unique self-adjoint extension.
From Theorem 3.3.3, we have that $\sigma(H) = \sigma_{\text{disc}}(H)$ and the eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ are such that $\lambda_k \to \infty$ as $k \to \infty$. The next couple of theorems will be aimed at calculating these eigenvalues and associated eigenfunctions.
The proof of the following theorem is just some straightforward algebraic computation.

Theorem 3.3.5. *Define the operators $A, A^\dagger : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by:*

$$A := \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right), \quad A^\dagger := \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

Call $T := A^\dagger A$. Then, $T = \frac{1}{2}(H-1)$, $TA^\dagger - A^\dagger T = [T, A^\dagger] = A^\dagger$ and $TA - AT = [T, A] = -A$.

We are more interested in the proof of the corollary below.

Corollary 3.3.6. *Let $\varphi_0 \in \mathcal{S}(\mathbb{R})$ such that*

$$\varphi_0(x) := \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R}.$$

If we define $\varphi_n(x) := (A^\dagger)^n \varphi_0$*, for* $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$*, then*

$$T\varphi_n = n\varphi_n, \quad n \in \mathbb{N}_0.$$

Moreover, we have for $k, \ell \in \mathbb{N}_0, k \neq \ell$ *that* $\langle \varphi_k, \varphi_\ell \rangle = 0$*.*

Proof. We will show the first part by induction.
Note that

$$\begin{aligned}
T\varphi_0 &= A^\dagger A \varphi_0 \\
&= \frac{1}{\sqrt{2}} A^\dagger \left(x + \frac{d}{dx} \right) \left(\frac{1}{\pi^{1/4}} e^{-\frac{1}{2}x^2} \right) \\
&= \frac{1}{\sqrt{2}\pi^{1/4}} A^\dagger \left(xe^{-\frac{1}{2}x^2} - xe^{-\frac{1}{2}x^2} \right) \\
&= 0.
\end{aligned}$$

Now assume the induction hypothesis $T\varphi_n = n\varphi_n$, for $n \in \mathbb{N}$. Let us compute:

$$\begin{aligned}
T\varphi_{n+1} &= TA^\dagger \varphi_n \quad \text{(by definition of } \varphi_{n+1}\text{)} \\
&= \left([T, A^\dagger] + A^\dagger T \right) \varphi_n \\
&= A^\dagger \varphi_n + A^\dagger n \varphi_n \\
&= A^\dagger (n+1) \varphi_n \\
&= (n+1) \varphi_{n+1}.
\end{aligned}$$

Thus we can apply the principle of mathematical induction to conclude $T\varphi_n = n\varphi_n$ for $n \in \mathbb{N}_0$.

Now assume $k \neq \ell$, $k, \ell \in \mathbb{N}_0$. Then

$$\begin{aligned}
\ell\langle\varphi_k, \varphi_\ell\rangle &= \langle\varphi_k, T\varphi_\ell\rangle \\
&= \langle T\varphi_k, \varphi_\ell\rangle \quad \text{(since } T \text{ is self-adjoint)} \\
&= k\langle\varphi_k, \varphi_\ell\rangle.
\end{aligned}$$

Since $k \neq \ell$, it follows that $\langle\varphi_k, \varphi_\ell\rangle = 0$. □

The final theorem in this section reads as follows:

Theorem 3.3.7. *Let*

$$H = -\frac{d^2}{dx^2} + x^2, \quad x \in \mathbb{R}.$$

Let $\lambda_k = 2k+1$, $k \in \mathbb{N}_0$ be the sequence of eigenvalues and $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be the associated eigenfunctions as in Corollary 3.3.6.

(a) Let λ be some eigenvalue of H and let $u \in \mathscr{D}(H)$ be an associated eigenfunction. Then $Au \in \mathscr{D}(H)$ and $H(Au) = (\lambda - 2)Au$.

(b) $\sigma(H) = \{\lambda_k : k \in \mathbb{N}_0\}$.

Proof. Let us first show (a).

$$\langle Hu, u\rangle = \left\langle \frac{d}{dx}u, \frac{d}{dx}u \right\rangle + \langle xu, xu\rangle \quad \text{(integration by parts)}$$

This shows that $Au \in L^2(\mathbb{R})$, since

$$\langle Au, Au\rangle = \langle A^\dagger Au, u\rangle = \frac{1}{2}\langle(H-1)u, u\rangle = \frac{1}{2}\langle Hu, u\rangle - \frac{1}{2}\langle u, u\rangle < \infty.$$

Now recall $\mathscr{D}(H) = \mathscr{D}(T)$ and

$$Tu = \frac{1}{2}(H-1)u = \frac{1}{2}(\lambda - 1)u.$$

Define $\mu = \frac{1}{2}(\lambda - 1)$, and fix $\varphi \in C_c^\infty(\mathbb{R})$.

$$\begin{aligned}
\langle Au, (T-\mu)\varphi\rangle &= \langle u, A^\dagger(T-\mu)\varphi\rangle \\
&= \langle u, (T-\mu)A^\dagger\varphi\rangle - \langle u, A^\dagger\varphi\rangle \\
&= \langle (T-\mu)u, A^\dagger\varphi\rangle - \langle Au, \varphi\rangle = -\langle Au, \varphi\rangle,
\end{aligned}$$

making $\langle Au, (T-\mu)\varphi\rangle = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R})$.

The operator $T - (\mu - 1)$ is essentially self-adjoint:

$$Au \in [\mathscr{R}(T-(\mu-1))]^\perp = \ker(T-(\mu-1)) \subset \mathscr{D}(T) = \mathscr{D}(H).$$

Thus $Au \in \mathscr{D}(H)$ and $TAu = (\mu-1)Au$. And since $T = \frac{1}{2}(H-1)$, we have $H = 2T+1$:

$$HAu = (2T+1)Au = 2TAu + Au = 2(\mu-1)Au + Au = (2\mu - 2 + 1)Au = (\lambda - 2)Au,$$

therefore, if u is an eigenfunction of H with eigenvalue λ, then Au is also an eigenfunction with eigenvalue $\lambda - 2$.
(2) now follows from (1).

$$H \geq 0 \Rightarrow \sigma(H) \subset (0, \infty)$$

Then, there exists $m \in \mathbb{N}$ such that $A^m u \neq 0$, and $A^{m+1}u = 0$. Otherwise, (1) will give us a sequence of eigenvalues that are not bounded from below.

$$\therefore \text{For such an } m, \quad HA^m u = (2T+1)A^m u = 2AA^m u + A^m u = 0 + A^m u = A^m u.$$

Since Au is an eigenfunction of H with eigenvalue $\lambda - 2$, by induction, $A^m u$ is also an eigenfunction of H with eigenvalue $\lambda - 2m$.

But the above calculation shows that $\lambda - 2m = 1$. In other words, there exists $m \in \mathbb{N}_0$ such that $\lambda = 2m + 1$.

This proves that any $\lambda \in \sigma(H)$ can be written as $2m+1$ for some $m \in \mathbb{N}$ uniquely. Hence, $\sigma(H) = \{\lambda_k : k \in \mathbb{N}_0\}$. $\square$

Remark 3.3.8. The eigenvalues $\lambda_k = 2k + 1$ of

$$H = -\frac{d^2}{dx^2} + x^2, \quad x \in \mathbb{R},$$

are simple, i.e., the eigenspace is spanned by exactly one eigenfunction.

The eigenfunctions of the operator H associated with each λ_k, $\varphi_k(x)$, have the form

$$\varphi_k(x) = c_k P_k(x) e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R}, \quad k \in \mathbb{N}_0.$$

Here, P_k are polynomials of degree k called Hermite polynomials, and c_k are constants such that

$$\langle \varphi_j, \varphi_k \rangle = \delta_{jk}, \quad k, j \in \mathbb{N}_0.$$

These functions $\varphi_k \notin C_c^\infty(\mathbb{R})$, but are elements of $S(\mathbb{R})$.

Remark 3.3.9. These results can be generalized to $-\Delta + |x|^2 \quad \text{on} \quad L^2(\mathbb{R}^n)$, using the notion of tensor product of operators.

3.4 $V(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$

Definition 3.4.1. An operator B is said to be **relatively bounded with respect to** A with relative bound ξ if there exists β such that for all $x \in \mathscr{D}(A)$,

$$\|Bx\| \leq \beta\|x\| + \xi\|Ax\|.$$

In this section, we will focus on operators $H = H_0 + V$ with $V(x) \to 0$ as $|x| \to \infty$ and such that M_V is relatively bounded with respect to H_0 with relative bound $\xi < 1$.
We will again assume the self-adjointness and the essential self-adjointness granted.

Theorem 3.4.2. *Let H be a self-adjoint operator on $\mathcal{H}$ and A be a symmetric, compact linear operator. Then*

$$\sigma_{ess}(H + A) = \sigma_{ess}(H).$$

Proof. We prove this by showing that $\{u_n\}$ is a singular sequence for H and A if and only if $\{u_n\}$ is a singular sequence for $H + A$ and A.
Let $\{u_n\}$ be a singular sequence for H and A. Then we have:

(i) $\|u_n\| = 1$

(ii) $u_n \rightharpoonup 0$ as $n \to \infty$

(iii) $\|(H - \lambda I)u_n\| \to 0$ as $n \to \infty$

We need to show $\|(H + A - \lambda I)u_n\| \to 0$ as $n \to \infty$, which is straightforward since

$$\|(H + A - \lambda I)u_n\| \leq \|(H - \lambda I)u_n\| + \|Au_n\| \to 0 \text{ as } n \to \infty.$$

Here, we have used the fact that A is compact.

For the converse, similarly, given $\|(H-\lambda I+A)u_n\| \to 0$ as $n \to \infty$,

$$\|(H-\lambda I)u_n\| = \|(H-\lambda I+A-A)u_n\| \leq \|(H-\lambda I+A)u_n\| + \|Au_n\| \to 0 \text{ as } n \to \infty.$$

This proves Weyl's theorem. $\square$

Theorem 3.4.3. *Let $V : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be continuous with $V(x) \to 0$ as $|x| \to \infty$. Let M_V be the multiplication operator associated with V and assume that M_V is relatively bounded with respect to H_0 with relative bound $\xi < 1$. Set $H = H_0 + V$. Then*

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty).$$

Proof. We will assume in this proof that V is bounded (relative bound 0), i.e., there exists $M > 0$ such that $|V(x)| \leq M$ for all $x \in \mathbb{R}^N$.

Recall the second resolvent equation: for an operator $T : \mathcal{H} \to \mathcal{H}$ and $\mu, \lambda \in \rho(T)$, we have

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu.$$

Here, call $T_\lambda = T - \lambda I$ and $R_\lambda = T_\lambda^{-1}$ (which exists for $\lambda \in \rho(T)$). By this equation, we can write for $c \in \mathbb{R}$,

$$(H + cI)^{-1} - (H_0 + cI)^{-1} = (H + cI)^{-1} V (H_0 + cI)^{-1}.$$

We aim to show that the RHS is compact.

Straightaway, $(H_0 + cI)^{-1}$ is bounded. We show next that $V(H_0 + cI)^{-1}$ is compact.

Choose a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^N)$ with $f_n \rightharpoonup 0$ as $n \to \infty$. Let $u_n := (H_0 + cI)^{-1} f_n$.

From the proof before, we have $u_n \xrightarrow{w} 0$ as $n \to \infty$.

Also note that

$$\|u_n\| + \|H_0 u_n\| \leq K_1,$$

for a constant $K_1 > 0$. Thus,

$$\|u_n\|_1^2 = \|u_n\|^2 + \|\nabla u_n\|^2 = \|u_n\|^2 + \langle H_0 u_n, u_n\rangle \quad \text{(integration by parts)},$$

$$\leq K_2,$$

for a constant $K_2 > 0$.

Therefore, if $\varphi \in C_c^\infty(\mathbb{R}^N)$, there exists $c_\varphi > 0$ such that

$$\|\varphi u_n\|_1^2 = \|\varphi u_n\|^2 + \|\nabla(\varphi u_n)\|^2 \leq c_\varphi^2.$$

Thus, $\{\varphi u_n\}_{n\in\mathbb{N}}$ is bounded.

Hence the sequence $\{\varphi u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$. Using Rellich's compactness theorem, there exists a subsequence (also denoted by φu_n for brevity) such that $\varphi u_n \to 0$ in $L^2(\mathbb{R}^N)$, for all $\varphi \in C_c^\infty(\mathbb{R}^N)$.

Now, given $\epsilon > 0$, we can choose $R > 0$ such that $|V(x)| \leq \epsilon$, $|x| > R$. We can now choose a cutoff function $\psi_R \in C_c^\infty(\mathbb{R}^N)$ with the property $0 \leq \psi_R \leq 1$, with ψ_R supported in $B(0, 2R)$ and $\psi_R|_{B(0,R)} = 1$.

