# Shape Optimization Problems on Polygons

**A Thesis**

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Sriram V

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2024

Supervisors: Dr. Anisa Chorwadwala and Dr. Sheela Verma

# Certificate

This is to certify that this dissertation entitled Shape Optimization Problems on Polygons towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sriram V at Indian Institute of Science Education and Research under the supervision of Dr. Anisa Chorwadwala, Associate Professor, Department of Mathematics and Dr. Sheela Verma, Assistant Professor, Department of Mathematics, Indian Institute of Technology (BHU), Varanasi during the academic year 2023-2024.

May, 2024.

Dr. Sheela Verma

May, 2024.

Dr. Anisa Chorwadwala

Committee:

Dr. Anisa Chorwadwala

Prof. Murali Krishna Vemuri

Dr. Sheela Verma

This thesis is dedicated to IISER Pune

# Declaration

I hereby declare that the matter embodied in the report entitled Shape Optimization Problems on Polygons are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anisa Chorwadwala and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and contributions.

Sriram.V

Sriram V
20191014

# Acknowledgments

I express my sincerest gratitude to my supervisors, Dr. Sheela Verma and Dr. Anisa Chorwadwala for their unconditional support throughout the whole project. They were instrumental in helping me develop the skills to tackle research problems in the future. I thank Prof. Murali Krishna Vemuri for his insights and suggestions regarding the project. I thank my friends at IISER Pune for their constant support throughout this journey.

I take this opportunity to acknowledge the financial support through the KVPY Fellowship by the Department of Science and Technology (DST), Govt. of India.

# Abstract

In this thesis, we aim to study the Pólya -Szegö conjecture, which states that the regular polygon with $n$ sides and fixed area minimizes the first Dirichlet eigenvalue among the family of simple polygons with $n$ sides in $\mathbb{R}^2$ and fixed area for $n \geq 3$. The conjecture for the cases $n = 3, 4$ was solved using Steiner symmetrization [15]. However, Steiner symmetrization fails to prove the conjecture when we consider $n \geq 5$. In 2022, Beniamin Bogosel and Dorin Bucur showed that the proof of the conjecture for $n \geq 5$ can be reduced to a finite number of numerical computations, assuming that the conjecture is true. They used the theory of shape derivatives and the finite element method to prove the local minimality of the regular polygon with a fixed area. They use surgery arguments further to prove the global minimality of the regular polygon with a fixed area.

# Contents

# Chapter 1

# Introduction

Shape optimization problems involve optimizing a shape functional $J$, defined as

$$J : \mathcal{O} \mapsto \mathbb{R}$$

where $\mathcal{O}$ is the family of open subsets of $\mathbb{R}^n$ $(n = 1, 2, \dots)$. Without loss of generality, we also consider a subfamily of open subsets of $\mathcal{O}$. These problems are inspired by natural phenomena like the shapes of drops, soap bubbles and circular drums. For example, a free-floating soap bubble will always be spherical because the bubble minimizes its surface area. More specifically, in cases like the bee honeycomb, polygonal structures arise as the optimal shapes [9]. In our case, we are specifically interested in studying the problems involving the shape functional, $\lambda_1$, the first eigenvalue of the Laplace operator with Dirichlet boundary condition.

The classical eigenvalue problem for the Laplace operator $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ in an open connected bounded domain $\Omega \subset \mathbb{R}^n$ has been studied extensively. The spectrum of the Laplace operator is known to be discrete, consisting of non-negative real eigenvalues converging to infinity. Laplacian eigenfunctions are of special interest to research since they appear in various physical scenarios. For example, they appear as electron wave functions in quantum waveguides and as vibration modes in acoustics. The drum frequencies of a thin membrane (a drum) are proportional to $\sqrt{\lambda_m}$, where $\lambda_m$ are the Dirichlet eigenvalues of the Laplacian[26]. The results in [18] discuss the various properties of Laplacian eigenvalues and eigenfunctions.

One of the most celebrated results in this direction is the Faber - Krahn inequality, which states that the ball of volume $c$ minimizes the first Dirichlet eigenvalue among all bounded sets of volume $c$, where $c > 0$, c fixed. The inequality motivates us to consider the problem of minimizing the first Dirichlet eigenvalue over a class of subsets, $P$, of $\mathcal{O}$ (class of open domains of $\mathbb{R}^d$). Let us consider $P = \mathcal{P}_n$, where $\mathcal{P}_n$ is the set of polygons with at most $n$ sides in $\mathbb{R}^2$. Because the ball is the most symmetric domain among all domains of fixed area, it was conjectured by Pólya and Szegö [15] that the regular $n$-sided polygon is the minimizer of the first Dirichlet eigenvalue over the class of polygons with at most $n$-sides.

For $n = 3, 4$, it was proven that the equilateral triangle and the square minimize $\lambda_1$ over the set of all triangles and quadrilaterals, respectively[15]. The proof uses the Steiner symmetrization method, which when applied to an arbitrary triangle converges to an equilateral triangle. Similarly, when Steiner symmetrization is applied to an arbitrary quadrilateral, it converges to a rectangle. However, when Steiner symmetrization is applied to an arbitrary polygon with $n \geq 5$, it produces a sequence of polygons with $n + 1$ sides. But a polygon with $n + 1$ sides is non-admissible in the set $\mathcal{P}_n$ ($\mathcal{P}_n$ is the family of simple polygons with $n$ sides in $\mathbb{R}^2$). This implies that Steiner symmetrization fails in the case $n \geq 5$.

In 2022, Beniamin Bogosel and Dorin Bucur showed that the proof of the conjecture for $n \geq 5$ can be reduced to a finite number of numerical computations [5]. We study the proof of the following main theorem proved in [5].

**Theorem** *For every $n \geq 5$, the proof of the Pólya - Szegö conjecture can be reduced to a finite number of numerical computations, provided that the conjecture is true.*

It is important to note that we prove the above theorem under the assumption that the conjecture is true! The conjecture remains unsolved.

The thesis is organised as follows:

Chapter 2 begins with a study of the properties of Sobolev spaces, the Laplace operator with Dirichlet boundary conditions, the theory of shape derivatives (required for the study of shape functionals like $\lambda_1$) and the theory of the Finite element method (required for estimation). Using the definition of the linear second-order elliptic operator and the first Dirichlet eigenvalue, we also study methods for finding lower and upper bounds for $\lambda_1$. Finally, we use Steiner symmetrization and Schwarz rearrangement to prove the Faber-Krahn inequality for triangles, quadrilaterals and general domains respectively. We will be brief with the proofs in this chapter.

Chapter 3 deals with the study of the shape derivatives of $\lambda_1$. We find the explicit expression of the derivative of $\lambda_1$ for a general domain $\Omega$.In particular, we define a perturbation field $\theta$ for polygons and find the expression of the derivative of $\lambda_1$ for a polygon. Using the expressions of first-order derivatives of $\lambda_1$ in the above cases, we find the explicit expression of second-order derivatives of $\lambda_1$ for general domains and polygons. We conclude the chapter by finding the modulus of continuity of the coefficients of the shape Hessian matrix of $\lambda_1$.

In Chapter 4, we study the eigenvalues of the shape Hessian matrix of the functional $|P|\lambda_1(P)$, where $P \in \mathcal{P}_n$, $|P|$ denotes the volume of $P$ and $\lambda_1(P)$ denotes the first Dirichlet eigenvalue of $P$. We use finite element methods to approximate the eigenvalues and find intervals containing the eigenvalues.

Chapter 5 starts with the results of the numerical simulations performed using the finite element method studied in Chapter 3 for some values of $n$. We also study a detailed procedure to prove the global minimality of the regular polygon to prove the conjecture.

**Original Contributions**: This thesis has no claims to any new results by the author. It is a literature review on the development made so far in an attempt to prove the Pólya - Szegö conjecture posed in the year 1951. The emphasis is on a clear, organised presentation of the literature. A sketch of the proof of the conjecture for $n = 3, 4$ is presented. We also give a clear derivation of the explicit estimates for eigenvalues and eigenfunctions of the Dirichlet Laplacian for the case $n \geq 5$. In the year 2022, Beniamin Bogosel and Dorin Bucur showed that the proof of the long-standing conjecture can be reduced to a finite number of numerical computations, provided the conjecture is true [5]. This thesis is an attempt to explain this long proof. This also means that the conjecture remains unsolved.

# Chapter 2

# Preliminaries

## 2.1 Sobolev spaces

The primary references for this subsection on Sobolev spaces are [8],[34] and supplemented by [31],[28].

**Notation** Let us denote $C_c^\infty(\Omega)$ to be the space of infinitely differentiable functions $\phi : \Omega \mapsto \mathbb{R}$ with compact support in $\Omega$. We define the space $L^1_{loc}(\Omega)$ by

$$L^1_{loc}(\Omega) := \left\{ \phi : \Omega \mapsto \mathbb{R} \text{ measurable} \mid \phi|_{\overline{\Omega}} \in L^1(\overline{\Omega}),\ \forall \overline{\Omega} \subset \Omega,\ \overline{\Omega} \text{ compact in } \Omega \right\}.$$

**Definition 2.1.1.** *Let $u, v \in L^1_{loc}(\Omega)$ and $\alpha$ be a multi-index. We define $v$ as the $\alpha$-th weak partial derivative of $u$,denoted by,*

$$D^\alpha u = v,$$

*if the condition*

$$\int_\Omega u D^\alpha \phi \ dx = (-1)^{|\alpha|} \int_\Omega v\phi \ dx,$$

*holds for all $\phi \in C_c^\infty(\Omega)$.*

**Definition 2.1.2.** *Let $k$ be a nonnegative integer and $p \geq 1$.The Sobolev space of index $(k, p)$*

*is defined by*

$$W^{k,p}(\Omega) := \{v \in L^p(\Omega) \mid D^\alpha v \in L^p(\Omega) \quad \textit{for all} \quad |\alpha| \leq k\},$$

*where $D^\alpha v$ denotes the $\alpha$-th weak partial derivative of $v$.*

We define the function $\|.\|_{k,p}$ on the space $W^{k,p}(\Omega)$ by

$$\|v\|_{k,p}^p := \sum_{|\alpha|\leq k} \|D^\alpha v\|_p^p,$$

where $\|.\|_p$ is the standard $L^p$ norm. It is immediate to see that the function $\|.\|_{k,p}$ is a norm on the space $W^{k,p}(\Omega)$. The space $W^{k,p}(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to $\|.\|_{k,p}$. We denote the closure of $C_0^\infty(\Omega)$ with respect to the same topology by $W_0^{k,p}(\Omega)$. For $p = 2$, we define the space $H^k(\Omega) := W^{k,2}(\Omega)$. Similarly, we denote $W_0^{k,2}(\Omega)$ by $H_0^k(\Omega)$.

**Example** We consider the Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

restricted to $(-1, 1)$. The weak derivative of $H(x)$ is the delta distribution, which is not integrable. Therefore $H \notin H^1((-1, 1))$.

It is important to note that $W^{k,p}(\Omega)$ is a Banach space with respect to the norm $\|.\|_p$ and $H^k(\Omega)$ is a Hilbert space with the inner product defined by

$$(u, v) := \sum_{|\alpha|\leq k} (D^\alpha u, D^\alpha v),$$

where

$$(D^\alpha u, D^\alpha v) = \int_\Omega D^\alpha u \; D^\alpha v \; dx.$$

**Definition 2.1.3.** *For $0 < s < 1$ and $1 \leq p < \infty$, we define*

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx \; dy < \infty \right\}.$$

The spaces $W^{s,p}(\Omega)$ are said to be fractional order Sobolev spaces and would be useful for studying the regularity of solutions of partial differential equations that arise later.

**Definition 2.1.4.** *For $k \in \mathbb{N}$, we define $W^{-k,p}(\Omega)$ as the dual space of $W_0^{k,p'}(\Omega)$ where $p'$ is the conjugate of $p$ i.e, $1/p + 1/p' = 1$*

Similarly, we define $W^{-k,2}(\Omega) = H^{-k}(\Omega) = (H_0^k(\Omega))'$ and we define the norm $\|.\|_{-1}$ on $H^{-1}(\Omega)$ by

$$\|f\|_{-1} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v\rangle}{\|v\|_1}$$

for $f \in H^{-1}(\Omega)$.

The following result on embeddings of Sobolev spaces connects the ideas of smoothness using 'classical' and 'weak' derivatives.

**Theorem 2.1.1.** *Let $1 \leq p \leq \infty, k \in \mathbb{Z}_+$ and $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$.*

*Case 1.* $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \textit{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

*Case 2.* $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \textit{for all } q \in [1, \infty).$$

*Case 3.* $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega}).$$

**Corollary 2.1.1.** *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain. Then we have*

$$H^1(\Omega) \hookrightarrow L^p(\Omega),$$

*for all $p$.*

## 2.2 Partial differential equations

The primary reference for this subsection is [22].

Let us consider bounded functions $a_{ij}, i, j = 1, \dots, N$, defined on $\Omega$ that satisfy the assumption:

$$\text{There exists } c > 0, \text{ such that } \forall \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N \ \forall x \in \Omega,$$

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \tag{2.1}$$

where $|\xi| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_N^2)^{1/2}$ denotes the Euclidean norm of $\xi$. The assumption (2.1) is called the uniform ellipticity assumption. We also assume that $a_{ij}$ satisfy the condition:

$$\forall x \in \Omega \text{ and } \forall i, j, \quad a_{ij}(x) = a_{ji}(x). \tag{2.2}$$

Let $a_0$ be a bounded function defined on $\Omega$.

**Definition 2.2.1.** *The linear elliptic operator $L$ is defined on $H^1(\Omega)$ by:*

$$Lu := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u. \tag{2.3}$$

*Here, derivatives are understood as distributions.*

An example of a linear elliptic operator is the Laplacian:

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}. \tag{2.4}$$

**Definition 2.2.2.** *Let $f \in L^2(\Omega)$. We define by $u$ a solution of the Dirichlet problem*

$$Lu = f \quad in \quad \Omega,$$

$$u = 0 \quad on \quad \partial\Omega \tag{2.5}$$

*if $u$ is the unique solution of the variational problem*

$$\begin{cases} u \in H_0^1(\Omega) \quad \text{and} \quad \forall v \in H_0^1(\Omega), \\ \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_\Omega a_0(x) u(x) v(x) dx = \int_\Omega f(x) v(x) dx. \end{cases} \tag{2.6}$$

We denote by $A_L$ the linear operator defined by:

$$A_L : L^2(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega).$$

$$f \mapsto u \quad \text{the solution of (2.6).} \tag{2.7}$$

We refer to the spectral theorem for self-adjoint, compact, positive operators on Hilbert spaces without proof:

**Theorem 2.2.1.** *[38] Let $H$ be a separable Hilbert space of infinite dimension and $T$ a positive, self-adjoint, compact operator on $H$. Then, there exists a sequence of real positive eigenvalues $(\mu_n), n \geq 1$ converging to 0 and a sequence of eigenvectors $(x_n), n \geq 1$, defining a Hilbert basis of $H$ such that $\forall n, Tx_n = \mu_n x_n$.*

In our case, we consider $H = L^2(\Omega)$ and the linear operator $T = A_L$. We prove that the operator $A_L$ is self-adjoint, compact, positive:

Let $f \in L^2(\Omega)$ and $u = A_L f$ be the solution of (2.6). We have

$$(f, A_L f) = \int_\Omega f(x) u(x) dx = \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx + \int_\Omega a_0(x) u^2(x) dx.$$

The condition (2.1) and the fact that $a_0(x)$ is positive implies that $A_L$ is positive.

Let $f, g \in L^2(\Omega)$ and $u = A_L f$,$v = A_L g$. We have:

$$(f, A_L g) = \int_\Omega f(x) v(x) dx = \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_\Omega a_0(x) u(x) v(x) dx.$$

Using assumption (2.2) and the equation (2.6), the right-hand side is equal to $\int_\Omega u(x) g(x) dx = (A_L f, g)$. This implies that $A_L$ is self-adjoint.

$A_L$ is compact due to the Rellich theorem. [22]

As a consequence of Theorem 1.2.1, there exists $(u_n)$ a Hilbert basis of $L^2(\Omega)$ and a sequence $\mu_n > 0$, converging to 0 such that $A_L u_n = \mu_n u_n$.

