# A Study of Heegaard Floer Homologies

**A Thesis**

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by

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# Certificate

This is to certify that this dissertation entitled A Study of Heegaard Floer Homologies towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Arijit Paul at Indian Institute of Science Education and Research under the supervision of Dr. Mainak Poddar, Professor, Department of Mathematics , during the academic year 2021-2022.

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This thesis is dedicated to my friends and family.

# Declaration

I hereby declare that the matter embodied in the report entitled A Study of Heegaard Floer Homologies are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Mainak Poddar and the same has not been submitted elsewhere for any other degree.

Arijit Paul

Arijit Paul

# Acknowledgments

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Finally, I would not be here if not for my family's emotional and financial support, and I would like to extend my deepest thanks to them as well.

# Abstract

This thesis aims to be a gateway into the theory of Heegaard Floer homology. These homology groups are powerful invariants for closed oriented 3-manifolds. The thesis initially deals with some of the theoretical background necessary to understand the definitions of Floer Homology. An outline of Morse theory and Morse homology is given at the very beginning of the thesis. Basics of symplectic topology are discussed in an attempt to gain an understanding of the significant tools being applied. Subsequently, different kinds of Floer homology theory are reviewed. The remainder of this thesis outlines a combinatorial algorithm due to Sarkar and Wang for calculating Heegaard Floer homology, coupled with an example to understand the concept better.

# Contents

# Introduction

Over the last couple of decades, Heegaard Floer homology has become a fruitful theory in the study of 3-dimensional and 4-dimensional topology. At its core, the Heegaard Floer homology groups are invariants for closed oriented 3-manifolds. First developed by Peter Ozsváth and Zoltán Szabó, the simplest version of Heegaard Floer homology associates to $M$ (a closed oriented 3-manifold) a finitely generated Abelian group $\widehat{HF}(M)$. The theory of Heegaard diagrams, along with Lagrangian Floer homology play a key role in this definition. Different variantions of this construction give related invariants, which are denoted by $HF^+(M)$, $HF^-(M)$ and $HF^\infty(M)$.

The primary objective of this project is to gain a proper understanding of Heegaard Floer homology. We start off with some preliminary results in Chapter 1. Here we briefly discuss Morse theory and Morse homology, which serve both as a prototypical example as well as a blueprint to the theory to follow. We also discuss some basic definitions and results in Symplectic topology. This discussion only helps us in appreciating the motivation behind Floer theory, but also enables us to understand its underlying details. We have also reviewed a few facts about Heegaard splittings and Heegaard diagrams of 3-manifolds, which will be useful in Chapter 2.

In Chapter 2, we dive into the world of Floer theory. After a brief discussion of Symplectic Floer theory, we move on to the study of Lagrangian and subsequently Heegaard Floer homology. Here we take care to chalk out the similarities with Morse homology, as well as point out the new complications that arise due to the infinite-dimensional nature of this theory. Although it borrows heavily from [11] and [7], we have tried to focus more on the intuition behind the concepts, rather than spending too much time on the complicated machinery being used.

Chapter 3 deals with a method to calculate Heegaard Floer homology. One cannot hope to utilize these invariants unless there is a nice way of calculating them. This chapter reviews one such method devised by Sarkar and Wang in [18]. Following a reformulation of Heegaard Floer theory by Lipshitz in [15], Sarkar and Wang give a combinatorial method for calculating these homology groups, which is the primary focus of this chapter. Towards the end, we also discuss an example to help us grasp this algorithm in a better way.

# Original Contribution

This thesis is entirely a literature review, and it attempts to provide an introduction into the world of Floer homology. As such it borrows heavily from [10], [7], [22], [18], [11] ,[15] and [3].

Many of the preliminary definitions in Chapter 1 come from [3] and [10], while portions of the section on Heegaard diagrams follow [7] and [6]. Chapter 2 mostly borrows from [7] and [11], while Chapter 3 deals with concepts explained in [15] and [18].

In this thesis, we have tried to provide an introductory exposition of the theory of Floer homologies. While this by no means is a novel attempt, we have tried to focus on examples and intuition, keeping the analytical complications in the background. We have tried to add many figures to help the reader visualize the concepts. We have also tried to provide intuition to a few proofs in Chapter 2 and made some ideas more concrete. We have tried to explain the example given in Section 3.2.2 concretely, which is not elaborated in [18] to a great extent.

Thus, even though the work in this thesis is not original, the author hopes that it will find a place as an introductory text among readers who wish to explore the subject.

# Chapter 1

# Preliminaries

## 1.1 Morse Theory and Morse Homology

As mentioned earlier, Morse homology serves as a finite-dimensional prototypical example of the theory developed by Floer and therefore it only made sense to start our project with a brief discussion of the major concepts involved in this theory. Here we outline some major definitions and theorems that will be useful to us in understanding the things that are to follow. Most of this section borrows from [1] and [3].

**Definition 1.1.1.** *Suppose that $f : M \to \mathbb{R}$ is a smooth function on a smooth manifold $M$. We say that a point $p \in M$ is a* ***critical point*** *of $f$ if $Df_p = 0$. Here $Df_p : T_pM \to T_p\mathbb{R}$ denotes the induced linear map of Tangent spaces.*

**Definition 1.1.2.** *Let $p$ be a critical point of $f$. We define the* ***Hessian*** *of $f$ at $p$ to be a symmetric bilinear functional $D^2 f_p : T_pM \times T_pM \to \mathbb{R}$ in the following way:*

$$D^2 f_p(u, v) = (U(V f))(p) = u(V f), \tag{1.1}$$

*where $U$ and $V$ are local vector field extensions of $u, v$ respectively in some neighbourhood of $p$.*

The fact that the the above definition is symmetric and well-defined follows from the argument below:

Let $[U, V]$ be the Lie brecket of $U$ and $V$. Then for a critical point $p$,

$$0 = Df_p([U, V]_p) = U(Vf)(p) - V(Uf)(p) \tag{1.2}$$

and hence, $U(Vf)(p) = V(Uf)(p)$, and we have proven that the definition is symmetric. Now, since $u(Vf)$ is independent of $U$ and similarly $v(Uf)$ is independent of $V$, we can conclude that the Hessian is independent of the extensions chosen.

**Remark 1.1.1.** *Let $(x_1, ..., x_n)$ be a local coordinate system in a neighbourhood $N$ of $p$. Then with respect to the basis $\frac{\partial}{\partial x_1}|_a, ..., \frac{\partial}{\partial x_n}|_a$, the Hessian is given by the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$.*

**Definition 1.1.3** (Morse Function)**.** *Let $f : M \to \mathbb{R}$ be a smooth map. We call a critical point $p$ **non-degenerate** provided that the Hessian of $f$ at that point is non-singular. We say that $f$ is a **Morse function** if all the critical points of $f$ are non-degenerate.*

We define the **Morse Index** of a function $f$ at a critical point $p$ in the following manner:

$$\text{Ind}(p) = \max\{dim(N) | N \text{ is a subspaces of } T_pM \text{ on which the Hessian is negative-definite}\}. \tag{1.3}$$

The Morse index can tell us quite a bit about the topology of the manifold. The following lemma depicts one such key scenario.

**Lemma 1.1.1** (Lemma of Morse, [1], Lemma 2.2)**.** *Suppose that $f : M \to \mathbb{R}$ is a smooth function and $p$ is a non-degenerate critical point of $f$. Then there exists a neighbourhood $U$ of $p$ and a system of local coordinates $(x_1, ..., x_n)$ on $U$ such that $x_k(p) = 0$ for all $k$. Moreover, the following identity holds throughout $U$:*

$$f = f(p) - (x_1)^2 - ... - (x_i)^2 + (x_{i+1})^2 + ... + (x_n)^2, \tag{1.4}$$

*where $i$ is Ind$(p)$. $U$ is called a **Morse chart**.*

One can find a detailed proof of the above in [1]. The above lemma immediately leads to the fact that the critical points of a Morse function must always be isolated and consequently, Morse functions on compact manifolds can only have a finite number of critical points.

For our purposes, we will often consider a special kind of Morse function.

**Definition 1.1.4.** *We say that a Morse function $f$ is **self-indexing** provided that for every critical point $p$ of $f$, we have $f(p) = Ind(p)$.*

We have the following theorem, the proof of which can be found in [1]:

**Theorem 1.1.2.** *There exists a self indexing Morse function on every smooth $n$-manifold $M$. Moreover, if $M$ is a connected manifold without boundary, then $f$ can be taken to have unique critical points of index $0$ and index $n$.*

We now move our discussion towards the notion of **gradient flow** in order to understand the concept of Morse homology. Let $\mathfrak{g}$ be a Riemannian metric on $M$. We denote the inner product of $V$ and $W$ with respect to $\mathfrak{g}$ by $< V, W >$. Then the **gradient** of a smooth function $f$ is defined to be the vector field $\nabla f$ determined by the identity

$$< V, \nabla f >= V(f), \tag{1.5}$$

where $V$ is any vector field on $M$.

**Definition 1.1.5.** *We define a gradient flow line to be a smooth map $\psi : I \to M$, (where $I$ is an open interval) that satisfies the following differential equation:*

$$\frac{d\psi}{dt} + (\nabla f)_{\psi(t)} = 0. \tag{1.6}$$

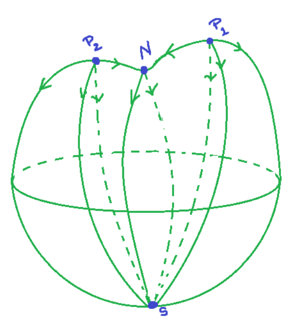


Figure 1.1: Gradient flow lines for the height function of a 'dented' sphere, assuming it to be embedded in $\mathbb{R}^3$ in the usual sense

The following theorem helps us to define the **flow map** of a smooth function:

**Theorem 1.1.3.** *Let $p$ be any point on $M$. Then there exists a unique gradient flow line $\psi_p$ such that $\psi_p(0) = p$.*

Combining all these flow lines together gives us the flow map $\Psi$, which is a smooth map $\Psi : M \times \mathbb{R} \to M$ given by $\Psi(p,t) = \psi_p(t)$.

**Theorem 1.1.4.** *Suppose that $\psi$ is a gradient flow line. We assume that the underlying smooth manifold $M$ in this case is compact. Then there exists points $a, b \in Crit(f)$ with the property that $\lim_{t\to-\infty} \psi(t) = a$ and $\lim_{t\to\infty} \psi(t) = b$. The points $a$ and $b$ are called the starting and ending points (denoted as $s(\psi)$ and $e(\psi)$ respectively) for the flow line $\psi$.*

Proofs for both of the above theorems can be found in [2].

**Definition 1.1.6.** *Suppose that $\Psi$ is the gradient flow of $f$, with $\psi_p$ being the gradient flow line through $p$. Let $a \in Crit(f)$. We define the* ***stable and unstable manifolds*** *of $a$ to be the following spaces:*

$$W^s(a) = \{p \in M | e(\psi_p) = a\}, \qquad W^u(a) = \{p \in M | s(\psi_p) = a\}. \tag{1.7}$$

Suppose that $a$ is a non-degenerate critical point. Then $W^u(a)$ is an embedded disc of dimension $\mathrm{Ind}(a)$ and $W^s(a)$ is an embedded disc with dimension $(dim(M) - \mathrm{Ind}(a))$. Theorem 4.2 in [4] provides a proof of this.

**Definition 1.1.7.** *Let $f$ be a smooth function and $\mathfrak{g}$ be a Riemannian metric on $M$. Suppose that $f$ is a Morse function and for every $a, b \in Crit(f)$, we have $W^u(a) \pitchfork W^s(b)$. The pair $(f, \mathfrak{g})$ is then said to be a* ***Morse-Smale pair****. Here, $\pitchfork$ denotes transversal intersection.*

The above might seem like a rather restrictive condition. However, that is not the case, as is depicted by the following theorem:

**Theorem 1.1.5** ((Smale Theorem)[17])**.** *Suppose that $M$ is a smooth manifold with boundary and $f$ is a Morse function on $M$. Also, suppose that the critical values of $f$ are distinct. Having fixed Morse charts in the neighbourhoods of each critical point of $f$, (we denote the union of these charts by $\Omega$) we consider a pseudo-gradient field $X$ on $M$ that is transversal to the boundary (Pseudo-gradient fields are defined below). Then there exists a pseudo-gradient field $X'$, which is close to $X$ in the $\mathcal{C}^1$-sense, equal to $X$ on $\Omega$ and for which we have $W^s_{X'}(a) \pitchfork W^s_{X'}(b)$ for all $a, b \in Crit(f)$.*

A **pseudo-gradient field (adapted to $f$)**, is defined to be a vector field $X$ on $M$ with the properties:

1. $(df)_a(X_a) \leq 0$, where the equality holds iff $a \in Crit(f)$.

2. Let $a \in Crit(f)$ and $U$ be a Morse chart around $a$. Then, $X$ is identical on $U$ to $-\nabla f$ for the cannonical metric on $M$, which is obtained by pulling back the standard metric on $\mathbb{R}^n$.

The importance of the Morse-Smale condition can be seen from the following standard theorem, the proof of which can be found in [3]

**Theorem 1.1.6.** *Let $W$ and $W'$ be two submanifolds of $M$ such that $W \pitchfork W'$. Then $W \cap W'$ is a submanifold of $M$ such that $codim(W \cap W') = codim(W) + codim(W')$.*

We will denote $W^u(a) \cap W^s(b)$ by $W(a,b)$, which is the space

$$W(a,b) = \{p \in M | s(\psi_p) = a, e(\psi_p) = b\}. \tag{1.8}$$

Thus when we have a Morse-Smale pair, $W(a,b)$ becomes a submanifold of dimension $\mathrm{Ind}(a) - \mathrm{Ind}(b)$. Now, $\mathbb{R}$ acts on $W(a,b)$ (because if $\psi(t)$ is a flow-line, so is $\psi(t+s)$ for all $s \in \mathbb{R}$) and hence, after quotienting $W(a,b)$ with $\mathbb{R}$, we obtain the following space:

$$\mathcal{M}(a,b) = W(a,b)/\mathbb{R} \tag{1.9}$$

which is a manifold of dimension $\mathrm{Ind}(a) - \mathrm{Ind}(b) - 1$.

We have the following theorem, which is essential to the definition of the Morse chain complex:

**Theorem 1.1.7** ([3], Chapter 3)**.** *Suppose that $M$ is a closed manifold and $(f, \mathfrak{g})$ is a Morse-Smale pair. Let $a, b \in Crit(f)$. Then $\mathcal{M}(a,b)$ defined above has a natural compactification to a smooth manifold with corners $\overline{\mathcal{M}(a,b)}$. The codimension-$k$ stratum of this compact space* *is given by:*

$$\overline{\mathcal{M}(a,b)}_k = \bigcup_{\substack{r_1,...,r_k \in Crit(f) \\ a,r_1,...,r_k,b \ distinct}} \mathcal{M}(a,r_1) \times \mathcal{M}(r_1,r_2) \times ... \times \mathcal{M}(r_{k-1},r_k) \times \mathcal{M}(r_k,b). \tag{1.10}$$

*For $k=1$, as oriented manifolds, we obtain:*

$$\partial\overline{\mathcal{M}(a,b)} = \bigcup_{\substack{r \in Crit(f) \\ a,r,b \ distinct}} \mathcal{M}(a,r) \times \mathcal{M}(r,b) \tag{1.11}$$

The proof of the above can be found in [3]. The primary idea is to look at the so-called 'broken trajectories'. For critical points $a$ and $b$, a broken trajectory from $a$ to $b$ is a collection of flow lines $\{u_0, ..., u_k\}$ for $-\nabla f$ such that $u_j$ connects $r_j$ to $r_{j+1}$, with $u_0$ connecting $a$ to $r_1$ and $u_k$ connecting $r_k$ to $b$. Then the space $\overline{\mathcal{M}(a,b)}$ is nothing but $\mathcal{M}(a,b) \cup \{\text{broken trajectories from } a \text{ to } b\}$. We note that since the index of critical points decrease when we travel along the flow lines, in the case when $\text{Ind}(b) - \text{Ind}(a) = 1$, there are no broken trajectories connecting $a$ and $b$ and the spaces $\overline{\mathcal{M}(a,b)}$ and $\mathcal{M}(a,b)$ are the same.

This compactification result is key to the definition of Morse homology. With this, we are now ready to talk about the Morse complex.

**Definition 1.1.8** (Morse complex)**.** *Let $Crit_k(f)$ be the set of critical points of $f$ which have Morse index $k$. We can then define the* ***Morse complex*** $(C_*^M(f,\mathfrak{g}), \partial^M)$ *in the following way:*
*The $k$-th chain module is the free $\mathbb{Z}/2\mathbb{Z}$ module generated by $Crit_k(f)$:*

$$C_k^M(f,\mathfrak{g}) := \mathbb{Z}/2\mathbb{Z}Crit_k(f). \tag{1.12}$$

*If $Ind(b) = Ind(a) - 1$ the moduli space $\mathcal{M}(a,b)$ is a $0$-dimensional manifold and the boundary map $\partial^M : C_k^M \to C_{k-1}^M$ is defined by counting the points in $\mathcal{M}(a,b)$ modulo $2$. For example, if $a \in Crit_k(f)$ then*

$$\partial^M(a) := \sum_{b \in Crit_{k-1}(f)} \#_2\mathcal{M}(a,b).b. \tag{1.13}$$

Here we are using $\mathbb{Z}/2\mathbb{Z}$ coefficients because we do not want to worry about orientaions and signs. A similar construction can be carried out with coefficients in $\mathbb{Z}$. The fact that $\partial^2 = 0$, i.e., the above construction indeed rsults in a chain complex follows from Theorem 1.1.7. Here we must look at the moduli space $\overline{\mathcal{M}(a,b)}$ which is a 1-dimensional compact manifold when $\mathrm{Ind}(a) - \mathrm{Ind}(b) = 2$. Since the boundary of a compact oriented 1-manifold has even number of points, we conclude that $\partial^2 = 0$. The homology of the above chain complex is known as **Morse homology**. It can be shown that Morse homology is isomorphic to singular homology. One can consult [5] for the details.

## 1.2 Basics of 3-Manifold Theory

### 1.2.1 Heegaard Splittings

In order to understand Heegaard Floer homology, we must first study the notions of Heegaard splittings and Heegaard diagrams. These concepts are central to the theory of 3-manifolds. A Heegaard splitting is nothing but a decomposition of a 3-manifold into 2 simpler parts called handlebodies. Most of the contents in this section follow [6], with occassional references to [7] whenever necessary. Throughout, our 3-manifolds are closed and orientation preserving, unless stated otherwise. We begin with the definition of a handlebody:

**Definition 1.2.1.** *In 3-dimensions, a $k$-**handle** is a 3-ball, thought of as $[0,1]^3$, that is glued (to some pre-existing submanifold) along $[0,1]^{3-k} \times \partial[0,1]^k$.*

More precisely, a 0-handle can be thought of as a 3-ball attached to the empty set. A 1-handle is a 3-ball attached along $[0,1]^2 \times \partial[0,1]$. Therefore, we can think of it as a solid cylinder attached to a pre-existing submanifold along its two bounding disks. Similarly, a 2-handle is a 3-ball attached along $[0,1] \times \partial[0,1]^2$ while a 3-handle is a 3-ball attached along $\partial[0,1]^3$, i.e. , along its entire boundary.

**Definition 1.2.2.** *Let $V$ be a compact connected orientable 3-manifold with boundary such that it has a handle decomposition made up of only 0-handles and 1-handles. Then $V$ is called a **handlebody**. The genus of a handlebody is the genus of its boundary surface. Thus, we can identify a genus $g$ handlebody with a closed regular neighbourhood of the wedge of $g$ circles in $\mathbb{R}^3$.*

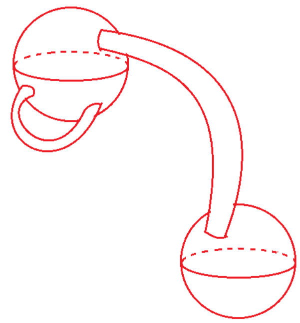

Figure 1.2: A genus 1 handlebody

**Definition 1.2.3.** *A **Heegaard splitting** of a closed* 3*-manifold* $M$ *is defined to be a decomposition* $M = V \cup_s W$ *where* $V, W$ *are handlebodies glued together at their boundary via* $s$*. Here,* $s : \partial V \to \partial W$ *is an orientation reversing homeomorphism. The genus of* $\partial V$ *or equivalently, the genus of* $\partial W$ *is known as the genus of the Heegaard splitting.*

The following theorem, by E. Moise, gives a lot of context to the study of Heegaard Splittings:

**Theorem 1.2.1** (Moise)**.** *There exists a Heegaard splitting for every closed oriented* 3*-manifold.*

**Proof.** We can prove the above by considering triangulations of the 3-manifold $M$ and constructing one handlebody by taking a regular neighbourhood of the 1-skeleton. The complement of this handlebody is again a handlebody, which is just a neighbourhood of the dual 1-skeleton, with edges that are lines perpendicular to the faces of the tetrahedra. □

Now, let $M$ be a closed manifold and suppose that we have a self-indexing Morse function (as defined in Definition 1.1.4) $f : M \to [0, 3]$ with the property that $f$ has only a single minimum and a single maximum. In this case, we can obtain a Heegaard decomposition of $M$ such that the Heegaard surface is given by $\Sigma = f^{-1}(3/2)$, while the handlebodies are given by $V = f^{-1}([0, 3/2])$ and $W = f^{-1}([3/2, 3])$. It can be also shown that if the genus of $\Sigma$ is $g$, then the number of critical points of $f$ with Morse index 1 is $g$, as is the number of critical points with Morse index 2.