Then,

$$\|Vu_n\| \leq \|V\psi_R u_n\| + \|V(1-\psi_R)u_n\|,$$

$$\leq M\|\psi_R u_n\| + \epsilon\|(1-\psi_R)u_n\|.$$

Therefore, we can find $N_\epsilon \in \mathbb{N}$ such that $\|\psi_R u_n\| \leq \frac{\epsilon}{M}$ for every $n \geq N_\epsilon$. For such large n, we observe that

$$\|Vu_n\| \leq \beta\epsilon,$$

where $\beta \geq 0$.

We can thus conclude that $Vu_n \to 0$ as $n \to \infty$, and hence $V(H_0 + ci)^{-1}$ is compact.

Now, using Weyl's theorem,

$$\sigma_{\text{ess}}((H + cI)^{-1}) = \sigma_{\text{ess}}((H_0 + cI)^{-1}),$$

$$\text{Thus if } \ (\mu + c)^{-1} \in \sigma_{\text{ess}}((H + cI)^{-1}), \text{ then } \mu \in \sigma_{\text{ess}}(H),$$

and vice versa. $\square$

Remark 3.4.4. If V is such that $V(x) \to 0$ for $|x| \to \infty$ and relatively bounded with H_0 with relative bound $\xi < 1$, we can construct singular sequences for H_0 and $H \geq 0$ with support outside a ball B_R with R arbitrarily large. Since $V(x) \to 0$ as $|x| \to \infty$, $\sigma_{\text{ess}}(H_0) \subset \sigma_{\text{ess}}(H)$. We only need to show the other side.
Since $\sigma_{\text{ess}}(H_0) = [0, \infty)$, if $\sigma_{\text{ess}}(H) \neq \sigma_{\text{ess}}(H_0)$, $\exists \lambda_0 > 0$ such that $\lambda \in \sigma_{\text{ess}}(H) \setminus \sigma_{\text{ess}}(H_0)$.
Let $\{\psi_k\}_{k \in \mathbb{N}}$ be any singular sequence for H and λ and $R \geq 0$ arbitrary. We claim that $\psi_k \to 0$ for $k \to \infty$.
If not, then $\exists$ a sequence $\{w_j\}_{j \in \mathbb{N}} \subset \{\psi_k\}_{k \in \mathbb{N}}$ and $\eta > 0$ such that $\|w_j\| \geq \eta \forall j \in \mathbb{N}$.
Since $\exists R > 0$ such that $\|V v_k\| \leq \epsilon, \|H_0 v_k\| \leq \|(H - \lambda I)v_k\| + \|V v_k\| + \epsilon \|u_n\|$,
$\Rightarrow \{\|v_k\|\}_{k \in \mathbb{N}}$ is bounded as well as $\{\|w_j\|\}_{j \in \mathbb{N}}$.
Using Rellich's compactness theorem, we find a subsequence $\{w_j\}_{j \in \mathbb{N}}$ and $u \in L^2(\mathbb{R}^N)$ such that $u_n \to u$.
But $\{\psi_k\}$ is a singular sequence, which forces $u = 0$, contradicting that $\|w_j\| \geq \eta \forall j \in \mathbb{N}$.
Therefore $\psi_k \to 0, k \to \infty$ in $L^2(\mathbb{R}^N)$.
For $\epsilon > 0$, choose $R_\epsilon > 0$ such that $|V(x)| < \epsilon$ for $|x| > R_\epsilon$. Then

$$\|(H_0 - I)v_k\| \leq \|(H - \lambda I)v_k\| + \|V v_k\| + \epsilon \|u_n\|.$$

Since V is continuous, we rule that

$$\limsup_{k \to \infty} \|(H_0 - \lambda I)v_k\| \leq \epsilon.$$

Therefore we have produced a singular sequence for H_0 and λ, with $\lambda < 0$, contradicting $\sigma_{\text{ess}}(H_0) = [0, \infty)$.

Thus we conclude that $\sigma_{\text{ess}}(H) \subset \sigma_{\text{ess}}(H_0)$. Therefore $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0)$.

Remark 3.4.5. We thus have $\sigma_{\text{ess}}(H+V) = [0, \infty)$. It is however possible that there exist discrete eigenvalues less than zero. In fact, if $\exists u \in \mathscr{D}(H)$ with $\langle Hu, u\rangle < 0$, then H has at least one negative eigenvalue since if not then $\sigma(H) \cap (-\infty, 0) = \emptyset$.
We then have $\sigma(H) \subset [0, \infty)$ and by spectral theorem we have $H \geq 0$, which is a contradiction.
It is also worthy to note that if at all there is an eigenvalue $\lambda < 0$, then $\lambda \in \sigma_{\text{disc}}(H)$.

Let us now look at examples of potentials V of the kind in this section. A particular example we will look at is

$$V(x) = V(r) = -\frac{1}{r}, \quad r > 0, r = |x|,$$

and obtain its negative eigenvalues and eigenfunctions. A natural approach with a radial potential such as ours is to use spherical coordinates. Note that we will just outline an approach to obtain said eigenvalues and not calculate them explicitly.
Consider the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on $L^2(S^{N-1})$, with S^{N-1} being the unit sphere in $\mathbb{R}^N$. This operator $-\Delta_{S^{N-1}}$ has a purely discrete spectrum (refer [4]), $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty, \quad j \to \infty$.
The eigenspaces associated with each λ_j are spanned by certain C^∞ functions $Y_{j,k}(x')$ called spherical harmonics[1].
Now, using separation of variables, we get

$$u(x) = f(r)Y_{j,k}(x'),$$

[1] Refer [19, Ch.4] for more details

where $x = rx'$, $x' \in S^{N-1}$, $r = |x|$, and eigenfunctions u associated with eigenvalue κ of the operator $-\Delta + V$ in the Hilbert space $L^2(\mathbb{R}^N)$ (for any potential V),

$$L^2(\mathbb{R}^N) = L^2((0,\infty), r^{N-1}dr) \otimes L^2(S^{N-1}, d\omega_{N-1}),$$

gives us the ODE

$$-v''(r) - \frac{N-1}{r}v'(r) + V(r)v(r) + \frac{\lambda_j}{r^2}v(r) = \kappa v(r)$$

for $r \in (0,\infty)$.

For $V(x) = -\frac{1}{r}$, we get

$$-v''(r) - \frac{N-1}{r}v'(r) - \left(\frac{1}{r} + \frac{\lambda_j}{r^2} + \kappa\right)v(r) = 0,$$

which is just a Bessel differential equation.

For $j \in \mathbb{N}, v = v_j \in L^2((0,\infty), r^{N-1}dr)$ becomes a solution for the Bessel differential equation. Accomplishing a unitary transformation $v \mapsto r^{N-1/2}v$, gives us an additional term including the factor $\frac{1}{r^2}$.

Therefore, in $\mathbb{R}^3$, for the Coulomb potential, we get an infinite sequence of negative eigenvalues. If V satisfies a certain speed of decay, i.e., decays faster than $-C_N(1+|x|^2)$ as $|x| \to \infty$, then the subspaces of $L^2(\mathbb{R}^N)$ spanned by the eigenfunctions are finite-dimensional.

The next example we will consider is a potential V of the form

$$V(x) = \frac{a}{|x|^2}, \quad a \geq -\left(\frac{N-2}{2}\right)^2, \quad \text{for } N \geq 3.$$

Call $L_a = -\Delta + \frac{a}{|x|^2}$ the Friedrichs extension of the same operator defined on $C_c^\infty(\mathbb{R}^N \setminus \{0\})$.

The restriction $a \geq -\left(\frac{N-2}{2}\right)^2$ makes the operator L_a positive semi-definite.

Consider $\sigma = -\frac{N-2}{2} - \frac{1}{2}(\sqrt{(N-2)^2 + 4a}$, with opposite sign to a.

If $L_a^\circ = L_a\big|_{C_c^\infty(\mathbb{R}^N\setminus\{0\})}$, then

$$\begin{aligned} L_a^\circ &= \left(-\nabla + \sigma\frac{x}{|x|^2}\right)\cdot\left(-\nabla + \sigma\frac{x}{|x|^2}\right), \\ &= -\Delta - \sigma(N-2-\sigma)\frac{1}{|x|^2}, \end{aligned}$$

where $a = -\sigma(N-2-\sigma)$.

If $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$,

$$\begin{aligned} \langle L_a^\circ\varphi, \varphi\rangle &= \int_{\mathbb{R}^N} L_a^\circ\varphi\cdot\overline{\varphi}\,dx, \\ &= \int_{\mathbb{R}^N}\left|\nabla\varphi(x) + \sigma\frac{x}{|x|^2}\varphi(x)\right|^2 dx \\ &\geq 0. \end{aligned}$$

upon substituting L_a° in.

Similarly, like with the Coulomb potential, L_a° is spherically symmetric, and we can use polar form.

Let $u(x) = f(r)Y_k(x')$ with $f \in C_c^\infty((0,\infty))$ and Y_k is a spherical harmonic of degree $k > 0$.

Applying L_a° to $f(r)Y_k(x')$ is equivalent to the Bessel operator:

$$f(r) \mapsto -\frac{d^2}{dr^2}f(r) - \frac{N-1}{r}f'(r) + \frac{a + k(k+N-2)}{r^2}f(r),$$

on the radial factor since $Y_{j,k}$ is harmonic.

This operator is also positive definite on $\mathbb{R}^N, L^2\left(r^{N-1}dr\right)$. Again, with the unitary transformation $g(r) = r^{\frac{N-1}{2}}f(r)$,

$$g(r) \mapsto -g''(r) + \frac{4\sigma^2 - 1}{4r^2}g(r),$$

which is a symmetric positive operator on $L^2(\mathbb{R}^N, dr)$.

The radial eigenfunctions of this operator have the asymptotics

$$u(x) = c|x|^{-\sigma}(1 + O(|x|))$$

as $|x| \to 0$. Decay is thus more rapid near the origin.

Remark 3.4.6. L_a° is essentially self-adjoint for $a \geq 1 - \left(\frac{N-2}{2}\right)^2$ on $C_c^\infty(\mathbb{R}^N \setminus \{0\})$, and L_a is its unique self-adjoint extension. But for $-\left(\frac{N-2}{2}\right)^2 \leq a < 1 - \left(\frac{N-2}{2}\right)^2$, the Friedrichs extension need not be the unique extension, but there exists a one-parameter class of extensions, that all differ from the Friedrichs one by just the change of one parameter. We will not delve into this further in this thesis, but [11] gives a comprehensive list of references that elaborate this operator in much greater detail.

3.5 Bounded and continuous V

Let $V : \mathbb{R}^n \to \mathbb{R}$ be bounded and (piecewise) continuous. Again, we assume that the operator $-\Delta + V|_{C_c^\infty(\mathbb{R}^n)}$ is essentially self-adjoint, leading to a unique self-adjoint extension:

$$H = H_0 + V = \overline{-\Delta + V|_{C_c^\infty(\mathbb{R}^n)}}.$$

Theorem 3.5.1. *Let $V : \mathbb{R} \to \mathbb{R}$ be defined by*

$$V(x) := \cos\sqrt{|x|},$$

and let $H = H_0 + V$. Then, we claim that

$$\sigma(H) = [-1, \infty).$$

Proof. First, observe that since $V(x) = \cos|x|$, we have $V(x) \leq 1$, so it follows immediately that $\sigma(H) \subseteq [-1, \infty)$.

For the converse, we will show that $[-1, 1] \subseteq \sigma(H)$. Specifically, we demonstrate that for each $\lambda \in [-1, 1]$, the operator H has essential spectrum at λ. We will construct a singular sequence for H at λ.

Let $\lambda \in [-1, 1]$ and choose $\varepsilon_n := \frac{1}{n}$. Then, for $n \in \mathbb{N}$, there exists a monotonically increasing sequence $\{x_n\} \subset \mathbb{R}$ such that

$$|V(x) - \lambda| < \varepsilon_n \quad \forall x \in B(x_n, 2n).$$

This follows from the fact that for $x > 0$,

$$V'(x) = -\frac{1}{2\sqrt{x}} \sin \sqrt{x} \to 0 \quad \text{as} \quad |x| \to \infty,$$

Thus, we also have that $\{x_n\}$ diverges to infinity. Also, assume that $B(x_n, 2n) \cap B(x_{n+1}, 2(n+1)) = \emptyset$ for all n.

Now, let $\psi_n \in C_c^\infty(\mathbb{R})$ be a cutoff function such that $0 \le \psi_n \le 1$, $\psi_n \equiv 1$ on $[-n, n]$, and ψ_n is supported on $(-2n, 2n)$. Define the sequence $\{\eta_n\}$ by

$$\eta_n(x) := \frac{\psi_n(x - x_n)}{\|\psi_n(x - x_n)\|_{L^2}},$$

so that $\|\eta_n\|_{L^2} = 1$. This sequence $\{\eta_n\}$ is compactly supported, and we claim it forms a singular sequence for H.