Setting $\lambda_n = \frac{1}{\mu_n}$ we have proved:

**Theorem 2.2.2.** *Let us consider a bounded open set $\Omega$ in $\mathbb{R}^N$. There exists a sequence of eigenfunctions (defining an orthonormal basis of $L^2(\Omega)$) and a sequence of corresponding positive eigenvalues (going to $+\infty$) that we will denote respectively $u_1, u_2, \dots$ and $0 < \lambda_1(L,\Omega) \leq \lambda_2(L,\Omega) \leq \dots$ satisfying:*

$$Lu_n = \lambda_n(L,\Omega) \quad in \quad \Omega,$$

$$u_n = 0 \quad on \quad \partial\Omega.$$

The above theorem implies the existence of an increasing sequence of eigenvalues and eigenfunctions for the Laplace operator with Dirichlet boundary conditions. The first Dirichlet eigenvalue of the Laplace operator, which we denote by $\lambda(\Omega)$(or $\lambda_1(\Omega)$ interchangeably), will be the topic of study throughout the whole thesis.

For the first Dirichlet eigenvalue, we have the following characterization using the Rayleigh quotient [35]:

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2 dx}{\int_\Omega v(x)^2 dx}.$$

## 2.3 Schwarz rearrangement

The primary reference for this subsection is [22] supplemented by [27].

**Definition 2.3.1.** *Let $\Omega$ in $\mathbb{R}^N$ be a measurable set. We denote by $\Omega^*$ the ball of the same volume as $\Omega$. Let $u$ be a non-negative measurable function defined on $\Omega$ and vanishing on its boundary $\partial\Omega$. We denote by $\Omega(c) = \{x \in \Omega \mid u(x) \geq c\}$ its level sets. The Schwarz* *rearrangement (or spherical decreasing rearrangement) of $u$ is the function $u^*$ defined on $\Omega^*$ by*

$$u^*(x) = sup\{c \mid x \in \Omega(c)^*\}$$

*In other words, $u^*$ is constructed from $u$ by rearranging the level sets of $u$ in balls of the same volume.*

### 2.3.1 Properties

**Theorem 2.3.1.** *Let $\Omega \subset \mathbb{R}^N$ be a measurable set, $u$ a non-negative measurable function defined on $\Omega$ and vanishing on its boundary. Let ,$u^*$ be the Schwarz rearrangement of $u$ defined on $\Omega^*$. Then:*

1. *$|\Omega| = |\Omega^*|$.*
2. *If $\Omega$ is open and if $u$ belongs to the space $H_0^1(\Omega)$, then $u^* \in H_0^1(\Omega^*)$ and*
$$\int_\Omega |\nabla u(x)|^2 dx \geq \int_{\Omega^*} |\nabla u^*(x)|^2 dx. \tag{2.8}$$
3. *If $\phi$ is any measurable function from $\mathbb{R}_+$ to $\mathbb{R}$, then*
$$\int_\Omega \phi(u(x)) dx = \int_{\Omega^*} \phi(u^*(x)) dx. \tag{2.9}$$

Proofs for the above properties can be found in [27]. Properties (2) and (3) would be used later in this chapter in proving the Faber-Krahn inequality for general domains.

## 2.4 Steiner symmetrization

The primary reference for this subsection is [22].

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a measurable set. Let us consider the hyperplane of symmetry to be $x_n = 0$. We denote by $\overline{\Omega}$ the projection of $\Omega$ on $\mathbb{R}^{n-1}$:

$$\overline{\Omega} := \{\overline{x} \in \mathbb{R}^{n-1} \quad \text{such that } \exists\ x_n \text{ with } (\overline{x}, x_n)\ \in \Omega\}.$$

Let us denote by $\Omega(\overline{x})$ the intersection of $\Omega$ with $\overline{x} \times \mathbb{R}$, where $\overline{x} \in \mathbb{R}^{n-1}$:

$$\Omega(\overline{x}) := \{x_n \in \mathbb{R} \quad \text{such that } (\overline{x}, x_n) \in \Omega\ , \overline{x} \in \overline{\Omega}\}.$$

**Definition 2.4.1.** *Let $\Omega \subset \mathbb{R}^N$ be a measurable set. Then we define the set*

$$\Omega^* := \{x = (\overline{x}, x_n) \quad \textit{such that } -\frac{1}{2}|\Omega(\overline{x})| < x_n < \frac{1}{2}|\Omega(\overline{x})|, \overline{x} \in \overline{\Omega}\}.$$

*the Steiner symmetrization of $\Omega$ with respect to $x_n = 0$.*

**Definition 2.4.2.** *Let $u$ be a non-negative measurable function defined on $\Omega$ and vanishing on its boundary. Let us denote by $\Omega(c) = \{x \in \Omega \mid u(x) \geq c\}$ its level sets. The Steiner symmetrization of $u$ is the function $u^*$ defined on $\Omega^*$ by*

$$u^*(x) = \sup\{c \mid x \in \Omega(c)^*\}.$$

Analogous to the Schwarz rearrangement, similar inequalities hold for Steiner symmetrization as well. We prove later in the chapter that repeated Steiner symmetrization of an arbitrary triangle or quadrilateral leads to an equilateral triangle or a rectangle respectively.

## 2.5 Hausdorff distance

The primary reference for this subsection is [22]

**Definition 2.5.1.** *Suppose $\Omega_1, \Omega_2$ are two non-empty compact sets in $\mathbb{R}^N$. We set*

$$\forall x \in \mathbb{R}^N, d(x, \Omega_1) := \inf_{y \in \Omega_1} |y - x|,$$

$$\rho(\Omega_1, \Omega_2) := \sup_{x \in \Omega_1} d(x, \Omega_2).$$

*Then the Hausdorff distance of $\Omega_1$ and $\Omega_2$ is defined by*

$$d^H(\Omega_1, \Omega_2) := \max(\rho(\Omega_1, \Omega_2), \rho(\Omega_2, \Omega_1)).$$

*We also have the following equivalent definition:*

$$d^H(\Omega_1, \Omega_2) = \inf\{\alpha > 0 : \Omega_2 \subset \Omega_1^\alpha \quad \text{and} \quad \Omega_1 \subset \Omega_2^\alpha\}$$

*where $\Omega^\alpha = \{x \in \mathbb{R}^N \mid d(x, \Omega) \leq \alpha\}$.*

**Definition 2.5.2.** *Denote by $\Omega_1, \Omega_2$ two open subsets of a compact set $B$. Then their Hausdorff distance is defined by:*

$$d_H(\Omega_1, \Omega_2) := d^H(B \setminus \Omega_1, B \setminus \Omega_2).$$

One of the most useful properties of the Hausdorff distance is the following compactness property :

**Theorem 2.5.1.** *Let $B$ be a fixed compact domain in $\mathbb{R}^N$ and $\Omega_n$ a sequence of open subsets of $B$. Then there exists an open set $\Omega \subset B$ and a subsequence $\Omega_{n_k}$ which converges for the Hausdorff distance to $\Omega$.*

The Hausdorff complement topology on open sets (using Definition 2.5.2) will used later in this chapter to study $\gamma$-convergence of solutions to PDEs in open sets, in the proof of Faber-Krahn inequality for triangles and quadrilaterals.

## 2.6 Faber - Krahn inequality

### 2.6.1 General Domains

The primary reference for this subsection is [23]

**Theorem 2.6.1.** *Let $B$ be the ball of volume $c$, where $c > 0$. Then,*

$$\lambda_1(B) = \min\{\lambda_1(\Omega), \Omega \subset \mathbb{R}^N, |\Omega| = c, \ \Omega \ \text{open}\}.$$

*Proof* Let $\Omega$ be a bounded open set of measure $c$ and $u_1$ be the eigenfunction corresponding to $\lambda(\Omega)$. Let us denote by $\Omega^* = B$ the ball of the same volume and $u_1^*$ the Schwarz rearrangement of $u_1$. From (2.8) and (2.9), we have,

$$\int_{\Omega^*} |\nabla u_1^*(x)|^2\,dx \le \int_{\Omega} |\nabla u_1(x)|^2\,dx. \quad \text{and} \quad \int_{\Omega^*} |u_1^*(x)|^2\,dx = \int_{\Omega} |u_1(x)|^2\,dx. \tag{2.10}$$

From the variational characterization of $\lambda_1$ , we have

$$\lambda_1(\Omega^*) \le \frac{\int_{\Omega^*} |\nabla u_1^*(x)|^2\,dx}{\int_{\Omega^*} u_1^*(x)^2\,dx} \quad \text{and} \quad \lambda_1(\Omega) = \frac{\int_{\Omega} |\nabla u_1(x)|^2\,dx}{\int_{\Omega} u_1(x)^2\,dx}. \tag{2.11}$$

Then (2.10) together with (2.11) yields the inequality in the case of general domains.

This theorem is called the Faber-Krahn inequality. The inequality was conjectured by Lord Rayleigh in 1877. It was proved by G. Faber in two dimensions in 1923 and Edgar Krahn extended the proof to any dimension of the Euclidean space in 1926.

### 2.6.2 Triangles and Quadrilaterals

The primary reference for this subsection is [32].

**Definition 2.6.1.** *For $\theta \in (0, \frac{\pi}{2}), h > 0,\ \xi \in \mathbb{R}^n$ and $\|\xi\| = 1$, we define*

$$C(\xi, h, \theta) = \{x \in \mathbb{R}^n : \|x\| < h, (x, \xi) > \|x\| \cos\theta\}$$

*to be the cone of height $h$, angle $\theta$ and axis $\xi$. The inner product $(,)$ is the usual inner product in $\mathbb{R}^n$.*

**Definition 2.6.2.** *A bounded domain $\Omega \subset \mathbb{R}^n$ is said to possess the uniform cone property if*

$$\forall x \in \partial\Omega, \exists C_x = C(\xi_x, h, \theta)$$

*such that*

$$\forall y \in B(x, r) \cap \Omega, y + C_x \subset \Omega,$$

*where $r$ is given such that $2r \leq h$ and $\theta, h$ fixed.*

We define the uniform cone property since we prove later in the chapter that triangles and quadrilaterals possess this property and we use the regularity of sets with this property.

We denote the collection of open subsets of the given set $D$ in $\mathbb{R}^n$ satisfying the uniform cone property by $\prod(\theta, h, r, D)$.

### Sobolev capacity of a set

**Definition 2.6.3.** *(Capacity) The Sobolev capacity of a set $\Omega \subset \mathbb{R}^n$ is defined as*

$$cap(\Omega) = \inf_{u \in \mathcal{U}(\Omega)} \|u\|^p_{W^{1,p}(\mathbb{R}^n)},$$

*where $\mathcal{U}(\Omega) = \{u \in W^{1,p}(\mathbb{R}^n) : u \geq 1$ in a neighbourhood of $\Omega\}$.*

While Lebesgue measure is the natural measure for functions in $L^p(\mathbb{R}^n)$, Sobolev capacity is the natural outer measure for functions in $W^{1,p}(\mathbb{R}^n)$.

**Definition 2.6.4.** *(Local capacity) Let $D$ be a bounded open set. The local capacity of a subset $\Omega$ in $D$ is*

$$cap(\Omega, D) = \inf \left\{ \int_D |\nabla u|^2 dx : u \in \mathcal{U}_\Omega \right\},$$

*where $\mathcal{U}_\Omega$ is the set of all functions $u$ of the Sobolev space $H^1_0(D)$ such that $u \geq 1$ almost everywhere in a neighborhood of $\Omega$.*

**Definition 2.6.5.** *For $c_0, r_0 > 0$ we say that an open set $\Omega$ has $(r_0, c_0)$-capacity density condition if $\forall x \in \partial\Omega, \forall t \in (0, r_0)$*

$$\frac{cap(B(x,t) \cap \Omega^c, B(x,2t))}{cap(B(x,t), B(x,2t))} \geq c_0.$$

*We denote the class of open subsets of $\Omega$ having the $(r, c)$ - capacity density condition by $\mathcal{O}_{c,r}(\Omega)$.*

**Proposition 2.6.1.** *$\prod(\theta, h, r, D) \subset \mathcal{O}_{c,r}(D)$.*

*Proof.* Since the cone property we consider is uniform, we have the same parameter values $\theta, h$ for the cone attached. The axis $\xi_{x_0}$ changes in direction with respect to the point $x_0$. Hence, all cones associated with $\partial\Omega$ are translations and rotations of $C(\xi_0, h, \theta)$. It is known that Sobolev capacity is invariant under translations and rotations.

Let us consider $r = h$ and

$$c = \frac{\operatorname{cap}(C(\xi_0, h, \theta) \cap B(0,t), B(0,2t))}{\operatorname{cap}(B(0,t), B(0,2t))}$$

where $\xi_0$ is a unit vector at the origin.

Thus, we have $\forall x \in \partial\Omega,\ \forall t \in (0,h)$,

$$\frac{\operatorname{cap}(B(x,t) \cap \Omega^c, B(x,2t))}{\operatorname{cap}(B(x,t), B(x,2t))} \geq c.$$

$\square$

### $\gamma$ convergence

Let us consider $\Omega$ a bounded set in $\mathbb{R}^n$. For the following boundary value problem,

$$\Delta u = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.12}$$

we denote by $u_{\Omega,f}$ the solution to the problem to show the dependence of the solution on $\Omega$ and $f$.

**Definition 2.6.6.** *($\gamma$ convergence) Let $B$ be a fixed ball in $\mathbb{R}^n$. Denote $(\Omega_n) \subset B$ a sequence of open sets in $\mathbb{R}^n$ and $\Omega \subset B$ an open set in $\mathbb{R}^n$. We say that $\Omega_n$ $\gamma$ - converges to $\Omega$ if, for every $f \in L^2(D)$, the sequence of solutions $u_{\Omega_n,f}$ to the problem (2.12) converges strongly in $L^2(D)$ to $u_{\Omega,f}$.*

**Theorem 2.6.2.** *Suppose $(\Omega_n) \subset \mathcal{O}_{c,r}(D)$ is a sequence which converges in the $H^c$-topology to an open set $\Omega$. Then $\Omega_n$ $\gamma$-converges to $\Omega$.*

*Proof.* We use the following results from [32] to prove the theorem and the proofs of the results are given in [32] and references therein.

- Let us consider a sequence $\Omega_n$ such that $\Omega_n \to \Omega$ in $H^c$ topology (Hausdorff complement topology) and let $f \in H^{-1}(D)$. There exists a subsequence of $(\Omega_n)$(denoted by $(\Omega_n)$),such that $u_{\Omega_n,f}$ weakly converges to $u$ in $H_0^1(D)$ and $u$ verifies the equation $-\Delta u = f$ in $H^{-1}(\Omega)$.

- Suppose $\Omega_n \to \Omega$ in $H^c$ - topology. Then the following statements are equivalent:

  1. for every $f \in H^{-1}(D)$, $u_{\Omega_n,f} \to u_{\Omega,f}$ strongly in $H_0^1(D)$.
  2. for $f \equiv 1$, $u_{\Omega_n,1} \to u_{\Omega,1}$ in $H_0^1(D)$.

- If $u \in H^1(\mathbb{R}^n)$ is a quasicontinuous function and $u = 0$ quasi everywhere in $\mathbb{R} \setminus \Omega$, then $u \in H_0^1(\Omega)$.

$\square$

We further refer to [23], supplemented by [32] for the study of properties of $H^c$ - topology, quasicontinuous functions and $\gamma$ - convergence. We also refer to [32] for the proof of the following corollary.