**Definition 1.2.4.** *Let* $V \cup_s W$ *and* $V' \cup_{s'} W'$ *be two Heegaard splittings for* $M$*. We say that these two are **isotopic** provided that there exists a map* $\Phi : M \times [0, 1] \to M$ *such that:*

1. $\Phi|_{M\times\{0\}} = id_M$,

2. $\Phi|_{M\times\{t\}}$ *is a homeomorphism* $\forall t$,

3. $\Phi|_{M\times\{1\}}$ *maps* $V$ *to* $V'$ *and* $W$ *to* $W'$.

**Definition 1.2.5.** *Suppose that* $V \cup_s W$ *is a genus* $g$ *Heegaard splitting of* $M$. *A* ***stabilization*** *of* $V \cup_s W$ *is defined to be a genus* $g+1$ *Heegaard splitting of* $M$, *which can be obtained as follows: We choose two points in* $\partial V$ *and join them with a small unknotted arc* $\eta$ *in* $W$. *Now, taking the union of* $V$ *with a closed tubular neighbourhood* $N(\eta)$ *of* $\eta$, *we obtain a genus* $g+1$ *handlebody* $V'$. *Similarly taking* $W' = \overline{W - N(\eta)}$ *(where the bar above denotes the closure of the corresponding space in* $M$*) gives a genus* $g+1$ *handlebody. This gives a genus* $g+1$ *Heegaard splitting* $V' \cup_{s'} W'$.


Figure 1.3: A stabilization, where we have denoted $\partial V = \partial W = S$, and $\partial V' = \partial W' = S'$

The operation described above is of central importance because of the following theorem. The proof can be found in [9]

**Theorem 1.2.2** (Reidemeister-Singer)**.** *Let* $H_1 = V \cup_s W$ *and* $H_2 = V' \cup_{s'} W'$ *be two Heegaard splittings for* $M$. *Then* $H_1$ *and* $H_2$ *will become isotopic after a finite sequence of stablizations.*

### 1.2.2 Heegaard Diagrams

We now shift our discussion to Heegaard diagrams, which are tools used to depict Heegaard splittings of 3-manifolds. These diagrams will play a pivotal role in our discussion of Heegaard Floer homology.

**Definition 1.2.6.** *Suppose $V$ is a genus $g$ handlebody. We define* ***a set of attaching circles*** *for $V$ to be a collection of $g$ closed embedded curves $(\alpha_1, ..., \alpha_g)$ in $\Sigma_g = \partial V$ such that it satisfies the following properties:*

1. *$\alpha_i$ do not intersect each other,*
2. *$\Sigma_g - \alpha_1 - ... - \alpha_g$ is connected,*
3. *Each $\alpha_i$ bounds a disjoint embedded disks in $V$.*

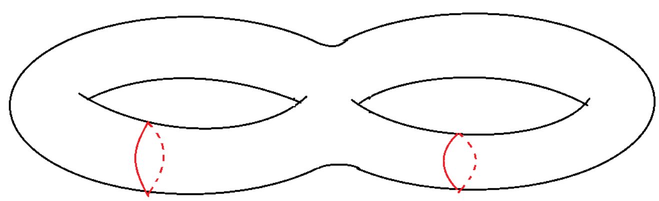

Figure 1.4: A set of attaching circles for a handlebody

**Definition 1.2.7.** *Suppose that $V \cup_s W$ is a genus $g$ Heegaard decomposition of $M$. A* ***Heegaard diagram*** *compatible to this decomposition is given by a triple $(\Sigma_g, \alpha, \beta)$ where $\Sigma_g = \partial V = \partial W$ and $\alpha = (\alpha_1, ..., \alpha_g)$ and $\beta = (\beta_1, ..., \beta_g)$ are sets of attaching circles for $V$ and $W$ respectively.*

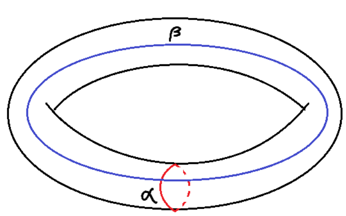


Figure 1.5: A Heegaard diagram compatible to the genus 1 Heegaard decomposition of $\mathbb{S}^3$

Clearly, the same Heegaard decomposition might admit multiple Heegaard diagrams. On the other hand, if we have a triple $(\Sigma_g, \alpha, \beta)$, where $\Sigma_g$ is a genus $g$ surface and $\alpha$ and $\beta$ satisfy the first two conditions of Definition 1.2.6, then it uniquely determines a Heegaard decomposition and therefore, uniquely determines a 3-manifold. Given such a triple $(\Sigma_g, \alpha, \beta)$ which is a Heegaard diagram for a 3-manifold $M$, we can re-build $M$ as follows:

We first need to thicken $\Sigma_g$ to $\Sigma_g \times [0, 1]$ and then attach thickened discs along $\alpha_i \times \{0\}$ for each $\alpha_i$ and along $\beta_i \times \{1\}$ for each $\beta_i$. This gives us a manifold with boundary which has two disjoint copies of 2-sphere as its boundary components, and we can fill in each of these with copies of 3-balls to obtain a 3-manifold. We know that if we have an orientation preserving homeomorphism from the 2-sphere to itself, then it is isotopic to the identity map. Thus, this filling in can be done in a unique way . It can be shown that the resulting 3-manifold turns out to be homeomorphic to $M$.

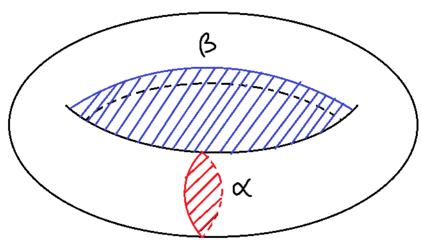


Figure 1.6: Recovering $\mathbb{S}^3$ from the Heegaard diagram in Figure 1.6

For a self-indexing Morse function, we have the following theorem:

**Theorem 1.2.3** ([7]). *Let $f$ be a self-indexing Morse function and let $\mathfrak{g}$ be a Riemannian metric on $M$. Then the pair $(f, \mathfrak{g})$ gives a Heegaard diagram for $M$.*

**Proof.** Let $-\nabla f$ denote the negative gradient of $f$ with respect to the metric $\mathfrak{g}$. Let $x \in M$, and we can study the flow line through $x$. Let $a_k$ and $b_k$ be the index 2 and index 1 critical points of $f$. Suppose $\alpha_i$ denote the points which end up at the critical point $b_i$ because of the flow and let $\beta_i$ denote those points whose flow lines start at $a_i$. Then since $f$ is self-indexing and the Morse lemma holds, $\alpha_i$ and $\beta_i$ are simple closed curves in $\Sigma = f^{-1}(3/2)$. It is not difficult to see that $\alpha_i$ and $\beta_j$ are sets of attaching circles for $V$ and $W$, as defined above. This gives us a Heegaard diagram for $M$. □

We will say that such a Heegaard diagram is compatible with the Morse function $f$.

We now talk about **Heegaard moves**, which are ways to alter the Heegaard diagram without changing the underlying 3-manifold. There are 3 such elementary moves: isotopy, handleslide and stabilization. We have already defined stabilizations for Heegaard splittings, stabilizations for Heegaard diagrams and the other two can be defined as follows:

**Definition 1.2.8.** *Suppose that $(\Sigma_g, \alpha, \beta)$ is a Heegaard diagram.*

1. *We say that the set of attaching circles $\alpha' = (\alpha'_1, ..., \alpha'_g)$ is **isotopic** to $\alpha$ provided that we can reach $\alpha'$ from $\alpha$ via an isotopy of $\Sigma_g$ so that at no point during the isotopy any of the $\alpha_i$'s intersect each other.*

2. *If we perform a **handleslide** of an attaching circle, say $\alpha_1$ over another one, say $\alpha_2$, we obtain another set of attaching circles $(\alpha'_1, \alpha_2, ..., \alpha_g)$ so that $\alpha'_1$ is a simple closed curve disjoint from $\alpha_1, ..., \alpha_g$. This move is possible only when $\alpha_1, \alpha_2$ and $\alpha'_1$ bound an embedded pair of pants in $\Sigma_g - \alpha_3 - ... - \alpha_g$. Figure 1.7 depicts such a move.*

3. *A **stabilization** of a Heegaard diagram $(\Sigma_g, \alpha, \beta)$ is very closely related to a stabilization of the corresponding Heegaard splitting. It leads to a Heegaard diagram $(\Sigma'_{g+1}, \alpha', \beta')$ where $\Sigma'_{g+1} = \Sigma_g \# \mathbb{T}^2$ ($\mathbb{T}^2$ is the standard 2-dimensional torus), $\alpha' = \alpha \cup \{\alpha_{g+1}\}$ and $\beta' = \beta \cup \{\beta_{g+1}\}$ ($\alpha_{g+1}$ and $\beta_{g+1}$ are two simple closed curves in $\mathbb{T}^2$ intersecting transversally at a single point. We also say that $(\Sigma_g, \alpha, \beta)$ is a destabilization of $(\Sigma'_g, \alpha', \beta')$*

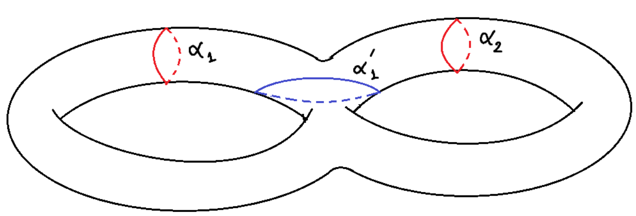


Figure 1.7: A handleslide of $\alpha_1$ over $\alpha_2$

It is obvious from the above definitions that while isotopies and handleslides keep the genus of the Heegaard diagram same, stabilizations increase it by 1. The following theorem asserts that these moves do not change the underlying manifold. A proof can be found in [7].

**Theorem 1.2.4.** *Suppose that* $(\Sigma_g, \alpha, \beta)$ *and* $(\Sigma'_{g'}, \alpha', \beta')$ *are two Heegaard diagrams for* $M$*. Then the two diagrams become homeomorphic after a finite sequence of the moves mentioned above. When we say that two diagrams are homeomorphic, we mean that there is a homeomorphism from one Heegaard surface onto the other which takes* $\alpha$ *circles to* $\alpha'$ *circles and takes* $\beta$ *circles to* $\beta'$ *circles.*

For our purposes it will be beneficial if we also consider pointed Heegaard diagrams. We define them below:

**Definition 1.2.9.** *A* ***pointed Heegaard diagram*** *is a tuple* $(\Sigma_g, \alpha, \beta, w)$ *where* $\Sigma_g, \alpha, \beta$ *are the same as before and* $w$ *is a point in the complement of the curves, i.e.,* $w \in \Sigma_g - \alpha - \beta$*.*

We will also consider **pointed Heegaard moves**, which are similar to the ones already mentioned, with slight differences:

1. For pointed isotopies, we do not allow the isotopies to cross $w$.

2. For pointed handleslides, we insist that $w$ lies outside the pair of pants over which the handleslide is taking place.

We have the following theorem analogous to theorem 1.2.3. Again, the proof can be found in [7].

**Theorem 1.2.5.** *Suppose that* $(\Sigma_g, \alpha, \beta, w)$ *and* $(\Sigma'_{g'}, \alpha', \beta', w')$ *are two pointed Heegaard diagrams for* $M$*. Then the two diagrams become homeomorphic after a finite sequence of pointed Heegaard moves. When we say that two diagrams are homeomorphic, we mean that there is a homeomorphism from one Heegaard surface onto the other which takes* $\alpha$ *circles to* $\alpha'$ *circles, takes* $\beta$ *circles to* $\beta'$ *circles and takes* $w$ *to* $w'$*.*

## 1.3 Basics of Symplectic Topology

Studying symplectic topology is essential to understand the definitions and constructions that follow. It also gives some context to the theory of Floer homology, which seems rather complicated at first sight. Symplectic topology is essential to understand Arnold's conjecture, which was the primary motivation behind the development of the theory. We start with a discussion on the motivations behind symplectic geometry and topology and then discuss some critical aspects of symplectic vector fields and symplectic manifolds, which we will repeatedly be using in upcoming sections. We will mostly follow the definitions and theorems in [10] for this section.

### 1.3.1 Motivation

Symplectic topology has its roots in classical mechanics. There, in order to obtain the equations of motion, we come across the problem of minimizing the **Action integral**

$$I(x) = \int_{t_0}^{t_1} L(t, x, \dot{x}) dt \tag{1.14}$$

over the set of paths $x \in C^1([t_0, t_1], \mathbb{R}^n)$ which satisfy the boundary condition $x(t_0) = x_0, x(t_1) = x_1$. The function $L$ is called the **Lagrangian**. It can be shown that a minimal path $x : [t_0, t_1] \to \mathbb{R}^n$ is a solution to the **Euler-Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{1.15}$$

Under the Legendre conditions (which is $det\big(\frac{\partial^2 L}{\partial \dot{x}_j \partial \dot{x}_k}\big) \neq 0$) we can obtain a system of first-order differential equations with $2n$ variables from equation 1.15. These equations are known as the **Hamiltonian differential equations**. These are:

$$\dot{x} = \frac{\partial H}{\partial y}, \dot{y} = -\frac{\partial H}{\partial x} \tag{1.16}$$

where $y_k := \frac{\partial L}{\partial \dot{x}_k}, k = 1, ..., n$ and $H := \sum_{j=1}^{n} y_j \dot{x}_j - L$.

Now suppose that $H$ is not dependent on $t$. Then 1.16 can be re-written as

$$\dot{z} = -J_0 \nabla H(z) \tag{1.17}$$

where $z = (x, y),\ J_0 := \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$, and $\nabla H$ is the gradient of $H$.

We call a diffeomorphism $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a **cannonical transformation** or a **symplectomorphism** provided that $d\psi(\eta)^T J_0 d\psi(\eta) = J_0$ for every $\eta \in \mathbb{R}^{2n}$. It can be shown that if $\psi$ is a cannonical transformation, and if $\eta(t)$ solves the Hamiltonian differential equation $\dot{\eta} = -J_0 \nabla H \circ \psi(\eta)$, then $z(t) = \psi(\eta(t))$ is a solution to $\dot{z} = -J_0 \nabla H(z)$.

Since the Hamiltonian differential equations behave in such a nice manner under cannonical transformations, it is natural in the study of Hamiltonian systems to introduce the notion of a manifold whose transition maps are cannonical transformations. Turns out this can be achieved by the introduction of a closed and non-degenerate 2-form on a smooth manifold. Such a manifold is called **symplectic**.

### 1.3.2 Symplectic Vector Spaces

We begin our study of symplectic topology with symplectic vector spaces. Later we will move on to the study of symplectic manifolds and Lagrangian submanifolds of such manifolds, which play a key role in the theory of Floer homology.

**Definition 1.3.1.** *Let $V$ be a finite-dimensional real vector space and let $\omega : V \times V \to \mathbb{R}$ be a bilinear form on $V$ that satisfies the following conditions:*

1. *(skew-symmetry) $\forall\ w \in V$, $\omega(v, w) = -\omega(w, v)$.*
2. *(non-degeneracy) For every $v \in V$, if we have $\omega(v, w) = 0\ \forall w \in V$ then $v = 0$.*

*Then the pair $(V, \omega)$ is called a **symplectic vector space.***

The vector space $V$ has to be an even dimensional space since if we have a real skew-symmetric matrix which has odd dimension, then it must have a kernel. We will denote the set of non-degenerate skew-symmetric bilinear forms on an even-dimensional vector space $V$ by $\Omega(V)$.

The Euclidean space $\mathbb{R}^{2n}$ with the skew symmetric form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j, \tag{1.18}$$

where $(x_1, ..., x_n, y_1, ..., y_n) \in \mathbb{R}^{2n}$ is the simplest example of a symplectic vector space. Another equally important example can be constructed in the following manner: We take a real vector space $V$ and denote its dual by $V^*$. Then the direct sum $V \oplus V^*$ is a symplectic vector space when equipped with the following form:

$$\omega((v, v^*), (w, w^*)) = w^*(v) - v^*(w). \tag{1.19}$$

**Definition 1.3.2.** *Suppose $\Phi : V \to V$ is a vector space isomorphism that preserves the symplectic structure, i.e., $\Phi^*\omega = \omega$, where $(\Phi^*\omega)(v, w) := \omega(\Phi v, \Phi w)$ for $v, w \in V$ . Such a map is called a* ***linear symplectomorphism*** *of $(V, \omega)$. These maps form a group. We will denote it by $Sp(V, \omega)$.*

**Definition 1.3.3.** *Suppose that $W$ is a linear subspace of $(V, \omega)$. Then $W^\omega = \{v \in V | \omega(v, w) = 0 \quad \forall w \in W\}$ is known as the* ***symplectic complement*** *of $W$. Based on the relation between $W$ and $W^\omega$, we define the following:*

1. *If $W \subset W^\omega$, we call $W$* ***isotropic****.*
2. *If $W^\omega \subset W$, we call $W$* ***coisotropic****.*
3. *If $W \cap W^\omega = \{0\}$, we call $W$* ***symmetric****.*
4. *If $W = W^\omega$, we call $W$* ***Lagrangian****.*

**Theorem 1.3.1** ([10], Lemma 2.2.1)**.** *Let $W$ be a subspace of $V$. Then, we have*

$$dim(W) + dim(W^\omega) = dim(V), \qquad (W^\omega)^\omega = W. \tag{1.20}$$

**Proof.** We first define the following map $\iota : V \to V^*$:

$$\iota(v)(w) = \omega(v, w). \tag{1.21}$$

The non-degeneracy of $\omega$ assures us that $\iota$ is an isomorphism, and it identifies $W^\omega$ with the annihilator of $W$ in $V^*$. We denote this annihilator by $W^\perp$ and since we know that $dim(W) + dim(W^\perp) = dim(V)$, we are essentially done. $\square$

The following theorem is one of the major results related to symplectic vestor spaces. It basically implies that all symplectic vector spaces of equal dimension are linearly symplectomorphic.

**Theorem 1.3.2** ([10], Theorem 2.1.3)**.** *Suppose that $(V, \omega)$ is a symplectic vector space of dimension $2n$. Then we can find a basis $\{s_1, ..., s_n, t_1, ..., t_n\}$ for $(V, \omega)$ such that*

$$\omega(s_j, s_k) = \omega(t_j, t_k) = 0, \qquad \omega(s_j, t_k) = \delta_{jk}. \tag{1.22}$$

*We call such a basis a symplectic basis. Moreover, we can find a vector space isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that $\Phi^*\omega = \omega_0$.*

**Proof.** We will prove this using induction over $n$. We know that $\omega$ is non-degenerate. Therefore, we can choose $s_1, t_1 \in V$ so that $\omega(s_1, t_1) = 1$. Thus the subspace spanned by $s_1$ and $t_1$ is symplectic. We denote the symplectic complement of this space by $W$. Then $(W, \omega)$ becomes a $2n - 2$ dimensional symplectic vector space. Now we apply the induction hypothesis to $W$ to find a symplectic basis of $W$, say $\{s_2, ..., s_n, t_2, ..., t_n\}$. Now the vectors $s_1, ..., s_n, t_1, ..., t_n$ form a symplectic basis of $V$. The linear map $\Phi : \mathbb{R}^{2n} \to V$ given by $\Phi(z) = \sum_{j=1}^n x_j s_j + \sum_{j=1}^n y_j t_j$ for $z = (x, y) \in \mathbb{R}^{2n}$, satisfies the condition we need. $\square$

We will also need to know a few details about the group $Sp(V, \omega)$. We will need this knowledge to define Maslov index and the first Chern class, both of which will be very important to us. All of the proofs can be found in [10]. We begin with a special case.

**Theorem 1.3.3** ([10], Lemma 1.1.12)**.** *$Sp(\mathbb{R}^{2n}, \omega_0)$ is a Lie group. Its Lie algebra is given by*

$$\mathfrak{sp}(\mathbb{R}^{2n}, \omega_0) := \{-J_0 S | S \in \mathbb{R}^{2n\times 2n}, S^T = S\}. \tag{1.23}$$

Now since Theorem 1.3.2 states that all symplectic vector spaces of equal dimension are isomorphic, the above theorem asserts that $Sp(V, \omega)$ is a Lie group as well, and in order to understand this group, it is enough to study the case of $Sp(\mathbb{R}^{2n}, \omega_0)$.

The elements of $Sp(\mathbb{R}^{2n}, \omega_0)$ can be thought of as real $2n \times 2n$ matrices $\Phi$ satisfying the equation $\Phi^T J_0 \Phi = J_0$. We also consider the usual identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ where the vector $z = (x, y)$ corresponds to the complex vector $x+iy$ for $x, y \in \mathbb{R}^n$ . Then multiplication by the matrix $J_0$ in $\mathbb{R}^{2n}$ corresponds to multiplication by $i$ in $\mathbb{C}^n$. This makes $GL(n, \mathbb{C})$ and $U(n)$ subgroups of $GL(2n, \mathbb{R})$ and $Sp(\mathbb{R}^{2n}, \omega_0)$, respectively.

The following two theorems are of paramount importance to us. The proofs, again, can be found in [10].