Indeed, for any $y \in L^2(\mathbb{R})$, we have

$$\langle \eta_n, y \rangle = \int_{\mathbb{R}} \frac{\psi_n(x - x_n)}{\|\psi_n(x - x_n)\|_{L^2}} y(x)\, dx \to 0 \quad \text{as} \quad n \to \infty,$$

since $x_n \to \infty$ as $n \to \infty$, and the support of ψ_n is compact.

Now, consider

$$\|(H_0 + V - \lambda I)\eta_n\| \le \|H_0 \eta_n\| + \|(V - \lambda I)\eta_n\|.$$

We know that $H_0\eta_n \to 0$ as $n \to \infty$, because the supports of η_n spread out, and the free Hamiltonian $H_0 = -\Delta$ on large regions of $\mathbb{R}$ tends to zero. Similarly, since $|V(x_n)-\lambda| < \varepsilon_n$ on $B(x_n, 2n)$, we have $\|(V-\lambda I)\eta_n\| \to 0$. Therefore, $\|(H-\lambda I)\eta_n\| \to 0$ as $n \to \infty$, implying that

$$\lambda \in \sigma_{ess}(H) \subseteq \sigma(H).$$

Next, we show that $[1, \infty) \subseteq \sigma(H)$. Let $\lambda \in [1, \infty)$. We take a sequence $\{x_n\}$ on $\mathbb{R}$ where $|V(x_n)| < \varepsilon_n$ on $B(x_n, 2n)$. Define the singular sequence $\{u_n\}$ for H_0 and λ that we used earlier to show that $\sigma_{ess}(H_0) = [0, \infty)$. Let $v_n := u_n(x - x_n)$. Then we have

$$\|(H - \lambda I)v_n\| \leq \|(H_0 - \lambda I)v_n\| + \|Vv_n\| \leq \|(H_0 - \lambda I)v_n\| + \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$\lambda \in \sigma_{ess}(H) \subseteq \sigma(H).$$

In conclusion, we have shown that $\sigma(H) = \sigma_{ess}(H) = [-1, \infty)$. $\square$

This brings us to the end of this chapter. In the next chapter, we will apply what was established in this chapter to solve certain kinds of partial differential equations.

Chapter 4

Introduction to Evolution Equations

4.1 Introduction

Throughout Chapters 1 and 2, we have established the necessary tools to tackle our main problem. This chapter contains more preliminary content, but this will be the base of our research problem to be introduced in Chapter 4. An evolution equation is a partial differential equation of the form

$$u_t = Au,$$

where $u = u(x, t)$, $u_t = \frac{\partial u}{\partial t}$, and A is a linear operator on a Hilbert space $\mathcal{H}$. Of course, the derivatives are in the non-classical/distributional sense. This chapter will be devoted to analyzing different examples of evolution equations.

Consider the initial value problem (I.V.P) for solution $u \in L^2(\mathbb{R})$,

$$\begin{cases} u_t & = Au, \quad A \in \mathbb{R},\, t > 0,\, x \in \mathbb{R}, \\ u(x,0) & = u_0(x). \end{cases}$$

This comprises of an evolution equation, with A being the 1×1 matrix, $[a]$. This is a usual ordinary differential equation and the solution to the I.V.P is given by

$$u(x,t) = u_0(x)e^{at}.$$

Consider the I.V.P. when A is a bounded operator on a Hilbert space.

$$\begin{cases} u_t & = Au, \quad t > 0,\, x \in \mathbb{R}^N, \\ u(x,0) & = u_0(x). \end{cases}$$

The solution is given by

$$u(x,t) = u_0(x)e^{At},$$

where e^{At} can be defined as

$$e^{At} = 1 + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^kA^k}{k!} + \ldots.$$

This expansion on the RHS is convergent whenever A is a bounded operator on $\mathcal{H}$. Hence, the major question is when A is an unbounded operator, say a differential operator. The rest of this chapter will be devoted to the cases of the examples we discussed in Chapter 2.

4.2 Semigroup Theory

Definition 4.2.1. Let $\mathcal{H}$ be a Hilbert space and let $\{E(t) : t \in \mathbb{R}\}$ be a collection of linear operators with the properties:

(a) $E(t+s) = E(t)E(s)$ for $s, t \in \mathbb{R}$,

(b) for all $f \in \mathcal{H}$ and all sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $t_n \to t$, one has the strong convergence $E(t_n)f \to E(t)f$,

(c) $E(0) = I$.

Such a collection is called a strongly continuous one-parameter group. A semigroup is half of this group, i.e.,

$$\{E(t) : t \geq 0\}.$$

Definition 4.2.2. A contraction semigroup is a semigroup with $\|E(t)\| \leq 1$ for all $t \geq 0$.

Definition 4.2.3. Let $\{E(t) : t \geq 0\}$ be a contraction semigroup and define the linear operator A as

$$Af := \lim_{t\to 0} \frac{E(t)f - f}{t},$$

with $\mathscr{D}(A) = \left\{ f \in \mathcal{H} : \lim_{t\to 0} \frac{E(t)f-f}{t} \text{ exists} \right\}$. Then A is called the infinitesimal generator of the contraction semigroup.

Why are we looking at semigroups? Consider the contraction semigroup $\{e^{tA} : t \geq 0\}$ when A is a bounded operator. Also consider the I.V.P.

$$\begin{cases} u_t &= Au, \quad t \geq 0, \\ u(x,0) &= u_0(x). \end{cases} \tag{4.1}$$

Clearly, if $E(t) = e^{tA}$, $E(t)u_0$ satisfies the I.V.P since

$$\frac{\partial}{\partial t} E(t)u_0 = Ae^{tA}u_0 = AE(t)u_0.$$

Therefore, if A is a bounded operator, it is precisely the infinitesimal generator of the contraction semigroup

$$\{E(t) = e^{tA}, t \geq 0\}.$$

A natural question to ask now, since we are usually given the operator A in the IVP, is when A generates a contraction semigroup.

Lemma 4.2.4. *Let $\{E(t) : t \geq 0\}$ be a contraction semigroup and A its infinitesimal generator. Then,*

(a) A is densely defined,

(b) A is a closed operator,

(c) $[0, \infty) \supseteq \rho(A)$,

(d) $\|R_\lambda\| \leq \frac{1}{\lambda}, \quad \lambda \in (0, \infty)$.

Hille and Yosida showed that if the operator A satisfies the properties of the lemma, then A generates a semigroup.

Theorem 4.2.5. ***(Hille-Yosida)*** *If an operator A is given, satisfying the properties of the lemma, then there exists a semigroup $\{E(t) : t \geq 0\}$ such that A is precisely its infinitesimal generator.*

The proofs of both the above results are tedious and add little to our discussion, and hence are omitted.
We know that $\{E(t) = e^{tA}\}$ is a straightforward choice for semigroup generated by A when A is a bounded operator. What do we do when A is unbounded? Stone's Theorem in the next section will help us with this.
The following theorem makes satisfying the conditions of the Hille-Yosida theorem easier.

Definition 4.2.6. An operator $A : \mathscr{D}(A) \to \mathcal{H}$ is said to be dissipative if $\mathrm{Re}\langle Au, u\rangle \leq 0 \quad$ for each $u \in \mathscr{D}(A)$.

Theorem 4.2.7. ***(Lumer-Phillips)*** *Let $A : \mathscr{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ be a densely defined operator.*

(a) If A is dissipative and $\mathscr{R}(A - \lambda_0 I) = \mathcal{H}$ for at least one $\lambda_0 > 0$, then A is the infinitesimal generator of a semigroup $\{E(t) : t \geq 0\}$.

(b) If A is the infinitesimal generator of a semigroup $\{E(t) : t \geq 0\}$, then $\mathscr{R}(A - \lambda I) = \mathcal{H}$ for all $\lambda > 0$ and A is dissipative.

4.3 Solution to the IVP

Now that we have established how we can verify if (4.1) has a solution, and what that solution is, let us characterize the solution, $E(t)u_0$.

Theorem 4.3.1 (M.H. Stone)**.** *Let $\{E(t) : t \in \mathbb{R}\}$ be a unitary, strongly continuous single-parameter group. Then, there is a unique self-adjoint operator T satisfying*

$$E(t) := e^{itT}, \quad t \in \mathbb{R}.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R})$ and $f \in L^2(\mathbb{R}^N)$. Define

$$f_\varphi := \int_{-\infty}^{\infty} \varphi(t)E(t)f \, dt.$$

This integral is a Riemann integral due to the strongly continuous nature of $E(t)$.
Define

$$\mathscr{D} := \{\text{all continuous linear combinations of functions } f_\varphi\}.$$

For a given $\epsilon > 0$, consider $\varphi_\epsilon \in C_c^\infty(\mathbb{R})$ with the properties $\varphi_\epsilon \geq 0$, $\varphi_\epsilon \subset [\epsilon, \epsilon]$ and $\int_{-\infty}^{\infty} \varphi_\epsilon(x)dx = 1$. Therefore,

$$\begin{aligned} \|f_\epsilon - f\| = \left\| \int_{-\epsilon}^{\epsilon} \varphi_\epsilon(t)(E(t)f - f) \, dt \right\| \leq \left(\int_{-\epsilon}^{\epsilon} \varphi_\epsilon(t) \, dt \right) \sup_{-\epsilon \leq t \leq \epsilon} \|E(t)f - f\| \\ \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0. \end{aligned}$$

This proves that $\mathscr{D}$ is dense in $L^2(\mathbb{R}^N)$.
Furthermore, for each $f_\varphi \in \mathscr{D}$,

$$\begin{aligned} \frac{1}{s}\left(E(s) - I\right) f_\varphi &= \int_{-\infty}^{\infty} \varphi(t) \frac{1}{s} \left(E(s+t) - E(t)\right) f \, dt \\ &= \int_{-\infty}^{\infty} \varphi'(t)E(t)f \, dt = f_{-\varphi'}, \end{aligned}$$

since

$$\frac{\varphi(t-s) - \varphi(t)}{s} \longrightarrow -\varphi'(t)$$

uniformly on supp φ as $s \to 0$.

Define the operator T for $f_\varphi \in \mathscr{D}$ as

$$Tf_\varphi := -if_{-\varphi'} = \lim_{s\to 0} \frac{1}{is}(E(s) - I)f_\varphi.$$

Note that T maps $\mathscr{D}$ onto itself:

$$\begin{aligned} E(t)f_\varphi &= E(t)\left(\int_{\mathbb{R}} \varphi(s)E(s)f\,ds\right) \\ &= \int_{\mathbb{R}} \varphi(s)E(s)E(t)f\,ds \\ &= \int_{\mathbb{R}} \varphi(s)E(s+t)f\,ds \\ &= (E(t)f)_\varphi. \end{aligned}$$

Also, note that T commutes with $E(t)$ on $\mathscr{D}$:

$$TE(t)f_\varphi = T(E(t)f)_\varphi = -i(E(t)f)_{-\varphi'} = E(t)(-if_{-\varphi'}) = E(t)Tf_\varphi.$$

Finally, note that T is symmetric on $\mathscr{D}$:

$$\langle Tf_\varphi, g_\varphi\rangle = \lim_{s\to 0}\left\langle \frac{1}{is}\left(E(s) - I\right)f_\varphi, g_\varphi\right\rangle = \lim_{s\to 0}\left\langle f_\varphi, \frac{1}{is}\left(I - E(-s)\right)g_\varphi\right\rangle = \langle f_\varphi, Tg_\varphi\rangle.$$

Let us now show that T is essentially self-adjoint on $\mathscr{D}$. We will show that

$$\ker(T^* + iI) = \{0\}.$$

Let $u \in \ker(T^* + iI) \subseteq \mathscr{D}(T^*)$. For all $\eta \in \mathscr{D} = \mathscr{D}(T)$,

$$\frac{d}{dt}\langle E(t)\eta, u\rangle = \langle iTE(t)\eta, u\rangle = i\langle E(t)\eta, T^*u\rangle = -\langle E(t)\eta, u\rangle \quad (\text{since}\, T^*u = -iu).$$

Therefore,

$$F(t) := \langle E(t)\eta, u\rangle \quad \text{satisfies} \quad F' = -F.$$

This is an ODE whose solution is

$$F(t) = F(0)e^{-t}.$$

Since

$$|F(t)| \le \|\eta\|\|u\|,$$

we conclude that $F(0) = 0$. Henceforth, $0 = F(0) = \langle E(0)\eta, u\rangle = \langle \eta, u\rangle$ for all $\eta \in \mathscr{D}$. This forces $u = 0$, since $u \in L^2$ but $\mathscr{D}$ is dense in $L^2(\mathbb{R})$.