**Corollary 2.6.1.** *If $(\Omega_n)$ $\gamma$-converges to $\Omega$, then $\lambda(\Omega_n) \to \lambda(\Omega)$.*

### Steiner symmetrization

**Algorithm**: Let us start with an arbitrary triangle $T_0 = \Delta A_0B_0C_0$. Pick any side, say $B_0C_0$. We consider the perpendicular to $B_0C_0$ from $A_0$. Denote this perpendicular $L_0$. Symmetrisation of $T_0$ with respect to $L_0$ yields an isosceles triangle. The vertex $A_0$ remains unaltered. Denote the vertices of the new triangle $T_1$ as $A_1, B_1$ and $C_1$. Suppose we pick $A_1B_1$ and symmetrise with respect to the perpendicular bisector of that side. The resulting triangle is $T_2$ with $A_2B_2 = B_2C_2$ and $A_2 = A_1$. Now, we symmetrise with respect to the perpendicular bisector of $A_2B_2$. In $T_3$, we have $A_3B_3 = B_3C_3$ and $A_3 = A_2$. Thus we have, $A_0 = A_1 = A_2 = A_3 \ldots$. We perform the steps repeatedly to get a sequence of triangles. The steps are given in detail in Figure 5.5. The figure is taken from [32]

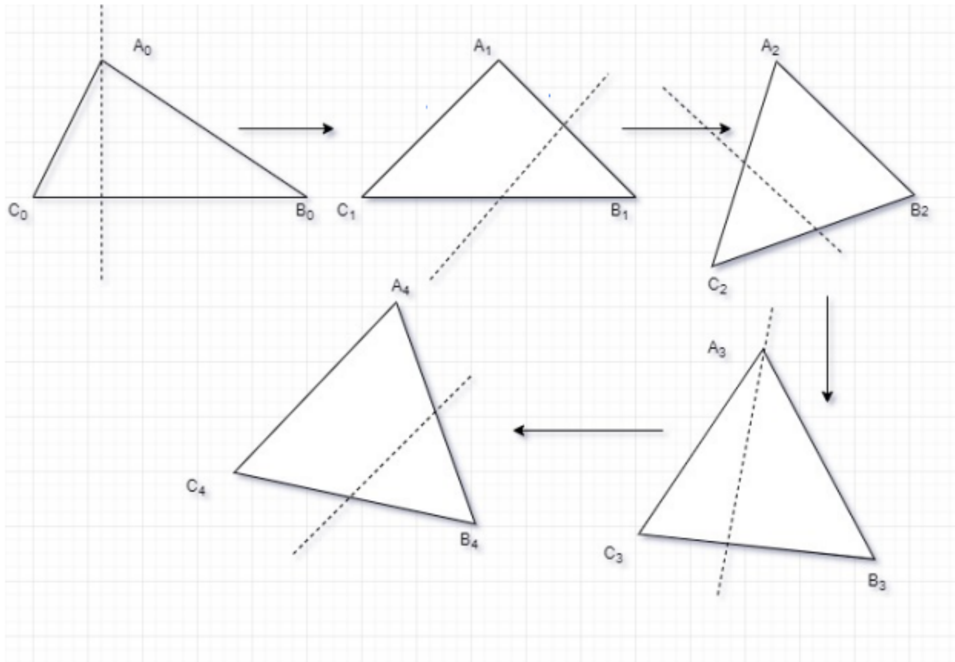


Figure 2.1: Symmetrisation sequence for triangles

Since Steiner symmetrisation conserves the area of the set being symmetrised, the entire sequence of triangles can be fit inside a ball of finite radius with centre at $A_0$. This implies that the sequence of triangles $T_n$ is a bounded sequence in $\mathbb{R}^2$, implying that the sequence of vertices $A_n, B_n, C_n$ are bounded sequences in $\mathbb{R}^2$. According to the Bolzano - Weierstrass theorem, there exists converging subsequences $B_{n_i}, C_{n_j}$ converging to points $B, C$ and $A_n$ is constant.

Since $A_n \to A, B_n \to B, C_n \to C$ in $\mathbb{R}^2$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0, d(A_n, A) < \varepsilon, d(B_n, B) < \varepsilon$. Therefore $T_n \subset T^\varepsilon$, $\forall n \geq n_0$, where $T^\varepsilon$ is given by

$$T^\varepsilon = \{x \in \mathbb{R}^N \mid d(x, T) \leq \varepsilon\}.$$

Similarly, we have $T \subset T_n^\varepsilon$, $\forall n \geq n_0$. Therefore $T_n \to T$ in Hausdorff distance.

**Claim 2.6.1.** *The Steiner symmetrization of an arbitrary triangle yields an equilateral triangle.*

*Proof.* Let us denote the sequence of side lengths of $T_n$ by $(a_n), (b_n)$and$(c_n)$. Since all the sequences are bounded sequences, by Bolzano - Weierstrass theorem, there exists a converging subsequence converging to $a, b$ and $c$ respectively. From the Steiner symmetrisation process, we have

$$b_1 = c_1, c_2 = a_2, b_3 = a_3, b_4 = c_4, a_5 = c_5, b_6 = a_6, \ldots$$

Therefore we have $a_{3k} = b_{3k}, b_{3k+1} = c_{3k+1}, a_{3k+2} = c_{3k+2}$. The sequences have a common subsequence, implying that they converge to the same limit. Hence $a = b = c$, i.e. the limit of the sequence of triangles is an equilateral triangle. $\square$

**Claim 2.6.2.** *The Hausdorff limit of the sequence $T_n$ is also an equilateral triangle.*

*Proof.* We have subsequences of vertices $A_{n_i}, B_{n_i}, C_{n_i}$ that converge in Hausdorff distance. Consider the side lengths of the triangles in this subsequences $a_{n_i}, b_{n_i}, c_{n_i}$. Since the vertices converge, the side lengths also converge to say $a', b', c'$ respectively. From Claim 2.6.1, we know that the sequences $a_n, b_n, c_n$ converge to the same limit, implying that their subsequences also have the same limit. Hence, we have $a' = a, b' = b$ and $c' = c$. Therefore, the Hausdorff limit of the sequence $T_n$ is also an equilateral triangle. $\square$

We note that the Steiner symmetrisation process for quadrilaterals yields a square and the arguments follow similar to arguments made for the proof of triangles.

**Claim 2.6.3.** *Triangles and quadrilaterals possess uniform exterior cone property.*

*Proof.* Suppose $T$ is a triangle contained in $D$. The sharp corners of polygons don't allow a cone with a nonzero opening angle to be contained in the interior of the polygon. However, cones of uniform height and angle at each point of $\partial T$ can be contained in the set $D \setminus T$. Similar arguments hold for quadrilaterals as well. This implies that triangles and quadrilaterals possess uniform exterior cone property.

$\square$

**Theorem 2.6.3.** *The equilateral triangle has the smallest principal eigenvalue $\lambda_1$ among all triangles of given area. The square has the smallest principal eigenvalue $\lambda_1$ among all quadrilaterals of a given area.*

*Proof.* Without loss of generality, we can assume that triangles and quadrilaterals of a given area can be fit inside a bounded open set $D$. By claim 2.6.3, we observe that both these sets of polygons have uniform exterior cone property. By claim 2.6.1, the Steiner symmetrisation process of triangles yields an equilateral triangle. This implies that the complement of these domains in $D$ converges to complements of the equilateral triangle and that of the square in the Hausdorff complement topology. By Proposition 2.6.1 and Theorem 2.6.1, the eigenvalues of the sequence of triangles and quadrilaterals converge to that of an equilateral triangle and a square respectively. By variational characterization $\lambda_1$, we conclude that the square and the equilateral triangle minimize $\lambda_1$ among all quadrilaterals and triangles of a given area respectively. □

Theorem 2.6.3 essentially proves the Faber-Krahn inequality for $n$-gons, where $n = 3, 4$. Since Steiner symmetrization for the case $n \geq 5$ leads to polygons with $n+1$ sides [22], we can't use the procedure for the case $n \geq 5$. Hence, we devise a new strategy in the next chapter to solve the conjecture for $n \geq 5$.

## 2.7 Bounds for $\lambda_1$

The primary references for this subsection are [30] and [14]. We study certain important methods to find bounds for $\lambda_1$, which would give us some important information regarding $\lambda_1$.

### 2.7.1 Lower bounds

**Theorem 2.7.1.** *Let $L$ be a second-order linear elliptic operator as defined by (2.3). Let $\lambda_1$ be the first eigenvalue for (2.5) with the Dirichlet boundary condition. Then there exists $\overline{\lambda} > 0$ such that $\overline{\lambda}$ is a lower bound for $\lambda_1$.*

*Proof.* We know from this chapter that the problem

$$Lu + \lambda u = 0 \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

is solvable for the value $\lambda = \lambda_1$, the first eigenvalue and it is solved by $u_1$, the first eigenfunction for $\lambda = \lambda_1$.

By variational characterization of $\lambda_1$, we have

$$\int_D \sum_{i,j=1}^{n} \left[ a_{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_j} - \lambda_1 u_1^2 \right] dV = 0. \tag{2.13}$$

Let us consider arbitrary $C^1$ functions $P_1, P_2, \ldots, P_n$ defined in $\Omega$. Then using Green's theorem, we get

$$\int \int \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (P_i u_1^2) dV = 0. \tag{2.14}$$

since $u_1 = 0$ on the boundary of the domain.

Summing equations (2.13) and (2.14), we have

$$\int_D \sum_{i,j=1}^{n} \left[ a_{ji} \frac{\partial u_1}{\partial x_j} \frac{\partial u_1}{\partial x_i} + 2 \sum_{i} P_i u_1 \frac{\partial u_1}{\partial x_i} + \left( \frac{\partial P_i}{\partial x_i} - \lambda_1 \right) u_1^2 \right] dV = 0.$$

Using the preceding equation, we define the following quadratic form in terms of $u$, $\frac{\partial u}{\partial x_i}$,

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2u \sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} + u^2 \left( \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} - \overline{\lambda} \right). \tag{2.15}$$

We observe that the quadratic form can be equivalently defined using the matrix M

$$M = \begin{pmatrix} a_{11} & \ldots & a_{1n} & P_1 \\ a_{21} & \ldots & a_{2n} & P_2 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} & P_n \\ P_1 & \ldots & P_n & \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} - \lambda \end{pmatrix} \tag{2.16}$$

Hence, if the expression (2.15) can be made positive for some selection of the functions $P_1, P_2, \ldots, P_n$ and some range of $\lambda$ say $0 \leq \lambda \leq \overline{\lambda}$ and, then $\overline{\lambda}$ will be a lower bound for the first eigenvalue $\lambda_1$. This is equivalent to the matrix $M$ being positive definite

The matrix M being positive definite is equivalent to the following condition

$$|M| = \det M > 0. \tag{2.17}$$

Suppose we denote by

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

We have

$$\det M = |A| \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} - \lambda |A| - \sum_{i,j=1}^{n} A_{ij} P_i P_j,$$

where $A_{ij}$ is the cofactor of the element $a_{ij}$ in the matrix $A$.

Using the fact that $L$ is elliptic, there exists $c_0$ and $c_1$ such that

$$c_0 \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq c_1 \sum_{i=1}^{n} \xi_i^2.$$

This implies that there exists $c_2$ and $c_3$ such that

$$c_2 \sum_{i=1}^{n} \xi_i^2 \leq \frac{1}{|A|} \sum_{i,j=1}^{n} A_{ij} \xi_i \xi_j \leq c_3 \sum_{i=1}^{n} \xi_i^2,$$

for all real $\xi_i$ and all $(x_1, \ldots, x_n)$ in $\Omega$. Hence, condition (2.17) implies

$$\sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} > c_2 \sum_{i=1}^{n} P_i^2 + \lambda.$$

If quantities $\lambda_1, \ldots, \lambda_n$ can be found such that the inequalities

$$\frac{\partial P_i}{\partial x_i} > c_2 P_i^2 + \lambda_i \quad i = 1, 2, \ldots, n \tag{2.18}$$

hold, then the inequality follows with $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. These inequalities can be solved explicitly using the theory of Riccati equations.

Suppose that the domain is contained in the cube $0 \leq x_i \leq \alpha, i = 1, 2 \ldots, n$. For the $P_i$ we make the selections

$$P_i = -\frac{C}{c_2} \cot C(x_i - \varepsilon),$$

where $\varepsilon > 0, \delta > 0$ are arbitrary and $C = \pi/(\alpha - \varepsilon) - \delta > 0$. With this choice, the inequalities (2.18) are satisfied for all $\lambda_i$ with

$$\lambda_i < \frac{C^2}{c_2}.$$

The lower bound $\overline{\lambda}$ for $\lambda_1$ essentially is given by $nC^2/c_2$.

$\square$

### 2.7.2 Upper bound for $\lambda_1$

**Theorem 2.7.2.** *Suppose $T$ is a triangle with sides of lengths $l_1 \leq l_2 \leq l_3$. Then we get*

$$\lambda_1(T) \leq \frac{\pi^2}{3A^2}(l_1^2 + l_2^2 + l_3^2),$$

*where $A$ denotes the area of $T$.*

**Remark 2.7.1.** *In this case, equality occurs for the equilateral triangle and this is the only instance where the bound agrees.*

*Proof.* In general, we obtain the upper bound for $\lambda_1$ by constructing a test function based on eigenfunctions of domains which are explicitly known. In our case, we shall use a test function based on the first eigenfunction of an equilateral triangle. We shall prove the result for a triangle $T'$ with vertices at $(0,0), (a,b)$ and $(1,0) \in \mathbb{R} \times \mathbb{R}^+$ and rescale to obtain the upper bound given in the theorem.

Let us consider the first eigenfunction, $u$, of the equilateral triangle [18] with vertices at the $(0,0), (1,0)$ and $(1/2, \sqrt{3}/2)$, that is,

$$u(x,y) = \sin\left(\frac{4\pi y}{\sqrt{3}}\right) - \sin\left[2\pi\left(x + \frac{y}{\sqrt{3}}\right)\right] + \sin\left[2\pi\left(x - \frac{y}{\sqrt{3}}\right)\right].$$

We perform a linear transformation on $u$ so that it satisfies the boundary conditions on $T'$. It is necessary to ensure that it satisfies the boundary condition so that it can be used as a test function for the first eigenvalue of $T'$. Thus, let

$$\phi(x,y) = u\left(x - \frac{a-1/2}{b}y, \frac{\sqrt{3}}{2b}y\right).$$

We have,

$$\int_{T'} |\nabla\phi(x,y)|^2 dx\ dy = \frac{2\pi^2}{b}[1 + (a-1)a + b^2] \quad \text{and} \quad \int_{T'} \phi(x,y)^2 dx\ dy = \frac{3b}{4}.$$

Using the characterization of $\lambda_1$ using Rayleigh quotient, we get

$$\lambda_1(T') \leq \frac{8\pi^2}{3b^2}[1 + (a-1)a + b^2].$$

if we now rescale $T'$ by $l_1$, we obtain a triangle, say $T$, with vertices at the origin, $(l_1, 0)$ and $(a_1, b_1) = l_1(a, b)$. With this notation we have

$$\lambda_1(T) \leq \frac{8\pi^2}{3b^2 l_1^2}[1 + (a-1)a + b^2] = \frac{8\pi^2}{3\beta^2}\left[1 + (\frac{a_!}{l_1} - 1)\frac{a_1}{l_1} + \frac{b_1^2}{l_1^2}\right]. \tag{2.19}$$

Using the relations,

$$l_2 = \sqrt{a_1^2 + b_1^2} \quad \text{and} \quad l_3 = \sqrt{(l_1 - a_1)^2 + b_1^2},$$

we get the bound

$$\lambda_1(T) \leq \frac{\pi^2}{3A^2}(l_1^2 + l_2^2 + l_3^2).$$

$\square$

## 2.8 Shape functional

The primary reference for this subsection is [29] supplemented by [2].

We denote any shape functional by $J(\Omega), J(.) : \Omega \mapsto J(\Omega) \in \mathbb{R}$ where $\Omega$ is a domain of class $C^k, k \geq 1$. We consider examples of non-smooth domains as well.

**Examples**:

$$J_1(\Omega) = \text{meas}(\Omega) = \int_{\Omega} dx,$$

$$J_2(\Omega) = \text{meas}(\partial\Omega) = \int_{\partial\Omega} dx,$$

In our case, the functional $J(\Omega)$, in general, takes the form

$$J(\Omega) = \int_{\Omega} F_1(x, y(x), \nabla y(x))dx + \int_{\partial\Omega} F_0(x, y(x), \nabla y(x))dx + \alpha\epsilon(\Omega),$$

where we denote by $y(\Omega)$ the solution of a boundary value problem defined on $\Omega$. We denote by $\epsilon(\Omega)$ the regularizing term that ensures the existence of optimal domain minimizing the functional $J(\Omega)$ over the appropriate class of domains and $\alpha > 0$ is a constant.

If $F_0 \equiv 0$, then $J$ is referred to as a distributed cost functional. If $F_1 \equiv 0$, then $J$ is called a boundary cost functional.

Since $\lambda(\Omega)$ is also a shape functional itself, we must study the properties of its derivatives and their regularity in non-smooth domains like polygons for the proof of the conjecture.

### 2.8.1 Shape derivatives

Let $\mathbb{P}(D)$ be the subset of open sets compactly contained in D, where $D \subset \mathbb{R}^n$ is assumed to be open and bounded.