**Theorem 1.3.4** ([10], Proposition 2.2.4)**.** *The unitary group $U(n)$ is a maximal compact subgroup of the symplectic linear group $Sp(\mathbb{R}^{2n}, \omega_0)$. The inclusion of $U(n)$ into $Sp(\mathbb{R}^{2n}, \omega_0)$ is a homotopy equivalence.*

**Theorem 1.3.5** ([10], Proposition 2.2.6)**.** $\pi_1(U(n)) \cong \mathbb{Z}$. *Here, $\pi_1(U(n))$ is the fundamental group of $U(n)$. The determinant map $det_{\mathbb{C}} : U(n) \to \mathbb{S}^1$ gives an isomorphism between the two groups.*

Thus $\pi_1(Sp(\mathbb{R}^{2n}, \omega_0)) \cong \mathbb{Z}$. This isomorphism can be made explicit via the **Maslov index**. We will define it axiomatically below. A proof can be found in [10]

**Theorem 1.3.6** ([10], Theorem 2.2.12)**.** *Given a loop $\Psi$ of symplectic matrices (i.e., $\Psi : \mathbb{R}/\mathbb{Z} \to Sp(\mathbb{R}^{2n}, \omega_0)$) there exists a unique function $\mu$ that gives us an integer $\mu(\Psi)$ and satisfies the following axioms. This function is called the* ***Maslov index****:*

1. *(homotopy) Two loops $\Psi_1$ and $\Psi_2$ are homotopic iff their Maslov indices are equal.*

2. *(product) For two loops $\Psi_1, \Psi_2$, we have $\mu(\Psi_1 \Psi_2) = \mu(\Psi_1) + \mu(\Psi_2)$.*

3. *(direct sum) If $n = n' + n''$ then we can identify $Sp(\mathbb{R}^{2n'}, \omega_0) \oplus Sp(\mathbb{R}^{2n''}, \omega_0)$ with a subgroup of $Sp(\mathbb{R}^{2n}, \omega_0)$. Then $\mu(\Psi' \oplus \Psi'') = \mu(\Psi') + \mu(\Psi'')$.*

4. *(normalization) The loop $\Psi_0 : \mathbb{R}/\mathbb{Z} \to Sp(\mathbb{R}^2, \omega_0)$, defined as*

$$\Psi_0(t) := \begin{bmatrix} cos(2\pi t) & -sin(2\pi t) \\ sin(2\pi t) & -cos(2\pi t) \end{bmatrix}$$

   *has Maslov index* 1.

Lagrangian subspaces will play a pivotal part in the theory coming up. We will denote the space of Lagrangian subspaces of a symplectic vector space $(V, \omega)$ by $L(V, \omega)$. It can be shown that the fundamental group of $L(\mathbb{R}^{2n}, \omega_0) \cong \mathbb{Z}$, and once again we can obtain an explicit map via the Maslov index:

**Theorem 1.3.7** ([10], Theorem 2.3.7)**.** *Let $\Lambda : \mathbb{S}^1 \to L(\mathbb{R}^{2n}, \omega_0)$ be a loop of Lagrangian subspaces. Then there exists a unique map $\mu : \pi_1(L(\mathbb{R}^{2n}, \omega_0)) \to \mathbb{Z}$ that sends $\Lambda$ to an integer and has the following properties:*

1. *(homotopy) Two loops $\Lambda_1$ and $\Lambda_2$ are homotopic iff their Maslov indices are equal.*

2. *(product) For two loops $\Lambda \in L(\mathbb{R}^{2n}, \omega_0), \Psi \in Sp(\mathbb{R}^{2n}, \omega_0)$, we have $\mu(\Lambda\Psi) = \mu(\Lambda) + 2\mu(\Psi)$.*

3. *(direct sum) If $n = n' + n''$ then we can identify $L(\mathbb{R}^{2n'}, \omega_0) \oplus L(\mathbb{R}^{2n''}, \omega_0)$ with a submanifold of $L(\mathbb{R}^{2n}, \omega_0)$. Then $\mu(\Lambda' \oplus \Lambda'') = \mu(\Lambda') + \mu(\Lambda'')$.*

4. *(zero) If $\Lambda$ is a constant loop, then $\mu(\Lambda) = 0$.*

5. *(normalization) The loop $\Lambda_0 : \mathbb{S}^1 \to L(\mathbb{R}^2, \omega_0)$, given by*

$$\Lambda_0(t) := e^{\pi i t}\mathbb{R} \subset \mathbb{C}$$

   *has Maslov index 1. Here, $\mathbb{R}^2$ is identified with $\mathbb{C}$ in the usual manner.*

**Definition 1.3.4.** *Let $L$ be a Lagrangian subspace of a general symplectic vector space $(V, \omega)$, and let $\Psi : (\mathbb{R}^{2n}, \omega_0) \to (V, \omega)$ be a liner symplectomorphism. Then, the Maslov index of a loop $\Lambda : \mathbb{S}^1 \to L(V, \omega)$ is defined to be the Maslov index of the loop $t \mapsto \Psi^{-1}\Lambda(t) \in L(\mathbb{R}^{2n}, \omega_0)$.*

It can be shown that this definition does not depend on the choice of $\Psi$.

We now define linear complex structures, which will be a prototypical case for our study of almost complex structures later on.

**Definition 1.3.5.** *An automorphism $J : V \to V$ of a real vector space is called a **linear complex structure** if $J^2 = -\mathbb{1}$.*

By defining such an automorphism, we can turn $V$ into a complex vector space, where $J$ serves the purpose of multiplying by $i$. Thus, the scalar multiplication is given by

$$\mathbb{C} \times V \to V : (a + ib, v) \mapsto av + bJv. \tag{1.24}$$

We denote the space of linear complex structures on $V$ by $\mathcal{J}(V)$. One of the basic examples for a linear complex structure is given by $\mathbb{R}^{2n}$ equipped with the automorphism induced by the following matrix

$$J_0 = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$$

.

**Definition 1.3.6.** *Let $(V, \omega)$ be a symplectic vector space. We say that a complex structure $J$ on $V$ is* ***compatible with*** *$\omega$ if the following holds:*

$$\omega(Jv, Jw) = \omega(v, w) \qquad \text{for all } v, w \in V \text{ and} \tag{1.25}$$

$$\omega(v, Jv) > 0 \qquad \text{for all } 0 \neq v \in V. \tag{1.26}$$

*A linear complex structure that satisfies only* 1.25 *is called $\omega$-**tame**.*

If $J$ is a complex structure that is compatible with $\omega$ then,

$$\mathfrak{g}_J(v, w) := \omega(v, Jw) \tag{1.27}$$

defines an inner product on $V$.

Next, we define symplectic vector bundles, and that will lead us to our definition of first Chern class.

**Definition 1.3.7.** . *Let $\pi : E \to M$ be a real vector bundle. Let $\omega_q : E_q \times E_q \to \mathbb{R}$ (where $q \in M$) be a family of symplectic bilinear forms on the fibres $E_q = \pi^{-1}(q)$. We also assume that the family varies smoothly with $q$. Then these $\omega_q$ come together to give rise to a smooth section $\omega$ of $E^* \wedge E^*$, where $E^* = Hom(E, \mathbb{R})$ is the dual bundle. This section $\omega$ is a nondegenerate skew-symmetric bilinear form on $E$ and the pair $(E, \omega)$ is called a* ***symplectic vector bundle****.*

**Definition 1.3.8.** *Two symplectic vector bundles $(E_1, \omega_1)$ and $(E_2, \omega_2)$ are called* ***isomorphic*** *if there exists a vector bundle isomorphism $\Phi : E_1 \to E_2$ such that $\Phi^*\omega_2 = \omega_1$.*

We have the following theorem, which relates the symplectic vector bundles with complex vector bundles in the case of $\omega$-tame complex structures.

As before, a **complex structure on a vector bundle** $E \to M$ is just an automorphism $J$ of $E$ with the property that $J^2 = \mathbb{1}$. We will call $J$ to be $\omega$-**compatible** (respectively, $\omega$-**tame**) provided that the induced complex structure $J_q$ on the fibre $E_q$ is $\omega$-compatible ($\omega$-tame) for all $q \in M$.

**Theorem 1.3.8** ([10], Theorem 2.6.3)**.** *Suppose that $(E_1, \omega_1)$ and $(E_2, \omega_2)$ are two symplectic vector bundles over $M$ and for $i \in \{1, 2\}$ let $J_i$ be an $\omega_i$-tame complex structure on $E_i$. Then the symplectic vector fields are isomorphic iff the complex vector bundles $(E_1, J_1)$ and $(E_2, J_2)$ are isomorphic.*

A proof of the above can be found in [10]. A triple $(\omega, J, \mathfrak{g})$, where $J$ is $\omega$-compatible and $\mathfrak{g} : E \times E \to \mathbb{R}$ is a symmetric and positive-definite bilinear form defined by $\mathfrak{g}(v, w) = \omega(v, Jw)$ is called a **Hermitian structure** on $E$. It can be shown that every symplectic vector bundle admits a Hermitian structure. For details, one can again consult [10].

We will now define the **first Chern class**. This is an element of the integral 2-dimensional cohomology of the base manifold. For vector bundles where the base is a closed oriented 2-dimensional manifold, the first Chern class $c_1$ is completely determined by the first Chern number, which is the value taken by $c_1$ on the fundamental 2-cycle of the base manifold. Therefore we will begin by describing this integer invariant for symplectic vector bundles over closed oriented 2-manifolds and generalize it afterwards. We can define it axiomatically as follows:

**Theorem 1.3.9** ([10], Theorem 2.7.1)**.** *There exists a unique functor $c_1$ from symplectic vector bundles over a closed oriented 2-manifold $\Sigma$ to $\mathbb{Z}$, called the first Chern number, that satisfies the following axioms:*

1. *(naturality) Suppose, $E$ and $E'$ are two symplectic vector bundles over $\Sigma$. Then they are isomorphic iff they have the same rank and the same $c_1$.*
2. *(functoriality) For any smooth map $\phi : \Sigma' \to \Sigma$ of closed oriented 2-manifolds and any symplectic vector bundle $E \to \Sigma$, we have $c_1(\phi^* E) = deg(\phi).c_1(E)$.*
3. *(additivity) For symplectic vector bundles $E$ and $E'$ over a closed oriented 2-manifold $\Sigma$, we have $c_1(E \oplus E') = c_1(E) + c_1(E')$.*

*4. (normalization) $c_1(T\Sigma) = 2 - 2g$, where $g$ is the genus of $\Sigma$.*

A proof can be found in [10]. We generalize the above concept as follows:

Suppose that $E$ is a symplectic vector bundle over any manifold $M$. Then to every smooth map $f : S \to M$ (where $S$ is a compact oriented 2-manifold without boundary), the first Chern number assigns an integer $c_1(f^*E)$. This integer can be shown to depend only on the homology class of $f$. Thus the first Chern number generalizes to a homomorphism $H^2(M;\mathbb{Z}) \to \mathbb{Z}$. This gives rise to a cohomology class $c_1(E) \in H^2(M;\mathbb{Z})/\text{torsion}$. There is a natural choice of a lift of this class to $H^2(M;\mathbb{Z})$ which is called the first Chern class.

### 1.3.3 Symplectic Manifolds

We are now ready to dive into the theory of symplectic manifolds. We will start with basic definitions and then go on to state Arnold's conjecture, which was the primary motivation for German mathematician Andreas Floer to develop his homology theory. Throughout this section we will assume that $M$ is a connected $C^\infty$-manifold, which (unless stated otherwise) has no boundary.

**Definition 1.3.9.** *Suppose that $M$ is a smooth manifold. We define a **symplectic structure** on $M$ to be a non-degenerate, closed 2-form $\omega \in \Omega^2(M)$.*

Here, non-degeneracy means that each tangent space $(T_qM, \omega_q)$ is a symplectic vector space.

For example, $\mathbb{R}^{2n}$ is a symplectic manifold when equipped with the symplectic form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$, i.e., if $\zeta = (\xi, \eta) \in \mathbb{R}^{2n}$ and $\zeta' = (\xi', \eta') \in \mathbb{R}^{2n}$ are two vectors in $\mathbb{R}^{2n}$, then

$$\omega_0(\zeta, \zeta') = \sum_{j=1}^{n} (\xi_j \eta_j' - \eta_j \xi_j') = -\zeta^T J_0 \zeta'. \tag{1.28}$$

Another example is given by $\mathbb{S}^2$(thought of as the standard unit sphere embedded in $\mathbb{R}^3$) eqiupped with the standard area form given by $\omega_x(\xi, \eta) = \langle x, \eta \times \xi \rangle$ for every $\xi, \eta \in T_x\mathbb{S}^2$.

**Definition 1.3.10.** *Let $(M, \omega)$ and $(M', \omega')$ be two symplectic manifolds. We call a smooth* *map $\psi : M \to M'$ a **symplectomorphism** provided that it is a diffeomorphism and it satisfies $\psi^*\omega' = \omega$.*

The celebrated theorem of Darboux states the following:

**Theorem 1.3.10** (Darboux)**.** *Every symplectic form $\omega$ on $M$ is locally symplectomorphic to the standard form $\omega_0$ on $\mathbb{R}^{2n}$.*

**Definition 1.3.11.** *A subspace $L$ of $M$ is called **Lagrangian** provided that for every $p \in L$, $T_pL$ is a Lagrangian subspace of the symplectic vector space $(T_pM, \omega_p)$. Similar definitions hold for symplectic, isotropic and coisotropic subspaces.*

For example, given a symplectic manifold $(M, \omega)$ the product $M \times M$ admits a natural symplectic structure $(-\omega) \oplus \omega$. The submanifolds $M \times \{p\}$ and $\{p\} \times M$ (where $p$ is a point on $M$) are symplectic while the diagonal is Lagrangian.

We now divert our attention to symplectic and hamiltonian isotopies, ideas which are central to Arnold's conjecture. The non-degeneracy of $\omega$ allows us to obtain a $1-1$ correspondence between vector fields and 1-forms on $M$, using the following map:

$$\chi(M) \to \Omega^1(M) : X \mapsto \iota(X)\omega. \tag{1.29}$$

Here $\iota(X)\omega = \omega(X, .)$ and we have denoted the space of vector fields on $M$ by $\chi(M)$.

**Definition 1.3.12.** *We call $X \in \chi(M)$ **symplectic** provided that $\iota(X)\omega$ is closed. We will use $\chi(M, \omega)$ to denote the space of symplectic vector fields.*

**Theorem 1.3.11** ([10], Proposition 3.1.5)**.** *Suppose that $M$ is a closed manifold. Let $t \mapsto \psi_t \in Diff(M)$ be a smooth family of diffeomorphisms which satisfy the following equation:*

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = id$$

*where $X_t \in \chi(M)$ for $t \in [0, 1]$. Then $\psi_t$ is a symplectomorphism for every $t$ iff $X_t$ is a symplectic vector field for every $t$. Moreover, if $X, Y \in \chi(M, \omega)$ then $[X, Y] \in \chi(M, \omega)$ and $\iota([X, Y])\omega = dH$, where $H = \omega(X, Y)$.*

**Proof.** We know that Cartan's formula for Lie derivative states:

$$\mathcal{L}_X\omega = \iota(X)d\omega + d(\iota(X)\omega).$$

Thus we have:

$$\begin{aligned}
\frac{d}{dt}\psi_t^*\omega &= \psi_t^*(\mathcal{L}_{X_t}\omega) \\
&= \psi_t^*(\iota(X_t)d\omega + d(\iota(X_t)\omega)) \\
&= \psi_t^* d(\iota(X_t)\omega).
\end{aligned}$$

Here $\omega$ is closed and therefore $d\omega = 0$. Also, we note that $X$ is a symplectic vector field iff $\mathcal{L}_X\omega = 0$. Now, suppose that $X$ and $Y$ be two symplectic vector fields and let $\psi_t$ and $\phi_t$ be the their flows. We then have $[X, Y] = \mathcal{L}_Y X = \frac{d}{dt}|_{t=0}\psi_t^* X$ and therefore:

$$\begin{aligned}
\iota([X, Y])\omega &= \frac{d}{dt}\Big|_{t=0}\iota(\psi_t^* X)\omega \\
&= \mathcal{L}_Y(\iota(X)\omega) \\
&= d\iota(Y)\iota(X)\omega \\
&= d(\omega(X, Y)).
\end{aligned}$$

The second equality above is due to the fact that $\psi_t^*\omega = \omega, \forall t$ and therefore $\iota(\psi_t^* X)\omega = \psi_t^*(\iota(X)\omega)$. This proves the theorem. $\square$

**Definition 1.3.13.** *Suppose that $\phi : M \to \mathbb{R}$ is a smooth function. Then we can define the* ***Hamiltonian vector field*** *for $\phi$ to be the vector field $X_\phi : M \to TM$ determined by the identity:*

$$\iota(X_\phi)\omega = d\phi.$$

*Now, suppose that $M$ is closed. In this case $X_\phi$ generates a smooth 1-parameter family of diffeomorphisms that satisfy the following equations:*

$$\frac{d}{dt}\psi_\phi^t = X_\phi \circ \psi_\phi^t, \qquad \psi_\phi^0 = id.$$

*This is known as the* ***Hamiltonian flow associated to*** *$\phi$.*

**Definition 1.3.14.** *Suppose, $\Psi : [0, 1] \times M \to M$ is a smooth map defined by $(t, q) \mapsto \psi_t(q)$ such that $\psi_t$ is a symplectomorphism on $M$ for every $t$ and $\psi_0 = id$. Such a family is known as a* ***symplectic isotopy*** *of $M$.*

For any such isotopy, we can find a unique family of vector fields $X_t : M \to TM$ so that $\frac{d}{dt}\psi_t = X_t \circ \psi_t$. Since each $\psi_t$ is a symplectomorphism, we have $X_t \in \chi(M, \omega)\ \forall t$. Therefore, $d\iota(X_t)\omega = 0$.

Now, suppose that all these 1-forms are exact. Then it is not difficult to see that in this case we obtain a smooth family of functions $\phi_t : M \to \mathbb{R}$ such that $\iota(X_t)\omega = d\phi_t$. Thus, for every $t$, $X_t$ becomes the Hamiltonian vector field associated to $\phi_t$ and the family $\psi_t$ is called a **Hamiltonian isotopy**.

**Definition 1.3.15.** *We call a symplectomorphism $\phi$ **Hamiltonian** if there is exists a Hamiltonian isotopy $\psi_t$ such that $\psi_0 = id$ and $\psi_1 = \phi$.*

We are now ready to define Arnold's conjecture.

**Conjecture 1.3.1** (Arnold's conjecture)**.** *Let $(M, \omega)$ be a compact symplectic manifold and suppose that $\psi : M \to M$ is a Hamiltonian symplectomorphism on $M$. Then the number of fixed points of $\psi$ is greater than or equal to the minimal number of critical points of a function on $M$. In the case where the fixed points are all non-degenerate then the number of fixed points is at least the minimal number of critical points for a Morse function on M.*

Since the minimal number of critical points of a function on $M$ is very difficult to estimate for arbitrary manifolds, one considers weaker versions of the conjecture: for example, the sum of the Betti numbers are used when the critical points are non-degenerate.

In order to fully dive into the theory, we still need a few more ingredients, one of them being the notion of an 'almost complex' structure. Below, we will define these terms.

**Definition 1.3.16.** *Let $M$ be a smooth manifold. An automorphism $J : TM \to TM$ is called an **almost complex structure** provided we have $J^2 = -\mathbb{1}$. The pair $(M, J)$ is known as an **almost complex manifold**.*

As we can see, this definition resembles that of linear complex structures for vector spaces rather closely. Therefore it comes as no surprise that the related definitions when we have a non-degenerate 2-form $\omega$, namely, $J$ being $\omega$-tame and $\omega$-compatible are also similar:

**Definition 1.3.17.** *Suppose that $\omega$ is a non-degenerate 2-form on $M$, and $J$ is an almost complex structure such that for every $p \in M$ and $0 \neq v \in T_pM$, we have $\omega(v, Jv) > 0$. Then*

*$J$ is called $\omega$-**tame**. Moreover, if $J$ additionally satisfies $\omega(Jv, Jw) = \omega(v, w)$ for every $q \in M$ and $v, w \in T_pM$ then $J$ is said to be $\omega$-**compatible**.*

We can show that $J$ is compatible with $\omega$ only if the bilinear form defined as $\langle v, w\rangle := \omega(v, Jw)$ gives a Riemannian metric on $M$. Next we look at the compatibility of a Riemannian metric with a complex structure.

**Definition 1.3.18.** *Let $M$ be a smooth manifold, $J$ be an almost complex structure on $M$ and $\mathfrak{g}$ be a Riemannian metric on $M$. We say that $\mathfrak{g}$ is **compatible** with $J$ provided that $\mathfrak{g}(Jv, Jw) = \mathfrak{g}(v, w)$ for $v, w \in T_pM$.*

In this case we can define a non-degenerate 2-form on $M$ via $\omega(v, w) := \mathfrak{g}(Jv, w)$, which turns out to be compatible with $J$ as well. We call such a triple $(\omega, J, \mathfrak{g})$ compatible. Thus, there is a close relation between Riemannian metrics, almost complex structures and non-degenerate 2-forms on $M$, which will be very useful to us in the future.

We will also need the definition of $J$-holomorphic curves, which is given below:

**Definition 1.3.19.** *Suppose that $(M, J)$ is an almost complex manifold and $(\Sigma, j)$ is a smooth Riemann surface. Then a smooth map $u : (\Sigma, j) \to (M, J)$ is called a $J$-**holomorphic curve** provided it satisfies the condition:*

$$J \circ du = du \circ j. \tag{1.30}$$

# Chapter 2

# Definition of Floer Homology

## 2.1 Symplectic Floer Homology

We begin this chapter with a very brief discussion of Symplectic Floer homology. This theory was developed by Floer and in many ways it paves the way for many variations of Floer homology, as it was one of the first situations where Floer adapted the concepts of Morse theory in an infinite dimensional setting in an attempt to prove Arnold's conjecture. We will skip most of the technical details in this section and focus more on the intuition behind the theory. Hopefully this will help us in not losing our focus when we do dive into some of the details later on, especially when we discuss Lagrangian and Heegaard Floer homology in the next two sections.