We have thus shown that T is essentially self-adjoint on $\mathscr{D}$.

Set $U(t) := e^{it\overline{T}}$, where $\overline{T}$ is the closure of T. Clearly, $\{U(t), t \in \mathbb{R}\}$ is a strongly continuous unitary group.

Finally, we show $U(t) = E(t)$ for all $t \in \mathbb{R}$. For every $\eta \in \mathscr{D} \subset \mathscr{D}(\overline{T})$,

$$\frac{d}{dt}U(t)\eta = i\overline{T}U(t)\eta.$$

Also, we have

$$E(t)\eta \in \mathscr{D} \subset \mathscr{D}(\overline{T}) \quad \text{for all}\, t \in \mathbb{R}.$$

Employing an age-old technique, define for each η,

$$\alpha(t) := E(t)\eta - U(t)\eta, \quad t \in \mathbb{R}.$$

Clearly, $\alpha(t)$ is strongly differentiable, where

$$\alpha'(t) = iTE(t)\eta - i\overline{T}U(t)\eta = iT\alpha(t).$$

Since T is essentially self-adjoint, $\overline{T}$ is self-adjoint.

We have,

$$\frac{d}{dt}\|\alpha(t)\|^2 = i\langle \overline{T}\alpha(t), \alpha(t)\rangle - i\langle \alpha(t), \overline{T}\alpha(t)\rangle = 0,$$

and $\alpha(0) = 0$.

This forces $\alpha(t) = 0$, and thereby $E(t)\eta = U(t)\eta$ for all $\eta \in \mathscr{D}, t \in \mathbb{R}$.

Since $\mathscr{D}$ is dense in $\mathcal{H}$, we have that

$$E(t) = U(t) \quad \forall t \in \mathbb{R}.$$

This proves Stone's theorem. □

This theorem tells us how the contraction semigroup from the Hille-Yosida theorem looks like by guaranteeing the existence of a unique self-adjoint operator T (which is the infinitesimal generator of the semigroup) and establishing a bijection between unitary strongly continuous groups $\{E(t) : t \in \mathbb{R}\}$ and operators of the form e^{itT}.

If A is unbounded, consider the Yosida approximation

$$A_\lambda := \lambda A R_\lambda = \lambda(A - \lambda I)^{-1} A \quad \text{(since } A \text{ commutes with } R_\lambda).$$

If A_λ is bounded and $(\frac{A}{\lambda} - I)^{-1}A \to A$ as $\lambda \to \infty$, $E_\lambda(t) = e^{tA_\lambda}$ exists and we call

$$\lim_{\lambda\to\infty} e^{tA_\lambda} = e^{tA},$$

and define the semigroup A generates to be $\{e^{tA} : t \geq 0\}$. This gives us one way of evaluating such an operator whenever A is unbounded.

Let us now look at examples.

Example 4.3.2. Consider the I.V.P.

$$\begin{cases} u_t & = u_x, \quad t \geq 0,\, x \in \mathbb{R}, \\ u(x,0) & = u_0(x),\, x \in \mathbb{R}, \end{cases}$$

on $\mathcal{H} = L^2(\mathbb{R})$ and choose $A\varphi = \frac{\partial^2 \varphi}{\partial x^2}$ with domain $\mathscr{D}(A) = H_0^1(\mathbb{R})$.

Note that

$$\langle A\varphi, \varphi \rangle = \langle \varphi', \varphi \rangle = -\langle \varphi, \varphi' \rangle \quad \text{(integration by parts)}.$$

Also,

$$\langle \varphi', \varphi \rangle = \overline{\langle \varphi, \varphi' \rangle},$$

which implies $\mathrm{Re}\langle A\varphi, \varphi \rangle = 0$, making A dissipative.
If we show that $\mathscr{R}(A - \lambda I) = \mathcal{H}$ for at least one $\lambda > 0$, by part (b) of the Lumer-Phillips theorem, A generates a semigroup.
Note that showing $\mathscr{R}(A - \lambda I) = \mathcal{H}$ is the same as showing for all $f \in L^2(\mathbb{R})$, there exists a solution to

$$\lambda\varphi - \varphi' = f, \quad \lambda > 0.$$

Using the integrating factor $e^{-\lambda x}$, we can solve the above ODE to get

$$\varphi(x) = \int_{-\infty}^{x} e^{-\lambda(x-s)} f(s)\, ds.$$

This is a unique solution which can be verified by standard methods. Therefore,

$$\mathscr{R}(A - \lambda I) = L^2(\mathbb{R}).$$

A hence generates a contraction semigroup $\{E(t) : t \geq 0\}$ and $u(x,t) = E(t)u_0(x)$. We have thus proven that the I.V.P has a solution. Here, we can explicitly write down the semigroup. For $v \in L^2(\mathbb{R})$,

$$E(t)v(x) = v(x-t).$$

Therefore, the solution is

$$u(x,t) = u_0(x-t).$$

Example 4.3.3. The Heat Equation

$$\begin{cases} u_t = \Delta u, & t > 0,\ x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

with $\mathcal{H} = L^2(\mathbb{R}^N)$, $A = \Delta$ with domain $\mathscr{D}(A) = H_0^2(\mathbb{R}^N)$. Again, let us show that A is dissipative. Using integration by parts,

$$\langle \Delta\varphi, \varphi\rangle = -\langle \nabla\varphi, \nabla\varphi\rangle = -\|\nabla\varphi\|^2 \leq 0.$$

We claim now that $\mathscr{R}(A-\lambda I) = \mathcal{H}$ for some $\lambda > 0$, or again as before, for each $f \in L^2(\mathbb{R}^N)$ we show that $\lambda u - \Delta u = f$, $x \in \mathbb{R}^N$, $\lambda > 0$ has a unique solution.
Taking Fourier transform, we have

$$\lambda\widehat{u}(\xi) = |\xi|^2\widehat{u}(\xi) = \widehat{f}(\xi), \quad \xi \in \mathbb{R}^N,$$

which, by Plancherel's formula and Fourier inversion, gives us a unique solution $u \in H^2(\mathbb{R}^N)$ for each $f \in L^2(\mathbb{R}^N)$.

Applying Lumer-Phillips, $A = \Delta$ generates a contraction semigroup and hence the IVP has a solution. Using the Yosida Approximation, we have that A generates $\{e^{tA} : t \geq 0\}$. By the Fourier transform, we can write the solution

$$u(x,t) = E(t)u_0(x) = e^{t\Delta}u_0(x)$$

where

$$e^{t\Delta}u_0(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy.$$

Example 4.3.4. Now consider a more abstract situation, but one we have encountered before, the I.V.P

$$\begin{cases} u_t = Au, & t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $A : \mathscr{D}(A) \subset \mathcal{H} \to \mathcal{H}$, $u_0 \in \mathcal{H}$. Assume that $-A$ is self-adjoint, positive definite linear operator with compact inverse. The operator $-\Delta + |x|^2$ falls into this category.

For such an operator, $\exists$ an orthonormal basis of eigenfunctions $\{\varphi_j\}$ and an increasing sequence of corresponding eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$$

with $\lambda_j \to \infty$ as $j \to \infty$.

Hence for $u \in \mathcal{H}$, we have the Fourier expansion

$$u = \sum_{j=1}^{\infty} \langle u, \varphi_j \rangle \varphi_j$$

and

$$-Au = \sum_{j=1}^{\infty} \lambda_j \langle u, \varphi_j \rangle \varphi_j.$$

Set $u(t) = \sum_{j=1}^{\infty} u_j(t)\varphi_j$, where $u_j(t) = \langle u(t), \varphi_j \rangle$. Therefore, by taking the inner product of the IVP with φ_j, we get

$$\begin{cases} \frac{d}{dt}u_j + \lambda_j u_j & = 0, t > 0 \\ u_j(0) & = u_{0,j} \end{cases}$$

where $u_{0,j} = \langle u_0, \varphi_j \rangle$.

Solving the above ODE gives us

$$u_j(x,t) = e^{-\lambda_j t} u_{0,j}(x),$$

which leads to

$$u(x,t) = E(t)u_0(x) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle u_0, \varphi_j \rangle \varphi_j.$$

The semigroup $\{E(t)\}$ generated is a semigroup by Lumer-Phillips with A as infinitesimal generator.

In particular, for $A = -\Delta + x^2$ on $L^2(\mathbb{R})$, we have

$$u(x,t) = \sum_{k=0}^{\infty} e^{-(2k+1)t} \langle u_0, \varphi_j \rangle \varphi_j,$$

where φ_j are the eigenfunctions of the operator A.

Example 4.3.5. An interesting question to now ask is if the wave equation, which is not traditionally in the form of the evolution equations we have been studying, can be manipulated in some way so we can apply the theory in this chapter.

Consider the I.V.P,

$$\begin{cases} u_{tt}(x,t) = \Delta u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \\ u_t(x,0) = u_1(x), & x \in \mathbb{R}^N. \end{cases}$$

The trick is to write this as a first-order problem in two dimensions, by setting

$$f(x,t) = u(x,t), \quad g(x,t) = u_t(x,t).$$

Then, the wave equation I.V.P transforms to

$$\frac{\partial}{\partial t}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \quad x \in \mathbb{R}^N, t > 0,$$

$$\begin{pmatrix} f(x,0) \\ g(x,0) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix}.$$

This falls into our category of evolution equations, on the Hilbert space

$$\mathcal{H} = H^1(\mathbb{R}^N) \oplus L^2(\mathbb{R}^N).$$

Define $A(u,v) = \langle v, \Delta u\rangle$ where $u, v \in D(A)$. It can then be shown again by methods much alike to the previous examples that A is an infinitesimal generator of a semigroup $\{E(t) : t \geq 0\}$, making the solution $E(t)\begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix}$.

Remark 4.3.6. The semigroup theory is a very important tool when it comes to showing the existence of solutions to a given initial value problem. However to find the solution explicitly, we need to use methods specific to each problem.

We have answered the question: How to get a solution for (4.1), let's ask a different question. In a finite-dimensional vector space, say of dimension N, if A is replaced by $m(A)$, where m is a measurable function, for example, a polynomial, we know that $m(A)$ has eigenvalues $m(\lambda_j)$, $\{j = 1, 2, ..., N\}$ where λ_j are eigenvalues of A. From there on, the solution to (4.1) just follows as in the case with A.

What do we do in the case when the vector space is infinite dimensional and our operator is self-adjoint in that space? Using all the theory we explored up until now, the spectral theorem, semigroup theory and the theory of self-adjoint extensions, how do we tackle this? This will be answered in the next chapter.

Chapter 5

Spectral Multipliers

5.1 Introduction

As teased in the previous chapter, we can define for a bounded measurable function m and a positive-definite self-adjoint operator T on a Hilbert space $\mathcal{H}$:

$$m(T) := \int_{\sigma(T)} m(\lambda)\, dP(\lambda),$$

where $\sigma(T)$ is the spectrum of T and P is the unique projection-valued measure that is the spectral decomposition of T.

Clearly, $m(T)$ is an operator again on the Hilbert space $\mathcal{H}$, but it need not necessarily be linear. Consider the example $m(\lambda) = \operatorname{sgn}(\lambda), \lambda \in \mathbb{R}$. This is therefore, new territory. The natural problem here is to ask when is $m(T)$ bounded.

For this, let us look at the case when $\mathcal{H} = L^2(\mathbb{R}^N)$. Then, for $f \in L^2(\mathbb{R}^N)$,

$$\begin{aligned}
\|m(T)f\|_2^2 &= \int_{\mathbb{R}^N} |m(T)f(x)|^2 \, dx \\
&= \int_{\mathbb{R}^N} \left| \left(\int_{\sigma(T)} m(\lambda) \; dP(\lambda) \right) f(x) \right|^2 \; dx \\
&\leq \|m\|_\infty^2 \int_{\mathbb{R}^N} |f(x)|^2 \, dx \\
&= \|m\|_\infty^2 \|f\|_2^2.
\end{aligned}$$

Therefore, $m(T)$ is a bounded operator on $L^2(\mathbb{R}^N)$.

Definition 5.1.1. An operator $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is called

(i) strong (p, q) if T is bounded, i.e.

$$\|T(f)\|_q \leq C_{p,q} \; \|f\|_p$$

for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$.

(ii) weak (p, q) if

$$\sup_{\lambda > 0} \; (\mu_f(\lambda))^{\frac{1}{q}} \leq C_{p,q} \; \|f\|_p,$$

where $\mu_f(\lambda) := \{(x \in \mathbb{R}^n : |Tf(x)| > \lambda)\}$, for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$.