We define for $k \geq 0$ and $0 \leq \alpha \leq 1$,

$$C_c^{k,\alpha}(D,\mathbb{R}^d) := \left\{\theta \in C^{k,\alpha}(D,\mathbb{R}^d) \mid \theta \text{ has compact support in } D\right\},$$

Let $\theta \in C_c^{0,1}(D,\mathbb{R}^d)$ be a vector field. We define the flow $T_\theta^t : D \to D, t \in [0,t_0]$ for each $x_0 \in D$ as $T_\theta^t(x_0) := x(t)$, where $x : [0,t_0] \to \mathbb{R}^d$ solves

$$\dot{x}(t) = \theta(x(t)) \text{ for } t \in [0,t_0], \quad x(0) = x_0.$$

For $\Omega \in \mathbb{P}(D)$, we consider the family of perturbed domains

$$\Omega_t := T_\theta^t(\Omega).$$

**Definition 2.8.1.** *Let $J : \mathbb{P}(D) \to \mathbb{R}$ be a shape functional.*

*(i) The Eulerian semiderivative of $J$ in direction $\theta \in C_c^{0,1}(D,\mathbb{R}^d)$ at $\Omega$ is defined by, when limit exists*

$$D_E J(\Omega)(\theta) := \lim_{t\downarrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

,

*(ii) $J$ is said to be shape differentiable at $\Omega$ if for all $\theta \in C_c^\infty(D,\mathbb{R}^d)$ it has a Eulerian semiderivative at $\Omega$ and the mapping*

$$D_E J(\Omega) : C_c^\infty(D,\mathbb{R}^d) \to \mathbb{R}, \quad \theta \mapsto D_E J(\Omega)(\theta)$$

*is linear and continuous, in which case $D_E J(\Omega)(\theta)$ is called the Eulerian shape derivative at $\Omega$.*

**Definition 2.8.2.** *Let $J : \mathbb{P}(D) \to \mathbb{R}$ be a shape functional, and assume that for all $t \in [0,t_0], D_E J(T_\theta^t(\Omega))$ exists in direction $\theta \in C_c^{0,1}(D,\mathbb{R}^d)$ at $T_\theta^t(\Omega)$*

*(i) The second order Eulerian semi derivative of $J$ in direction $\gamma \in C_c^{0,1}(D,\mathbb{R}^d)$ at $\Omega$ is defined by, when the limit exists,*

$$D_E^2 J(\Omega)(\theta,\gamma) := \lim_{t\downarrow 0} \frac{D_E J(T_\theta^t(\Omega))(\gamma) - D_E J(\Omega)(\gamma)}{t}.$$

*(ii) $J$ is said to be twice shape differentiable at $\Omega$ if it has a second order Eulerian semiderivative for all $\theta, \gamma \in C_c^\infty(D, \mathbb{R}^d)$ at $\Omega$ and the mapping*

$$D_E^2 J(\Omega) : (C_c^\infty(D, \mathbb{R}^d))^2 \to \mathbb{R}, \quad (\theta, \gamma) \mapsto D_E^2 J(\Omega)(\theta, \gamma)$$

*is bilinear and continuous, in which case $D_E^2 J(\Omega)(\theta, \gamma)$ is called the second order Eulerian shape derivative at $\Omega$.*

We also require the explicit expression of the Frechet derivative of $\lambda(\Omega)$ along with the Eulerian derivative, which has been explained by Laurain[29] in his paper.

## 2.9 Finite element method

The primary reference for this subsection is [41] supplemented by [16].

The finite element method is used to approximate the solution of the equation (2.20) using discrete approximation. We explain an example of the method below from [16]:

**Example**:

We consider the Laplace eigenvalue problem in an interval, say $[0, \pi]$. In variational form, the Laplace eigenvalue problem reads as

$$\text{Find } \lambda \in \mathbb{R} \text{ and } u \in H_0^1(\Omega) \text{ such that } (\nabla u, \nabla v) = \lambda(u, v)\ \forall v \in H_0^1(\Omega). \tag{2.20}$$

over the infinite dimensional space $V := H_0^1(\Omega)$. *The classical method is to replace $H_0^1(\Omega)$ in this variational problem by a sequence of finite dimensional subspaces $V^h$ contained in* $H_0^1(\Omega)$. The elements of $v^h$ of $V^h$ are called trial functions.

In our example, the admissible space is $H_0^1(\Omega)$. This implies that we can't consider piecewise constant functions as elements of $V^h$. Therefore the simplest choice that we can consider for $V^h$ is the space of functions which are linear over each interval $[(j-1)h, jh]$, continuous at the nodes $x = jh$ and zero at $x = 0$. The derivative of such a function is piecewise constant; thus $V^h$ is a subspace of $V$. We call these trial functions as linear elements.

For $j = 1, \ldots, N$, let $\phi_j^N$ be the function which equals 1 at the node $x = jh$ and vanishes at all others. These hat functions constitute a basis for the subspace since every member of $V^h$ can be written as a combination

$$v^h(x) = \sum_{i}^{N} q_j \phi_j^h(x).$$

**Discrete version of the eigenvalue problem**:

Let $T^h$ be a triangulation of domain $\Omega$ (we refer to the triangulation as a 'mesh' in Chapter 5). We use the piecewise-continuous linear finite element space $V^h(\subset V)$ as the approximation space. The basis of the space $V^h$ is given by the hat function. Suppose we denote $\dim(V^h) = n$. We use the classical method to solve the variation problem (2.20) in $V^h$,

$$\text{Find } \lambda^h \in \mathbb{R} \text{ and } u_h \in V^h \text{ s.t } (\nabla u_h, \nabla v_h) = \lambda^h (u_h, v_h) \ \forall v_h \in V^h. \tag{2.21}$$

Since the dimension of $V^h$ is finite, the problem (2.21) has finitely many eigenpairs. We denote by $\{\lambda_i^h, u_i^h\}_{i=1}^n$ the eigenpairs for the problem 2.21. We also assume that the eigenvalues $\lambda_i^h$ are increasing and the eigenfunctions $u_i^h$ form an orthogonal basis. Let us consider the basis function of $V^h$ as $\{\phi_i\}_{i=1}^n$. This implies that the problem (2.21) is to solve the generalized eigenvalue problem of $n \times n$ matrices $S^h$ and $T^h$,

$$S^h x = \lambda^h T^h x, \quad \text{where} \quad S_{i,j}^h = (\nabla\phi_i, \nabla\phi_j),\ T_{i,j}^h = (\phi_i, \phi_j).$$

For $u \in H^2(\Omega)$, we define $\pi_{1,h} u \in V^h$ as the Lagrange interpolation of $u$, that is ,

$$(\pi_{1,h} u)(a_i) = u(a_i) \quad \text{for each node } a_i \text{ of triangulation} \quad T^h.$$

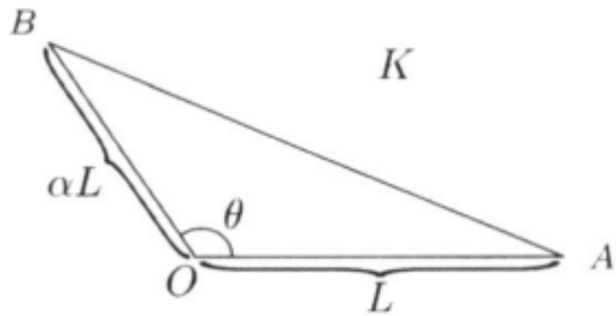


Figure 2.2: Representation of a triangle element by the parameters $\alpha, \theta, L$

For functions $u \in H^2(\Omega)$, we have

$$|u - \pi_{1,h} u|_{H^1} \leq C_1 h |u|_{H^2} \quad \text{for } u \in H^2(\Omega),$$

where $C_1$ is defined as below and $h$ is the mesh size to be specified.

$$C_1 := \max_{T \in \mathcal{T}^h} C_1(T)/h.$$

The estimate (2.9) would be useful in Chapter 4 for the estimate of eigenvalues.

Here, $C_1(T)$ is defined on a triangle element $T$ (Figure 2.2 from [41] gives a representation of the triangle element) of $\mathcal{T}^h$:

$$C_1(T) := \sup_{v \in H^2(T), v(O)=v(B)=v(A)=0} \frac{|v|_{H^2(T)}}{|v|_{H^1(T)}}.$$

For each $T$, let $L$ be such that, $|OB| \leq L = |OA| \leq |AB|$, $\theta$ be the maximum angle, and $\alpha L$ $(0 < \alpha \leq 1)$ be the smallest edge length.

Let us denote the projection $P_h : V \mapsto V^h$ such that for $v \in V$,

$$\int_\Omega (\nabla v - \nabla p_v, \nabla v_h)\, dx = 0 \quad \forall v_h \in V^h.$$

The projection operator would be useful in estimating eigenvalues of the Shape Hessian matrix over the finite element space in Chapter 4.

# Chapter 3

# Pólya - Szegö Conjecture

We denote by $\mathcal{P}_n$ the family of simple polygons with $n$ sides in $\mathbb{R}^2$ and for every $n \geq 3$, we consider the problem

$$\min_{P \in \mathcal{P}_n, |P|=\pi} \lambda_1(P). \tag{3.1}$$

**Polya - Szego conjecture** The regular polygon with $n$ sides and area $\pi$ is the unique solution to the problem (3.1).

In Chapter 2, we showed the proof of the conjecture for $n = 3, 4$. In this chapter, we devise the following step-by-step procedure to prove the conjecture for $n \geq 5$, assuming the conjecture is true. We show Step 1 and Step 2 of the procedure in detail in this Chapter. Step 3 and Step 4 of the procedure are shown in Chapters 4 and 5, respectively.

**Step 1** (Formal computation of the shape Hessian matrix): We use the theory of shape derivatives from [29] to find the first-order and second-order shape derivative of $\lambda_1(\Omega)$ for a bounded, Lipschitz domain $\Omega$. We also find the expression of the first-order and second-order shape derivative of $\lambda_1(P)$, where $P \in \mathcal{P}_n$, in terms of the coordinates of the vertices of the polygon. From [23], we show that the minimization of $\lambda_1(P)$, where the polygon $P$ has a fixed area, is equivalent to the minimization of the scale-invariant functional $l_n(P) := |P|\lambda(P)$, over the family of polygons $\mathcal{P}_n$. Using the explicit expression of the first-order shape derivative of the functional $l_n$ for a polygon, we verify that the regular polygon is a critical point for $l_n$. We also find the explicit expression for the second-order shape derivative of the functional $l_n$.

**Step 2** (Stability analysis of the shape Hessian matrix): The objective is to prove that the regular polygon is a local minimum for the functional $l_n$. Since we have verified that the regular polygon is a critical point for the functional $l_n$, it is necessary to verify that the coefficients of the shape Hessian matrix of $l_n$, $\mathbf{M}^\lambda$, are continuous for perturbations of the regular polygon. To verify the continuity of the coefficients, we require the $L^\infty$ estimates of eigenfunctions, gradients of eigenfunctions, the first Dirichlet eigenvalue and the gradient of the first Dirichlet eigenvalue. We also require the $L^\infty$ estimates of functions, which are the solutions of an elliptic PDE, for which we use Gagliardo-Nirenberg type interpolation inequalities. The stability analysis will also be required later to prove that the proof of the conjecture can be reduced to a finite number of numerical computations.

**Step 3** (Study of eigenvalues of the shape Hessian matrix, $\mathbf{M}^\lambda$ ): After verifying the continuity of the coefficients of the shape Hessian matrix, $\mathbf{M}^\lambda$, it is necessary to prove that the matrix is positive definite. We show in Chapter 4 that it is enough to prove that the $2n-4$ eigenvalues of $\mathbf{M}^\lambda$ are strictly positive to prove the positive definiteness of $\mathbf{M}^\lambda$ (four eigenvalues of $\mathbf{M}^\lambda$ are equal to zero). We employ finite element methods to approximate the eigenvalues of the shape Hessian matrix. We use interval arithmetic to find an interval in which the eigenvalues lie and verify that zero doesn't lie in the interval. This proves that the eigenvalues are strictly positive and the regular polygon is a local minimum for the functional $l_n$.

**Step 4** (Proof of the conjecture): We assume that the conjecture is true and our objective is to prove that the proof of the conjecture can be reduced to a finite number of numerical computations when assumed true. The strategy is to find a compact set in $\mathbb{R}^{2n-4}$ outside which an optimal polygon that minimizes the functional $l_n$ cannot be found. To find such a compact set $P'$, we find the maximum diameter, $D_{max}$, of an optimal polygon and the minimum edge length, $e_{min}$, of an optimal polygon using surgery arguments from [7]. From the proof for local minimality of the regular polygon, we know of a local neighbourhood of the regular polygon, denoted by $\mathcal{L}_n$, where it is the unique polygon minimizing $l_n$. This implies that it is sufficient to prove that there exists no optimal polygon in the compact set $P' \setminus \mathcal{L}_n$.

## 3.1 Computation of the shape Hessian matrix $\mathrm{M}^{\lambda}$

### 3.1.1 General domains

The primary references for the following subsections are [5] and [29].

We use the notations and equations describing tensor calculus from [29]. For second-order tensors $S, T : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$, we define the product

$$S : T = \sum_{i,j=1}^{n} S_{ij} T_{ij}.$$

Suppose $\Omega$ is a bounded, Lipschitz domain and $\theta, \xi \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Let $u$ be the $L^2$-normalized eigenfunction associated with the first Dirichlet eigenvalue of $\Omega$, $\lambda(\Omega)$.

We denote by $\Omega_\theta = (I + \theta)(\Omega)$ for a given vector field $\theta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\|\theta\|_{W^{1,\infty}} < 1$. We denote by $\tilde{u}(\theta)$ the material derivative as defined in [29]. We suppose $\theta$ is small enough such that $\lambda(\Omega_\theta)$ is still a simple eigenvalue.

**Theorem 3.1.1.** *The first order distributed shape derivative of $\lambda(\Omega)$ is given by*

$$\lambda'(\Omega)(\theta) = \int_\Omega \boldsymbol{S}_1^{\lambda} : D\theta \, dx,$$

*such that*

$$\boldsymbol{S}_1^{\lambda} = (|\nabla u|^2 - \lambda(\Omega) u^2)\boldsymbol{Id} - 2(\nabla u \otimes \nabla u).$$

*where* $\boldsymbol{Id}$ *represents the identity operator.*

*Proof.* We define by $u_\theta$, the solution of

$$\int_{\Omega_\theta} \nabla u_\theta \cdot \nabla v_\theta \, dx = \lambda(\Omega_\theta) \int_{\Omega_\theta} u_\theta v_\theta \, dx, \quad \forall v_\theta \in H_0^1(\Omega_\theta), \tag{3.2}$$

with the normalization condition

$$\int_{\Omega_\theta} (u_\theta)^2 \, dx = 1.$$

We consider the change of variables, $u^\theta = u_\theta \circ (I+\theta) \in H_0^1(\Omega)$, so that $u_\theta = u^\theta \circ (I+\theta)^{-1}$ and we get,

$$\nabla u_\theta = (I + D\theta^T)^{-1} \nabla u^\theta \circ (I+\theta)^{-1}. \tag{3.3}$$

The change of variables in (3.2) implies

$$\int_\Omega B(\theta) \nabla u^\theta \cdot \nabla v \, dx = \lambda(\Omega_\theta) \int_\Omega u^\theta v \det(I + D\theta) \, dx, \quad \forall v \in H_0^1(\Omega). \tag{3.4}$$

where $B(\theta) = \det(I + D\theta)(I + D\theta)^{-1}(I + D\theta^T)^{-1}$.

Differentiating (3.4) at 0, we obtain for all $v \in H_0^1(\Omega)$,

$$\int_\Omega B'(0)(\theta) \nabla u \cdot \nabla v \, dx + \int_\Omega \nabla \tilde{u}(\theta) \cdot \nabla v \, dx = \lambda'(\Omega)(\theta) \int_\Omega uv \, dx + \lambda(\Omega_\theta) \int_\Omega \tilde{u}(\theta) v + uv \, \mathrm{div}\theta \, dx, \tag{3.5}$$

where $B'(0)(\theta) = \mathrm{div}\theta I - D\theta - D\theta^T$.