Suppose that $(M, \omega)$ is a closed symplectic manifold, and let $\phi : M \to M$ be a Hamiltonian symplectomorphism. This means that we have a family of symplectomorphisms $\psi_t$ where $t \in [0, 1]$ and a family of functions $H_t : M \to \mathbb{R}$ with the properties:

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \iota(X_t)\omega = dH_t, \quad \psi_0 = id, \quad \psi_1 = \phi$$

We are interested in estimating the number of fixed points of $\psi_1 = \phi$. This version of the theory deals with the case when $H_0 = H_1$. This enables us to define an 1-periodic function $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$, where $H(t, .) = H_t = H_{t+1}$ and instead of looking at the fixed points of

$\phi$ we can alternatively look at periodic solutions for the following differential equations:

$$\dot{z}(t) = X_t(z(t)), \qquad z(t+1) = z(t),$$

which are nothing but Hamilton's differential equations. Moreover, this case only deals with contractible solutions of the aforementioned equations.

We denote the space of smooth and contractible loops on $M$ by $\mathcal{L}(M)$. The tangent space of $\mathcal{L}(M)$ at $\gamma$ is the space of vector fields along $\gamma$. We can define the following 1-form on $\mathcal{L}(M)$:

$$\delta\mathcal{A}_H(\gamma)(\hat{\gamma}) := \int_0^1 \omega(\dot{\gamma}(t) - X_{H_t}(\gamma(t)), \hat{\gamma}(t))dt \tag{2.1}$$

where $\gamma \in \mathcal{L}(M)$ and $\hat{\gamma} \in T_\gamma\mathcal{L}(M)$. The above 1-form is closed and if it is exact, then we can find a function $\mathcal{A}_H$ such that Equation 2.1 gives its differential (hence the notation).

Thus, in this scenario, the periodic solutions we are interested in are given by the critical points of the above function. Here we see the resemblance with Morse theory, and Floer wanted to develop a theory analogous to Morse theory using the negative gradient flow of $\mathcal{A}_H$

It can be shown (after choosing a suitable almost complex structure (which is $\omega$-compatible) and defining a Riemannian metric on $\mathcal{L}(M)$) that the $-ve$ gradient flow-lines of $\mathcal{A}_H$ are contractible maps $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ that satisfy the following PDE (called the Floer equation):

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0. \tag{2.2}$$

Here $J_t$ is an appropriate family of almost complex structures on $M$, chosen earlier.

Now, provided that all these contractible periodic solutions of Hamilton's equations are non-degenerate, $\mathcal{A}_H$ is a Morse function. We denote the set of critical points of $\mathcal{A}_H$ by $Crit(H)$. For $z^+, z^- \in Crit(H)$ we define the set of Floer trajectories connecting $z^-$ to $z^+$

by:

$$\mathcal{M}(z^-, z^+; H, J) := \{u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M | u \text{ satisfies (2.2)}, \lim_{s\to-\infty} u(s,t) = z^-(t),$$
$$\lim_{s\to+\infty} u(s,t) = z^+(t)\}.$$

Now we are ready to construct the Floer chain complex.We define

$$CF_*(H) := \bigoplus_{z\in Crit(H)} \mathbb{Z}/2\mathbb{Z}\langle z\rangle, \tag{2.3}$$

and

$$\partial\langle z\rangle := \sum_{w\in Crit(H)} n_2(z,w)\langle w\rangle. \tag{2.4}$$

Here $n_2(z,w) := \#_2\{u \in \mathcal{M}(z,w;H,J) | \mu(w) - \mu(z) = 1\}$, where $\mu(z)$ denotes the so called relative Morse index of $\mathcal{A}_H$ at $z$. The homology of this complex is known as the Floer homology of $M$.

Of course, there are a number of theoretical details that we have skipped. Many aspects such as the fact that the moduli spaces are finite dimensional, the matter of compactness of the moduli spaces along with certain gluing theorems need to be proved before defining the chain complex. We also need to confirm that the groups defined above do not depend on the many choices made by us in the definition. These were proved by Floer in a series of papers. However, we steered clear to these topics to depict a barebones version of the concepts and understand the underlying intuition of the theory along with its similarity with Morse theory.

## 2.2 Lagrangian Floer Homology

Having talked a little about the origins of Floer theory, we now move to another variation of the theory, called the Lagrangian Floer theory. Also devised by Floer in [11] as another attempt at proving Arnold's conjecture for a specific case, this theory provides the framework of Heegaard Floer homology to a large extent. In what follows, we will try to provide a general idea of the definition of Lagrangian Floer homology. Many of the ideas follow parts of [3], [12], [13] and [11], with occasional references to other resources as necessary.

### 2.2.1 Preparatory Theorems and Definitions

As mentioned above, Arnold's conjecture deals with the fixed points of Hamiltonian symplectomorphisms. We have already defined what symplectomorphisms and Hamiltonian symplectomorphisms are. The following theorem emphasizes the importance of Lagrangian submanifolds in this context.

**Theorem 2.2.1.** *Suppose that $\phi : (M, \omega) \to (M, \omega)$ is a symplectomorphism ($M$ has dimension $2n$) and let $G$ denote its graph. Then, $G$ is an embedded Lagrangian submanifold in $(M \times M, -\omega \oplus \omega)$. Here $-\omega \oplus \omega$ is nothing but the symplectic form $-\pi_1^*\omega + \pi_2^*\omega$ where $\pi_1$ and $\pi_2$ are projections to the first and last $2n$ coordinates respectively.*

**Proof.** We have used $G$ to denote the graph of $\phi$. Let $\omega' = -\omega \oplus \omega$. Then $G$ is the image of the embedding $\gamma : M \to M \times M$ given by $x \mapsto (x, \phi(x))$. Then we have:

$$\begin{aligned}
\gamma^*(\omega') &= \gamma^*(\pi_1^*(-\omega)) + \gamma^*(\pi_2^*(\omega)) \\
&= -\omega + \phi^*(\omega) \\
&= -\omega + \omega \\
&= 0
\end{aligned}$$

□

Another important fact we need to point out is that for $\phi$ as above and a Lagrangian submanifold $L$, $\phi(L)$ is Lagrangian as well.

**Definition 2.2.1.** *Suppose that we have a Lagrangian submanifold $L$ with the property that given any Hamiltonian symplectomorphism $\phi$, we have $L \cap \phi(L) \neq \emptyset$. Then $L$ is said to be **non-displaceable**. Otherwise, we call $L$ **displaceable**.*

If we bring all the above discussion together, we can see that when $\phi$ is a symplectomorphism of $M$, the graph of $\phi$ can be considered to be the image of the diagonal $\Delta$ of $M \times M$ under the symplectomorphism $1 \times \phi$ of $M \times M$.

**Theorem 2.2.2.** *Let $(M, \omega)$ be a closed symplectic manifold. For a Hamiltonian symplectomorphism $\phi : (M, \omega) \to (M, \omega)$ we have $(1 \times \phi)(\Delta) \cap \Delta \neq \emptyset$. We can find a $1-1$ correspondance between the intersection points and the fixed points of $\phi$. In this case $(1 \times \phi)$ is also a Hamiltonian symplectomorphism of $(M \times M, -\omega \oplus \omega)$.*

We will prove the above theorem in the special case when $X_t = X \forall t$, i.e., $X_t$ is independent of time.

**Proof.** (When $X_t$ does not depend on time) It is enough to prove that $\phi$ has a fixed point as the rest of the statements follow easily from that. $\phi$ is a Hamiltonian symplectomorphism, so there exists a family of symplectomorphisms $\psi_t$ and a function $H : M \to \mathbb{R}$ with the properties:

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \iota(X_t)\omega = dH_t, \quad \psi_0 = id, \quad \psi_1 = \phi$$

Now, because the manifold is compact, $H$ (which obviously does not depend on $t$ in this case) has some critical point and since $\omega$ is non-degenerate, the set of critical points of $H$ is the same as the zero set of $X$. These points are precisely the fixed points of the flow $\psi_t$ and consequently the fixed points of $\phi$. Thus, when $X_t$ is independent of $t$, the set of fixed points of $\phi$ is non-empty. □

Therefore, the problem of finding out how many fixed points a Hamiltonian symplectomorphism has is nothing but a special case of the problem where we try to estimate how many intersection points there are between two Lagrangian submanifolds. This is what makes Lagrangian Floer theory so important in the context of the proof of Arnold's conjecture.

### 2.2.2 The Lagrangian Floer Chain Complex

The way Lagrangian Floer Homology is defined is not very different from the construction of Morse homology. However, there are certain extra complications that arise in this case. The manifold we consider in this scenario is a certain infinite dimensional space of trajectories, and the function defined on it is an action functional. Certain constant trajectories play the role of critical points, and these generate the chain complex in this case.

Suppose that $L_0$ and $L_1$ are two compact Lagrangian submanifolds in $(M, \omega)$ such that

$L_1 \pitchfork L_2$. We will look at the space of smooth trajectories from $L_0$ to $L_1$ with the $C^\infty$-topology, i.e.,

$$\mathcal{P}(L_0, L_1) := \{\eta : [0,1] \to M | \eta \text{ is smooth}, \eta(0) \in L_0, \eta(1) \in L_1\} \tag{2.5}$$

For our convinience, we will write $\mathcal{P}$ for $\mathcal{P}(L_0, L_1)$ from now on. Clearly, the constant paths in $\mathcal{P}$ correspond to the intersection points $L_0 \cap L_1$.

Now, the space of trajectories defined above might not be a connected space. Therefore, we will work with connected components of $\mathcal{P}$. Let $\widehat{\eta}$ be an element of $\mathcal{P}$. We will look at the connected component that contains $\widehat{\eta}$, and call it $\mathcal{P}(\widehat{\eta})$. The universal covering of $\mathcal{P}(\widehat{\eta})$ consists of points of the form $[\eta, w]$, where $w : [0,1] \times [0,1] \to M$ is a smooth path in $\mathcal{P}(\widehat{\eta})$ from $\widehat{\eta}$ to $\eta$. We denote this space as $\widetilde{\mathcal{P}(\widehat{\eta})}$. Thus $w \in \widetilde{\mathcal{P}(\widehat{\eta})}$ implies

$$w(s, .) \in \mathcal{P}(\widehat{\eta}) \;\; \forall s \in [0,1], \quad w(0, .) = \widehat{\eta}, \quad w(1, .) = \eta. \tag{2.6}$$

Now, in its full generality, the action functional in this theory is actually defined on the **Novikov covering** of $\mathcal{P}(\widehat{\eta})$ and not on $\widetilde{\mathcal{P}(\widehat{\eta})}$ described above. However, in order to reduce complications we will impose some conditions on $(M, \omega)$ and $L_0$ and $L_1$. This will make the action functional well-defined on $\widetilde{\mathcal{P}(\widehat{\eta})}$ and we can follow the blueprint of Morse theory.

We start by assuming that our manifold is **symplectically aspherical** (defined below) and both $L_0$ and $L_1$ are simply connected.

**Definition 2.2.2.** *We say that a symplectic manifold $(M, \omega)$ is* ***symplectically aspherical*** *provided that*

$$\int_{\mathbb{S}^2} f^*\omega = 0 \tag{2.7}$$

*for every $[f] \in \pi_2(M)$, where $f$ is a smooth representative.*

Under these hypotheses, the action functional becomes well-defined on $\widetilde{\mathcal{P}(\widehat{\eta})}$ and another phenomenon called bubbling is ruled out. Now, we can define the action functional $\widetilde{\mathcal{P}(\widehat{\eta})} \to \mathbb{R}$ as the symplectic area of $w(I^2) \in (M, \omega)$ in the following way: (here $I^2 := [0,1] \times [0,1]$)

$$\mathcal{A}([\eta, w]) = \int_{I^2} w^*\omega \tag{2.8}$$

**Theorem 2.2.3.** *The above definition is well-defined.*

**Proof.** Suppose that $(\eta, w)$ and $(\eta, w')$ are representatives from the same class $[\eta, w]$. This enables us to define a map on the cylinder $\bar{w}\#w' : \mathbb{S}^1 \times [0,1] \to M$ so that $\bar{w}\#w'(s,0)$ and $\bar{w}\#w'(s,1)$ are loops in $L_0$ and $L_1$ respectively. Here, the bar over $w$ denoted the loop $-w$. We have assumed that $L_i$ ($i \in \{0,1\}$) are simply connected, thus there exist 2-disks $D_i$ in $L_i$ such that the boundary of $D_0$ is the loop $\bar{w}\#w'(s,0)$ and the boundary of $D_1$ is the loop $\bar{w}\#w'(s,1)$. Thus, we cap the cylinder and get a 2-sphere. By the symplectically aspherical property, this sphere has zero symplectic area. Moreover, $L_0$ and $L_1$ being Lagrangian, $\omega$ vanishes on each $L_i$. Therefore, the portions of this sphere lying on $L_0$ and $L_1$ also have zero symplectic area. Consequently, the symplectic area of the cylinder $\bar{w}\#w' : \mathbb{S}^1 \times [0,1] \to M$ is equal to zero and we have:

$$
\begin{aligned}
&\int_{\mathbb{S}^1\times[0,1]} \bar{w}\#w'^*\omega = 0 \\
\Rightarrow &\int_{I^2} \bar{w}^*\omega + \int_{I^2} w'^*\omega = 0 \\
\Rightarrow &-\int_{I^2} w^*\omega + \int_{I^2} w'^*\omega = 0 \\
\Rightarrow &\int_{I^2} w^*\omega = \int_{I^2} w'^*\omega
\end{aligned}
$$

□

In local coordinates, this action functional takes the following form:

$$\mathcal{A}([\eta, w]) = \int_0^1 \int_0^1 \omega\Big(\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}\Big) ds dt. \tag{2.9}$$

In Morse homology, the Riemannian metric played a crucial part in the calculation of the gradient and so if we want to follow in those footsteps, we are required to specify a Riemannian metric on $\widetilde{\mathcal{P}(\widehat{\eta})}$. We will define that now.

Let $\mathcal{J}$ be the space of almost complex structures on $(M, \omega)$ and let $J = \{J_t\}_{0\le t\le 1}$ (each $J_t \in \mathcal{J}$) be a smooth family of $\omega$-compatible almost complex structures on $(M, \omega)$. This gives us smooth family of Riemannian metrics $\{\mathfrak{g}_t\}_{0\le t\le 1}$. Then we can define a Riemannian metric on $\widetilde{\mathcal{P}(\widehat{\eta})}$ as:

$$\langle\langle \xi_1, \xi_2 \rangle\rangle := \int_0^1 \mathfrak{g}_t(\xi_1(t), \xi_2(t))dt \tag{2.10}$$

where $\xi_1, \xi_2 \in T\widetilde{\mathcal{P}(\widehat{\eta})}$.

We are now ready to calculate the gradient of $\mathcal{A}$ with respect to the above metric. We have,

$$\begin{aligned} d\mathcal{A}_{([\eta,w])}(\xi) &= \int_0^1 \omega\Big(\frac{\partial \eta}{\partial t}, \xi(t)\Big)dt \\ &= \int_0^1 \omega\Big(\frac{\partial \eta}{\partial t}, J_t(-J_t\xi(t))\Big)dt \\ &= \int_0^1 \mathfrak{g}_t\Big(\frac{\partial \eta}{\partial t}, -J_t\xi(t)\Big)dt \\ &= \langle\langle \frac{\partial \eta}{\partial t}, -J_t\xi(t) \rangle\rangle \\ &= \langle\langle J_t\frac{\partial \eta}{\partial t}, \xi(t) \rangle\rangle. \end{aligned}$$

So from the definition of gradient, we have:

$$\nabla \mathcal{A} = \mathrm{grad}(\mathcal{A}([\eta, w]) = J_t\frac{\partial \eta}{\partial t} \tag{2.11}$$

We are interested in the critical points of $\mathcal{A}$, which are points $[\eta, w]$ where $\nabla\mathcal{A} = 0$. Now, $J_t$ is an automorphism of $TM$ for each $t$. Therefore, $\nabla\mathcal{A}$ is zero at $[\eta, w]$ iff $\frac{\partial \eta}{\partial t} = 0$, i.e., when $\eta$ is a constant path. As mentioned before, these correspond to the points of intersection between $L_0$ and $L_1$. Just in the Morse theory case, we can now define flow lines of $-\nabla\mathcal{A}$ connecting critical points $p$ and $q$: such a flow line is a smooth map $u : \mathbb{R} \to \widetilde{\mathcal{P}(\widehat{\eta})}$ that satisfies the following:

$$\frac{\partial u}{\partial s} + J_t\frac{\partial u}{\partial t} = 0, \qquad \lim_{s\to-\infty} u(s) = p, \qquad \lim_{s\to+\infty} u(s) = q. \tag{2.12}$$

We can see that the above equation is nothing but the Cauchy-Riemann equation for the $\omega$-compatible almost complex structure $\{J_t\}$. We say that a smooth map $u : \mathbb{R}\times[0, 1] \to (M, \omega)$ is a $J$-holomorphic strip in $(M, \omega)$ provided that the first equation in 2.12 holds. Then we can denote the space of flow lines (i.e., $J$-holomorphic strips in $(M, \omega, J)$) connecting $p$ with

$q$ as:

$$\mathcal{M}_J(p,q,L_0,L_1) := \{u : \mathbb{R}\times[0,1] \to (M,\omega) | u \text{ satisfies equ. 2.12, and } u(s,.) \in \mathcal{P}(L_0,L_1)\}$$

For a non-compact $(M,\omega)$ that is still exact and assuming that $L_0, L_1$ are compact, we can carry out a similar procedure, provided that we add one more condition on the flow lines. In this case, $u$ must have finite energy as well,

$$\int_{\mathbb{R}\times[0,1]} u^*\omega < \infty \tag{2.13}$$

We do not need to worry about this condition in the case of a compact $M$ because compactness ensures that provided $u$ satisfies the necessary limit conditions described before, $u$ has finite energy. One can find the details in [14].

Let $p, q \in L_0 \cap L_1$. The space of smooth maps $u : \mathbb{R}\times[0,1] \to M$ satisfying the limit conditions in 2.12 is denoted by $C^\infty(\mathbb{R}\times[0,1], M; L_0, L_1)$. This gives us a bundle map $\cup_u C^\infty(u^*(TM)) \mapsto C^\infty(\mathbb{R}\times[0,1], M; L_0, L_1)$, where $C^\infty(u^*(TM))$ is the space of vector fields along $u(\mathbb{R}\times[0,1])$. Now, equation 2.12 gives a section of this bundle, which is $u \mapsto \bar{\partial}_J(u)$. Here, the moduli space $\mathcal{M}_J(p,q,L_0,L_1)$ is the zero locus of this section.

The finite-dimensionality of $\mathcal{M}_J(p,q,L_0,L_1)$ is ensured when $\bar{\partial}_J$ intersects the zero-section transversally. However, transversality is one of the newer complications that arise while trying to define Lagrangian Floer homology and it is a delicate issue of the subject. The details can be found in [11].

Instead of looking at smooth maps $u$, we can actually consider elements of the Sobolev space $W^k_p(\mathbb{R}\times[0,1], M; L_0, L_1)$ for $k > p/2$ and $p > 1$. It can be shown that for $u \in W^k_p(\mathbb{R}\times[0,1], M; L_0, L_1)$ with $\bar{\partial}_J(u) = 0$, $u$ is smooth.

An important result regarding this is the fact that we can find a dense subset $\mathcal{J}_{\text{reg}}(L_0, L_1)$ of $C^\infty([0,1], \mathcal{J})$ (where $\mathcal{J}$ is the set of $\omega$-compatible almost complex structures) with the property that for $J = \{J_t\} \in \mathcal{J}_{\text{reg}}(L_0, L_1)$ and every $u \in \mathcal{M}_J(p,q,L_0,L_1)$ the operator

$$D(\bar{\partial}_J)_u : \{\xi \in W^k_p(u^*(TM)) | \xi(s,0) \in L_0, \xi(s,1) \in L_1\} \to W^{k-1}_p(u^*(TM)) \tag{2.14}$$

is surjective and Fredholm. It can then be shown that the kernel of $D(\bar{\partial}_J)_u$ is finite dimensional and we can identify it with $T\mathcal{M}_J(p,q,L_0,L_1)$, where $T\mathcal{M}_J$ denotes the tangent space at $u$. Thus $\mathcal{M}_J$ is finite dimensional as well.

Just like the Morse theory scenario, $\mathcal{M}_J(p,q,L_0,L_1)$ admits an $\mathbb{R}$-action. We can therefore quotient it out by this $\mathbb{R}$-action and obtain the unparametrized moduli space $\widetilde{\mathcal{M}}_J(p,q,L_0,L_1)$.

Now, suppose that $\beta \in \pi_2(M, L_0 \cup L_1)$. Let $u$ be an element of the unparametrized moduli space defiend above such that the homotopy class of $u$ is given by $\beta$. Then the space of such $u$'s is denoted by $\widetilde{\mathcal{M}}_J(p,q,L_0,L_1;\beta)$.

The dimension of $\mathcal{M}_J(p,q,L_0,L_1)$ is given by the so-called Maslov index $\mu(\beta)$, which is an integer, defined in the following way:

Let $(M,\omega)$ be a symplectic manifold and let $L$ be a Lagrangian submanifold. We consider smooth maps of the complex unit disk into $M$ with the property that the boundary of the disk is mapped to $L$. It can be shown that this leads to a fibration $u^*(TM) \cong \mathbb{R}^{2n} \times \mathbb{D}^2$ which is trivial as symplectic bundles, and when restricted to the boundary of the disk, we obtain a loop of Lagrangian subspaces of $\mathbb{R}^{2n}$ with the standard symplectic form. We are now in the realm of Theorem 1.3.7, and can define the Maslov index in accordance with that.