Remark 5.1.2. Although we have defined the above notions for $\mathbb{R}^n$ as the ambient space, we can most definitely generalize these definitions for an arbitrary ambient space X.
Now that we have shown that $m(T)$ is a bounded operator on $L^2(\mathbb{R}^n)$, it is only to ask if it can be bounded on $L^p(\mathbb{R}^n)$, $p \neq 2$.

A natural way one might go about this is to prove the result for $L^2 \cap L^p(\mathbb{R}^n)$. Then use a limiting argument. However, there are several intricacies one must account for, and tackling these is the objective of this chapter.

5.2 Fourier Multipliers

Definition 5.2.1. Let $\lambda \leq p \leq \infty$ and $m \in S'(\mathbb{R}^N)$. Then m is said to be a Fourier multiplier on $L^p(\mathbb{R}^N)$ if $T_m f \ := \check{m} * f \in L^p(\mathbb{R}^N)$ for all $f \in S(\mathbb{R}^N)$ and

$$\sup_{\|f\|_p=1} \|\check{m} * f\|_p < \infty,$$

where m is the inverse Fourier transform of m.

The set of all such m is called $\mathcal{M}_p(\mathbb{R}^N)$ and it is a Banach space with respect to the norm $\|m\|_{\mathcal{M}_p(\mathbb{R}^N)} = \sup_{\|f\|_p=1} \|m * f\|_p$. The following is a very important measure theoretic fact that will be useful througout the remainder of this thesis. Refer to [15, Ch. 4, Sec. 4.3] for a proof.

Theorem 5.2.2 (**Tchebychev**). *Let X be a Banach space. Then,*

$$\mu_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}) \leq \lambda^{-p}\|f\|_p^p$$

for all $f \in L^p(X)$.

Corollary 5.2.3. *If T is strong (p, q) then it is weak (p, q).*

Proof. We are given that $\exists\ C_{p,q}$ such that

$$\|Tf\|_q \leq C_{p,q}\ \|f\|_p \quad \forall f \in L^p(\mathbb{R}^N).$$

Notice by Tchebychev's inequality,

$$\big|\{x \in \mathbb{R}^N : |Tf(x)| > \lambda\}\big| \leq \lambda^{-q}\|Tf\|_q^q.$$

Then,

$$\lambda^q \left|\{x \in \mathbb{R}^N : |Tf(x)| > \lambda\}\right| \leq \|Tf\|_q^q,$$

implying that

$$\lambda \left|\{x \in \mathbb{R}^N : |Tf(x)| > \lambda\}\right|^{\frac{1}{q}} \leq C_{p,q} \, \|f\|_p.$$

Since the RHS is independent of λ,

$$\sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^N : |Tf(x)| > \lambda\}\right|^{\frac{1}{q}} \leq C_{p,q} \, \|f\|_p.$$

which proves that T is weak (p, q). $\square$

The next two results will form the basis of proof techniques in proving any operator of the form $m(T)$ is L^p bounded, $1 < p < \infty$, whenever it is linear.

Definition 5.2.4. An operator T is said to be *sublinear* if $T(f + g)$ is determined by the values of Tf, Tg and

$$|T(f + g)| \leq |Tf| + |Tg|.$$

The definition of strong and weak (p, q) can be extended to such sublinear operators. This leads to a theorem that harnesses a strong method of proof: duality.

Theorem 5.2.5 (**Marcinkiewicz Interpolation**)**.** *Let $1 < r \leq \infty$ and T be a sublinear operator. If T is weak $(1, 1)$ and strong (r, r), then T is strong (p, p) for any $1 < p < r$.*

Refer [6, Ch. 6, Sec. 6.5] for a proof of this well-known theorem.

The following provides a very useful decomposition of L^1-functions.

Theorem 5.2.6 (Calderón-Zygmund Decomposition). *Let $f \in L^1(\mathbb{R}^N)$. For every $\eta > 0$, f can be decomposed as $f = g + b = g + \sum_{j=1}^{\infty} b_j$ such that*

$$|g(x)| \leq 2^N \eta \quad a.e.\ x \in \mathbb{R}^N, \tag{5.1}$$

where supp $(b_j) \in Q_j$, Q_j is a cube with $\int b_j dx = 0$. The Q_j have disjoint interior,

$$\sum_{j=1}^{\infty} |Q_j| \leq \eta^{-1} \|f\|_1, \tag{5.2}$$

and

$$\|g\|_1 + \sum_{j=1}^{\infty} \|b_j\|_1 \leq 6\|f\|_1. \tag{5.3}$$

Proof. It is enough to show the theorem for $f \geq 0$, as otherwise write $f = f^+ - f^-$ and perform the decomposition separately for f^+ and f^-.
Notice that since $f \in L^2(\mathbb{R}^N)$, given η, we can find $l \geq 0$ such that for every cube of side length l, $|Q|^{-1} \int f dy < \eta$. Make an l-sided cube mesh of $\mathbb{R}^N$ and choose Q^0 among these cubes.
Subdivide Q^0 into 2^N cubes of side $\frac{l}{2}$ each and let Q^1 be one cube among them: $\exists$ two possibilities:

(i) $|Q^1|^{-1} \int_{Q^1} f \ dy < \eta$

(ii) $|Q^1|^{-1} \int_{Q^1} f \ dy \geq \eta$.

We don't need to do any further subdivision in the case (i), since we have our estimate. Collect all such cubes Q_j in a sequence. In case of (ii), continue the subdivision. Therefore, if $x \notin \bigcup_j Q_j$, by Lebesgue Differentiation theorem, we get

$$f(x) \leq \eta, \quad \text{a.e. } x \in \mathbb{R}^N \setminus \bigcup_j Q_j.$$

Define:

$$g(x) = \begin{cases} \frac{1}{|Q_j|} \int_{Q_j} f \, dy & x \in Q_j; \\ f(x) & x \notin Q_j \end{cases}$$

$$b_j(x) = (f(x) - g(x))\chi_{Q_j}(x), \quad j \in \mathbb{N}.$$

Clearly, $|g(x)| \leq 2^N \eta$ a.e. $x \in \mathbb{R}^N$,

$$\int_{Q_j} b_j(x) \, dx = \int_{Q_j} f(x) - g(x) \, dx = 0,$$

and from (i), we get $\sum_{j=1}^{\infty} |Q_j| \leq \frac{1}{\eta} \|f\|_1$, concluding our proof. □

We conclude this section with a multiplier theorem [9].

Theorem 5.2.7 (**Mikhlin-Hörmander**). *Let $m \in L^\infty(\mathbb{R}^N)$ and s be the least integer $> \frac{N}{2}$. Then for all $|\alpha| \leq s$, if*

$$\sup_{R>0} R^{-N+2|\alpha|} \int_{R<|\xi|<2R} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} m(\xi) \right|^2 d\xi < A^2, \tag{5.4}$$

then $m(t)$ is an L^p multiplier for any $p \in (1, \infty)$.

Moreover, T_m is weak (1,1).

Remark 5.2.8. The condition (5.4) is due to Hörmander. However, the theorem was first proved by Mikhlin for a stronger assumption,

$$\left| \frac{\partial^\alpha}{\partial \xi^\alpha} m(\xi) \right| \leq B_\alpha |\xi|^{-|\alpha|} \tag{5.5}$$

for all multi-indices α. Let us show that if a bounded measurable function satisfies the Mikhlin's condition, it satisfies the Hörmander condition too.

$$
\begin{aligned}
R^{-N+2|\alpha|}\int_{R<|\xi|<2R}\left|\frac{\partial^\alpha}{\partial\xi^\alpha}m(\xi)\right|^2 d\xi &\leq R^{-N+2|\alpha|}B_\alpha^2\int_{R<|\xi|<2R}|xi|^{-2|\alpha|}d\xi\\
&= R^{-N+2|\alpha|}\omega_{N-1}B_\alpha^2\int_R^{2R} r^{-2|\alpha|}\ r^{N-1}\ dr\\
&= R^{-N+2|\alpha|}\omega_{N-1}B_\alpha^2\frac{r^{-2|\alpha|+N}}{-2|\alpha|+N}\Big|_R^{2R}\\
&= A_\alpha < \infty.
\end{aligned}
$$

This final constant A_α is independent of R, and hence if we take $\sup_{R>0}$ on both sides of the inequality and see that since we are considering only a finite number of multi-indices α, we can find A such that $A_\alpha < A^2$ for all $|\alpha| \leq s$.

Proof. **(Mikhlin-Hörmander)** If we show that T_m is weak $(1,1)$, i.e. for every $f \in L^p(\mathbb{R}^n)$, if we have

$$\sup_{\lambda>0}\lambda\ \mu_f(\lambda) \leq c\ \|f\|_1,$$

then since T_m is strong $(2,2)$, using Marcinkiewicz-Interpolation theorem, we have that T_m is strong (p,p) for $1<p<2$. Since $T_m : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a bounded linear map, $\exists!\ \ T_m^* : L^{p'}(\mathbb{R}^N) \to L^{p'}(\mathbb{R}^N)$ with $\frac{1}{p}+\frac{1}{p'}=1$ such that

$$\langle T_m f, g\rangle = \langle f, T_m^* g\rangle, \quad \forall\ f \in L^p(\mathbb{R}^N), g \in L^{p'}(\mathbb{R}^N).$$

The adjoint operator T_m^* is a bounded operator with the same norm as T_m, and in particular, it is exactly equal to $T_{\overline{m}}$, where $\overline{m}$ is the complex conjugate of m: This is because for any $f \in L^2 \cap L^p(\mathbb{R}^N)$ and $g \in L^2 \cap L^{p'}(\mathbb{R}^N)$, we have

$$\begin{aligned}
\langle T_m f, g\rangle &= \int_{\mathbb{R}^N} \mathcal{F}(T_m f)(\xi)\ \overline{\hat{g}(\xi)}\ d\xi, && \text{(by Parseval)}\\
&= \int_{\mathbb{R}^N} m(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi,\\
&= \int_{\mathbb{R}^N} \hat{f}(\xi)\overline{\overline{m(\xi)}\ \hat{g}(\xi)} d\xi\\
&= \langle f, T_{\overline{m}} g\rangle. && \text{(again by Parseval)}
\end{aligned}$$

By the uniqueness of the adjoint and the denseness of $L^2 \cap L^p$ in L^p, we have that $T_m^* = T_{\overline{m}}$. This will conclude that $T_m : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a bounded operator for all $p \in (1, \infty)$. All that remains is to show that T_m is weak (1,1).

To show T_m is weak $(1, 1)$, we need to show for any $\lambda > 0$,

$$\mu_{T_m f}(\lambda) = \left|x \in \mathbb{R}^N : |T_m f(x)| > \lambda\right| \leq \frac{c}{\lambda}\|f\|_1.$$

Note that if $f = g + b$ is the Calderón-Zygmund decomposition of f, then

$$\mu_{T_m f}(\lambda) \leq \mu_{T_m g}\left(\frac{\lambda}{2}\right) + \mu_{T_m b}\left(\frac{\lambda}{2}\right)$$

We know that $b = \sum_{j=1}^{\infty} b_j$ such that $\exists$ a collection of cubes Q_j with disjoint interior with supp $b_j \subseteq Q_j$ $\forall$ j. Call $\Omega = \bigcup_j Q_j$. Define another measurable set $\Omega^* = \bigcup_j Q_j^*$ where Q_j^* is Q_j but with twice the side length.

Then,

$$\begin{aligned}
\mu_{T_m f}(\lambda) &\leq \mu_{T_m g}\left(\frac{\lambda}{2}\right) + \mu_{T_m b}\left(\frac{\lambda}{2}\right), \\
&\leq \mu_{T_m g}\left(\frac{\lambda}{2}\right) + \left|\{x \notin \Omega^*\ :\ |T_m b(x)| > \frac{\lambda}{2}\}\right| + |\Omega^*|\\
&= \mu_1 + \mu_2 + \mu_3.
\end{aligned} \tag{5.6}$$

Let us estimate μ_3 first.

$$\mu_3 = |\Omega^*| \leq \sum_{j=1}^{\infty} |Q_j^*| \leq 2^N \sum_{j=1}^{\infty} |Q_j| \leq \frac{2^n c}{\lambda}. \tag{5.7}$$

Now let us tackle μ_1.

$$\begin{aligned}
\mu_1 = \mu_{T_m g}\left(\frac{\lambda}{2}\right) &= \left|\left\{\alpha \in \mathbb{R}^N : |T_m g| > \frac{\lambda}{2}\right\}\right| \\
&\leq c\left(\frac{2}{\lambda}\right)^2 \|T_m g\|_2^2 && \text{(Tchebychev)} \\
&\leq \frac{c}{\lambda^2}\|g\|_2^2 && (T_m \text{ is strong } (2,2)) \\
&\leq \frac{c}{\lambda^2}\|g\|_1\|g\|_\infty && \text{(Hölder)} \\
&\leq \frac{c}{\lambda^2}\|g\|_1 2^N\lambda && (5.1) \\
&= \frac{c}{\lambda}\|f\|_1.
\end{aligned}$$

For brevity we incorporate all constants together at the end, into c.