The corresponding derivative of the normalization condition evaluated at zero is given by

$$\int_\Omega 2u\tilde{u}(\theta) + u^2 \mathrm{div}\theta \; dx = 0. \tag{3.6}$$

Since $u$ is the eigenfunction associated to $\lambda(\Omega)$, we get

$$\int_\Omega \nabla u . \nabla \tilde{u}(\theta) \; dx = \lambda(\Omega) \int_\Omega u \tilde{u}(\theta) \; dx.$$

Using the fact that $\int_\Omega u^2 \, dx = 1$, and taking $v = u$ in (3.5) we obtain

$$\lambda'(\Omega)(\theta) = \int_\Omega [\left(|\nabla u|^2 - \lambda(\Omega)u^2\right)\mathbf{Id} - 2(\nabla u \otimes \nabla u) : D\theta]\, dx = \int_\Omega \mathbf{S}_1^\lambda : D\theta \; dx. \tag{3.7}$$

$\square$

**Theorem 3.1.2.** *The second order distributed shape derivative of $\lambda_1(\Omega)$ is given by*

$$\lambda_1''(\Omega)(\theta,\xi) = \int_\Omega K^\lambda(\theta,\xi)\, dx,$$

*with*

$$K^\lambda(\theta,\xi) = -2(\nabla\tilde{u}(\theta)\cdot\nabla\tilde{u}(\xi)) + 2\lambda(\Omega)\tilde{u}(\theta)\tilde{u}(\xi) + [\lambda(\Omega)(\theta)\, div\;\xi + \lambda(\Omega)(\xi)\, div\;\theta]u^2$$

$$+\boldsymbol{S}_1^\lambda : (D\theta\, div\;\xi + D\xi\, div\;\theta)$$

$$+(-|\nabla u|^2 + \lambda u^2)(div\;\xi\, div\;\theta + D\theta^T : D\xi) + 2(D\theta D\xi + D\xi D\theta + D\xi D\theta^T)\nabla u\cdot\nabla u.$$

*Proof.* To compute the second-order derivative of the eigenvalue, we consider the expression for the first-order derivative and differentiate w.r.t $\xi$ at the point 0. Denote by $\Omega_\xi = (I+\xi)(\Omega)$. Let us assume that $\xi$ is small enough such that $\lambda(\Omega_\xi)$ is an eigenvalue of multiplicity 1. Denote by $u_\xi \in H_0^1(\Omega_\xi)$ the eigenfunction associated to $\lambda(\Omega_\xi)$. We have

$$\lambda'(\Omega_\xi)(\theta) = \int_{\Omega_\xi} [\left(|\nabla u_\xi|^2 - \lambda(\Omega_\xi)u_\xi^2\right)\mathbf{Id} - 2(\nabla u_\xi \otimes \nabla u_\xi)] : D(\theta\circ(I+\xi)^{-1})dx. \tag{3.8}$$

We again consider a change of variables, $u^\xi = u_\xi\circ(I+\xi) \in H_0^1(\Omega)$.

We obtain the following initial formula for the second-order derivative:

$$\lambda''(\Omega)(\theta,\xi) = \int_\Omega \mathbf{S}_1^\lambda : (D\theta \text{ div } \xi) - \mathbf{S}_1^\lambda : (D\theta D\xi)\, dx$$

$$-\int_\Omega [\lambda'(\Omega)(\xi)u^2 + 2\lambda(\Omega)u\tilde{u}(\xi)] \text{ div } \theta\, dx$$

$$+\int_\Omega [(-D\xi - D\xi^T)\nabla u\cdot\nabla u + 2(\nabla u\cdot\nabla\tilde{u}(\xi))]\mathrm{div}\theta\ dx$$

$$+\int_\Omega \left[4(D\xi^T\nabla u\odot\nabla v) : D\theta - 4(\nabla\tilde{u}(\xi)\odot\nabla u) : D\theta\right]\ \mathrm{div}\ \theta\, dx.$$

We have,

$$\mathbf{S}_1^\lambda : (D\theta D\xi) = -2(D\theta D\xi\nabla u)\cdot\nabla u + \left(|\nabla u|^2 - \lambda^2\right) D\theta^T : D\xi.$$

We also simplify the terms involving the symmetric tensor product. Using the definition of the derivative and the normalization condition (3.6), we obtain

$$\lambda''(\Omega)(\theta,\xi) = -2\int_\Omega (\nabla\tilde{u}(\theta)\cdot\nabla\tilde{u}(\xi) - \lambda(\Omega)\tilde{u}(\theta)\tilde{u}(\xi))\, dx + \int_\Omega \mathbf{S}_1^\lambda : (D\theta\ \mathrm{div}\xi)\, dx$$

$$-\int_\Omega \lambda'(\Omega)(\theta)u^2\mathrm{div}\ \xi + \lambda'(\Omega)(\xi)u^2\mathrm{div}\theta\, dx$$

$$-\int_\Omega 2(D\xi\nabla u)\cdot\nabla u\ \mathrm{div}\theta\, dx + \int_\Omega (4D\xi^T\nabla u\odot\nabla u) : D\theta\, dx$$

$$+\int_\Omega (-|\nabla u|^2 + \lambda(\Omega)u^2)D\theta^T : D\xi + 2(D\theta D\xi\nabla u)\cdot\nabla u\ dx.$$

We have

$$2(D\xi^T\nabla u\odot\nabla u) : D\theta = ((D\xi D\theta + D\xi D\theta^T)\nabla u)\cdot\nabla u.$$

Which gives

$$
\begin{aligned}
\lambda''(\Omega)(\theta,\xi) = &-2\int_\Omega (\nabla\widetilde{u}(\theta)\cdot\nabla\widetilde{u}(\xi)) - \lambda(\Omega)\widetilde{u}(\theta)\widetilde{u}(\xi)\ dx \\
&+2\int_\Omega ((D\theta D\xi + D\xi D\theta + D\theta D\xi^T)\nabla u)\cdot\nabla u\ dx \\
&+\int_\Omega (-|\nabla u|^2 + \lambda u^2)(\mathrm{div}\theta \mathrm{div}\ \xi + D\theta^T : D\xi)\ dx \\
&+\int_\Omega \mathbf{S}_1^\lambda : (D\theta\ \mathrm{div}\ \xi + D\xi\ \mathrm{div}\theta)\ dx \\
&-\int_\Omega (\lambda'(\Omega)(\theta)\ \mathrm{div}\xi + \lambda'(\Omega)(\xi)\ \mathrm{div}\theta)\ u^2\ dx.
\end{aligned}
$$

This finishes the proof of the theorem. □

The above results give an explicit expression of the shape derivative of $\lambda_1(\Omega)$ for a bounded, Lipshcitz domain $\Omega$. We define a specific perturbation $\theta$ for polygons in the next subsection, using which we find the expressions of derivatives for polygons.

### 3.1.2 Polygons

To consider the perturbations for a polygon, we extend the geometric perturbation of vertices to a global perturbation for the polygon as follows:

Let us denote by $a_i$ $(i = 0, \ldots, n-1)$ the vertices of the polygon. For each vertex $a_i$ of the polygon, we consider the vector field $\theta_i \in W^{1,\infty}(\mathbb{R}^2)$ to be the corresponding perturbation. We consider a triangulation $\mathcal{T}$ of the polygon $\Omega$ such that the edges of the polygon are complete edges of some triangles in the triangulation. We also consider a family of globally Lipschitz functions $\phi_i$ such that the functions $\phi_i$ are piecewise affine on each triangle of $\mathcal{T}$ and they satisfy

$$
\phi_i(a_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.9}
$$

Several choices for the triangulation are possible as shown in Figure 3.1. The figure has

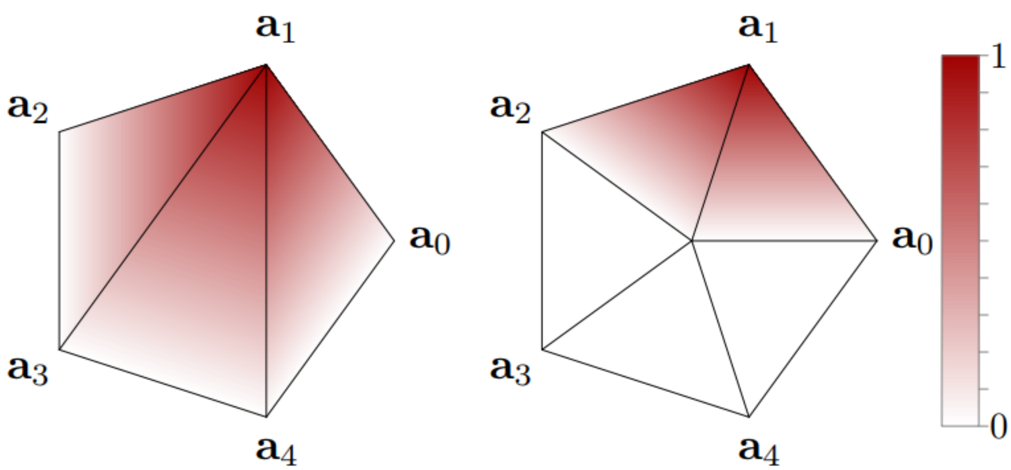


Figure 3.1: A graphical view of the function $\phi_2$ on two examples of admissible triangulations used for defining perturbations on a polygon.

been taken from [5]. We construct a global perturbation $\theta$ for a polygon in $\mathbb{R}^2$, given by

$$\theta = \sum_{i=0}^{n-1} \theta_i \phi_i \in W^{1,\infty}(\mathbb{R}^2). \tag{3.10}$$

For a non - degenerate polygon whose vertices are denoted by $(x_i, y_i) \in \mathbb{R}^2$ and whose edges are oriented in the anti-clockwise direction, the coordinates are given by

$$\mathbf{x} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1}) = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}), \tag{3.11}$$

where $\mathbf{a}_i$ denote the vertices of the polygon.

The area of the polygon with vertices $(x_i, y_i)$ is given by

$$\mathcal{A}(\mathbf{x}) = \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i).$$

We obtain the following expressions for the gradient of area functional and coefficients of the Hessian matrix of area functional [[29]] :

$$\nabla\mathcal{A}(\mathbf{x}) = \left(\int_\Omega \nabla\phi_i \, dx\right)_{i=0,\dots,n-1}, \tag{3.12}$$

$$D^2\mathcal{A}(\mathbf{x}) = (\mathbf{A}_{ij})_{0\le i,j\le n-1}. \tag{3.13}$$

where the blocks $(\mathbf{A}_{ij})$ are given by

$$\mathbf{A}_{ij} = \int_\Omega [\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i]. \tag{3.14}$$

**Theorem 3.1.3.** *The gradient of the first Dirichlet eigenvalue $\lambda(\Omega)$, when $\Omega$ is a polygon with coordinates $\mathbf{x}$ is given by*

$$\nabla\lambda(\mathbf{x}) = \left(\int_\Omega \boldsymbol{S}_1^\lambda \nabla\phi_i \, dx\right)_{i=0,\dots,n-1} = -\int_{\partial\Omega} |\nabla u|^2 \phi_i \boldsymbol{n} \, dx.$$

**Remark 3.1.1.** *We study the regularity results for eigenfunctions of non-smooth domains like polygons ([10],[1]) to prove that the boundary expression is valid. The eigenfunction has $H^{2+\delta}$ regularity away from corners in a polygon and a singular part near the corners:*

$$u = u_{reg} + u_{sing}$$

*where $u_{reg} \in H^{2+\delta}(\Omega)$ for some $\delta > 0$ and*

$$u_{sing} = \sum_{i=0}^{n-1} C_i \phi_i r^{\frac{\pi}{\omega_i}} \sin\left(\frac{\pi}{\omega_i}\theta\right) \tag{3.15}$$

*where $\omega_i$ are the angles of the polygon considered, $C_i$ are constants, $\phi_i$ is the cutoff function equal to 1 in a neighbourhood of the vertex $\boldsymbol{a}_i$ and $(r, \theta)$ are the polar coordinates around the angle $i$.*

*This result is also useful for studying the $H^{2+s}$ regularity of $u_1$ in Chapter 4.*

*Proof.* The proof of the first equality follows from [29]. We prove the validity of the boundary expression:

The gradient of $u$ is point-wise defined on $\partial\Omega$ in a classical way, except at the vertices. To show the validity of the expression around the vertices, we use the regularity of $u$. We fix a vertex $j$ and define

$$\Omega_\epsilon = \Omega \backslash (B(a_{j-1}, \epsilon) \cup B(a_j, \epsilon) \cup B(a_{j+1}, \epsilon)),$$

$$\Gamma_\epsilon = \Omega \cap (B(a_{j-1}, \epsilon) \cup B(a_j, \epsilon) \cup B(a_{j+1}, \epsilon)).$$

Since $u|_{\partial\Omega} \in H^2(\Omega_\epsilon)$, it is immediate that div $\mathbf{S}_1^\lambda = 0$ on $\Omega_\epsilon$. The gradient $\nabla u$ is colinear with the normal vector $\mathbf{n}$ on $\partial\Omega$, since $u = 0$ on $\partial\Omega$. This implies that $(\nabla u \otimes \nabla u)\ \mathbf{n} = (\mathbf{n} \cdot \nabla u)\ \nabla u = |\nabla u|^2 \mathbf{n}$. This further implies that $\mathbf{S}_1^\lambda \mathbf{n} = -|\nabla u|^2 \mathbf{n}$. Therefore we obtain

$$\int_{\Omega_\epsilon} \mathbf{S}_1^\lambda \nabla \phi_i \, dx = -\int_{\partial\Omega_\epsilon \backslash \Gamma_\varepsilon} |\nabla u|^2 \phi_i \mathbf{n} \, dx + \int_{\Gamma_\varepsilon} \mathbf{S}_1^\lambda \mathbf{n} \phi_i.$$

**Claim 3.1.1.** *The term* $\int_{\Gamma_\varepsilon} \boldsymbol{S}_1^\lambda \boldsymbol{n}\ \phi_i$ *converges to* $0$ *for* $\varepsilon \to 0$.

Since $u_{reg} \in H^{2+\delta}(\Omega)$, the gradient of $u_{reg}$ is bounded. From the expression of $u_{sing}$, we get $|\nabla u_{sing}| \leq C r^{\frac{\pi}{\omega_i} - 1}$ for some constant C independent on $\varepsilon$, from which we conclude that

$$\int_{\Gamma_\varepsilon} |\nabla u_{reg}|^2 + |\nabla u_{sing}|^2 \to 0,$$

for $\varepsilon \to 0$

This proves the claim and the claim implies that the boundary expression is valid.

□

We introduce the functions $\mathbf{U}_i \in H_0^1(\Omega, \mathbb{R}^2)$ $(i = 0, \ldots, n-1)$ such that

$$\widetilde{u}(\theta) = \sum_{i=0}^{n-1} \theta_i \cdot \mathbf{U}_i.$$

From (3.5), we have the following two PDEs: $\mathbf{U}_i \in H_0^1(\Omega, \mathbb{R}^2)$ ,

$$\int_\Omega (D\mathbf{U}_i \nabla v - \lambda(\Omega)\mathbf{U}_i v)\, dx = \int_\Omega [-(\nabla\phi_i \otimes \nabla u)\nabla v + 2(\nabla u \odot \nabla v)\nabla\phi_i]\; dx$$

$$+ \int_\Omega \mathbf{S}_1^\lambda \phi_i \int_\Omega uv\; dx + \lambda(\Omega)\int_\Omega uv\nabla\phi_i\, dx. \tag{3.16}$$

Similarly, we get,

$$\int_\Omega (2u\mathbf{U}_i + u^2\nabla\phi_i)\; dx = 0. \tag{3.17}$$

from the normalization condition (3.6).

*The functions $\boldsymbol{U}_i$ can be essentially described as the "directional" shape derivatives of the function $u$ along the perturbation $\theta_i$ where $\theta$ represents the perturbation of the domain.*

**Theorem 3.1.4.** *The Hessian matrix $\mathcal{N}^\lambda \in \mathbb{R}^{2n\times 2n}$ of $\lambda_1$ with respect to the coordinates of the $n$-gon is given by the following $n \times n$ block matrix*

$$\mathcal{N}^\lambda = (\mathcal{N}_{ij}^\lambda)_{0\le i,j\le n-1},$$

*where the $2 \times 2$ blocks are given by*

$$\mathcal{N}_{ij}^\lambda = \int_\Omega \left(-2D\,\boldsymbol{U}_i D\,\boldsymbol{U}_j^T + 2\lambda(\Omega)\,\boldsymbol{U}_i\,\boldsymbol{U}_j^T + \nabla\phi_i \otimes (\boldsymbol{S}_1^\lambda \nabla\phi_j) + (\boldsymbol{S}_1^\lambda\nabla\phi_i)\otimes\nabla\phi_j\right)\, dx,$$

$$+ \int_\Omega \left(-|\nabla u|^2 + \lambda(\Omega)u^2\right)(2\nabla\phi_i \odot \nabla\phi_j)\, dx$$

$$+2\int_\Omega ((\nabla\phi_i \cdot \nabla u)(\nabla\phi_j \otimes \nabla u) + (\nabla\phi_j\cdot\nabla u)(\nabla u \otimes \nabla\phi_i) + (\nabla\phi_i\cdot\nabla\phi_j)(\nabla u\otimes\nabla u))\; dx$$

$$-\int_\Omega (u^2\nabla\phi_i)\otimes\left(\int_\Omega \boldsymbol{S}_1^\lambda\nabla\phi_j\, dx\right)\, dx$$

$$-\int_\Omega u^2\left(\int_\Omega \boldsymbol{S}_1^\lambda\nabla\phi_i dx\right)\otimes\nabla\phi_j\, dx.$$

*Proof.* The proof of the theorem follows from [29]. We use the formula for $\mathcal{K}^\lambda$ (see Theorem 3.1.2) for the second-order shape derivative of $\lambda(\Omega)$, with $\theta$ given by (3.10).