We then have the following theorem the proof of which can be found in [11]:

**Theorem 2.2.4.** *Suppose that $L_0$ and $L_1$ are compact Lagrangian submanifolds of $(M,\omega)$ with $L_0 \pitchfork L_1$. Then there exists a dense subset $\mathcal{J}_{reg}(L_0,L_1)$ in $C^\infty([0,1],\mathcal{J})$ of $\omega$-compatible almost complex structures, such that for $J = \{J_t\} \in \mathcal{J}_{reg}(L_0,L_1)$, $p$ and $q$ in $L_0 \cap L_1$, and $\beta \in \pi_2(M, L_0 \cup L_1)$, the space $\widetilde{\mathcal{M}}_J(p,q,L_0,L_1,\beta)$ is a smooth manifold and its dimension is given by the Maslov index $\mu(\beta)$.*

Even though we have made extra assumptions on $L_0, L_1$ and $(M,\omega)$ for making the theory easier to understand, all of the theory discussed until now holds for any closed symplectic manifold and compact Lagrangian submanifolds, provided that we make the corresponding adaptations. However the following few results do not hold in general. In fact, we cannot define Lagrangian Floer homology for arbitrary Lagrangian submanifolds of arbitrary symplectic manifolds.

In order to proceed further, we need to impose specific conditions on $M, L_0$ and $L_1$. As an example, we can require that the Lagrangians submanifolds of $(M, \omega)$ are monotone and insist that their Maslov index must be greater than 2. Monotone Lagrangian submanifolds are defined below:

**Definition 2.2.3.** *Let $L$ be a Lagrangian submanifold of $(M, \omega)$. Then, $L$ is said to be monotone provided that we can write the symplectic form $\omega$ in the following way:*

$$\omega = \lambda \mu_L.$$

*Here, $\lambda > 0$ and $\mu_L$ denotes the so-called Maslov class.*

If the above conditions are met, we can define Lagrangian Floer homology. However, these conditions are by no means the least restrictive ones where the homology groups are well-defined. The advantage of these asumptions is that this leads to a very similar situation as in Morse thoery: the complex is the free abelian group generated by the points in $L_0 \cap L_1$ with coefficients in $\mathbb{Z}_2$; and the sum in the definition of the boundary operator is a finite sum.

From now on we will assume that the symplectic area of any disk with boundary in $L_0$ or $L_1$ is zero. In other words, for a closed $M$ and closed Lagrangian submanifolds $L_0$ and $L_1$, we have assumed that $[\omega].\pi_2(M, L_j) = 0$ for $j = 0, 1$. Under these conditions, when $\mu(p, q, \beta) = 1$ , $\widetilde{\mathcal{M}}_J(p, q, L_0, L_1; \beta)$ can be shown to be a compact manifold of dimension 0. The proof uses Gromov's compactness arguments and the fact that our restrictions do not allow the existence of holomorphic disks and spheres with boundary in $L_0$ or $L_1$.

Again, we are using the field $\mathbb{Z}_2$ to bypass the hassle of taking care of orientations. For a $J = \{J_t\}$ with the properties mentioned in Theorem 2.2.4, we denote the number of points in the unparametrized moduli space $\widetilde{\mathcal{M}}_J(p, q, L_0, L_1; \beta)$ module 2 by $n_2(p, q, L_0, L_1, \beta)$.

However, before we define the boundary operator, there is another aspect that we have to consider. Since the solutions of 2.12 might give us infinitely many homotopy classes of $\pi_2(M, L_0 \cup L_1)$, and therefore, we need to make sense of these infinitely many homotopy classes of connecting orbits. We define the notion of Novikov field over $\mathbb{Z}_2$ for this purpose, which is:

$$\Lambda := \Big\{ \sum_{j=0}^{\infty} a_j T^{c_j} | a_j \in \mathbb{Z}_2, c_j \in \mathbb{R}, \lim_{j \in \infty} c_j = \infty \Big\}. \tag{2.15}$$

We are now ready to define the chain complex. Let $CF(L_0, L_1)$ be the free $\Lambda$-module generated by the points in $L_0 \cap L_1$, (there are finitely many such points since $L_0$ and $L_1$ are compact and $L_0 \pitchfork L_1$). Then, for $p \in L_0 \cap L_1$ and $J = \{J_t\}$ as in Theorem 2.2.4, the boundary operator can be defined by

$$\partial_J(p) := \sum_{\substack{q \in L_0 \cap L_1, \beta \in \pi_2(M, L_0 \cup L_1) \\ \mu(p,q,\beta)=1}} n_2(p, q, L_0, L_1, \beta) T^{\omega(\beta)} q. \tag{2.16}$$

We have the following theorem, whose proof can be found in [11].

**Theorem 2.2.5.** *Suppose that $L_0$ and $L_1$ are closed Lagrangian submanifolds of $(M, \omega)$ such that $L_0 \pitchfork L_1$. We further assume that $(M, \omega)$ is closed and $\{J_t\}$ is an almost complex structure with the properties mentioned in Theorem 2.2.4. If $[\omega].\pi_2(M, L_j) = 0$ for $j \in \{0, 1\}$, then $\partial_J : CF(L_0, L_1) \to CF(L_0, L_1)$ satisfies $\partial_J \circ \partial_J = 0$.*

When theorem 2.2.5 holds, the complex $(CF(L_0, L_1), \partial_J)$ is called the Lagrangian Floer chain complex of $(L_0, L_1)$. The Lagrangian Floer Homology of $(L_0, L_1)$ is defined as the homology this complex,

$$HF(L_0, L_1, J) := \frac{\ker \partial_J}{\operatorname{im} \partial_J}. \tag{2.17}$$

It is proved in [11] that these homology groups do not depend on the almost complex structure chosen. Moreover, if we have a symplectomorphism $\phi$, then $HF(L_0, L_1) \simeq HF(L_0, \phi(L_1))$, provided that the transversality requirements are still fulfilled. This gives us a way to address Arnold's conjecture, which is depicted in the following theorems:

**Theorem 2.2.6.** *Suppose that $(M, \omega)$ is a closed symplectic manifold, $L$ is a Lagrangian submanifold and $\phi$ is a Hamiltonian symplectomorphism with the property that $L \pitchfork \phi(L)$. Furthermore, suppose that $[\omega].\pi_2(M, L) = 0$, then*

$$\#(L \cap \phi(L)) \geq \sum_{j=0}^{n} \mathit{Rank}\ H_j(L, \mathbb{Z}_2). \tag{2.18}$$

**Theorem 2.2.7.** *Suppose that $(M, \omega)$ is a closed symplectic manifold and $\phi$ is a Hamiltonian* *symplectomorphism that is also a Morse function. When $\pi_2(M) = 0$, we have*

$$|Fix\ (\phi)| \geq \sum_{j=0}^{n} Rank\ H_j(M, \mathbb{Z}_2). \tag{2.19}$$

## 2.3 Heegaard Floer Homology

We will now finally dive into Heegaard Floer homology. We start off with the notions of symmetric products and totally real tori, which will be the backbone of our theory. Then after a brief discussion on holomorphic discs and spin$^c$ structures, we move on to the definition of the chain complex as stated by Ozsváth and Szabó. We will almost exclusively focus on the case when our space is a rational homology sphere, as that makes some of the details much easier to understand. This chapter follows [7] and [22] quite closely.

### 2.3.1 Symmetric Products

For a pointed Heegaard diagram $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$, we can define the symmetric product of the surface $\Sigma_g$ as follows:

**Definition 2.3.1.** *The $g$-fold symmetric product of $\Sigma_g$ is given by the $g$-fold cartesian product quotiented by $S_g$, where $S_g$ is the symmetric group of $g$ elements. Thus,*

$$Sym^g(\Sigma_g) := \Sigma_g \times ... \times \Sigma_g / S_g \tag{2.20}$$

*Basically, $Sym^g(\Sigma_g)$ is the set of unordered $g$-tuples in $\Sigma_g$ where we allow the points to repeat multiple times.*

Clearly, symmetric product can be defined for any surface. The next theorem gives us some context to the study of such objects.

**Theorem 2.3.1.** *$Sym^g(\Sigma_g)$ as defined above is a smooth manifold.*

**Proof.** We will only provide a sketch of the proof, the details can be found in [19]. Suppose that $D$ is the set of points in $\mathrm{Sym}^g(\Sigma_g)$ which corresponds to those $g$ tuples in $(\Sigma_g)^g =$

$\Sigma_g \times ... \times \Sigma_g$ ($g$ times) with at least 2 same entries. We will call $D$ the diagonal of $\mathrm{Sym}^g(\Sigma_g)$. Given a point $p \in \mathrm{Sym}^g(\Sigma_g)$, we wish to show that it has a neighbourhood holomorphic to $\mathbb{C}^g$. When $p$ does not lie on $D$, this is a straightforward matter as given that the neighbourhood $N$ of $p$ is small enough, the preimage of $N$ under the natural quotient map $\pi : (\Sigma_g)^g \to \mathrm{Sym}^g(\Sigma_g)$ will be a disjoint union of open sets in $(\Sigma_g)^g$, each of which will be holomorphic to $\mathbb{C}^n$.

On the other hand, when $p \in D$, we can invoke geometric invariant theory (G.I.T) to obtain our desired results. one can consult Chapter 10 in [20] for details.

Locally, we know that $\mathrm{Sym}^g(\Sigma_g)$ can be identified with $\mathbb{C}^n/S_n$. $\mathbb{C}^n$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[x_1, ..., x_n])$ and it can be shown via G.I.T that

$$\mathbb{C}^n/S_n \cong \mathrm{Spec}(\mathbb{C}[x_1, ..., x_n]^{S_n})$$

where $(\mathbb{C}[x_1, ..., x_n]^{S_n})$ denotes the space of complex polynomials that are invariant under the action of $S_g$. However, this space can in turn be generated by elementary symmetric polynomials in $[x_1, ..., x_n]$, i.e., we have

$$\mathbb{C}^n/S_n \cong \mathrm{Spec}(\mathbb{C}[p_1, ..., p_n])$$

where $p_1, ..., p_n$ are the elementary symmetric polynomials. Now, since $\mathrm{Spec}(\mathbb{C}[p_1, ..., p_n]) \cong \mathbb{C}^n$, we are essentially done. $\square$

The next two theorems, characterize the first two homotopy groups of $\mathrm{Sym}^g(\Sigma_g)$, and thus tell us a lot about the topology of this manifold:

**Theorem 2.3.2** ([7], Lemma 2.6). *Let $\Sigma_g$ be a genus $g$ surface. Then $\pi_1(Sym^g(\Sigma_g)) \simeq H_1(Sym^g(\Sigma_g)) \simeq H_1(\Sigma_g)$.*

**Proof.** Let $D$ be the diagonal as defined in the proof of Theorem 2.3.1. Now, the inclusion of $\Sigma_g \times \{x\} \times ... \times \{x\}$ (where $x \in \Sigma_g$) into $\mathrm{Sym}^g(\Sigma_g)$ induces a map $H_1(\Sigma_g) \to H_1(\mathrm{Sym}^g(\Sigma_g))$. To prove the theorem, we need show that $\pi_1(\mathrm{Sym}^g(\Sigma_g))$ is Abelian and that we can invert the above map.

We notice that any curve $\gamma$ in $\mathrm{Sym}^g(\Sigma_g)$ gives a map of a $g$-fold covering of $\mathbb{S}^1$to $\Sigma_g$, and that in turn gives us a homology class in $H_1(\Sigma_g)$. This gives us a map $H_1(\mathrm{Sym}^g(\Sigma_g) \to$

$H_1(\Sigma_g)$, which is inverse to the map induced by the inclusion above.

We now need to show that $\pi_1(\mathrm{Sym}^g(\Sigma_g))$ is Abelian. Suppose that $\gamma : \mathbb{S}^1 \to \mathrm{Sym}^g(\Sigma_g)$ is a null-homologous curve that does not intersect $D$. Then just like before, it gives a map $\widehat{\gamma}$ of a $g$ fold cover of $\mathbb{S}^1$ into $\Sigma_g$. This map is null-homologous, which means that we can find a map $i : V \to \Sigma_g$ (where $V$ is a 2-dimensional manifold with boundary) such that $i|_{\partial V} = \widehat{\gamma}$. By choosing a $V$ with appropriate genus, we can extend this to a branched $g$-fold covering of the 2-dimensional disk $p : V \to D$. In this scenario, let $s$ be the map that sends $z \in D$ to the image of $p^{-1}(z)$ under $i$. Then $s$ induces the null-homotopy of $\gamma$ that we are looking for. $\square$

**Theorem 2.3.3** ([7], Proposition 2.7)**.** *Suppose that $\Sigma_g$ is a Riemann surface of genus $g > 1$. Now, if we quotient $\pi_2(Sym^g(\Sigma_g))$ by the action of $\pi_1(Sym^g(\Sigma_g))$, then the resulting group, $\pi_2'(Sym^g(\Sigma_g)) \cong \mathbb{Z}$. Here, both $\pi_1$ and $\pi_2$ are calculated using the same basepoint. When $g > 2$, $\pi_1(Sym^g(\Sigma_g))$ acts trivially on $\pi_2(Sym^g(\Sigma_g))$ and we have $\pi_2(Sym^g(\Sigma_g)) = \mathbb{Z}$.*

**Proof.** (Sketch) Roughly speaking, the isomorphism $\pi_2^{'}(\mathrm{Sym}^g(\Sigma_g)) \simeq \mathbb{Z}$ is obtained by the intersection number of the generator of $\pi_2'$ with the submanifold $\{x\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$, where $x$ is a generic point. The actual proof requires the use of hyperelliptic structures and hyperelliptic involutions and we refer to [7] for details.

The basic idea in the case when $g > 2$ is to consider a sphere in the kernal of the above isomorphism and use techniques related to hyperelliptic involutions and so-called 'splicing' to make sure that we miss the space $\{x\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$, where $x$ is a generic point and thus reduce the problem to calculating the $\pi_2$ of the space $\mathrm{Sym}^g(\Sigma_g - x)$ and claiming that this is zero.

One way of proving the above claim is to notice that $\Sigma_g - x$ is just the wedge of $2g$ circles. This space is a deformation retract of the space $\mathbb{C} - \{p_1, ..., p_{2g}\}$. Then, we can identify $\mathrm{Sym}^g(\mathbb{C} - \{p_1, ..., p_{2n}\}$ with the space of monic polynomials $q$ in one variable of degree $g$, such that $q(p_i) \neq 0 \forall i$. It can be shown that this is same as lookin at the complement of $2g$ hyperplanes in $\mathbb{C}^{2g}$. Our claim now follows from a theorem by Hattori (see [21] for details).

The case $g = 2$ is handled differently. It can be shown that the symmetric product in this case is nothing but the 4-torus blown up at a point, and $\pi_2'$ in this case is calculated directly.

□

When $g > 2$ we can calculate the generator $S$ of $\pi_2(\mathrm{Sym}^g(\Sigma))$ using the equation below:

$$\langle c_1(\mathrm{Sym}^g(\Sigma_g)), [S]\rangle = 1 \tag{2.21}$$

Here $c_1$ denotes the first Chern class.

Having defined symmetric products, we now divert our attention to certain subspaces of these manifolds, called the totally real tori embedded in $\mathrm{Sym}^g(\Sigma_g)$. These spaces play a role similar to the Lagrangian subspaces in Lagrangian Floer theory in the sense that our chain modules will be generated by the points of intersection of two of these subspaces.

**Definition 2.3.2.** *Suppose that $M$ is a complex manifold with complex structure $J$, and let $L$ be a submanifold. Then, we say that $L$ is a totally real submanifold provided that no tangent space of $L$ contains a $J$-complex line. In other words, we want $T_\lambda L \cap JT_\lambda L = \{0\}$ for every $\lambda \in L$.*

**Theorem 2.3.4.** *Let $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ be a Heegaard diagram. The spaces $\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times ... \times \beta_g$, which are $g$-dimensional tori embedded in $Sym^g(\Sigma)$ are totally real submanifolds.*

**Proof.** We will focus on the case of $\mathbb{T}_\alpha$ as the case for $\mathbb{T}_\beta$ is identical. Let $(\Sigma_g)^g$ denote the cartesian product of $g$ copies of $\Sigma_g$, endowed with the product complex structure. Since the $\alpha_i$'s do not intersect each other, $\mathbb{T}_\alpha$ does not intersect the diagonal $D$ (as defined in the proof of Theorem 2.3.2). Then away from the set $D$, the projection map $p : (\Sigma_g)^g \to \mathrm{Sym}^g(\Sigma_g)$ is a holomorphic local diffeomorphism. Now, it is not difficult to show from the definitions that $\alpha_1 \times ... \times \alpha_g \subset (\Sigma_g)^g$ is a totally real submanifold. Therefore, as $\mathbb{T}_\alpha$ does not intersect $D$, the local diffeomorphism $p$ ensures that $\mathbb{T}_\alpha$ is totally real as well. □

The following theorem gives context to the study of such submanifolds. Apart from being the cornerstone of Heegaard Floer theory, they also enable us to understand the topology of the underlying manifold.

**Theorem 2.3.5.**

$$\frac{H_1(Sym^g(\Sigma_g))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma_g)}{[\alpha_1], ..., [\alpha_g], [\beta_1], ..., [\beta_g]} \cong H_1(M; \mathbb{Z})$$

**Proof.** In light of Theorem 2.3.2, the first isomorphism should not be a surprise, as we have already proven that $\mathrm{Sym}^g(\Sigma_g) \cong H_1(\Sigma_g)$, and $\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g, \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g$. To see the second isomorphism, we recall the procedure of obtaining the 3-manifold from a given Heegaard diagram. In that process, we attach thickened disks along the $\alpha$ and the $\beta$ circles and fill in 3-balls to obtain the manifold. For each $\alpha_i$, we attach one thickened disk $D_i \times [0, 1]$ so that $\alpha_i$ lies along $\partial D_i \times \{\frac{1}{2}\}$. We do a similar procedure for each $\beta_i$ as well. Now, adding such disks makes the homology classes of these cycles trivial, which is akin to quotienting out these circles from the homology group. On the other hand, adding 3-balls do not affect $H_1$ as it depends only on the 2-skeleton of any manifold, if we look at $M$ as a CW-complex. Thus, we can obtain $H_1(M)$ from $H_1(\Sigma_g)$ by quotienting the later out with the $\alpha$ and $\beta$ circles, and the second isomorphism is justified. □

It can be shown that the two totally real tori can be made to intersect transversally and consequently, their intersection is a set of discrete points.The chain complex that we are concerned with, $\widehat{CF}(\mathcal{H})$, is freely generated over $\mathbb{Z}/2\mathbb{Z}$ by these intersection points. These points are just unordered $g$-tuples of the points of intersection between the $\alpha$- and $\beta$-circles where none of the $\alpha$-circles, (or $\beta$-circles), appear more than once.

In case of a pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, w)$, we will also need to consider the following space, which has complex codimension 1 and is disjoint from from $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$:

$$V_w := \{w\} \times \mathrm{Sym}^{g-1}(\Sigma_g) \tag{2.22}$$

Thus $V_w$ consists of unordered $g$-tuples of points in $\Sigma_g$ where at least one point is $w$.

Now we need to define the boundary operator $\partial : \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$. In order to do that, however, we will need to study certain holomorphic disks in $\mathrm{Sym}^g(\Sigma_g)$.

**Definition 2.3.3.** *Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and let $e_\alpha$ and $e_\beta$ denote the arcs in the boundary $\partial\mathbb{D}$ with $Re(z) \geq 0$ and $Re(z) \leq 0$ respectively. Let $x$ and $y$ be two intersection points of the tori. A Whitney disk from $x$ to $y$ is defined to be a continuous map $\phi : \mathbb{D} \to Sym^g(\Sigma_g)$ with the following properties:*

1. $\phi(-i) = x$,
2. $\phi(i) = y$,
3. $\phi(e_\alpha) \subset \mathbb{T}_\alpha$ *and*

4. $\phi(e_\beta) \subset \mathbb{T}_\beta$.

*We denote the set of homotopy classes of Whitney disks from $x$ to $y$ by $\pi_2(x, y)$.*

Suppose that we have two Whitney disks, one of them from $x$ to $y$, and the other one from $y$ to $z$. Then we can glue these disks together to obtain a Whitney disk from $x$ to $z$. This gives us a multiplicative structure $* : \pi_2(x, y) \times \pi_2(y, z) \to \pi_2(x, z)$.

We can enrich our study of topological disks in $\mathrm{Sym}^g(\Sigma_g)$ by looking at the so-called 'Domains' in $\Sigma_g$. We make this idea concrete in the following discussion:

Suppose that $x$ and $y$ are two intersection points of the two totally real tori. Let $w \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$ be any point in the compliment of the attaching circles. We also note that the attaching circles divide the surface $\Sigma_g$ into disjoint components, the closures of which we will call **regions** and denote them by $D_1, ..., D_m$.

Let $n_w : \pi_2(x, y) \to \mathbb{Z}$ be the algebraic intersection number:

$$n_w(\phi) = \#\phi^{-1}(V_w), \tag{2.23}$$

where $V_w$ is same as in 2.22.

**Definition 2.3.4.** *Given $\phi \in \pi_2(x, y)$, the following formal linear combination is called the domain associated to $\phi$:*

$$\mathcal{D}(\phi) = \sum_{i=1}^{m} n_{w_i}(\phi) D_i.$$

*Here $w_i$ is an interior point of $D_i$ . If $n_{w_i}(\phi) \geq 0 \ \forall i$, then we write $D(\phi) \geq 0$.*

Let $x, y, z \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\phi_1 \in \pi_2(x, y)$ and $\phi_2 \in \pi_2(y, z)$. Then it can be shown that the following holds:

1. $\mathcal{D}(\phi_1 * \phi_2) = \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2)$ and
2. $\mathcal{D}(S * \phi) = \mathcal{D}(\phi) + \sum_{i=1}^{n} D_i$, where $S$ is the positive generator of $\pi_2(\mathrm{Sym}^g(\Sigma_g))$.