Finally, we tackle $\mu_2 = \left|\left\{\alpha \in \Omega^* : |T_m b(x)| > \frac{\lambda}{2}\right\}\right|$. We need to produce a c such that $\mu_2 \leq \frac{c}{\lambda}\|f\|_1$.

It is enough if we show that:

$$\int_{x \notin \Omega^*} |T_m b(x)| \, dx \leq c\|b_j\|_1, \tag{5.8}$$

where c does not depend on j.

This is because if this claim is true, we have

$$\int_{x \notin \Omega^*} |T_m b(x)| \, dx \leq \sum_{j=1}^{\infty} \int_{x \notin Q_j} |T_m b_j(x)| \, dx \leq c \sum_{j=1}^{\infty} \|b_j\|_1 \leq 6c\|f\|_1,$$

by (5.3). By Tchebychev,

$$\mu_2 = \left| \left\{ x \notin \Omega^* : |T_m b(x)| > \frac{\lambda}{2} \right\} \right| \leq \frac{2}{\lambda} \int_{x \notin \Omega^*} |T_m b(x)| \, dx$$
$$= \frac{c}{\lambda} \|f\|_1.$$

This will prove the theorem. All we need to show is the local estimate.

Let $\varphi \in C_c^\infty(B(0,2))$ such that $\varphi(\xi) = 1 \ \ \forall \ |\xi| \leq 1$.

Define now:

$$\beta(\xi) = \varphi(\xi) - \varphi(2\xi).$$

This then implies, $\sum_{l=-\infty}^{\infty} \beta(2^{-l}\xi) = 1$, since it gives us a telescoping sum for $\xi \neq 0$.

Let $m_l(\xi) = \beta(\xi) \ m(2^k\xi)$. Then,

$$\int_{\mathbb{R}^N} |(1-\Delta)|^{s/2} \, m_l(\xi)|^2 \, d\xi < c, \tag{5.9}$$

by (5.4) for $s > 0$.

If $K_l(x)$ is the kernel associated with $m_l(x)$, then by Plancherel's identity,

$$\int_{\mathbb{R}^N} (1 + |x|^2)^2 \ |K_l(x)|^2 \, dx < c.$$

Let $A_R = \{x \in \mathbb{R}^N : \max_m |x_m| > R\}$. Let χ_R be the characteristic function of A_R. Then by Cauchy Schwarz,

$$\int_{A_R} |K_l(x)| \, dx \leq \left(\int_{A_R} (1 + |x|^2)^{-s} \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} (1 + |x|^2)^s |K_l(x)|^2 \, dx \right)^{\frac{1}{2}}$$
$$= cR^{\frac{N}{2} - s},$$

which gives us a good estimate for very large R.

Let us now reapply the same technique to $\xi_k m_l(\xi)$, where ξ_k is the k-th co-ordinate of ξ, i.e. By (5.9),

$$\int |(1-\Delta)|^{\frac{s}{2}}\ \xi_k m_l(\xi)|^2\, d\xi < c.$$

By Plancherel,

$$\int (1+|x|^2)^s |\nabla K_l(x)|^2\, d\xi < c.$$

Again using the Cauchy Schwarz inequality,

$$\int |\nabla K_\xi(x)|\, dx \le \left(\int_{\mathbb{R}^N} (1+|x|^2)^{-s}\, dx\right)^{\frac{1}{2}} \sqrt{c} = c.$$

Note that by Taylor expansion, $K_l(x+y) - K_l(x) \approx \nabla K_l(x) \cdot y + O(|y|^2)$. For small y, higher order terms will be negligible.

Now by Fundamental Theorem of Calculus, for all k,

$$\int_0^{y_k} \frac{d}{dx_k} K_l(x+t)\, dx_k \sim K_l(x+y) - K_l(x),$$

for small y. Therefore,

$$\int_{\mathbb{R}^N} |K_l(x+y) - K_l(x)|\, dx \le |y| \int_{\mathbb{R}^N} |\nabla K_l(x)|\, dx \le c|y|.$$

Also note, as a tempered distribution,

$$K(x) := \sum_{l=-\infty}^{\infty} 2^{Nl} K_l(2^l x) = \sum_{l=-\infty}^{\infty} m_l(2^{-l}\xi).$$

Now if Q_j is a cube of side R centred at the origin, then

$$\begin{aligned}
\int_{x\notin Q_j^*} |2^{Nl}K_l(2^l.\cdot) * b_j|\, dx &\le \int_{x\in Q_j}\int_{x\notin Q_j^*} |2^{Nl}K_l(2^l(x-y))||b_j(y)|\, dx\, dy \\
&\le \|b_j\|_1 \int_{A_{2^l R}} |K_l(x)|\, dx \\
&\le c(2^l R)^{\frac{N}{2}-s}\|b_j\|_1. \qquad (5.10)
\end{aligned}$$

Since $\int_{Q_j} b_j\, dy = 0$,

$$\int_{x\notin Q_j}\int_{y\in Q_j} K_l(2^l(x-y))\, b_j(y)\, dy\, dx = \int_{x\notin Q_j}\int_{y\in Q_j} \Big[K_l(2^l(x-y)) - K_l(2^l x)\Big]\, b_j(y)\, dy\, dx.$$

Therefore,

$$\begin{aligned}
\int_{x\notin Q_j^*} |2^{Nl}K_l(2^{Nl}.\cdot) * b_j|\, dx &\le \int_{y\in Q_j}\int_{x\notin Q_j^*} 2^{Nl}|K_l(2^l(x-y)) - K_l(2^l x)||b_j(y)|\, dx\, dy \\
&\le c(2^l R)||b_j||_1. \qquad (5.11)
\end{aligned}$$

We use the estimate in (5.10) for $2^l R > 1$, and the estimate in (5.11) for $2^l R \le 1$. Henceforth,

$$\int_{x\notin Q_j} |T_m b_j(x)|\, dx \le c||b_j||_1,$$

completing the proof. $\square$

Remark 5.2.9. It may be clear for some readers as to how (5.9) came from the Hörmander condition, for which here is an explanation. For any $\varphi \in C_c^\infty(\mathbb{R}_+)$, we establish that the Sobolev norm $\sup_{t>0}\|\varphi m_t\|_s$, $m_t(\xi) = m(t\xi)$, is equivalent to the classical Hörmander condition, i.e.

$$\sup_{R\ge 0} R^{-N+2|\alpha|}\int_{R<|\xi|<2R} \left|\frac{\partial^\alpha}{\partial \xi^\alpha}\varphi m_t(\xi)\right|^2 d\xi < \infty.$$

Recall the classical definition of the Sobolev norm, $s \in \mathbb{Z}$, $s > \frac{N}{2}$:

$$\begin{aligned}
\|\varphi m_t\|_s &= \sum_{|\alpha|\le s} \int_{\mathbb{R}^N} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \varphi m_t(\xi) \right|^2 d\xi,\ t > 0 \\
&= \sum_{|\alpha|\le s} \int_{\mathbb{R}^N} \left| \sum_{\beta+\gamma=\alpha} m_t^{(\gamma)}(\xi)\ \varphi^{(\beta)}(\xi) \right|^2 d\xi \\
&\le \sum_{|x|\le s} \sum_{\beta+\gamma=\alpha} \int_{R<|\xi|<2R} \left| m_t^{(\gamma)}(\xi) \varphi^{(\beta)}(\xi) \right|^2 d\xi \\
&\le \sum_{|\alpha|\le s} \sum_{\beta+\gamma=\alpha} \int_{R<|\xi|<2R} t^{2|\gamma|} \left| m^{(\gamma)}(t\xi)\ \varphi^{(\beta)}(\xi) \right|^2 d\xi \\
&\le C_{N,\varphi} \sum_{|\alpha|\le s} \int_{tR<|\xi|<2tR} t^{2|\alpha|} t^{-N} \left| m^{(\alpha)}(\xi) \right|^2 d\xi \\
&= C_{N,\varphi} \sum_{|\alpha|\le s} (tR)^{-N+2|\alpha|} \int_{tR<|\xi|<2tR} \left| m^{(\alpha)}(\xi) \right|^2 d\xi \cdot R^{N+2|\alpha|} \\
&\le C_{N,\varphi} R^{N+2s} \sum_{|\alpha|\le s} (tR)^{-N+2|\alpha|} \int_{tR<|\xi|<2tR} \left| m^{(\alpha)}(\xi) \right|^2 d\xi \\
&\le C_{N,\varphi}\ c,
\end{aligned}$$

where $(tR)^{-N+2|\alpha|} \int_{tR<|\xi|<2tR} \left| m^{(\alpha)}(\xi) \right|^2 d\xi < c, \ \ \forall\ |\alpha| \le s,\ t > 0.$,

For converse inequality, let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that supp $\varphi \subseteq B(0,2)$ and $\varphi \equiv 1$ whenever

$|\xi| \leq 1$. Let $|\alpha| \leq s$. We have for $R > 0$,

$$\begin{aligned}
R^{-N+2|\alpha|} \int_{R<|\xi|<2R} \left| m^{(\alpha)}(\xi) \right|^2 d\xi &\leq \sum_{|\alpha| \leq s} R^{-N+2|\alpha|} \int_{R<|\xi|<2R} \left| m^{(\alpha)}(\xi) \right|^2 d\xi \\
&= \sum_{|\alpha| \leq s} R^{-N+2|\alpha|} \int_{1<|\xi'|<2} \left| m^{(\alpha)}(R\xi') \right|^2 R^N d\xi' \\
&= \sum_{|\alpha| \leq s} \int_{1<|\xi'|<2} \left| m_R^{(\alpha)}(\xi') \right|^2 d\xi' \\
&\leq \sum_{|\alpha| \leq s} \int_{1<|\xi'|<2} \left| \sum_{\beta+\gamma=\alpha} m_R^{(\gamma)}(\xi') \phi^{(\beta)}(\xi') \right|^2 d\xi' \\
&\leq \|\varphi m_R\|_s^2 \\
&\leq c^2,
\end{aligned}$$

which is a uniform bound for all $|\alpha| \leq s$.

5.3 Spectral Multipliers as Fourier Multipliers

Earlier, we have defined the operator,

$$m(-\Delta) = \int_{\sigma(-\Delta)} m(\lambda)\ dP(\lambda),$$

where P is the unique projection-valued measure that is the spectral decomposition of $-\Delta$. Then the following corollary follows directly from Theorem 5.2.7. Define $m(-\Delta)$ as a Fourier multiplier:

$$\mathcal{F}(m(-\Delta)f) = m(|\xi|^2)\ \hat{f}(\xi). \tag{5.12}$$

Corollary 5.3.1. *Let $m \in L^\infty(\mathbb{R}_+)$. For $s > \frac{N}{2}$ if*

$$\sup_{t>0} \|\varphi m_t\|_s < \infty, \tag{5.13}$$

then $m(-\Delta)$ is a bounded L^p-multiplier.

Proof. This just follows from the fact that since m abides to the regularity condition, so does $m(|\xi|^2)$. $\square$

Let us now move to Schrödinger Operators A that are densely defined on $L^2(\mathbb{R}^N)$. We now ask the question, when is $m(A)$ a bounded L^p- multiplier for bounded measurable functions m.

William Hebisch [7] gives a sufficient condition similar to Corollary 5.3.1 as to when $m(A)$ is a bounded L^p-multiplier.

Hebisch considers operators of the kind $T = -\Delta + V$ such that V is a non-negative potential, i.e., $V \geq 0$.

Theorem 5.3.2 (**Hebisch**)**.** *Suppose that $T = -\Delta + V$, $V \geq 0$ is a densely defined Schrödinger Operator. Let m be a bounded measurable function on the real line. If for some $\varepsilon > 0$, non-zero $\varphi \in C_c^\infty(\mathbb{R}_+)$ and constant C, we have the Sobolev norm*

$$\|\varphi m_t\|_{\frac{N+!}{2}+\varepsilon} \leq C, \tag{5.14}$$

then $m(T)$ is weak $(1,1)$ and bounded on L^p for $1 < p < \infty$.

Before we jump into the proof of this theorem, we need a few lemmas.

Lemma 5.3.3 (**Segal**)**.** *For semi-positive-definite self-adjoint operators A and B such that B is relatively bounded with respect to A with relative bound β, i.e., $\|Bx\| \leq \beta\|x\| + \gamma\|Ax\|$,*

$\gamma < 1$, then $A+B$ is self-adjoint on $\mathscr{D}(A)$ and essentially self-adjoint on any core for A. Moreover,

$$\|e^{-(A+B)}\| \leq \|e^{-A}e^{-B}\|. \tag{5.15}$$

The proof of this lemma uses the Kato-Trotter formula, which can be seen from [12].