Hence, using the expression for $\mathcal{K}^\lambda$, we obtain

$$
\begin{aligned}
\mathcal{N}^\lambda_{ij} &= \int_\Omega \left(-2D\mathbf{U}_i D\mathbf{U}_j^T + 2\lambda(\Omega)\mathbf{U}_i\mathbf{U}_j^T\right) dx \\
&+ \int_\Omega \left(|\nabla u|^2 - \lambda(\Omega)u^2\right)\left(\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i\right) dx \\
&-2\int_\Omega \left(\nabla u \otimes \nabla u\right)\left(\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i\right) dx \\
&+2\int_\Omega \left(\nabla\phi_i \cdot \nabla\phi_j\right)\left(\nabla u \otimes \nabla u\right) dx \\
&-\int_\Omega (u^2\nabla\phi_i) \otimes \left(\int_\Omega \mathbf{S}_1^\lambda \nabla\phi_j \; dx\right) \; dx \\
&-\int_\Omega \left(u^2 \int_\Omega \mathbf{S}_1^\lambda \nabla\phi_i \; dx\right) \otimes \nabla\phi_j \; dx.
\end{aligned}
$$

A direct computation shows that

$$
\begin{aligned}
(\nabla u \otimes \nabla u)(\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i) + (\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i)(\nabla u \otimes \nabla u) \\
= |\nabla u|^2(\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i).
\end{aligned}
$$

Using the above equality, the expression simplifies to

$$
\begin{aligned}
\mathcal{N}^\lambda_{ij} &= \int_\Omega \left(-2D\mathbf{U}_i D\mathbf{U}_j^T + 2\lambda(\Omega)\mathbf{U}_i\mathbf{U}_j^T\right) dx \\
&+ \int_\Omega \left(-|\nabla u|^2 - \lambda(\Omega)u^2\right)\left(\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i\right) dx \\
&+2\int_\Omega \left(\nabla\phi_i \cdot \nabla\phi_j\right)\left(\nabla u \otimes \nabla u\right) dx \\
&-\int_\Omega (u^2\nabla\phi_i) \otimes \left(\int_\Omega \mathbf{S}_1^\lambda \nabla\phi_j dx\right) \; dx \\
&-\int_\Omega u^2 \left(\int_\Omega \mathbf{S}_1^\lambda \nabla\phi_i dx\right) \otimes \nabla\phi_j \; dx.
\end{aligned}
$$

□

**Theorem 3.1.5.** *Let us suppose that $\Omega$ is a regular $n$ - gon and the triangulation $\mathcal{T}$ defining $\phi_i$ is symmetric. Then the Hessian matrix of $\lambda_1(\Omega)|\Omega| = \mathcal{A}(\mathbf{x})\lambda_1(\mathbf{x})$ in terms of coordinates of the polygon has the $2 \times 2$ blocks $\boldsymbol{M}_{ij}^{\lambda}$ given by*

$$\boldsymbol{M}_{ij}^{\lambda} = |\Omega| \int_{\Omega} (-2D\,\boldsymbol{U}_i D\,\boldsymbol{U}_j^T + 2\lambda(\Omega) U_i U_j^T) dx$$

$$-\lambda(\Omega) \int_{\Omega} (\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i) dx$$

$$+2|\Omega| \int_{\Omega} (\nabla\phi_i \cdot \nabla\phi_j)(\nabla u \otimes \nabla u)\ dx.$$

*Proof.* Simplifying the Hessian of the product $\mathcal{A}(\mathbf{x})\lambda_1(\mathbf{x})$ , we have

$$\text{Hess}(\lambda_1(\mathbf{x})\mathcal{A}(\mathbf{x})) = (|\Omega| \text{ Hess } \lambda_1(\mathbf{x})) + (\nabla\lambda_1(\mathbf{x}) \otimes \nabla\mathcal{A}(\mathbf{x})) + (\nabla\mathcal{A}(\mathbf{x}) \otimes \nabla\lambda_1(\mathbf{x})) + (\lambda_1(\mathbf{x}) \text{ Hess } \mathcal{A}(\mathbf{x})).$$

A direct computation using the definition of the functions $\phi_i$ implies that

$$\int_{\Omega} \left(-|\nabla u|^2 - \lambda(\Omega)u^2\right) (\nabla\phi_i \otimes \nabla\phi_j - \nabla\phi_j \otimes \nabla\phi_i)\, dx = \frac{\lambda_1(\Omega)}{|\Omega|} \mathbf{A}_{ij},$$

where $\mathbf{A}_{ij}$ are the blocks of the Hessian matrix of the area given in (3.14) and $\Omega$ is the regular polygon.

Using the definition of the functions $\phi_i$, it is immediate that $\nabla\phi_i$ is a piecewise constant on the triangles $T_k$. We have

$$\int_{T_k} u_1^2\, dx = \frac{1}{n} \int_{\Omega} u_1^2\, dx = \frac{1}{n}.$$

for $k = 0, ..., n-1$ by the symmetry of the eigenfunction. Therefore

$$\int_{T_k} u_1^2 \nabla\phi_i\, dx = \frac{1}{|\Omega|} \int_{T_k} \nabla\phi_i\, dx = \frac{1}{|\Omega|} \nabla\mathcal{A}(x).$$

This implies that the last term in the shape Hessian $N_{ij}^{\lambda}$ can be simplified as

$$\int_{\Omega} u_1^2 \left[\nabla\phi_i \otimes \left(\int_{\Omega} \mathbf{S}_1^{\lambda} \nabla\phi_j\, dx\right) + \left(\int_{\Omega} \mathbf{S}_1^{\lambda} \nabla\phi_i\, dx\right) \otimes \nabla\phi_j\right] dx = \frac{2}{|\Omega|} (\nabla\mathcal{A}(\mathbf{x}) \otimes \nabla\lambda_1(\mathbf{x})).$$

Substituting the above-simplified expressions in the expression of $\mathcal{N}_{ij}^{\lambda}$ and substituting the expression of $\mathcal{N}_{ij}^{\lambda}$ in the expression for the Hessian matrix of $\mathcal{A}(\mathbf{x})\lambda(\mathbf{x})$ proves the theorem. □

The matrices $\mathcal{N}^{\lambda}$ and $\mathbf{M}^{\lambda}$ represent the shape Hessian matrices of $\lambda_1(\mathbf{x})$ and $\lambda_1(\mathbf{x})\mathcal{A}(\mathbf{x})$, respectively, for polygons with coordinates $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The expression of the shape Hessian matrices will be used to prove the local minimality of the regular polygon in the family of simple polygons with a fixed area.

### 3.1.3 Criticality of the regular polygon

**Proposition 3.1.1.** *[22] Let $c > 0$. The problems*

$$(L_1) : \min_{\substack{|P|=|\mathbb{P}_n|,\\ P\in\mathcal{P}_n}} \lambda_1(P),$$
$$(L_2) : \min_{P\in\mathcal{P}_n} |P|\lambda_1(P),$$

*have the same solutions, up to rescalings.*

*Proof.* Let $H_t$ be the homothety of origin O and ratio $t$: $H_t(x) = tx$. Since $H_t \circ \Delta = t^2\Delta \circ H_t$, we get

$$\lambda_n(H_t(P)) = \frac{\lambda_n(P)}{t^2}.$$

The bijective correspondence between solutions of these two problems is given by:

- The definition of $(L_1)$ implies every solution of $(L_1)$ is a solution of $(L_2)$,
- If $P$ is a solution of $(L_2)$ with volume $c_1$, then $H_t(P)$ with $t = \sqrt{\frac{|\mathbb{P}_n|}{c_1}}$ is a solution of $(L_1)$.

This implies that the problems $(L_1)$ and $(L_2)$ have the same solutions, up to rescalings. □

Proposition 3.1.1 is important in the sense that this allows us to work with the problem $(L_2)$ in this thesis. This also implies that we can work with the matrix $\mathbf{M}_{ij}^{\lambda}$, which has a simplified expression instead of $\mathcal{N}_{ij}^{\lambda}$.

The following result on criticality of the regular polygon can be shown using ideas from [24].

**Theorem 3.1.6.** *The regular polygon is a critical point for* $\mathbf{x} \to \mathcal{A}(\mathbf{x})\lambda_1(\mathbf{x})$.

*Proof.* Let us denote by $\mathbb{P}_n$ the regular polygon inscribed in the unit circle with $\mathbf{a}_0 = (1, 0)$ and denote by $\lambda_1$ its first Dirichlet eigenvalue. Consider the functions $\phi_i, i = 0, \ldots, n-1$ defined in (3.9) on a symmetric triangulation $\mathcal{T}$, as in Figure 3.1. For $i \in \{0, \ldots, n-1\}$, the components $2i, 2i+1$ of the gradient of $|P|\lambda_1(P)$ are given by

$$\lambda_1 \int_{\mathbb{P}_n} \nabla\phi_i + |\mathbb{P}_n|\mathbf{S}_1^{\lambda}\nabla\phi_i. \tag{3.18}$$

Given the symmetry of the first eigenfunction and the symmetric triangulation we consider for $\mathbb{P}_n$, it is enough to compute the gradient for $i = 0$.

Using the definition of the functions $\phi_i$ from (3.9), we have $\phi_0 = (1, -1/\tan(\theta))1_{T_+} + (1, 1/\tan(\theta))1_{T_-}(T_0 = T_+, T_- = T_{n-1})$. This implies that

$$\lambda_1 \int_{\mathbb{P}_n} \nabla\phi_0 = \frac{2\lambda_1}{n}\begin{pmatrix}1\\0\end{pmatrix}. \tag{3.19}$$

Using the expression of $\nabla\lambda_1(x)$ and the expression

$$\int_{T_j} \nabla u_1 \cdot \nabla v = \lambda_1 \int_{T_j} u_1 v, \quad \forall v \in H_0^1(\mathbb{P}_n). \tag{3.20}$$

we have

$$|\mathbb{P}_n| \int_{\mathbb{P}_n} \mathbf{S}_1^{\lambda}\nabla\phi_0 = -4|\mathbb{P}_n|\left(\int_{T_+}\left((\partial_x u_1)^2 - \frac{1}{\tan\theta}\partial_x u_1 \partial_y u_1\right)\right). \tag{3.21}$$

Denote $A_{xx} = \int_{T_+}(\partial_x u_1)^2$, $A_{yy} = \int_{T_+}(\partial_y u_1)^2$, $A_{xy} = \int_{T_+}\partial_x u_1 \partial_y u_1$. By the symmetry of the eigenfunction, we have that $A_{xx} + A_{yy} = \lambda_1/n$. When transferred from $T_-$ to $T_+$. the gradients undergo a rotation. This implies the matrix equality

$$R_\theta \begin{pmatrix} A_{xx} & -A_{xy} \\ -A_{xy} & A_{yy} \end{pmatrix} R_\theta^T = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{pmatrix}. \tag{3.22}$$

where $R_\theta$ is given by

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

It is immediate that $-A_{xx}\sin\theta + A_{yy}\sin\theta + 2A_{xy}\cos\theta = 0$. Since $A_{xx} + A_{yy} = \frac{\lambda_1}{n}$ and $-A_{xx}\sin\theta + A_{yy}\sin\theta + 2A_{xy}\cos\theta = 0$, we have

$$\lambda_1 = 2n\left(A_{yy} - \frac{\cos\theta}{\sin\theta}A_{xy}\right). \tag{3.23}$$

Using (3.23) in (3.21), we simplify the expression (3.21) to

$$|\mathbb{P}_n|\mathbf{S}_1^\lambda \nabla\phi_0 = -\frac{2\lambda_1}{n}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.24}$$

Summation of (3.19) and (3.24) shows that the first two components of the gradient of $\mathbf{x} \to \mathcal{A}(\mathbf{x})\lambda_1(\mathbf{x})$ are zero. By symmetry, all the other components are also zero. This proves that $\mathbb{P}_n$ is a critical point of $\lambda_1(\mathbf{x})\mathcal{A}(\mathbf{x})$. □

The above theorem is crucial in showing the proof of the conjecture. Since we have shown that the regular polygon is a critical point, it remains to prove that the regular polygon is a local minimizer of $\lambda_1$ among the family of simple polygons in $\mathbb{R}^2$. This requires a study of the properties of the shape Hessian matrix $\mathbf{M}_{ij}^\lambda$.

## 3.2 Stability of the shape Hessian matrix $\mathrm{M}^{\lambda}$

The primary reference for the following section is [5], supplemented by [17].

As explained in step 2 of the strategy at the beginning of the chapter, in this section we study the continuity of the coefficients of the shape Hessian matrix $\mathcal{N}_{ij}^{\lambda}$. We also estimate the modulus of continuity of the coefficients of $\mathcal{N}_{ij}^{\lambda}$ in this section, which would be used to show the proof of the conjecture in Chapter 5.

### 3.2.1 Basic quantitative estimates of eigenvalues and eigenfunctions

In this section, we estimate the variation of the first Dirichlet eigenvalue $\lambda_1$ and the corresponding eigenfunction $u_1$ for the regular polygon $\mathbb{P}_n$ and a perturbed polygon $P$.

Let $\Omega \in \mathbb{R}^2$ be a bounded Lipschitz connected domain and $f \in H^{-1}(\mathbb{R}^2)$. We consider the problem

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Let $\Omega_\alpha$ , $\alpha \in \{a, b\}$ be two such domains. We denote by $u_\alpha$ the solution of (3.25) on $\Omega_\alpha$ for the right hand side $f_\alpha$. In case $f_\alpha = 1$, we denote by $w_\alpha$ the solution of (3.25) and $w_\alpha$ is called the torsion function. We denote by $u_{1,\alpha}$ the $L^2$ normalized eigenfunctions on $\Omega_\alpha$ corresponding to the Dirichlet eigenvalue $\lambda_{1,\alpha}$.

We initially consider the estimates in the case of domains with lesser regularity, like domains with uniform $(\rho, \varepsilon)$ (see Definition 2.6.2) cone condition from [17]. We use the convexity of polygons to deduce the estimates for eigenfunctions and eigenvalues in our case. Assume that $\Omega_a, \Omega_b \in \mathbb{R}^2$ satisfy uniform $(\rho, \varepsilon)$ cone condition [17]. Let us denote by $d_H(\Omega_1, \Omega_2)$ the Hausdorff distance between the sets $\Omega_1$ and $\Omega_2$.

**Proposition 3.2.1.** *Let $\Omega_a, \Omega_b \subseteq \mathbb{R}^2$ satisfy the uniform $(\rho, \varepsilon)$ condition. Suppose we consider $f_a = f_b := f \in L^2(\mathbb{R}^2)$. There exists a constant $C$ such that the following estimates* *hold:*

$$\int_{\mathbb{R}^2} |\nabla u_a - \nabla u_b|^2 \leq C\|f\|_{L^2}^{\frac{1}{2}}\|f\|_{H^{-1}}^{\frac{1}{2}} \left(\frac{d_H(\Omega_a,\Omega_b)}{\rho\sin\varepsilon}\right)^{\frac{1}{2}}. \tag{3.26}$$

$$\int_{\mathbb{R}^2} |u_a - u_b|^2 \leq C\|f\|_{H^{-1}} \left(\frac{d_H(\Omega_a,\Omega_b)}{\rho\sin\varepsilon}\right)^{\frac{1}{2}}. \tag{3.27}$$

*where $C$ depends only on the diameters and $d_H$ denotes the Hausdorff distance (see Definition 2.5.1).*

A detailed proof of the estimates (3.26) and (3.27) is given by Savaré-Schimperna [17]. These estimates would be used in our estimates of eigenfunctions.

Before proceeding on to the proof for estimates of eigenvalues and eigenfunctions in the case of polygons, we consider a proof for estimates of functions in the case where the sets are convex and bounded, in which case the level sets of the eigenfunctions and the torsion function are convex.