**Definition 2.3.5.** *Let $x, y$ be two intersection points and define the formal sum $\mathcal{A} := \sum_{i=1}^{n} a_i D_i$ such that $\partial\mathcal{A}$ gives a path from $x$ to $y$ through the $\alpha$ curves and a path from $y$ to $x$ through the $\beta$ curves. Then we will say that $\mathcal{A}$ connects $x$ to $y$.*

When $\phi \in \pi_2(x, y)$ it is not difficult to see that $\partial(\mathcal{D}(\phi))$ gives a path from $x$ to $y$ through the $\alpha$ curves and a path from $y$ to $x$ through the $\beta$ curves. For $g > 1$, we have the following theorem, which in some sense serves as a converse:

**Theorem 2.3.6.** *Let $g > 1$ and $x, y$ be two intersection points. Suppose that $\mathcal{A}$ connects $x$ to $y$ according to the definition above. Then we can find $\phi \in \pi_2(x, y)$ such that $\mathcal{D}(\phi) = \mathcal{A}$. Moreover, if $g > 2$, then $\phi$ is uniquely determined by $\mathcal{A}$.*

We also note that when $g = 2$, $\pi_2'(x, y)$ can be defined by quotienting out $\pi_2(x, y)$ with the following relation: We will consider $\phi_1 \sim \phi_2$ provided $\mathcal{D}(\phi_1) = \mathcal{D}(\phi_2)$.

Given two intersection points $x$ and $y$, Theorem 2.3.6 gives a situation where there exists Whitney disks from $x$ to $y$. We now discuss the general condition when Whitney disks cannot exist between two intersection points.

**Definition 2.3.6.** *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and suppose that $a$ and $b$ are paths from $x$ to $y$ such that $a$ lies in $\mathbb{T}_\alpha$ while $b$ lies in $\mathbb{T}_\beta$. Then $l = a - b$ is a loop in $Sym^g(\Sigma_g)$, and under the isomorphism given by Theorem 2.3.5, $l$ maps to $H_1(M, \mathbb{Z})$. We call this image $\epsilon(x, y)$.*

$\epsilon(x, y)$ can be shown to be independent of the paths $a$ and $b$. Now, we can identify the path $a$ with a collection of arcs in $\mathbb{T}_\alpha$ whose boundary is given by $\partial a = y_1 + ... + y_g - x_1 - ... - x_g$ (where $x = \{x_1, ..., x_g\}$ and $y = \{y_1, ..., y_g\}$). We can identify $b$ similarly, and thus $l = a - b$ gives a closed cycle in $\Sigma_g$ and $\epsilon(x, y)$ denotes the image of this cycle in $H_1(M, \mathbb{Z})$.

We also have an additive structure on $\epsilon$ in the following sense:

$$\epsilon(x, y) + \epsilon(y, z) = \epsilon(x, z) \tag{2.24}$$

For $g > 2$, it can be shown that for two intersection points $x$ and $y$, there does not exist any Whitney disk if $\epsilon(x, y) \neq 0$. In the case when $\epsilon(x, y) = 0$, $\pi_2(x, y)$ is isomorphic to $\mathbb{Z} \oplus H^1(M, \mathbb{Z})$. A proof can be found in [7]. We can define an equivalence relation on the intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ by the following relation: $x \sim y$ provided that $\epsilon(x, y) = 0$.

To move our discussion about these equivalence classes forward, we require the definition of Spin$^c$ structures. We will start of with an oriented closed 3-manifold $M$. Since the Euler characteristics of $M$ is trivial, it admits vector fields that are nowhere vanishing.

**Definition 2.3.7.** *Let $M$ be as above and suppose that $v_1$ and $v_2$ are two vector fields on $M$ that vanish nowhere. Suppose that there exists a ball $B$ in $M$ such that $v_1|_{M-B}$ is homotopic to $v_2|_{M-B}$. Then $v_1$ is said to be homologous to $v_2$. This partitions the nowhere vanishing vector fields into equivalence classes, and the space of Spin$^c$ structures over $M$ is defined to the the space of nowhere vanishing vector fields modulo this equivalence relation.*

The space of Spin$^c$ structures over $M$ will be denoted by Spin$^c(M)$.

Now, suppose that $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ is a Heegaard diagram for $M$ and $f$ is a Morse function compatible with $\mathcal{H}$. Let $x$ be an intersection point of the totally real tori. Then we get a $g$-tuple of trajectories (for $\nabla f$) from the critical points of index 1 to those with index 2. Similarly, we get trajectories that join the critical point of index 0 with the critical points with index 3 due to $w$. We can then delete regular neighbourhoods of these trajectories to obtain the space $M - \{\text{disjoint union of 3-balls}\}$. The gradient vector field is non-zero in this region, and on all the boundary spheres it has index 0. Therefore, this vector field can be extended as a nowhere vanishing vector field over $M$. We denote by $s_w(x)$ the Spin$^c$ structure that is obtained by the homology class of this nowhere vanishing vector field.

It can be shown that for two intersection points $x$ and $y$, the difference $s_w(y) - s_w(x)$ is the same as the Poincaré dual of $\epsilon(x, y)$ defined above. It follows that $s_w(y) = s_w(x) \iff \pi_2(x, y) \neq \emptyset$. Proofs for both these statements can be found in [7]

### 2.3.2 The Moduli space of Holomorphic Disks

Suppose that we have a pointed Heegaard diagram $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ for $M$. A complex structure on $\Sigma_g$ defines a complex structure on the cartesian product $(\Sigma_g)^g$, which in turn lends a complex structure to $\text{Sym}^g(\Sigma_g)$. Now, let $\phi \in \pi_2(x, y)$ be a Whitney disk. Then the space of holomorphic representatives of $\phi$ form a Moduli space, which we denote by $\mathcal{M}(\phi)$. It can be proven that under suitably generic conditions, $\mathcal{M}(\phi)$ is smooth. We direct the reader to [7] for the details.

The moduli space $\mathcal{M}(\phi)$ admits an $\mathbb{R}$ action just like the previous cases. This $\mathbb{R}$ action corresponds to complex automorphisms of $\mathbb{D}^2 \subset \mathbb{C}$ which preserve $i$ and $-i$. Applying the Riemann mapping theorem, we can transform $\mathbb{D}^2$ into $[0,1] \times i\mathbb{R} \subset \mathbb{C}$. Then, $e_1$ gets mapped to $1 \times i\mathbb{R}$ and $e_2$ gets mapped to $0 \times i\mathbb{R}$, and the automorphisms we are interested in are nothing but vertical translations. So, if $u \in \mathcal{M}(\phi)$ then pre-composition with any of these transformations gives us another holomorphic representative $u'$. We will obtain the quotient of $\mathcal{M}(\phi)$ by this $\mathbb{R}$ action, and call the resulting space the unparametrized Moduli space. Thus, $\widehat{\mathcal{M}}(\phi) := \frac{\mathcal{M}(\phi)}{\mathbb{R}}$.

Let $x$ be an intersection point and $\phi \in \pi_2(x,x)$ be a constant Whitney disk. Then it is easy to see that $\mathcal{M}(\phi)$ is just a point. In this situation the action defined above is not free. However, it can be shown that the above $\mathbb{R}$-action is indeed free in all other cases.

The expected dimension of $\mathcal{M}(\phi)$ is called the Maslov index. It correcponds to the index of an elliptic operator and we denote it by $\mu(\phi)$. The Maslov index has the following properties:

1. Suppose that the complex structure of $\mathrm{Sym}^g(\Sigma_g)$ is changed in an $n$-dimensional family. Then the parametrized moduli space obtained has dimension $n+\mu(\phi)$ around solutions which are cut out smoothly by the defining equation.

2. For $\phi_1 \in \pi_2(x,y), \phi_2 \in \pi_2(y,z)$ we have:

$$\mu(\phi_1 * \phi_2) = \mu(\phi_1) + \mu(\phi_2).$$

3. When $\phi \in \pi_2(x,x)$ is the homotopy class of the constant map, we have $\mu(\phi) = 0$.

Now, the fact that the above moduli space has a compactification was shown by Ozsváth and Szabó in [7]. Having studied energy bounds and analyzing Gromon limits, they proved the following theorem:

**Theorem 2.3.7.** *Let $g > 2$ and let $\phi \in \pi_2(x,y)$ . Then, there exists a family of perturbations that (provided $\mu(\phi) = 1$) makes $\widehat{\mathcal{M}}(\phi)$ into a compact oriented manifold of dimension $0$. A similar result is true in the $g = 2$ case, with the difference being that now we take $\phi \in \pi_2'(x,y)$.*

### 2.3.3 The Heegaard Floer Chain Complex

Now we are ready to define the chain complexes corresponding to the various Heegaard Floer homology groups. In order to simplify some of the arguments, we will assume $M$ to be a rational homology 3-sphere throughout this section, and only elude to the general case very briefly at the end.

Suppose that $M$ is a rational homology 3-sphere and $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ is a pointed Heegaard diagram with genus $g > 0$. Also, suppose $t \in \mathrm{Spin}^c(M)$. Let $Q$ be a subset of the set of the intersection points of the totally real tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ with the additional property that for every $x \in Q$, we have $s_w(x) = t$. Then, we define $\widehat{CF}(\alpha, \beta, t)$ to be the free Abelian group generated by the elements of $Q$.

We can provide this group with a relative grading in the following manner:

$$\mathrm{gr}(x, y) = \mu(\phi) - 2n_w(\phi). \tag{2.25}$$

Here, $\phi \in \pi_2(x, y)$, and $\mu$ denotes the Maslov index. It can be shown that this integer does not depend on the homotopy class $\phi$.

**Definition 2.3.8.** *Having chosen a perturbation as in Theorem* 2.3.7*, we pick two intersection points $x$ and $y$ and choose $\phi \in \pi_2(x, y)$. Now, if $\mu(\phi) = 1$, let $c(\phi)$ denote the number of points in the unparametrized moduli space $\widehat{\mathcal{M}}(\phi)$. Otherwise we define $c(\phi) = 0$. We define $\partial : \widehat{CF}(\alpha, \beta, t) \to \widehat{CF}(\alpha, \beta, t)$ to be the map given by:*

$$\partial x := \sum_{\{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \phi \in \pi_2(x,y) | s_w(y) = t, n_w(\phi) = 0\}} c(\phi).y \tag{2.26}$$

The fact that $\partial^2 = 0$, i.e., the above map, along with $\widehat{CF}(\alpha, \beta, t)$ indeed defines a chain complex is proved in [7] through the analysis of Gromov compactification of $\widehat{\mathcal{M}}(\phi)$.

**Definition 2.3.9.** *The homology groups of the complex $(\widehat{CF}(\alpha, \beta, t), \partial)$ are known as the Heegaard Floer homology groups $\widehat{HF}(\alpha, \beta, t)$.*

One of the primary results in [7] deals with the fact that the homology group $\widehat{HF}(\alpha, \beta, t)$ does not depend on the choices made in the definition. This can be proved by analysing now the pointed Heegaard moves (isotopies, handle slides and stabilizations) affect our homology groups. A proof can be found in [7]. This together with Theorem 1.2.5, imples:

**Theorem 2.3.8.** *Let* $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ *and* $\mathcal{H}' := (\Sigma'_{g'}, \alpha', \beta', w')$ *be two pointed Heegaard diagrams for* $M$ *, and let* $t \in Spin^c(M)$*. Then* $\widehat{HF}(\alpha, \beta, t) \cong \widehat{HF}(\alpha', \beta', t)$*.*

Using the above theorem, we can finally define:

$$\widehat{HF}(M, t) = \widehat{HF}(\alpha, \beta, t). \tag{2.27}$$

We will now try and understand some variations of the theory. The definition of $\widehat{HF}(M, t)$ uses $w$, the basepoint, making it special. In that case we only counted holomorphic disks disjoint form $V_w$ when calculating the boundary map $\partial$. Therefore it makes sense to modify the theory in a way where all the holomorphic disks are utilized.

Let $x$ be an intersection point and let $t \in \mathrm{Spin}^c(M)$. For $i \in \mathbb{Z}$, we denote the free Abelian group generated by $[x, i]$ as $CF^\infty(\alpha, \beta, t)$. Again, we can give a relative grading to these generators:

$$\mathrm{gr}([x, i], [y, j]) = \mathrm{gr}(x, y) + 2i - 2j. \tag{2.28}$$

Now, let $\partial : CF^\infty(\alpha, \beta, t) \to CF^\infty(\alpha, \beta, t)$ be the following map:

$$\partial[x, i] = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y)} c(\phi).[y, i - n_w(\phi)]. \tag{2.29}$$

This again turns out to be a chain complex, and its homology groups are known as the Heegaard Floer homology groups $HF^\infty(M, t)$.

Here we can define an isomorphism $U$ on $CF^\infty(\alpha, \beta, t)$ in the following way:

$$U([x, i]) = [x, i - 1] \tag{2.30}$$

$U$ decreases the grading by two. It can be shown (see [8]) that when $M$ is a rational homology three-sphere, $HF^\infty(M, t) \cong \mathbb{Z}[U, U^{-1}]$.

We now consider the elements $[x, i]$ with $i < 0$. Suppose that $CF^-(\alpha, \beta, t)$ is the abelian group that is generated freely over these points. Then it is not difficult to see that

$CF^-(\alpha,\beta,t)$ is a subgroup of $CF^\infty(\alpha,\beta,t)$ and we denote the quotient group $CF^\infty(\alpha,\beta,t)/CF^-(\alpha,\beta,t)$ by $CF^+(\alpha,\beta,t)$.

**Lemma 2.3.9.** *The group $CF^-(\alpha,\beta,t)$ is a subcomplex of $CF^\infty(\alpha,\beta,t)$. Consequently, we obtain a short exact sequence of chain complexes:*

$$0 \to CF^-(\alpha,\beta,t) \xrightarrow{\iota} CF^\infty(\alpha,\beta,t) \xrightarrow{\pi} CF^+(\alpha,\beta,t) \to 0$$

Now, $U$ restricted to $CF^-(\alpha,\beta,t)$ gives an endomorphism. Thus, it gives an endomorphism on $CF^+(\alpha,\beta,t)$ as well, and this leads to a short exact sequence:

$$0 \to \widehat{CF}(\alpha,\beta,t) \xrightarrow{\iota} CF^+(\alpha,\beta,t) \xrightarrow{U} CF^+(\alpha,\beta,t) \to 0$$

where $\iota(x) = [x,0]$.

Again, both of these turn out to be chain complexes.

**Definition 2.3.10.** *The Floer homology groups of $(CF^+(\alpha,\beta,t),\partial)$ and $(CF^-(\alpha,\beta,t),\partial)$ are denoted by $HF^+(\alpha,\beta,t)$ and $HF^-(\alpha,\beta,t)$ respectively.*

It can again be proved (see [7]) that these groups remain unchanged under pointed isotopies, handle slides and stabilizations. Thus we can define:

$$HF^\pm(M,t) = HF^\pm(\alpha,\beta,t). \tag{2.31}$$

We will now briefly discuss the case when $b_1(M) > 0$. In this case $\pi_2(x,y)$ becomes larger and we end up with infinitely many homotopy classes with Maslov index 1 in the definition of $\partial$. In order for the definition to make sense, we need to make sure that only finitely many of these homotopy classes support holomorphic disks. We ensure this by using special kinds of Heegaard diagram, called 'admissible' Heegaard diagrams. With these changes, most of the constructions from this section carries through and we can define the Heegaard Floer homology groups.

# Chapter 3

# Calculating Heegaard Floer Homology

Having understood the theory, we now need to calculate the Heegaard Floer homology groups for certain spaces. However, given the amount of theoretical complications in the definition, it should not be a surprise that these groups are especially difficult to calculate. In this section, we talk about a combinatorial algorithm for computing these homology groups. This algorithm, first given by Sarkar and Wang in [18], describes a way to obtain 'nice' Heegaard diagrams and uses those to calculate the Heegaard Floer homology of spaces. However, before we get there, we need to understand a reformulation of the theory, given by Lipshitz in [15], which will be used to describe this algorithm.

## 3.1 A Cylindrical Reformulation of the Theory

In this section we give a somewhat different description of the Heegaard Floer homology groups, which will be helpful in giving a combinatorial method for computation of these homologies. This reformulation was given by Lipshitz in [15] and they produce groups isomorphic to the ones in [7]. We will mostly refer to [15] for the proofs as well.

The definition of Heegaard Floer homology discussed in the last chapter involves holomorphic disks in $\mathrm{Sym}^g(\Sigma_g)$. In this reformulation, we ditch $\mathrm{Sym}^g(\Sigma_g)$ in favour of $\Sigma \times [0,1] \times \mathbb{R}$ and look at holomorphic curves in this space. Contrary to $\mathrm{Sym}^g(\Sigma_g)$, this space can be visualized, and that already gives us an advantage. Apart from this, when we consider the technical details, the reformulation has a few more advantages when compared to the original definitions.

Just like in the previous chapter, we consider a pointed Heegaard diagram $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$. Let $W := \Sigma_g \times [0,1] \times \mathbb{R}$. We will refer to points in $W$ as $(p, s, t)$, where $p \in \Sigma_g, s \in [0,1]$ and $t \in \mathbb{R}$. We will also consider the usual projections $p_{\mathbb{D}} : W \to [0,1] \times \mathbb{R}, p_{\mathbb{R}} : W \to \mathbb{R}$ and $p_{\Sigma} : W \to \Sigma_g$. Moreover, we will denote the union of all the $\alpha$ circles as $\bar{\alpha}$ and the union of all $\beta$ circles as $\bar{\beta}$. In this reformulation, we will define our boundary operator by counting holomorphic curves which have their boundary on $C_\alpha \cup C_\beta$, (where $C_\alpha = \bar{\alpha} \times \{1\} \times \mathbb{R}$ and $C_\beta = \bar{\beta} \times \{0\} \times \mathbb{R}$) and have appropriate asymptotics at $\pm\infty$.

From now on, we will take $g > 1$, where $g$ is the genus of $\Sigma_g$. This does not affect the generality of the theory in any manner because we can always obtain a Heegaard diagram with a higher genus through stabilization. Also, we will only focus on $\widehat{HF}$, as the other variations can be reformulated in a similar way.

**Definition 3.1.1.** *We define an intersection point to be a set of $g$ distinct points $x = \{x_1, ..., x_g\}$ in $\bar{\alpha} \cap \bar{\beta}$ so that on each attaching circle there is only one $x_i$.*

This corresponds to an intersection point in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ mentioned previously in Section 2.3. The generators of our reformulated chain complex (which we again call $\widehat{CF}$) are these intersection points.

$\Sigma_g$ is divided into connected components by the $\alpha$ and $\beta$ curves. These connected components will be called **regions**. We define a 2-**chain** to be a formal sum of regions with coefficients in $\mathbb{Z}$. For our purposes, we will be interested in special 2-chains:

**Definition 3.1.2.** *Let $x$ and $y$ be two generators, as defined above. We call a 2-chain $\phi$ a domain if $\partial(\partial(\phi)|_\alpha) = y - x$. The collection of all domains will be denoted by $\pi_2(x, y)$.*

Let $p$ be a point on $\Sigma_g$ that is disjoint from the attaching circles, i.e., $p$ lies in a region, say $D_p$. Then, we denote the coefficient of $D_p$ in a domain $\phi$ as $n_p(\phi)$. We say that $\phi$ is **positive** provided that $n_p(\phi) \geq 0$ for all such $p$. We also give a special notation to the space of those domains which vanish on the region containing the basepoint $w$, i.e.,

$$\phi \in \pi_2^0(x, y) := \{\phi \in \pi_2(x, y) | n_w(\phi) = 0\} \tag{3.1}$$

**Definition 3.1.3.** *Suppose that $\mathcal{H}$ is a Heegaard diagram such that for any intersection point $x$, any positive domain $\phi \in \pi_2^0(x,x)$ is trivial. Then $\mathcal{H}$ is known as an admissible Heegaard diagram.*

We define $J$ to be an almost complex structure on $W$ with the following properties:

1. (J1) $J$ is tamed by $\omega$.
2. (J2) Let $U_{p_i}$ be a cylindrical neighborhood of $\{p_i\} \times [0,1] \times \mathbb{R}$ Then $J = J_\Sigma \times J_\mathbb{D}$ is split. (Here, $U_{p_i}$ is such that the closure of $U_{p_i}$ does not intersect $(\bar{\alpha} \cup \bar{\beta}) \times [0,1] \times \mathbb{R}$).
3. (J3) $J$ is translation invariant in the $\mathbb{R}$–factor.
4. (J4) $J(\partial/\partial t) = \partial/\partial s$.
5. (J5) $J$ preserves $T(\Sigma_g \times \{(s,t)\}) \subset TW$ for all $(s,t) \in [0,1] \times \mathbb{R}$.

$(J1)$ is required to ensure that the moduli spaces we define are compact. $(J2)$ ensures the so-called 'positivity of domains', which we will define in a bit. $(J3)$ and $(J4)$ make $W$ 'cylindrical' (see [16]). $(J5)$ makes sure that $J$ is symmetric and adjusted to $\omega$ (see [16]).

It is worth noting that we can view $J$ as a path $J_s$ of complex structures on $\Sigma_g$. Moreover, $C_\alpha$ and $C_\beta$ are Lagrangian to $W$ with respect to $\omega$.