Lemma 5.3.4. *Write for a kernel K,*

$$\|K\|_b^* = \max\left\{\sup_x \int |K(x,y)|(1+|x-y|^b)\ dy,\ \ \sup_y \int |K(x,y)|(1+|x-y|^b)\ dx\right\}. \tag{5.16}$$

If supp $m \subseteq [-1,4]$, $\varepsilon > 0$, $b \geq 0$, then

$$\|m(T)\|_b^* \leq C\|m\|_{\frac{N+!}{2}+\varepsilon+b},$$

where the norm on the right is a Sobolev norm. The constant C is independent of m and T.

Proof. Set $K(\lambda) = \lambda^{-1}\ m(-\log\lambda)$. Since supp $m \in [-1,4]$, then clearly supp $K \subseteq [e^{-4}, e]$. Note that

$$\|K\|^2_{\frac{N+1}{2}+\varepsilon+b} = \int_{\mathbb{R}} |\hat{K}(\xi)|^2(1+|\xi|^2)^{\frac{N+1}{2}+\varepsilon+b}\ d\xi.$$

For $m \in L^1$,

$$\int |K(\lambda)|\ d\lambda = \int_{e^{-4}}^{e} \frac{|m(-\log\lambda)|}{\lambda}\ d\lambda = \int_{-4}^{1} |m(-u)|\ du = \|m\|_1.$$

This basically tells us that K is also L^1 if m is L^1, and $|\hat{K}(\xi)| \leq \|K\|_1 = \|m\|_1$. We can translate this to any $\frac{N+1}{2}+\varepsilon+b$ using the Fourier transform definition, and we get,

$$|\hat{K}(\xi)|^2 \leq \|K\|^2_{\frac{N+1}{2}+\varepsilon+b} = \sum_{i\leq\frac{N+1}{2}+\varepsilon+b} \int_{-\infty}^{\infty} \left|\frac{\partial^i}{\partial\lambda^i}K(\lambda)\right|^2 \, d\lambda$$
$$\leq C_1\|m\|^2_{\frac{N+1}{2}+\varepsilon+b},$$

if $\frac{N+1}{2}+\varepsilon+b \in \mathbb{Z}$. Now let $K(\lambda) = \sum_{n\in\mathbb{Z}} \hat{K}(n)e^{in\lambda}$, $e_n = e^{ine^{-T}}e^{-T}$. Combining this with the definition of K, we get

$$m(T) = K(e^{-T})e^{-T} = \sum \hat{K}(n)e_n.$$

We now claim that $\|e_n\|_b^* \leq C_2(1+|n|)^{\frac{N}{2}+b}$. If we show this, then,

$$\begin{aligned}
\|m(T)\|_b^* &\leq C_2 \sum |\hat{K}(n)|(1+|n|)^{\frac{N}{2}+b} \\
&\leq C_2 \left\{\sum |\hat{K}(n)|^2(1+|n|)^{2b+N+1+\varepsilon}\right\}^{\frac{1}{2}} \left\{\sum (1+|n|)^{-1-\varepsilon}\right\}^{\frac{1}{2}} \\
&\leq C_2'\|K\|_{\frac{N+1+\varepsilon}{2}+b} \\
&\leq C_3\|m\|_{\frac{N+1}{2}+\varepsilon+b},
\end{aligned}$$

proving what we need. Let us prove the claim.

Since T is self-adjoint, we get these inequalities for which the proof is given just after the proof of this lemma:

$$\int |e^{-T}(x,y)| \, e^{|x-y|} \, dx \leq C_0 \tag{5.17}$$

$$\sup_x \int |e^{-T}(x,y)|^2 \, dx \leq C_0' \, . \tag{5.18}$$

Let $\ell = \gamma^{-1}|n|\|e^{-T}\|_\gamma$, for $\gamma < 1$, where $\|e^{-T}\|_\gamma = \max\{\sup_x \int |e^{-T}(x,y)|e^{\gamma|x-y|}(1+|x-y|)^b \, dy, \sup_y \int |e^{-T}(x,y)|e^{\gamma|x-y|}(1+|x-y|)^b \, dx\}$.

Then, $\|e^{-T}\|_\gamma \leq C\sup_y \int |e^{-T}(x,y)|e^{|x-y|} \, dx \leq CC_0$. Also note that $e^{ine^{-T}}$ being a unitary operator, wlog, we get,

$$
\begin{aligned}
\|e_n\|_b^* &\leq \int_{\mathbb{R}^N} |e^{ine^{-T}} e^{-T}(x,y)|(1+|x-y|)^b \ dx \\
&= \int_{|x-y|<\ell} + \int_{|x-y|>\ell} \\
&\leq (1+\ell)^b \ell^{\frac{N}{2}} |B(0,1)|^{\frac{N}{2}} \|e^{ine^{-T}} e^{-T}(\cdot,y)\|_2 + e^{-\gamma\ell} \|(1+|x-y|)^b e^{\gamma|x-y|} e^{ine^{-T}} e^{-T}(\cdot,y)\|_1 \\
&\leq (1+\ell)^{b+\frac{N}{2}} \sqrt{C_3} \ \|e^{-T}(\cdot,y)\|_2 + e^{-\gamma\ell} \|e^{-T}\|_\gamma \ e^{|n|\|e^{-T}\|_\gamma} \\
&\leq (1+\ell)^{b+\frac{N}{2}} \sqrt{C_3} C_0' + e^{-\gamma\ell} \|e^{-T}\|_\gamma e^{\gamma\ell} \\
&\leq C_2 (1+|n|)^{b+\frac{N}{2}},
\end{aligned}
$$

completing the proof of this lemma. □

Proof. **(Hebisch)** By the Spectral Theorem (Theorem 2.5.6), $m(T)$ is bounded on $L^2(\mathbb{R}^N)$. Again, as in Theorem 5.2.7, we show that $m(T)$ is weak $(1,1)$. Then, we use the Marcinkiewicz Interpolation Theorem (Theorem 5.2.5) and duality, we prove that $m(T)$ is strong (p,p) for $1 < p < \infty$.

By Lemma 5.3.3, we have $\|e^{-tT}\| \leq \|e^{t\Delta} e^{-tV}\|$. Then for any $f \in L^2(\mathbb{R}^N)$, we have

$$
\begin{aligned}
0 \leq \|e^{-tT} f(x)\|_2^2 \leq \|e^{t\Delta} e^{-tV} f(x)\|_2^2 &= (4\pi t)^{-\frac{N}{2}} \int \left| \int e^{-\frac{|x-y|^2}{4t}} e^{-tV(y)} f(y) \ dy \right|^2 \ dx \\
&\leq (4\pi t)^{-\frac{N}{2}} \int \left| \int e^{-\frac{|x-y|^2}{4t}} f(y) \ dy \right|^2 \ dx \\
&= \|e^{t\Delta} f(x)\|_2^2,
\end{aligned}
$$

for all $f \in L^2(\mathbb{R}^N)$. Therefore using Plancherel Theorem, if $e^{-tT}(x,y)$ denotes the Heat kernel for T, then

$$
0 \leq e^{-tT}(x,y) \leq e^{t\Delta}(x,y), \tag{5.19}
$$

where $e^{t\Delta}(x,y) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}$. We now prove the following inequalities for $s, t > 0$.

$$\int e^{-tT}(x,y)\ e^{s|x-y|}\ dx \le Ce^{Cs^2 t} \tag{5.20}$$

$$\int_{\mathbb{R}^N} |e^{-tT}(x,y)|^2\ dx \le Ct^{-\frac{N}{2}} \tag{5.21}$$

$$\sup_{x,y} |e^{-tT}(x,y)| \le Ct^{-\frac{N}{2}} \tag{5.22}$$

Let us do the proofs of the latter two inequalities first. Note, using (5.19)

$$\begin{aligned}\int_{\mathbb{R}^N} |e^{-tT}(x,y)|^2\ dx &\le (4\pi t)^{-N} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{2t}}\ dx \\ &\le Ct^{-N} t^{\frac{N}{2}} \\ &= Ct^{-\frac{N}{2}},\end{aligned}$$

proving (5.21). Sinilarly, using (5.19), we prove (5.22) as follows,

$$\begin{aligned}\sup_{x,y} |e^{-tT}(x,y)| &\le \sup_{x,y} e^{-\frac{|x-y|^2}{4t}} (4\pi t)^{-\frac{N}{2}} \\ &\le Ct^{-\frac{N}{2}}.\end{aligned}$$

Let us now come to inequality (5.20). For $s, t > 0$,

$$\begin{aligned}\int e^{-tT}(x,y)\ e^{s|x-y|}\ dx &\le (4\pi t)^{-\frac{N}{2}} \int e^{-\frac{|x-y|^2}{4t}} e^{s|x-y|}\ dx \\ &= (4\pi t)^{-\frac{N}{2}} \int e^{-\frac{(|x-y|-2ts)^2}{4t}} e^{s^2 t}\ dx \\ &= (4\pi t)^{-\frac{N}{2}} \omega_{N-1} e^{s^2 t} \int_{-\infty}^{\infty} e^{-\frac{(r-2ts)^2}{4t}} r^{N-1}\ dr \\ &\le Ce^{Cs^2 t}.\end{aligned}$$

These inequalities also prove the inequalities (5.17) and (5.18) for $s = t = 1$. Let us now state and prove one final lemma before the rest of the proof of this theorem.

Lemma 5.3.5. *For every $k > 0$, there exists $\zeta, \theta > 0$ such that if $m \in H^\zeta$, supp $m \subset [-1, 4]$, then*

$$|m(T)(x,y)| \leq C\|m\|_\zeta(1+|x-y|)^{-k} \tag{5.23}$$

for all $x, y \in \mathbb{R}^N$.

Proof. Set $v(\lambda) = m(\lambda)e^\lambda$, and $\zeta = \frac{N}{2} + k + 1$. By the same logic as in Lemma 5.3.4, $\|v\|_\zeta \leq C_1\|m\|_\zeta$. This then, by Lemma 5.3.4,

$$\|v(T)\|_k^* \leq C_2\|v\|_\zeta.$$

Using the inequalities (5.19),(5.22), we get

$$\begin{aligned} |(1+|x-y|)^k \; m(T)(x,y)| &= \left| \int v(T)(x,s) \; e^{-T}(s,y) \; (1+|x-y|)^k \; ds \right| \\ &\leq \int |v(T)(x,s)| \; (1+|x-y|)^k \; |e^{-T}(s,y)| \; (1+|s-y|)^k \; ds \\ &\leq \|v(T)\|_k^* \; \sup_x (4\pi)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4}} (1+|x|)^k. \end{aligned}$$

Taking $y \to 0$, the lemma follows. □

Clearly (5.14) is true for all $\varphi \in C_c^\infty(\mathbb{R}_+)$. Fix $\varphi, \psi \in C_c^\infty(\mathbb{R})$, supp $\varphi \subseteq [\frac{1}{4}, 2]$, $\sum_{k\in\mathbb{Z}} \varphi(2^{2k}x) = 1$ for all $x > 0$ and supp $\psi \subseteq [-1, 1]$ with $\psi(x) = 1$ for all $x \in [0, \frac{1}{2}]$. Then define:

$$m_k(\lambda) := \varphi(2^{2k}\lambda)m(\lambda) \; ; \; \psi_k(\lambda) = \psi(2^{2k}\lambda).$$

We then get some more inequalities for $b < \varepsilon$, i.e., there exists C such that

$$\|(\psi_k m_k)(T)\|_{L^1,L^1} \leq C \tag{5.24}$$

$$\int |m_k(T)(x,y)|(1+2^{-k}|x-y|)^b \; dx \leq C \tag{5.25}$$

$$|\psi_k(T)(x,y)| \leq C \; 2^{-kN}(1+2^{-k}|x-y|)^{-N-1} \tag{5.26}$$

Let us now prove these inequalities. (5.25) is an easy consequence of the properties of dilations, and using Lemma (5.3.4) and replacing ε by $\varepsilon - b$. Similarly, (5.26) is derived from using Lemma 5.3.5. Once these two are done, (5.24) is obvious from them.

Now, let us move on to the Calderón Zygmund Decomposition. Let $f \in L^1$, and apply the Calderón Zygmund decomposition at height λ. Refer Theorem 5.2.6.