**Lemma 3.2.1.** *Let $\Omega_a, \Omega_b$ be convex bounded open sets. Suppose $f_a = f_b = f \in L^\infty(\mathbb{R}^2), f \geq 0$. Then we have*

$$\|\nabla v_a - \nabla v_b\|_{L^2(\mathbb{R}^2)}\, dx \leq C\left(|\Omega_\alpha|\; diam(\Omega_a)| + |\Omega_b|\; diam(\Omega_b)|\right), \tag{3.28}$$

*where $C := d_H(\partial\Omega_\alpha, \partial\Omega_b)\|f\|_\infty^2$.*

*Proof.* Let $\hat{\Omega} = \Omega_\alpha \cap \Omega_b$. This implies that $d_H(\partial\hat{\Omega}, \partial\Omega_\alpha) \leq d_H(\partial\Omega_\alpha, \partial\Omega_b)$. Let us denote by $\tilde{v}$ the solution of (3.25) in $\hat{\Omega}$. Then, we get,

$$\int_{\Omega_\alpha} |\tilde{v} - v_\alpha|^2\, dx = \int_{\Omega_\alpha} f(\tilde{v} - v_\alpha)\, dx \leq \|f\|_\infty |\Omega_\alpha| \max_{x\in\Omega_\alpha} (v_\alpha(x) - \tilde{v}(x)).$$

To estimate the quantity $\max_{x\in\Omega_\alpha}(v_\alpha(x) - \tilde{v}(x))$., we use the harmonicity of $v_\alpha - \tilde{v}$ on $\hat{\Omega}$. This implies

$$\max_{x\in\hat{\Omega}}(v_\alpha(x) - \tilde{v}(x)) \leq \max_{x\in\partial\Omega_\alpha\oplus B_\varepsilon} v_\alpha(x).$$

where $\varepsilon = d_H(\partial\Omega_a, \partial\Omega_b)$ and $(\partial\Omega_\alpha \oplus B_\varepsilon)$ denotes the Minkowski sum of boundary of $\Omega_\alpha$ and ball of radius $\varepsilon$. In order to estimate $v_\alpha(x)$ , we have

$$v_\alpha(x) \le \|f\|_\infty w_\alpha(x) \le \varepsilon \|f\|_\infty^2 \|\nabla w_\alpha\|_\infty,$$

for every $x \in \partial\Omega_\alpha \oplus B_\varepsilon$ , where $w_\alpha$ is the torsion function.

We get

$$|\nabla w_\alpha(x)| \le W_x/2.$$

where $W_x$ is the width using the classical barrier method. This implies

$$\int_{\Omega_\alpha} |\tilde{v} - v_\alpha|^2 \, dx \le \varepsilon \|f\|_\infty^2 |\Omega_\alpha| \frac{\mathrm{diam}(\Omega_\alpha)}{2}.$$

Summing the estimates for $v_a$ and $v_b$ proves the lemma.

□

The above lemma in particular uses the convexity of the sets $\Omega_a$ and $\Omega_b$. We observe in Proposition 3.2.2 that the convexity of polygons would be used in deriving the estimate of the difference of eigenvalues.

**Perturbations of the regular polygon**.

We denote by $P$ a perturbation of $\mathbb{P}_n$ i.e polygon $\mathbf{a}_0\mathbf{a}_1\ldots\mathbf{a}_{n-1}$ with $n$ sides such that for every $i = 0, \ldots, n-1$ we have $|\mathbf{a}_i\mathbf{a}_i^*| \le \varepsilon$.

We denote by $\lambda_k, \lambda_k^*$ the $k$-th eigenvalues and by $u_k$ and $u_k^*$ the corresponding normalized eigenfunctions on $P, \mathbb{P}_n$, respectively.

**Estimate for eigenvalue difference**:

**Proposition 3.2.2.** *Under the previous hypotheses, we have*

$$|\lambda_1 - \lambda_1^*| \le \int_{\mathbb{R}^2} |\nabla u_1 - \nabla u_1^*|^2 \, dx \le 2\ (L_1 + L_3), \tag{3.29}$$

*where*

$$L_1 = \varepsilon (\lambda_1^*)^2 \, \|u_1^*\|_{L^2}^2 \, (2\pi + 2\pi(1+\varepsilon)^3),$$

$$L_2 = \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(\frac{(r_n+\varepsilon)^4 - r_n^4}{r_n^4}\right) + \frac{2\lambda_2}{\lambda_2-\lambda_1}\left(\frac{L_1}{\lambda_1(\mathbb{P}_n \oplus B_\varepsilon)}\right)^{\frac{1}{2}},$$

$$L_3 = \frac{2\lambda_1}{1+\alpha_1}L_2 + \lambda_1^*\left(1-\left(\frac{r_n}{r_n+\varepsilon}\right)^2\right) + \lambda_1^*\left(\frac{L_1}{\lambda_1(\mathbb{P}_n \oplus B_\varepsilon)}\right)^{\frac{1}{2}}.$$

*where* $\mathbb{P}_n \oplus B_\varepsilon$ *denotes the Minkowski sum of* $\mathbb{P}_n$ *and* $B_\varepsilon$.

*Proof.* We introduce the problem $\phi \in H_0^1(P), -\Delta\phi = \lambda_1^* u_1^*$ in $D'(P)$.

Using Lemma 3.2.1 , we have

$$\int_{\mathbb{R}^2} |\nabla\phi - \nabla u_1^*|^2\, dx \leq L_1. \tag{3.30}$$

We consider the decomposition $\phi = \sum_{i=1}^{+\infty} \alpha_i u_i$ (eigenfunctions constitute a Hilbert basis in $H_0^1(P)$).

We also have the following estimate:

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\phi - \nabla u_1|^2\, dx &= \lambda_1^* \int_{\mathbb{R}^2} u_1^*\phi\, dx - 2\lambda_1 \int_{\mathbb{R}^2} u_1\phi\, dx + \lambda_1 \\ &= \lambda_1^* + \lambda_1^* \int_{\mathbb{R}^2} u_1^*(\phi - u_1^*)\, dx - 2\lambda_1\alpha_1 + \lambda_1 \\ &= 2\lambda_1(1-\alpha_1) + \lambda_1^* - \lambda_1 + \lambda_1^*\|\phi - u_1^*\|_2^2.\end{aligned}$$

In order to estimate the term $(1-\alpha_1)$, we consider the following inequalities:

$$\int_{\mathbb{R}^2} |\nabla\phi|^2\, dx = \lambda_1^* \int_{\mathbb{R}^2} \phi u^* \leq \lambda_1^*\|\phi\|_2 \leq \frac{\lambda_1^*}{(\lambda_1)^{\frac{1}{2}}}\|\nabla\phi\|_2.$$

(We use Poincare's inequality for the second inequality)

$$\sum_i \alpha_i^2\lambda_i = \int_{\mathbb{R}^2} |\nabla\phi|^2 dx.$$

We note the following inclusion

$$\frac{r_n}{r_n+\varepsilon}P \subseteq \mathbb{P}_n \subseteq \frac{r_n}{r_n-\varepsilon}P.$$

Due to the monotonicity of eigenvalues, the inclusion implies

$$\left(\frac{r_n+\varepsilon}{r_n}\right)^2 \lambda_1 \geq \lambda_1^* \geq \left(\frac{r_n-\varepsilon}{r_n}\right)^2 \lambda_1. \tag{3.31}$$

Consequently, we have $\alpha_1^2\lambda_1 + \lambda_2 \sum_{i=2}^{\infty} \alpha_i^2 (\lambda_i^*)^2 \leq \frac{(\lambda_1^*)^2}{\lambda_1} \leq \left(\frac{r_n+\varepsilon}{r_n}\right)^4 \lambda_1$ so

$$\alpha_1^2\lambda_1 + \lambda_2 \left(\int_{\mathbb{R}^2} \phi^2\, dx - \alpha_1^2\right) \leq \left(\frac{r_n+\varepsilon}{r_n}\right)^4 \lambda_1.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^2} \phi^2\, dx &\geq \int_{\mathbb{R}^2} (u_1^*)^2\, dx - 2\int_{\mathbb{R}^2} u_1^*(u_1^* - \phi)\, dx \\ &\geq 1 - 2\|u_1^* - \phi\|_2^2 \geq 1 - 2\left(\frac{L_1}{\lambda_1(\mathbb{P}_n \oplus B_\varepsilon)}\right)^{\frac{1}{2}},\end{aligned}$$

which, after elementary computations leads to

$$1-\alpha_1^2 \leq \frac{\lambda_1}{\lambda_2-\lambda_1}\left(\frac{(r_n+\varepsilon)^4 - r_n^4}{r_n^4}\right) + \frac{2\lambda_2}{\lambda_2-\lambda_1}\left(\frac{L_1}{\lambda_1(\mathbb{P}\oplus B_\varepsilon)}\right)^{\frac{1}{2}} := L_2.$$

Using the above estimate for $(1-\alpha_1^2)$, (3.30) and (3.31), we have

$$\int_{\mathbb{R}^2} |\nabla\phi - \nabla u_1|^2\, dx \leq \frac{2\lambda_1 L_2}{1+\alpha_1} + \lambda_1^*\left(1 - \left(\frac{r_n}{r_n+\varepsilon}\right)^2\right) + \lambda_1^*\left(\frac{L_1}{\lambda_1^*(\mathbb{P}_n \oplus B_\varepsilon)}\right)^{\frac{1}{2}} := L_3. \tag{3.32}$$

By summation of (3.32) and (3.30), the inequality (3.29) follows. □

The above theorem gives an estimate of the difference between eigenvalues, which would be used in estimating the difference $\|N_{ij} - N_{ij}^*\|_{L^\infty}$.

**Estimates for eigenfunction difference**:

**Proposition 3.2.3.** *There exists a constant $C > 0$ such that for all $0 < \varepsilon < \varepsilon_0$*

$$\|\nabla u_1\|_{L^\infty} \leq C \quad \text{and} \quad \|u_1 - u_1^*\|_{L^\infty} \leq C\|u_1 - u_1^*\|_{H^1}^{\frac{1}{3}} \leq L_5,$$

$$\text{where} \quad L_5 := C\left(L_4 + L_1 + L_3 + \frac{L_1}{\lambda_1(\mathbb{P}_n \oplus B_\varepsilon)}\right),$$

*and $L_1, L_2, L_3, L_4$ are given by Proposition 3.2.2.*

*Proof.* The first inequality is a consequence of the barrier method.

The second inequality is a consequence of the Gagliardo-Nirenberg inequality ([33]), given by,

$$\|u_1 - u_1^*\|_{L^\infty} \leq C\|\nabla u_1 - \nabla u_1^*\|_{L^3}^{\frac{2}{3}}\|u_1 - u_1^*\|_{L^3}^{\frac{1}{3}}.$$

Then we use the first inequality and the continuous embedding $H^1(B_2) \subseteq L^3(B_2)$.

We consider the estimates $\|\nabla u_1 - \nabla u_1^*\|_{L^2}, \|\phi - u_1\|_{L^2}, \|\phi - u_1^*\|_{L^2}$ combining which gives the estimate for $\|u_1 - u_1^*\|_{H^1}$.

$$\begin{aligned}
\int_{\mathbb{R}^2} (\phi - u_1)^2\, dx &= (1 - \alpha_1)^2 + \sum_{i=2}^{\infty} \alpha_i^2 = (1 - \alpha_1)^2 + \int_{\mathbb{R}^2} \phi^2\, dx - \alpha_1^2 \\
&\leq (1 - \alpha_1)^2 + (1 + \|\phi - u_1^*\|_2^2) - \alpha_1^2 \\
&\leq 2\ (1 - \alpha_1) + 2\|\phi - u_1^*\|_2 + \|\phi - u_1^*\|_2^2 \\
&\leq \frac{2}{1 + \alpha_1} L_2 + 2\left(\frac{L_1}{\lambda_1(\mathbb{P}_n \oplus B_\varepsilon)}\right)^{\frac{1}{2}} + \frac{L_1}{\lambda_1(\mathbb{P}_n \oplus B_\varepsilon)} := L_4.
\end{aligned}$$

Using Poincare inequality and (3.30) , we get

$$\int_{\mathbb{R}^2} |\phi - u_1^*|^2\, dx \leq \frac{L_1}{\lambda_1(\mathbb{P}_n \oplus B_\varepsilon)}.$$

From Proposition 3.2.2, we have

$$\|\nabla u_1 - \nabla u_1^*\|_{L^2} \leq 2\ (L_1 + L_3).$$

The summation of the above estimates proves the theorem. $\square$

### 3.2.2 Estimates of variation of coefficients of $\mathcal{N}_{ij}^{\lambda}$

The primary reference for the following subsection is [5], supplemented by [11],[10].

Let us denote by $(\mathcal{N}_{ij})$ and $(\mathcal{N}_{ij}^*)$ the coefficients of shape Hessian matrix of the Dirichlet eigenvalue of $P$ and $\mathbb{P}_n$ respectively. The objective is to find an estimate for $\|\mathcal{N}_{ij} - \mathcal{N}_{ij}^*\|_\infty$ and we require $L^\infty$ estimates of $u_1, \lambda_1, \phi, \mathbf{U}_i$ and their gradients for the perturbed polygon $P$ and the regular polygon $\mathbb{P}_n$.

Since the expression of the coefficients of $\mathcal{N}_{ij}$ involves the solutions of elliptic PDEs ((3.16),(3.17)) satisfied by the functions $\mathbf{U}_i$ with data in $H^{-1+\gamma}$, we require quantitative estimates of the perturbation of eigenfunction in $H^2$.

**1. Estimates for terms** $\phi, \nabla\phi, u_1, u_1^*, \nabla u_1, \nabla u_1^*$:

Let $\phi^* : T^* \to \mathbb{R}, \phi : T \to \mathbb{R}$ be the functions defined in (3.10) on a symmetric triangulation $\mathcal{T}$ of $\mathbb{P}_n$ and $P$ respectively. We assume that $\forall i = 0, \dots n-1, |\mathbf{a}_i\mathbf{a}_i^*| \leq \varepsilon$ (which implies $d_H(\partial P, \partial\mathbb{P}_n) \leq \varepsilon$). Then using the definition of $\phi$, we have:

$$\|\phi^*\|_{L^\infty} \leq 1, \quad \|\phi\|_{L^\infty} \leq 1, \quad \|\nabla\phi^*\|_{L^\infty} \leq \frac{1}{2\sin\frac{2\pi}{n}}, \quad \|\nabla\phi\|_{L^\infty} \leq \frac{1}{2\sin\frac{2\pi}{n} - 2\varepsilon},$$

The following inequalities for $u_1, u_1^*$ follow from the fact that the fact that level sets of $u_1, u_1^*$ are convex.

$$\|u_1^*\|_{L^\infty} \leq \sqrt{\lambda_1^*}, \quad \|u_1\|_{L^\infty} \leq \sqrt{\lambda_1},$$

$$\|\nabla u_1^*\|_{L^\infty} \leq (\lambda_1^*)^{\frac{3}{2}}, \quad \|\nabla u_1\|_{L^\infty} \leq (\lambda_1)^{\frac{3}{2}}(1+\varepsilon).$$

**2. Estimate for $\mathbf{U}_i, \mathbf{U}_i^*$**:

From (3.16), we write $\forall i = 0, \dots, n-1$,

$$\begin{cases} -\Delta \mathbf{U}_i - \lambda \mathbf{U}_i = f & \text{in } P \\ \mathbf{U}_i = 0 & \text{on } \partial P \\ \displaystyle\int_P u_1 \mathbf{U}_i \, dx = 0, \end{cases} \tag{3.33}$$

where $f \in H^{-1}(P, \mathbb{R}^2)$ is defined by the right-hand side of (3.16) . The term $f$ consists of the following type of terms (possibly multiplied by other geometric quantities):

$$\lambda_1 u_1 1_T \nabla\phi, \nabla\phi D^2 u, \frac{\partial \phi}{\partial n} \mathcal{H}^1 | L,$$

where $L$ is an edge of $T$ and $n$ is the normal. We note the following :

- It is required to show that the function $\mathbf{U}_i \in H^{1+s}$. This fact will be required to derive the $H^1$ estimate of $(\mathbf{U}_i - \mathbf{U}_i^*)$.
- We show that the eigenfunction $u_1 \in H^{2+s}$ for $s \in [0, 1)$, implying that $D^2 u \in H^s$. The $H^{2+s}$ regularity of $u_1$ is required for proving $H^{1+s}$ regularity of $\mathbf{U}_i$.
- Lemma 3.14 and Lemma 3.15 in [5] are required for the $H^{1+s}$ estimate of $\mathbf{U}_i$.
- It is important to understand the behaviour of $u_1$ in the complement of $P$. To understand the $L^p, H^1$ properties of the extension of $u_1$, it is enough to extend the function by 0 in the complement. However, to understand the $H^2$ property, we extend $u_1$ using the Stein extension operator $E_P$. The objective here is to find an estimate of the form $\|E_P(u_1) - E_{P_n}(u_1^*)\|_{H^2(\mathbb{R}^2)}$.
- It is important to note from [10] that the $H^s$ estimates of $u_1, U_i$ are used for finite element approximation of $u_1, \mathbf{U}_i$ and for using the Aubin-Nitsche lemma.