For such a $J$, we will look at certain $J$-holomorphic embeddings, which satisfy certain conditions:

Let $x$ and $y$ be two generators and $\phi \in \pi_2^0(x,y)$ be a domain. Let $S$ be a surface with boundary, such that it has $2g$ marked points on its boundary. We denote them as $(a_1, ..., a_g, b_1, ..., b_g)$. The naming is done in such a way that $a_i$ and $b_i$ alternate when we go round the boundary in a counter-clockwise fashion. These points divide the boundary $\partial S$ into $2g$ arcs, and we name these arcs as $l_j$ and $m_j$ (where $1 \leq j \leq g$), again in an alternating fashion. Let $p_1$ and $p_2$ denote the projection maps from $\Sigma_g \times \mathbb{D}^2$ onto $\Sigma_g$ and $\mathbb{D}^2$ respectively.

Now, we consider maps $u : S \to \Sigma_g \times \mathbb{D}^2$ with the following properties:

1. the image of $p_1 \circ u$ coincides with $\phi$,

2. the image of $p_2 \circ u$ is $g\mathbb{D}^2$ in homology,

3. the $a_i$'s and $b_i$'s are mapped injectively to $x_i$'s and $y_i$'s respectively via $p_1 \circ u$ and to $-i$ and $i$ respectively via $p_2 \circ u$ and

4. $l_j$ and $m_j$ are mapped to the $\alpha$ and $\beta$ arcs respectively via $p_1 \circ u$ and to the arcs $e_1$and $e_2$ arcs in $\partial(\mathbb{D}^2)$ via $p_2 \circ u$.

Here, as before, the arc $e_1$ joins $-i$ to $i$ in the half-plane $Re(z) > 0$, and the arc $e_2$ joins $i$ to $-i$ in half-plane $Re(z) < 0$.

We also need to ensure that the required transversality conditions are satisfied within the class of almost complex structures satisfying the conditions $(J1) - (J5)$ mentioned above. Most of the details can be found in [15].

Let $\mathcal{M}(\phi)$ denote the moduli space of the holomorphic embeddings $u$ satisfying the above conditions. The expected dimension of this space is given by the Maslov index, denoted by $\mu(\phi)$. We can compute it combiantorially in the following way:

Let $x$ be an intersection point and $\phi \in \pi_2(x, y)$. Now, let $x_i$ lie on $\alpha_i$ and $\beta_i$. Then around $x_i$ there are 4 regions, say $D_1, D_2, D_3$ and $D_4$. Then the average of the coefficients of these $D_j$'s in $\phi$ is denoted as $\mu_{x_i}(\phi)$. Then, we can define the point measure $\mu_x(\phi)$ as $\sum \mu_{x_i}(\phi)$. On the other hand, we have the Euler measure $e(\phi)$ which we can calculate once we specify a metric on $\Sigma_g$ such that all the $\alpha$ and $\beta$ circles become geodesics and intersect each other at right angles. In this scenario, it can be shown that $e(\phi)$ is given by $(1/2\pi)$ times the integral of the curvature on $\phi$. The Euler measure as defined above turns out to respect addition, and for a $2n$-gon region $D$, $e(D) = 1 - \frac{n}{2}$. Then we have:

**Theorem 3.1.1** ([15] ). *Let $\phi \in \pi_2(x, y)$. Then the Maslov index can be calculated via the following:*

$$\mu(\phi) = e(\phi) + \mu_x(\phi) + \mu_y(\phi). \tag{3.2}$$

*, where $\mu_x$ and $\mu_y$ denote the point measures of $x$ and $y$ respectively.*

For a non-trivial $\phi$, $\mathbb{R}$ acts freely on $\mathcal{M}(\phi)$ just as in the previous cases and we can define the unparametrized moduli space $\widehat{\mathcal{M}}(\phi)$ as the quotient of $\mathcal{M}(\phi)$ by this $\mathbb{R}$-action. Moreover, when $\mu(\phi) = 1$, $\widehat{\mathcal{M}}(\phi)$ becomes a zero-dimensional manifold. This allows us to count the number of holomorphic embeddings of the form described above. Let $c(\phi)$ be this number modulo 2. We then define the boundary map $\partial \widehat{CF} \to \widehat{CF}$ in the following way:

$$\partial x = \sum_{y} \sum_{\{\phi \in \pi_2^0(x,y) | \mu(\phi)=1\}} c(\phi).y \tag{3.3}$$

It is proved in [15] that $\partial^2 = 0$ and thus $(\widehat{CF}, \partial)$ is a chain complex, and we can define the corresponding homology groups. Similar constructions can be carried out using coefficients in $\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$ and in that case, we obtain homology groups that are isomorphic to the ones defiens in section 2.3. This isomorphism is proved by [15] in the following theorem:

**Theorem 3.1.2.** *The Heegaard Floer homology group defined using the symmetric products in [7] are isomorphic to the corresponding groups defined using the cylindrical reformulation in [15].*

In order to get a basic understanding of the theorem above, we can try and discuss the proof in a simplified case. The arguments here follow the ones given by Lipshitz in [15].

The Heegaard Floer homology groups, along with the corresponding concepts defined in Chapter 2 will be denoted with a supersctipt $S$ (signifying the importance of Symmetric products, for example $\widehat{HF}^S$), while the ones defined in this chapter will be denoted using a superscript $C$ (for Cylindrical). With this said, we note that the generators corresponding to the chain groups, specifically the intersection points can be identified with each other, and it can be shown that for two such generators $x$ and $y$, $\pi_2^S(x, y)$ can be identified with $\pi_2^C(x, y)$ as well.

Now we make a few assumptions on the Heegaard diagram $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ to simplify the details. We assume that $\mathcal{H}$ is such that the product complex structure $J^C := J_{\Sigma_g} \times J_{\mathbb{D}}$ satisfies all the transversality requirements for all index 1 homology classes in the cylindrical reformulation. Moreover, the complex structure $J^S$ on $\mathrm{Sym}^g(\Sigma_g)$ induced from $\Sigma_g$ does the same for index 1 domains in the theory using the symmetric products. Now, we will try to construct a map between the moduli spaces of holomorphic representatives $\widehat{\mathcal{M}}^S$ and $\widehat{\mathcal{M}}^C$.

For a holomorphic (with respect to $J^C$) map $u : S \to W$ we will define a corresponding map $u' : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$ which would be holomorphic with respect to $J^S$. We do this in the following way:

We pick a point $q$ in $\mathbb{D}$. Now, we denote the preimages of $q$ in $S$ under the map $(p_{\mathbb{D}} \circ u)$ as $(p_{\mathbb{D}} \circ u)^{-1}(q)$. Here, $p_{\mathbb{D}}$ denotes the projection map defined din the beginning of the section. Including multiplicities, there are $g$ such preimages. Then, if we define $u'$ such that $u'(q) = p_{\Sigma_g} \circ u((p_{\mathbb{D}} \circ u)^{-1}(q))$, it will give us a map of $\mathbb{D}$ into $\mathrm{Sym}^g(\Sigma_g)$. It can be shown that $u'$ is $J^S$ holomorphic, and this gives a homeomorphism of the moduli spaces $\widehat{\mathcal{M}}^S$ and $\widehat{\mathcal{M}}^C$. We refer to [15] for the details.

## 3.2 A Combinatorial Method for Calculating Heegaard Floer Homologies

In the reformulation described above, only $c(\phi)$ does not have a combinatorial description. Therefore, if we device a way to calculate this function, we can calculate the Heegaard Floer homology of a space in a purely combinatorial manner. That is what we are aiming for in this section. But first, we will need the following theorem, which states that the only domains that can have holomorphic representatives are the positive ones.

**Theorem 3.2.1.** *Let $\phi \in \pi_2^0(x, y)$ be such that $\phi$ has a holomorphic representative. Then $\phi$ is positive.*

**Proof.** Let, $J$ denote the product complex structure on $W$. Suppose that $\phi$ has a holomorphic representative. This means that there is a map $u$ of the kind discussed before, which is a holomorphic embedding. Thus, for a point $p$ which does not lie on the attaching circles, $n_p(\phi)$ is nothing but the intersection number of $u(S)$ with $\{p\} \times \mathbb{D}^2$. These two objects have a $+ve$ intersection number, i.e., $n_p(\phi) \geq 0$ for all $p$, due to the fact that both of these are holomorphic objects with respect to $J$. $\square$

Now, it can be shown that when there exists a holomorphic representative for $\phi$, we can calculate the Maslov index using the following formula:

$$
\begin{aligned}
\mu(\phi) &= 2e(\phi) + g - \chi(S) \\
&= e(\phi) + b + \frac{1}{2}(g - t).
\end{aligned}
$$

Here $b$ counts how many branch points $p_1 \circ u$ has, while $t$ counts the number of components of $S$ whose image under $p_1 \circ u$ is a single point.

We now take a look at how one can calculate the count function for certain Heegaard diagrams called 'nice' Heegaard diagrams, and later we will explore algorithms to transform any Heegaard diagram into a 'nice' Heegaard diagram.

**Definition 3.2.1.** *Suppose that $M$ has a pointed Heegaard diagram $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ such that every region except the one that contains the basepoint $w$ is a bigon or a square. We will call $\mathcal{H}$ nice in this case.*

Suppose that $M$ is a closed oriented 3-manifold with a nice admissible Heegaard diagram $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$. Let $J$ be a complex structure on $W$ satisfying (J1)-(J5).

**Definition 3.2.2.** *For a nice admissible Heegaard diagram $\mathcal{H}$ as above, let $\phi \in \pi_2^0(x, y)$ be such that:*

1. *All of its coefficients are either $0$ or $1$,*
2. *Topologically, $\phi$ is an embedded disk and on its boundary, there are $2n$ marked points. We call these points vertices of $\phi$.*
3. *At each vertex $v$, we have $\mu_v(\phi) = \frac{1}{4}$ and*
4. *None of the $x_i$'s or $y_i$'s lie in the interior of $\phi$.*

*Then $\phi$ is called an an empty embedded $2n$-gon.*

The theorems that follow give us a way of calculating $c(\phi)$ for nice admissible Heegaard diagrams and thus enabling us to calculate the Heegaard Floer homology of $M$.

**Theorem 3.2.2** ([18], Theorem 3.3)**.** *Let $\mathcal{H}$ be nice and admissible. Suppose that $\phi \in \pi_2^0(x, y)$, $\mu(\phi) = 1$ and there exists a holomorphic representative of $\phi$. Then, $\phi$ must be an empty embedded $2n$-gon with $n = 1$ or $n = 2$.*

**Proof.** We will only give an outline of the proof, the details can be found in [18]. Also, for obvious reasons, when $n = 1$ we will call the $2n$-gon a bigon and when $n = 2$, we will call it a square. Let $\phi = \sum_i a_i D_i$, where none of the $D_i$'s contain any $x_i$'s or $y_i$'s. By Theorem 3.2.1, $a_i \geq 0, \forall i$. Moreover, since $\mathcal{H}$ is nice, all $D_i$'s are bigons or squares and consequently, $e(D_i) \geq 0, \forall i$ (This is because, as discussed before, the Euler measure of a $2n$-gon region is given by $(1 - \frac{n}{2})$). Thus $e(\phi) \geq 0$, and from 3.2, and the fact that $\mu(\phi) = 1$, we get $0 \leq \mu_x(\phi) + \mu_y(\phi) \leq 1$.

Now, if $\partial\phi \neq 0$ on some part of an $\alpha$ cycle, say $\alpha_i$, we will say that $\phi$ 'hits' $\alpha_i$. Since $\phi$ cannot be $n\Sigma_g$ (as $\Sigma_g$ contains $x$ and $y$), $\phi$ hits at least one such $\alpha$ cycle. Without loss of generality, we denote it by $\alpha_1$. Now, since $\partial(\partial\phi\_\alpha) = y - x$, we have $\mu_{x_1}, \mu_{y_1} \geq \frac{1}{4}$. It can be argued that if $\phi$ does not hit some $\alpha$-circle, then it lies outside $\phi$.

Now, we recall our discussion after Theorem 3.2.1 and use the formula mentioned there along with the observation that $e(\phi)$ can only take the values 0 or $\frac{1}{2}$ to list all the possibilities that can occur. We then try to inspect the nature of the surface $S$ that maps to $\Sigma_g \times \mathbb{D}$, and analyse the image of $S$ under the projection $p_1 : \Sigma_g \times \mathbb{D} \mapsto \Sigma_g$. It can be shown that after disregarding trivial disks (which are the components of $S$ that map to points in $\Sigma_g$ under the above projection) the rest of $S$, (which we denote by $F$) is either a double branched cover over the unit disk with Euler characteristics 1 and one branch point (in which case it can be shown that it gives an empty embedded square) or a single cover of the unit disk (which results in an empty embedded bigon). □

**Theorem 3.2.3** ([18], Theorem 3.4)**.** *Let $\phi \in \pi_2^0(x, y)$ be an empty embedded $2n$-gon, where $n = 1$ or 2. Then the product complex structure $J$ on $\Sigma_g \times \mathbb{D}^2$ satisfies the transversality conditions for $\phi$ under a perturbation of the $\alpha$ and the $\beta$ circles. Moreover, $\mu(\phi) = c(\phi) = 1$.*

**Proof.** The proof of the first statement (regarding transversality) can be found in [15]. Here we provide a sketch for the proof of the other statements. The details can be found in [18].

For an $2n$-gon $\phi$, we must have some $x_i$ or some $y_i$ at each of its corners, while at every other coordinate $x_j$ (or, $y_j$), $\mu_{x_j}(\phi) = 0$ (or, $\mu_{y_j}(\phi) = 0$). Thus, $\mu_x(\phi) + \mu_y(\phi) = 2n.(1/4) =$

$(n/2)$. On the other hand, $e(\phi) = 1 - (n/2)$ and thus $\mu(\phi) = 1$.

Now, when $n = 1$ we can choose $F$ (as defined in the proof of Theorem 3.2.2) so that it is a disk which has 2 points marked at its boundary, and is mapped to $\phi$ via a diffeomorphism. If we consider the complex structure on $F$ which is induced by $\Sigma_g$, upto reparametrization, there exists a unique holomorhic map from $F$ to the unit disk. Thus $\phi$ has a holomorphic representative and it can be proved that this is the only holomorphic representative of $\phi$. Thus $c(\phi) = 1$. The case when $n = 2$ is similar, except in this case we take $F$ to be a disk such that it has 4 points marked in its boundary. □

Thus, for $\phi \in \pi_2^0(x, y)$ with $\mu(\phi) = 1$, if $\phi$ is an empty embedded $2n$-gon, where $n \in \{1, 2\}$, we get $c(\phi) = 1$. Otherwise, we have $c(\phi) = 0$.

We shall now try to explain the algorithm which will give us a nice admissible Heegaard diagram from an admissible one. We will achieve this by performing Heegaard moves on the $\beta$ curves. These moves were explained earlier (isotopies and handleslides).

### 3.2.1 A combinatorial algorithm to obtain nice Heegaard diagrams

Suppose that $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ is a pointed Heegaard diagram. We wish to ensure that all the regions in $\mathcal{H}$ are either bigons and squares, except for the region that contains the basepoint $w$. Given a Heegaard diagram, we say that bigons and squares are good, while all other regions are bad. For this purpose, we will do certain operations on the Heegaard diagram, and for notational ease, we will continue to call it $\mathcal{H}$.

**Step 1 (Ensuring all the regions are disks) :**

At this first step, we want to make sure that all the regions in $\mathcal{H}$ are disks. For that purpose, we need every $\alpha$ circle to intersect some $\beta$ circle and vice-versa. So suppose there exists $\alpha_k$ which does not intersect any $\beta$ circles. In this scenario, we take an arc $c$ connecting $\alpha_k$ to some $\beta_j$ in such a way that $c$ avoids all other intersection points and intersects $\beta_j$ only at its endpoint. Now, we can do a 'finger' move along $c$, (shown in the figure below) and that makes sure that $\alpha_k$ intersects $\beta_j$.


Figure 3.1: Making sure $\alpha$ circles intersect $\beta$ circles

In the figure above and all the others that follow in this section, we will use red to denote $\alpha$ circles and blue to denote $\beta$ circles.

Now, let there be a $\beta_k$ that does not intersect any of the $\alpha$ circles. In this case, we can take an arc connecting $\beta_k$ to some $\alpha_j$ (we denote it by $c$) such that $c$ only intersects $\alpha_j$ at its endpoint and it does not intersect any other $\alpha$ circle. Then we can do multiple finger operations as shown in the figure below and ensure that $\beta_k$ intersects $\alpha_j$, while still retaining the properties of attaching circles.


Figure 3.2: Making sure $\beta$ circles intersect $\alpha$ circles

Since there are only finitely many $\alpha$ and $\beta$ circles, we can be assured that after repeating the above process a few times we will have every $\alpha$ circle intersecting some $\beta$ circle and vise-versa.

Killing off non-disk regions is a simple matter now. Since the complement of the $\alpha$ curves can be identified with a punctured sphere, all the regions are planar surfaces, and every boundary component of every region must contain both $\alpha$ and $\beta$ circles (due to the previous steps). Thus to kill regions that have more than one boundary component (i.e., the non-disc regions) we can again perform finger moves. The specific kind is depicted in the figure below. Repeating this multiple times, we get a Heegaard diagram all of whose regions are disks.


Figure 3.3: Finger move to kill non-disk regions

**Step 2 (Ensuring all the regions are bigons or squares (excluding the one containing the basepoint $w$)):**

Having made sure that $\mathcal{H}$ only has disk regions, we now proceed with the next step, which is to make sure that $\mathcal{H}$ is nice.

We first define a few quantities called **badness, distance and distance d complexity** for an $2n$-gon disk region in the following manner:

**Definition 3.2.3.** *Let $D_w$ be the region that contains the basepoint $w$ and let $D$ be any other region. Let $w' \in D$ be an interior point. Let $\gamma$ be a curve that connects $w$ to $w'$ in such a way that $\gamma$ does not intersect the $\alpha$ circles. If in addition, $\gamma$ is such that the number of intersection points between it and the $\beta$ circles is less than any other such curve $\gamma'$, then the number of times $\gamma$ intersects the $\beta$ circles is called the* ***distance*** *of the region $D$. We will denote it by $d(D)$.*

**Definition 3.2.4.** *We define the* ***badness*** *of a $2n$-gon disk region $D$ to be $b(D) = max\{n-2, 0\}$. A region $D$ is bad if $b(D) \neq 0$.*

**Definition 3.2.5.** *For $\mathcal{H} := (\Sigma_g, \alpha, \beta, w)$ which has only disk regions, the distance of $\mathcal{H}$ (denoted by $d(\mathcal{H})$) is given by the largest distance of bad regions.*

**Definition 3.2.6.** *The distance $d$ complexity of $\mathcal{H}$ is denoted by $c_d(\mathcal{H})$ and is given by:*

$$c_d(\mathcal{H}) = \Big(\sum_{i=1}^{m} b(D_i), -b(D_1), -b(D_2), ..., -b(D_m)\Big), \tag{3.4}$$

*where each $D_i$ is a distance $d$ bad region, arranged in non-increasing order of their badness. The first term of $c_d(\mathcal{H})$ is called the total badness of distance $d$ of $\mathcal{H}$ and is denoted by $b_d(\mathcal{H})$.*

Clearly, $c_d(\mathcal{H}) = 0$ when there are no bad regions of distance $d$. We put the lexicographic order on the set of distance $d$ complexities. This will enable us to keep track of the complexities and ensure that we are on the right track. We now state the following theorem, which gives us our desired algorithm:

**Theorem 3.2.4** ([18], Lemma 4.1)**.** *Let $\mathcal{H}$ be a pointed Heegaard diagram all of whose regions are disks and $d(\mathcal{H}) = d$. Let $c_d(\mathcal{H}) \neq 0$. Then we can do a series of isotopies and handleslides on $\mathcal{H}$ to obtain a new Heegaard diagram $\mathcal{H}'$ such that $\mathcal{H}'$ has only disk regions and it satisfies $d(\mathcal{H}') \leq d(\mathcal{H})$ and $c_d(\mathcal{H}') < c_d(\mathcal{H})$.*

**Proof.** Having ordered the bad regions with distance $d$ as before, we focus on $D_m$. Since it is a bad region, it is a $2n$ gon with $n \geq 3$. We now pick a region right next to $D_m$, which has distance $d-1$ and has a common $\beta$ edge with $D_m$. We call it $D_*$. We denote one of their common $\beta$ edges as $b_*$, and name the $\alpha$ edges of $D_m$ as $a_1, ..., a_n$ starting from $b_*$ and going in a counterclockwise fashion.

Now we will attempt a finger move on $b_*$ through $a_2$, as shown in the figure below. Our finger divides $D_m$ in two regions, we call them $D_{m_1}$ and $D_{m_2}$.

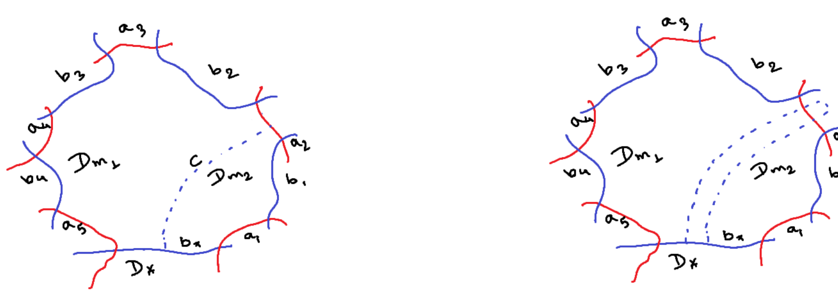

Figure 3.4: The beginning of our finger move through $a_2$

Now, a few scenarios can arise when we consider the position of the end of the finger. Firstly, if we end up in a square region of distance greater than or equal to $d$, we just travel forward and come out of the region through the opposite $\alpha$ side. It is worth noting that this keeps the distance of any of the bad regions unchanged since all bad regions have distance less than or equal to $d$.