Let Q_j^* be the ball with the same centre as Q_j and radius $2d_{Q_j}$ where d_{Q_j} is the diameter of Q_j. Choose a compactly supported L^1 function such that supp $h \subset \{x : \; |x| < 1\} =: \mathcal{B}$. Then, using (5.25),

$$\begin{aligned}
\int_{|x|>2} |m_k(T)h(x)| \; dx &\leq \int_{|x|>2} |m_k(T)(x,y)| \; |h(x)| \; dx \\
&\leq \|h\|_1 \sup_{y \in \mathcal{B}} \int |m_k(T)(x,y)| \; dx \\
&\leq \|h\|_1 \sup_{y} \int_{|x-y|>1} |m_k(T)(x,y)| \; dx \\
&\leq \|h\|_1 2^{kb} \sup_{y} \int |m_k(T)(x,y)| \; (1+2^{-k}|x-y|)^b \; dx \\
&\leq C 2^{kb} \|h\|_1.
\end{aligned}$$

Also,

$$\sum_{k \geq 0} \int_{|x|>2} |m_k(T)h(x)| \; dx \leq C \sum_{k \geq 0} 2^{kb} \|h\|_1 \; \leq C_1 \|h\|_1.$$

Therefore, using dilation and applying the above for each b_j,

$$\sum_{i\leq k_j}\int_{(Q_j^*)^c}|m_i(T)b_j(x)|\ dx\leq C\|b_j\|_1. \tag{5.27}$$

Let us recall what we need to prove: We need to show $m(T)$ is weak $(1,1)$, i.e., for $f\in L^1$, $\mu_{m(T)f}(\lambda)\leq\frac{C}{\lambda}\|f\|_1$. It is the 'bad part' that usually gives us trouble, so let us tackle that first. One final lemma will help us with that.

Lemma 5.3.6. *There exists C such that*

$$\|\sum_j\psi_{k_j}(T)f_j\|_2^2\leq C\lambda\|f\|_1. \tag{5.28}$$

Proof. Observe that there exists C_0 such that if $Q=\{x:\max_j\{x_j\}\leq 1\}$ then for all x, we have

$$\sup_{y\in Q}\ (1+|x-y|)^{-N-1}\leq C_0\inf_{y\in Q}\ (1+|x-y|)^{-N-1}.$$

By dilation, for all j,

$$\sup_{y\in Q_j}\ (1+2^{-k_j}|x-y|)^{-N-1}\leq C_0\inf_{y\in Q_j}\ (1+2^{-k_j}|x-y|)^{-N-1}. \tag{5.29}$$

Now fix j and set the centre of Q_j to be y_0. Then by (5.29) and (5.26),

$$\begin{aligned}|\psi_{k_j}(T)b_j(x)|&\leq C\int 2^{-k_jN}\ (1+2^{-k_j}|x-y|)^{-N-1}|b_j(y)|\ dy\\&\leq 2^{-k_jN}\lambda C_1|Q_j|\ (1+2^{-k_j}|x-y_0|)^{-N-1}\\&\leq\lambda C_2\int 2^{-k_jN}\ (1+2^{-k_j}|x-y|)^{-N-1}\chi_{Q_j}(y)\ dy\\&=\lambda C_3\ (2^{-k_jN}(1+2^{-k_j}|\cdot|)^{-N-1}*\chi_{Q_j})(x).\end{aligned}$$

If $h \in L^2$, then

$$\begin{aligned}|\langle h, 2^{-k_j N}(1+2^{-k_j}|\cdot|)^{-N-1} * \chi_{Q_j}\rangle_{L^2}| &= |\langle (1+2^{-k_j}|\cdot|)^{-N-1} * h, \chi_{Q_j}\rangle_{L^2}| \\ &\leq C_4 \langle Mh, \chi_{Q_j}\rangle,\end{aligned}$$

where M is the Hardy-Littlewood Maximal function (Refer Appendix). Since M is bounded on L^2,

$$\begin{aligned}\left| \left\langle h, \sum_j \psi_{k_j}(T) b_j \right\rangle_{L^2} \right| &\leq C_5 \langle Mh, \sum_j \lambda \chi_{Q_j}\rangle \\ &\leq C_6 \|h\|_2 \left\| \sum_j \lambda \chi_{Q_j} \right\|_2 \\ &\leq C_6 \|h\|_2 \sum_j \lambda^2 |Q_j| \\ &\leq C\lambda \|f\|_1,\end{aligned}$$

completing the proof of the lemma. □

Therefore, if $\ell > k$, by the definition of ψ_k, we have $\psi_k m_\ell = 0$ so, $\psi_k(T)\ m_\ell(T) = 0$, since the supports of ψ_k and m_ℓ are disjoint.
Similarly, for $\ell > k$, by the definition of ψ_k, $\psi_k(T)\ m_\ell(T) = m_\ell(T)$.

Now we go back to the Calderón-Zygmund Decomposition.

$$
\begin{aligned}
m(T)f &= \sum_{j,\ell} m_\ell(T)b_j \;+\; m(T)g \\
&= \sum_j \left(\sum_{\ell \le k_j} m_\ell(T)b_j + \sum_{\ell > k_j} m_\ell(T)b_j \right) + m(T)g \\
&= \sum_j \sum_{\ell \le k_j} m_\ell(T)b_j + m(T) \left(\sum_j \psi_{k_j}(T)b_j + g \right) - \sum_j m_{k_j}(T)\psi_{k_j}(T)b_j. \qquad (5.30)
\end{aligned}
$$

We use the same logic as in Theorem 5.2.7, and divide $\mu_{m(T)f}(\lambda)$ into three parts, corresponding to the three terms in (5.30), $\mu_{m(T)f}(\lambda) = \mu_1 + \mu_2 + \mu_3$.
As before, define $\Omega^* = \bigcup_j Q_j^*$. Then, by (5.27),

$$
\begin{aligned}
\mu_1 &= \left| \left\{ x : \left| \sum_j \sum_{\ell \le k_j} m_\ell(T)b_j \right| > \frac{\lambda}{3} \right\} \right| \\
&\le |\Omega^*| + \frac{\lambda}{3} \int_{\Omega^*} \left| \sum_j \sum_{\ell \le k_j} m_\ell(T)b_j(x) \right| \, dx \\
&\le \frac{C}{\lambda} \|f\|_1 + \frac{C}{\lambda} \sum_j \|b_j\|_1 \\
&\le \frac{C}{\lambda} \|f\|_1.
\end{aligned}
$$

Next, using Lemma 5.3.6,

$$
\left\| \sum_j \psi_{k_j}(T)b_j + g \right\|_2^2 \le C\lambda \|f\|_1.
$$

And since $m(T)$ is strong $(2,2)$ and using Tchebychev's Inequality,

$$\begin{aligned}
\mu_2 &= \left|\left\{x : \left|m(T)\left(\sum_j \psi_{k_j}(T)b_j + g\right)\right| > \frac{\lambda}{3}\right\}\right| \\
&\leq \left(\frac{C}{\lambda^2}\right)\left\|\sum_j \psi_{k_j}(T)b_j + g\right\|_2^2 \\
&\leq \frac{C}{\lambda}\|f\|_1.
\end{aligned}$$

Finally, by (5.24) and Tchebychev's Inequality,

$$\begin{aligned}
\mu_3 &= \left|\left\{x : \left|\left(\sum_j m_{k_j}(T)\psi_{k_j}(T)b_j\right)\right| > \frac{\lambda}{3}\right\}\right| \\
&\leq \frac{3}{\lambda}\left\|\sum_j m_{k_j}(T)\psi_{k_j}(T)b_j\right\|_1 \\
&\leq \frac{C}{\lambda}\sum_j \|b_j\|_1 \\
&\leq \frac{C}{\lambda}\|f\|_1,
\end{aligned}$$

completing the proof of the theorem. $\square$

5.4 Next Steps

Let us revisit the Schródinger Operator of the form $\mathscr{L}_a = -\Delta + \frac{a}{|x|^2}$ that we briefly touched upon in [7, Sec. 2.3] detailing Theorem 5.3.2 proves that the operators of the form $m(\mathscr{L}_a)$ are bounded L^p-multipliers for $a \geq 0$.

[11] establishes the L^p boundedness of $m(\mathscr{L}_a)$ for $-\frac{(N-2)^2}{4} \leq a \leq 0$, $N \geq 3$. The operator is semi-positive-definite in this interval. The operator does not have a unique self-adjoint extension for $\frac{-(N-2)^2}{4} < a < 1 - \frac{(N-2)^2}{4}$, $N \geq 3$. The result was proved considering the Friedrich's extension for this region. The proof does follow a similar pathing, with the Marcinkiewicz Interpolation and Calderón-Zygmund Decomposition again taking centre stage.

Possible next steps include the analysis of $m(\mathscr{L}_a)$ on other related Banach spaces.

Bibliography

[1] S. Axler. *Linear Algebra Done Right*. Undergraduate Texts in Mathematics. Springer New York, 1997. ISBN: 9780387982595. URL: https://books.google.co.in/books?id=ovIYVIlithQC.

[2] S. Axler. *Measure, Integration & Real Analysis*. Graduate Texts in Mathematics. Springer International Publishing, 2019. ISBN: 9783030331436. URL: https://books.google.co.in/books?id=DQ3BDwAAQBAJ.

[3] D. Borthwick. *Spectral Theory: Basic Concepts and Applications*. Graduate Texts in Mathematics. Springer International Publishing, 2020. ISBN: 9783030380021. URL: https://books.google.co.in/books?id=d3zWDwAAQBAJ.

[4] F. Dai and Y. Xu. *Approximation Theory and Harmonic Analysis on Spheres and Balls*. Springer Monographs in Mathematics. Springer New York, 2013. ISBN: 9781461466604. URL: https://books.google.co.in/books?id=8SA_AAAAQBAJ.

[5] L.C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN: 9780821849743. URL: https://books.google.co.in/books?id=Xnu0o_EJrCQC.

[6] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013. ISBN: 9781118626399. URL: https://books.google.co.in/books?id=wI4fAwAAQBAJ.

[7] Waldemar Hebisch. "A multiplier theorem for Schrödinger operators". In: *Colloquium Mathematicum* 60 (1990), pp. 659–664. URL: https://api.semanticscholar.org/CorpusID:118822241.

[8] Waldemar Hebisch. "Almost everywhere summability of eigenfunction expansions associated to elliptic operators". eng. In: *Studia Mathematica* 96.3 (1990), pp. 263–275. URL: http://eudml.org/doc/218994.

[9] Lars Hörmander. "Estimates for translation invariant operators in L^p spaces". In: *Project Euclid* (1960).

[10] S. Kesavan. *Functional Analysis*. Texts and Readings in Mathematics. Hindustan Book Agency, 2009. ISBN: 9789386279422. URL: https://books.google.co.in/books?id=tfJdDwAAQBAJ.

[11] R. Killip, C. Miao, M. Visan, J. Zhang, and J. Zheng. "Sobolev spaces adapted to the Schrödinger operator with inverse-square potential". In: *Mathematische Zeitschrift* 288 (2018), pp. 1273–1298. URL: https://doi.org/10.1007/s00209-017-1934-8.

[12] M.G. Krein, I. Gohberg, and M. Kac. *Topics in Functional Analysis: Essays Dedicated to M. G. Krein on the Occasion of His 70th Birthday*. Advances in mathematics. Academic Press, 1978, pp. 185–195. ISBN: 9780122871504. URL: https://books.google.co.in/books?id=eNnuAAAAMAAJ.

[13] J.R. Munkres. *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000. ISBN: 9780131816299. URL: https://books.google.co.in/books?id=XjoZAQAAIAAJ.

[14] R.S. Pathak. *A Course in Distribution Theory and Applications*. CRC Press, 2001. ISBN: 9780849309816. URL: https://books.google.co.in/books?id=KnifEqXZQQUC.

[15] H.L. Royden and P. Fitzpatrick. *Real Analysis*. Prentice Hall, 2010. ISBN: 9780131437470. URL: https://books.google.co.in/books?id=0Y5fAAAACAAJ.

[16] W. Rudin. *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1991. ISBN: 9780070542365. URL: https://books.google.co.in/books?id=Sh_vAAAAMAAJ.

[17] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1976. ISBN: 9780070856134. URL: https://books.google.co.in/books?id=kwqzPAAACAAJ.

[18] K.B. Sinha and S. Srivastava. *Theory of Semigroups and Applications*. Texts and Readings in Mathematics. Springer Singapore, 2017. ISBN: 9789811048654. URL: https://books.google.co.in/books?id=uA3vvgEACAAJ.

[19] E.M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Monographs in harmonic analysis. Princeton University Press, 1971. ISBN: 9780691080789. URL: https://books.google.co.in/books?id=YUCV678MNAIC.

[20] K. Yosida. *Functional Analysis*. Classics in Mathematics. Springer Berlin Heidelberg, 2012. ISBN: 9783642618598. URL: https://books.google.co.in/books?id=yj4mBQAAQBAJ.