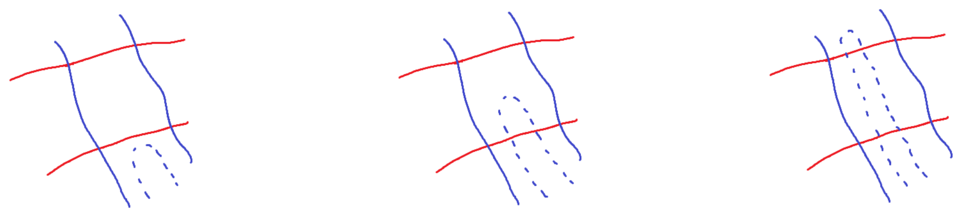

Figure 3.5: When our finger encounters a square region

Continuing in this fashion, we may encounter one of the following scenarios: we might reach a bigon region, or a region which has distance less than $d$; we might even reach another distance $d$ bad region (i.e., $D_i$ such that $i < m$) or we might end up back in $D_m$. We will consider each of these cases separately.

If we finally reach a bigon region, we end up forming a square and a new bigon region, as depicted in the following figure:

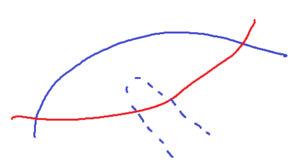

Figure 3.6: When our finger encounters a bigon region

In this case $D_{m_2}$ is already good, and we have $b(D_{m_1}) = b(D_m) - 1$. Thus $b_d(\mathcal{H}') =$

$b_d(\mathcal{H}) - 1$. Thus in this case $d(\mathcal{H}') \leq d(\mathcal{H})$ and $c_d(\mathcal{H}') < c_d(\mathcal{H})$.

Now suppose we reach a region $D'$ which has $d(D') = d' < d$. We have depicted the situation in the figure below:

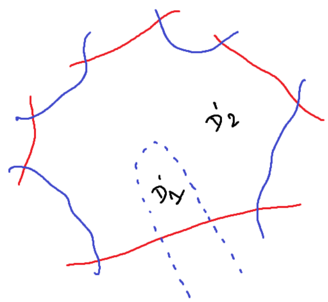


Figure 3.7: When our finger encounters a region with distance less than $d$

We have already covered the case when $D'$ is a bigon. In this scenario as well, we have $D_{m_2}$ is good and $b(D_{m_1}) = b(D_m) - 1$. Due to our finger move, the region $D'$ gets separated into a bigon $D'_1$ and another region $D'_2$, which might be a bad region, but if so, it must be a bad region whose distance is less than $d$. Thus, we again have $d(\mathcal{H}') \leq d(\mathcal{H})$ and $c_d(\mathcal{H}') < c_d(\mathcal{H})$, potentially at the expense of increasing distance $d'$ complexity.

We now look at the case when our finger reaches another bad region $D_i$ which has distance $d$ as well. In this case as well, we get a bigon, say $D_{i_1}$ and another bad region $D_{i_2}$ with distance $d$. Now $b(D_{i_2}) = b(D_i) + 1$ but $b(D_{m_1}) = b(D_m) - 1$. Even though the total badness remain the same, we have managed to decrease the distance $d$ complexity by moving the badness to an earlier bad region. Hence, we have $d(\mathcal{H}') = d(\mathcal{H})$ and $c_d(\mathcal{H}') < c_d(\mathcal{H})$.

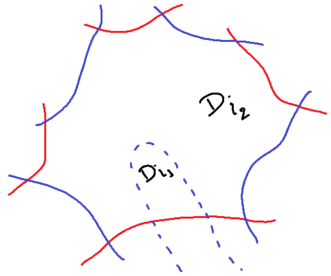

Figure 3.8: When our finger encounters a bad region with dictance $d$

We are now left with the case when we return to $D_m$ with our finger. Suppose that we return via an adjacent edge, say $a_1$. The situation is depicted in the following figure. We note that we have an entire $\beta$ circle, say $\beta_i$ along one side of the boundary of our finger. Let $b_* \subset \beta_j$. In this case we do not perform the finger move. Instead, we perform a handleslide of $\beta_j$ over $\beta_i$, as shown in the figure below:

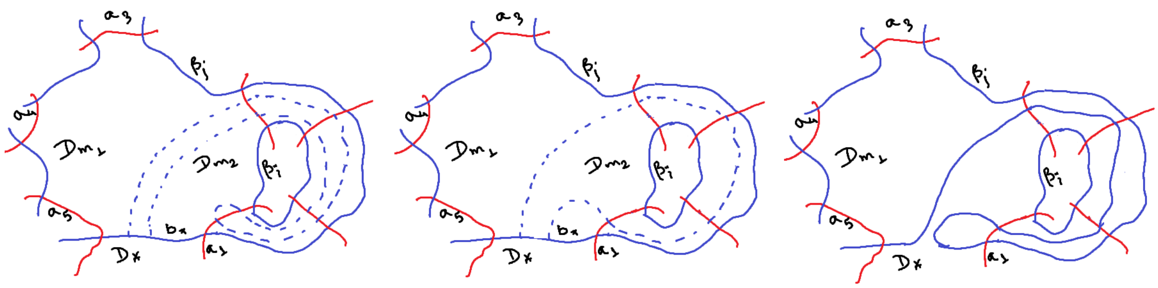

Figure 3.9: When we return to $D_m$ via an adjacent side: a handleslide of $\beta_j$ over $\beta_i$

Here as well, we have $d(\mathcal{H}') \leq d(\mathcal{H})$ and $c_d(\mathcal{H}') < c_d(\mathcal{H})$. This is because we have not increased the distance of any bad regions and although we ended up increasing the badness of $D_*$, $d(D_*) = d - 1$, while $D_{m_2}$ is a bigon now and $b(D_{m_1}) = b(D_m) - 1$.

Now suppose that we return to $D_m$ through a non-adjacent edge, say, $a_k$, wher $3 < k \leq n$. Then, we ditch the finger move through $a_2$, and instead perform a finger move through $a_3$. In doing so, we might reach one of the previous cases. If that happens, we follow the arguments made before to decrease the distance $d$ complexity. However, we may return back to $D_m$

again, say, through $a_i$. Now, $3 < i < k$ (we give an argument for this below) and in this case, we once again do not perform this finger move through $a_3$ and instead perform the move through $a_4$. Repeating the above again, we will have a situation where we either do not come back to $D_m$, or come back to $D_m$ through an adjacent edge, in which case we do a handlesilde similar to the case when we returned through $a_1$. We get $b(D_{m_1}) = \max\{n - j - 1, 0\}$ and $b(D_{m_2}) = \max\{j - 2, 0\}$. Moreover, $b(D_{m_1}) + b(D_{m_2}) \leq n - 3$. Thus, after the handleslide, we have managed to decrease the total badness of distance $d$ and in this case $d(\mathcal{H}') \leq d(\mathcal{H})$ and $c_d(\mathcal{H}') < c_d(\mathcal{H})$.

Now we give an argument in favour of our claim $3 < i < k$ in the above case. Our finger cannot come back through $a_3$ or $a_k$. Now if $i > k$ or $i < 3$ we can obtain two simple closed curves $c_1$ and $c_2$ which intersect transversally at one point as shown in the figure below:

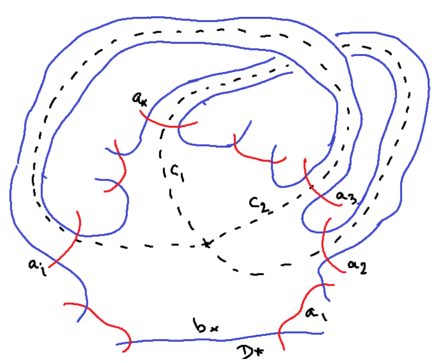


Figure 3.10: When we return to $D_m$ via a non-adjacent side: $c_1$ and $c_2$ denote the cores of each finger move

Both $c_1$ and $c_2$ lie in the complement of the $\beta$ circles, which is nothing but a punctured sphere. If we attach disks to it to obtain a sphere, we will have $H_1(\mathbb{S}^2) \simeq 0$. But as homology classes, we get $[c_1].[c_2] = 1$, giving us a contradiction. □

After repeating the above process enough times, we will eventually reach $c_d(\mathcal{H}) = 0$, and obtain a nice Heegaard diagram. However, we still need to ensure that this algorithm does not change the admissibility of the initial diagram. This is discussed in [18] in detail.

### 3.2.2 An Example

As an example, we will take a look at a Heegaard diagram of the Poincaré Homology Sphere, denoted by $\Sigma(2,3,5)$. A Heegaard diagram for this space is given below.

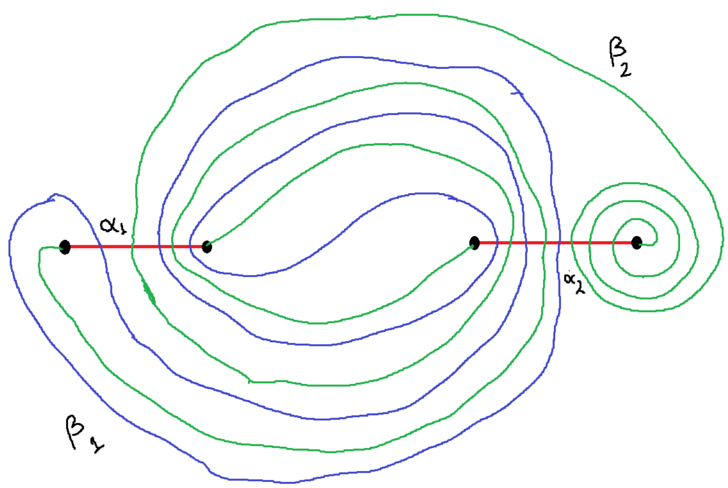


Figure 3.11: A Heegaard diagram $\mathcal{H}$ for the Poincaré Homology Sphere $\Sigma(2,3,5)$

In this diagram, the two black circles on the left are identified together. We do this by attaching a 1-handle to the two points. both $\alpha_1$ and $\beta_2$ pass through this handle. This makes $\alpha_1$ into a closed curve and does the same for $\beta_2$. Similarly, the two black circles on the right are also connected via a 1-handle. Here we have used different colours for the two $\beta$ circles so that we can distinguish between the two easily.

However, this depiction is not very easy to work with. For this reason, we cut the diagram along the $\alpha$ circles and obtain the following diagram, which is planar in nature.

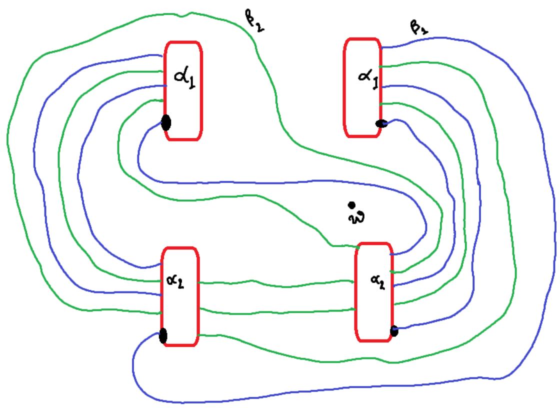


Figure 3.12: Redrawing the Heegaard diagram for $\Sigma(2,3,5)$

At first glance, the diagram might seem overwhelming. In order to simplify matters, below we have coloured each region with a different colour. We have also depicted the basepoint $w$, which resides in the grey region.

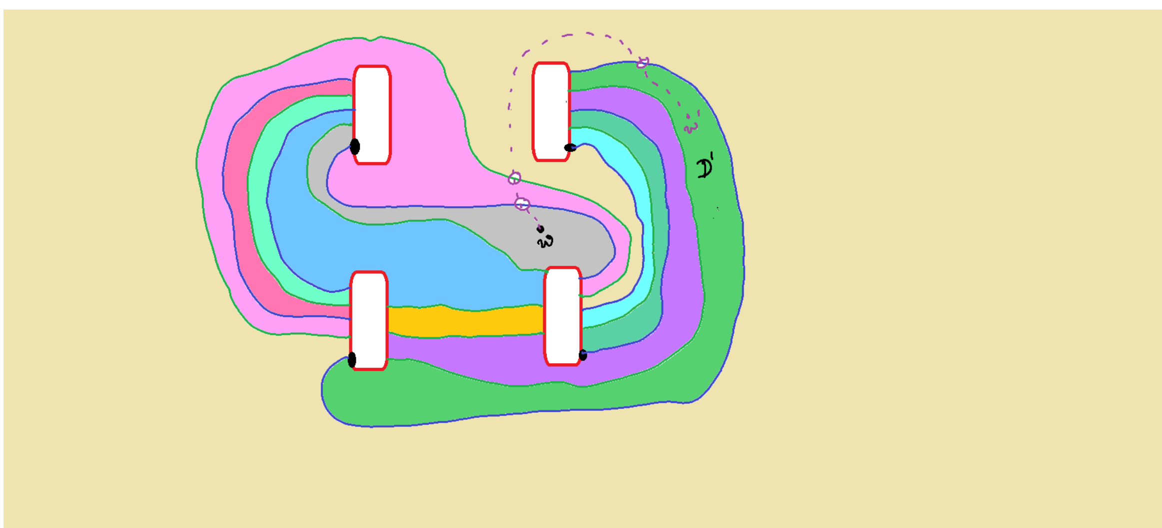


Figure 3.13: Depicting the regions in $\mathcal{H}$, and calculating the distance

Since all the regions are already disks, now we have to calculate the distance and the badness of each region. We note that in order to calculate the distance, we need to consider those curves from $w$ to the interior of a region that minimize the intersections with the $\beta$ circles. In the diagram above, we have depicted one such curve that can be used to calculate the distance of the green region $D'$. We can see that the dotted curve intersects the $\beta$ circles at three points, so $d(D') = 3$. Similarly, we have to calculate the badness of $D'$. Clearly, $D'$ is a square, so its badness $b(D') = 0$.

Having calculated badness and distance for each region, we have depicted them in the diagram below:

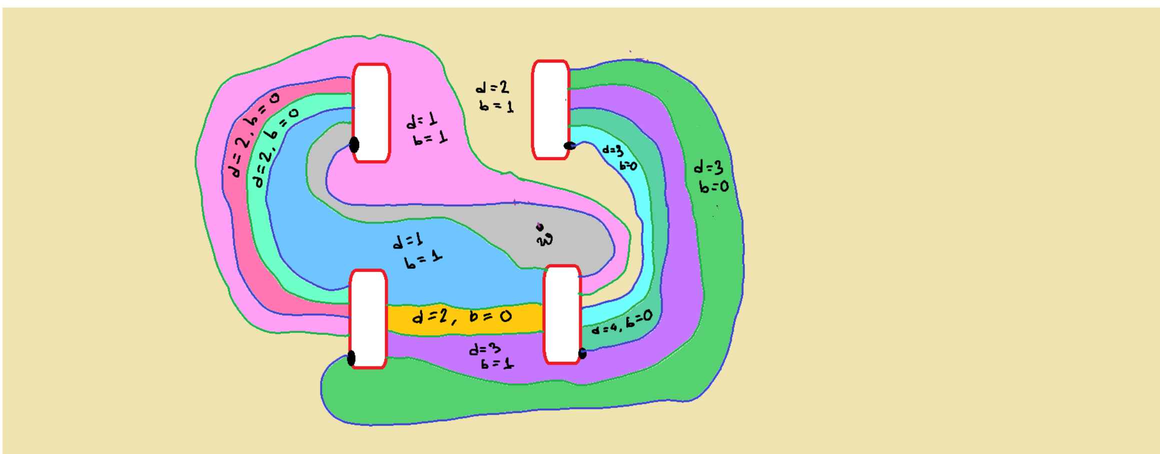


Figure 3.14: Distances and badness of the different regions

We see the there are no distance 4 bad regions. So we focus on distance 3 bad regions. In what follows, we will focus on the purple region, denoted by $D_m$. For notational convenience, we have named the edges as $b_*, a_1, b_1, a_2, b_2$ and $a_3$, going counterclockwise, just like in the algorithm.

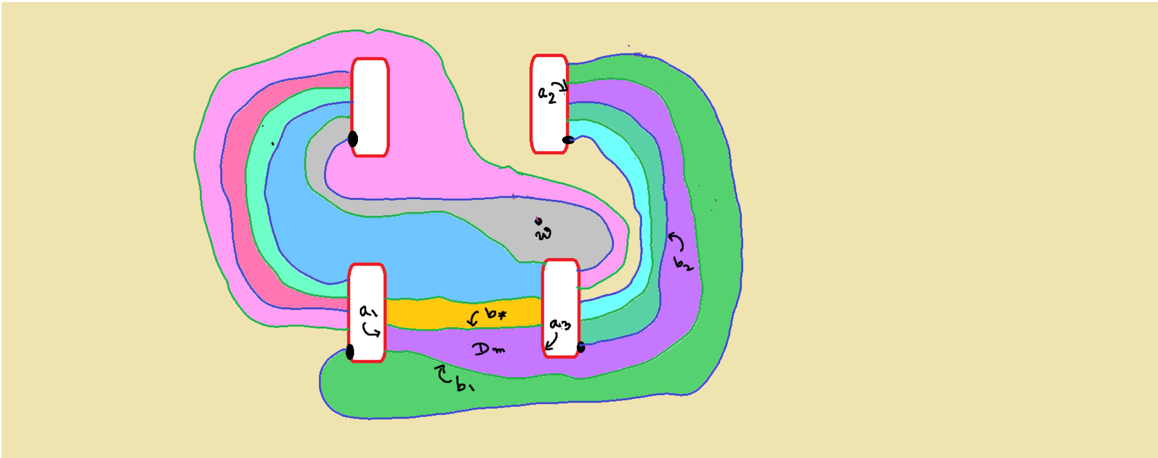


Figure 3.15: Preparing for a finger move in $D_m$

Now we are ready to do the finger move through $a_2$. Doing so leads us to the teal region. As explained in the diagram, we have divided the region $D_m$ into two regions $D_{m_1}$ and $D_{m_2}$. In our case both of these are good, as both are square regions. On the other hand, the teal region is divided into two regions as well, one of them being $D_1'$, which is a bigon, and the other being $D_2'$. During our process, we have actually increased the badness of the teal region as well as the yellow region, but since these were distance 2 bad regions to begin with, we have successfully reduced the distance 3 complexity.

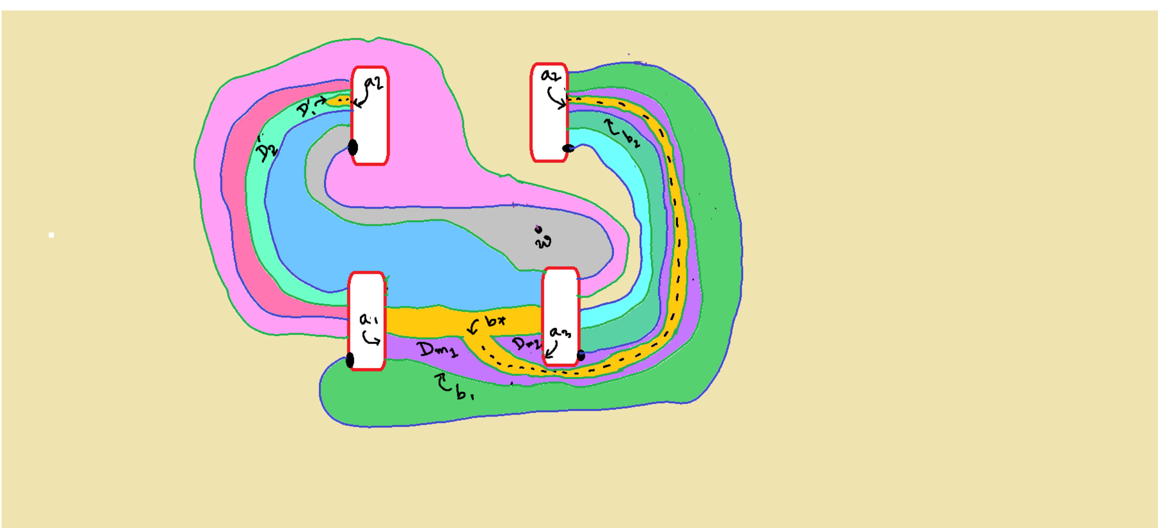


Figure 3.16: Aftermath of the finger move

In order to obtain a nice Heegaard diagram, we need to continue this for quite a number of steps, but since now we understand the primary mechanism, carrying out those steps are little more than mechanical work.

# Chapter 4

# Concluding remarks

## 4.1 Conclusion

In conclusion, through this thesis, we have tried to provide a comprehensive introduction to the theory of Floer homology. We have attempted to provide the necessary theoretical background before reviewing the significant aspects of the theory. While discussing the theory as well, we have tried to focus more on intuition rather than the technical details (for which enough resources are cited so that readers can explore that aspect as well, if they choose to do so). Thus, we hope this thesis would help out those who wish to explore the world of Floer homology but are intimidated by the complexity of the theory or the vastness of the literature.

## 4.2 The Path Forward

This thesis by no means encompasses all of Heegaard Floer homology theory, and there is still quite a lot of the theory left to study. We had to gloss over some of the very technical details in our thesis, sometimes due to the lack of time and sometimes because of the lack of the required theoretical background. Therefore, revisiting some of the topics after building a more solid foundation would certainly be fruitful. Having understood the theory, we can then move on to study its applications of Heegaard Floer homology in various fields. One such aspect is its application to knot theory, which yields invariants known as Knot Floer Homologies. We can also look into efficient ways to calculate these invariants and try and contribute to the literature eventually.

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