**2.1 Uniform $H^{2+s}$ regularity of the eigenfunctions**:

We recall the following result from [1].

**Lemma 3.2.2.** *Suppose we denote by $P$ a perturbation of the regular polygon $\mathbb{P}_n$, as above. Let $0 < \gamma \leq \frac{\pi}{\omega_0}$. Then, the solution $v$ of (3.25) in $P$ satisfies*

$$\|v\|_{H^{1+\gamma}(P)} \leq C\|f\|_{H^{-1+\gamma}(P)}.$$

*for every $f \in H^{-1+\gamma}(P)$. The constant $C$ depends on $\gamma$ but is independent of $f$ and $P$. We denote by $\omega_0 = \max \omega_i$, where $\omega_i$ represent the interior angles of the polygon $P$.*

The proof of the lemma involves a study of the equation (3.15) in Section 3.1. A more detailed proof is given by Laurain[29] using the equation (3.15).

Let us denote by $s_0 = \frac{\pi}{\omega_0} - 1$ and let $0 \leq s \leq s_0$.

**Corollary 3.2.1.** *Under the previous hypotheses and notations, we have*

$$u_1 \in H^{2+s}(P), \quad \|u_1\|_{H^{2+s}(P)} \leq C,$$

*with $C$ depends on $s$ but is independent on the perturbation.*

*Proof.* To estimate $\|f\|_{H^s(P)}$, we use the Gagliardo Nirenberg interpolation inequality [20]. In our case, $f = \lambda_1 u_1$. Since we know $H^1$ and $L^2$ estimates of $u_1$, this gives a bound for $f$, thus proving the corollary. □

### 3.Construction of the Stein extension operator

We describe the Stein extension operator from [11] and refer the reader to the same for detailed proof of the observations below.

The operator extends a function $\varphi$ defined on a closed set $\Omega \subset \mathbb{R}^2$ over the whole of $\mathbb{R}^2$. We can extend the function $\varphi$ over the whole of $\mathbb{R}^2$ by defining $\varphi$ to be equal to zero in the complement of the set $\Omega$. However, to make the extension of $\varphi$ more regular, we define a regularized distance function $\Delta(x)$ and we divide the complement of $\Omega$ into a mesh of cubes $Q_k$.

It is important to note that the size of the cubes $Q_k$ depends on the distance of $Q_k$ from the set $\Omega$. We also define the regularized distance function $\Delta(x)$ based on the diameter of the cubes $Q_k$ that the point $x$ belongs to. We also observe that the function $\Delta(x)$ is differentiable and the derivatives of $\Delta(x)$ are bounded.

Define the function $\tau$ by

$$\tau(t) = \frac{e}{\pi t}\mathrm{Im}\left[\exp\left(-(t-1)^{1/4}\exp\left(-\frac{i\pi}{4}\right)\right)\right] \tag{3.34}$$

Then it is immediate that

$$\int_1^\infty \tau(t)\,dt = 1, \quad \text{for } k = 1, 2, \ldots, \tag{3.35}$$

$$\int_1^\infty t^k \tau(t)\,dt = 0, \quad \text{as } t \to \infty \text{ we have } O(t^{-k}). \tag{3.36}$$

Let $c > 0$ be a constant such that

$$\forall (x, y) \in \mathbb{R}^2 \setminus P, \quad c\Delta(x, y) > \phi_j(x) - y, \tag{3.37}$$

The extension operator $E_P$ is defined for $x \in [m_j, M_j]$ and $y < \phi_j(x)$, by

$$E_P(\psi_j u_1)(x, y) = \int_1^\infty \psi_j(x, y + 2c\Delta(x, y)u_1(x, y + 2c\Delta(x, y))\tau(t)\,dt, \tag{3.38}$$

where $\psi_j$ denotes the cutoff function supported in $B(\mathbf{a}_j^*, \frac{3}{4}l_n)$ and $\mathbf{a}_j^*$ is a vertex of $\mathbb{P}_n$.

It is important to note that (3.35) and (3.36) ensure that the integral expression in (3.38) is well-defined. We also observe that the inequality (3.37) ensures that

$$y + 2c\Delta(x, y) > \phi_j(x),$$

which is necessary so that $u_1(x, y + 2c\Delta(x, y))$ is well defined.

We also consider a similar construction for the regular polygon $\mathbb{P}_n$ as well. Further proof for the construction of the Stein extension operator is given by Stein[11],[5].

We denote $\mathbf{U}, \mathbf{U}^*$ the solutions of (3.16) corresponding to $P, \mathbb{P}_n$ respectively. We denote by $\mathcal{N}_{ij}^{\lambda}$ and $(\mathcal{N}_{ij}^{\lambda})^*$ the shape Hessian matrices of $\lambda_1(P)$ and $\lambda_1(\mathbb{P}_n)$ respectively. Without loss of generality, we assume that $P \subseteq \mathbb{P}_n$.

**Lemma 3.2.3.** *For every $s \in [0, \frac{1}{2} \bigwedge (\frac{\pi}{\omega_0} - 1)]$, we have that*

$$\| \boldsymbol{U} \|_{H^{1+s}(P)} \leq C_s,$$

*for a constant $C_s > 0$.*

*Proof.* The case $s = 0$ follows from Poincare inequality :

$$\|\mathbf{U}\|_{H_0^1(P)} \leq \frac{\lambda_2 + 1}{\lambda_2 - \lambda_1} \|f\|_{H^{-1}(P)}.$$

We note that $u_1 \in H^{2+s}(P)$ and all the quantities in $f \in H^{-1}(P, \mathbb{R}^2)$ as defined in (3.17) are controlled for our perturbation, in a norm which is at least $H^{-1+s}$. Using the bound for $\lambda_2 - \lambda_1$ from [4], we get the bound for the case $s = 0$.

For $s > 0$, we use Lemma 3.2.2 to get the bound. $\square$

**Lemma 3.2.4.** *There exist positive constants $\nu, C > 0$ such that for every admissible perturbation*

$$\| \boldsymbol{U} - \boldsymbol{U}^* \|_{H_0^1(B_2, \mathbb{R}^{\not{k}})} \leq C\varepsilon^{\nu}.$$

*Proof.* We introduce the following auxiliary problem

$$\begin{cases} -\Delta V = \lambda^* U^* + f^* \quad \text{in} \quad P \\ V = 0 \quad \text{on} \quad \partial P. \end{cases} \tag{3.39}$$

Using the result from [17], we have

$$\|V - \mathbf{U}^*\|_{L^2} \leq C \|\lambda^* \mathbf{U}^* + f^*\|_{H^{-1}(B_2)} \varepsilon^{\frac{1}{2}}.$$

From lemma 3.2.3, it is clear that both $V, \mathbf{U}^*$ belong to $H^{1+s}$ with controlled norm. Finally, we get,

$$\|V - \mathbf{U}^*\|_{H^1(\mathbb{P}_n)} \leq C\varepsilon^{\nu}.$$

(See [5] for detailed proof of the above estimate.)

We introduce the function $\tilde{V} = V - (\int_P V u_1 dx) u_1 \in H_0^1(P)$. Then we have,

$$\|V - \tilde{V}\|_{H_0^1(P)} \leq C\varepsilon^\nu.$$

Since both $\mathbf{U}, \tilde{V}$ are orthogonal to $u_1$, using Poincare inequality, we get

$$\|\mathbf{U} - \tilde{V}\| \leq \frac{\lambda_2}{\lambda_2 - \lambda_1} \|f - \tilde{f}\|_{H^{-1}}.$$

Summing the estimates proves the lemma. $\square$

**Remark 3.2.1.** *Using Gagliardo - Nirenberg interploation inequality [33], we have*

$$\| \boldsymbol{U} - \boldsymbol{U}^* \|_{L^\infty} \leq C \| \boldsymbol{U} - \boldsymbol{U}^* \|_{H^1}^{\frac{1}{3}},$$

*for some* $C > 0$. *(See Proposition 3.2.3)*

**Theorem 3.2.1.** *Let us consider a polygon* $P \in \mathbb{P}_n$ *such that* $|a_i a_i^*| \leq \varepsilon \leq \varepsilon_0$. *Then there exists* $C, \nu > 0$ *such that, we have,*

$$\|\mathcal{N}_{ij}^\lambda - (\mathcal{N}_{ij}^\lambda)^*\|_{L^\infty} \leq C\varepsilon^\nu,$$

$$\forall k = 1, \dots, 2n, \ |\lambda_k(\mathcal{N}^\lambda) - \lambda_k((\mathcal{N}^\lambda)^*)| \leq C\varepsilon^\nu.$$

*Proof.* The first inequality follows by using the estimates of $\lambda_1, \lambda_1^*, u_1, u_1^*$ and Remark 3.2.1. The second inequality follows from Weyl inequality on the stability of eigenvalues of a Hermitian matrix under perturbation. $\square$

**Remark 3.2.2.** *We know that the Hessian matrix of the area of a polygon is constant. Hence, a similar estimate holds for the coefficients of the shape Hessian* $\boldsymbol{M}^\lambda$.

Theorem 3.2.1 essentially gives a modulus of continuity for the coefficients of the shape Hessian, depending on $C, \nu$. This is important in proving that the regular polygon is a global minimizer, assuming the conjecture is true.

# Chapter 4

# Study of eigenvalues of $\mathrm{M}^\lambda$

The primary reference for this chapter is [5], supplemented by [41].

This chapter deals with the study of eigenvalues of $\mathbf{M}^\lambda$, the shape Hessian matrix of the functional $|P|\lambda_1(P)$, for a polygon P.

## 4.1 Computation of eigenvalues of $\mathrm{M}^\lambda$

We assume the notation $\lambda(\mathbf{x})$ and $\mathcal{A}(x)$ to denote the first Dirichlet eigenvalue of a polygon and the area of a polygon, respectively, with vertex coordinates given by $\mathbf{x} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1})$. We denote by $\mathcal{N}^\lambda$ and $\mathbf{M}^\lambda$, the shape Hessian matrix of $\lambda_1(P)$ and $|P|\lambda_1(P)$, respectively, for a polygon $P$.

We denote by $\mathbf{t}_x = (1, 0, \ldots, 1, 0) \in \mathbb{R}^{2n}$, $\mathbf{t}_y = (0, 1, \ldots, 0, 1) \in \mathbb{R}^{2n}$.

**Proposition 4.1.1.** *The vectors $\boldsymbol{t}_x, \boldsymbol{t}_y \in \mathbb{R}^{2n}$ are eigenvectors of $\boldsymbol{M}^\lambda$ corresponding to the zero eigenvalue.*

*Proof.* Simplifying the Hessian of the product $\mathcal{A}(\mathbf{x})\lambda(\mathbf{x})$, we have

$$\mathrm{Hess}(\lambda_1(\mathbf{x})\mathcal{A}(\mathbf{x})) = (\mathcal{A}(\mathbf{x})\,\mathrm{Hess}\,\lambda_1(\mathbf{x})) + (\nabla\lambda_1(\mathbf{x})\otimes\nabla\mathcal{A}(\mathbf{x})) + (\nabla\mathcal{A}(\mathbf{x})\otimes\nabla\lambda_1(\mathbf{x})) + (\lambda_1(\mathbf{x})\mathrm{Hess}\,\mathcal{A}(\mathbf{x})).$$

Suppose we consider a triangulation of the polygon with no interior vertices as in Figure 3.1. Then $\sum_{i=1}^{n} \phi_i = 1$, which implies that $\sum_{i=0}^{n} \nabla\phi_i = 0$. From Theorem 3.1.3, we have

$$\sum_{i=0}^{n} \mathbf{S}_1^{\lambda} \nabla\phi_i = 0,$$

implying $\nabla\lambda(\mathbf{x}) \cdot \mathbf{t}_x = \nabla\lambda(\mathbf{x}) \cdot \mathbf{t}_y = 0$.

By the same choice of triangulation for the definition of $\phi_i$, we get that $\sum_{i=1}^{n} \mathbf{U}_i = 0$. We immediately have $\mathcal{N}^{\lambda}\mathbf{t}_x = \mathcal{N}^{\lambda}\mathbf{t}_y = 0$ from Theorem 3.1.4.

It remains to show that the vectors $\mathbf{t}_x$ and $\mathbf{t}_y$ are eigenvectors of the Hessian matrix of $\mathcal{A}(\mathbf{x})$. The Hessian matrix of the area of the polygon is given by the constant $2n \times 2n$ block matrix

$$D^2\mathcal{A}(\mathbf{x}) = (\mathbf{A}_{ij})_{0 \le i,j \le n-1},$$

where the non-zero $2 \times 2$ blocks are explicitly given by $\mathbf{A}_{ij} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$ if $j = i+1$ and $\mathbf{A}_{ij} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$ if $i = j+1$.

From the expression of Hessian of area, $\mathbf{t}_x$ and $\mathbf{t}_y$ are in the kernel of the Hessian matrix of area of a polygon. This finishes the proof of the proposition. $\square$

A similar proof of the fact that $\mathbf{r} = (1, 0, \cos(\frac{2\pi}{n}), \sin(\frac{2\pi}{n}), \ldots, \sin(\frac{2(n-1)\pi}{n}))$ and $\mathbf{s} = (0, -1, \sin(\frac{2\pi}{n}), -\cos(\frac{2\pi}{n}), \ldots, -\cos(\frac{2(n-1)\pi}{n}))$ are eigenvectors of $\mathbf{M}^{\lambda}$ corresponding to the zero eigenvalues is given in ([5], Proposition 4.12).

We denote by $\mathbb{P}_n = [\mathbf{a}_0\mathbf{a}_1 \ldots \mathbf{a}_{n-1}]$ the regular polygon with $n$- sides, centred at the origin, with the vertex $\mathbf{a}_0$ at the point $(1, 0)$. We also denote by $\lambda_1 := \lambda_1(\mathbb{P}_n)$ its first Dirichlet eigenvalue and $u_1 := u_1(\mathbb{P}_n)$, the $L^2$ normalized eigenfunction, corresponding to the eigenvalue $\lambda_1$. We consider $\theta = 2\pi/n$ in this chapter.

We assume that the functions $\phi_j$ in (3.10) are defined on a symmetric triangulation $\mathcal{T}$ (see Figure 3.1) and is made of the triangles $T_j$ having vertices $(0,0), (\cos j\theta, \sin j\theta), (\cos(j+$

$1)\theta, \sin(j+1)\theta)$, $0 \leq j \leq n-1$. We also use the notation $T_+ = T_0, T_- = T_{n-1}$.

### Change of basis

We perform a change of basis on the matrix $\mathbf{M}^\lambda$ such that for each vertex, the basis directions correspond to the radial and tangential directions (see Figure 3 in [5]). The change of basis gives more information about the eigenvalues of $\mathbf{M}^\lambda$.

The Hessian matrix in the new basis is given by the formula $\mathbf{H}^\lambda = \mathbf{T}^T\mathbf{M}^\lambda\mathbf{T}$, where $\mathbf{T} = (\mathbf{T}_{ij})_{1\leq i,j\leq n}$ is a $2\times 2$ block matrix with $\mathbf{T}_{jj} = \begin{pmatrix} \cos(j-1)\theta & -\sin(j-1)\theta \\ \sin(j-1)\theta & \cos(j-1)\theta \end{pmatrix}$. This naturally implies that $\mathbf{M}^\lambda$ and $\mathbf{H}^\lambda$ have same eigenvalues.

We also observe that the resulting matrix $\mathbf{H}^\lambda$ is circulant with respect to its $2\times 2$ blocks:

$$\mathbf{H}^\lambda = \begin{pmatrix} \mathbf{H}_0 & \mathbf{H}_1 & \dots & \mathbf{H}_{n-1} \\ \mathbf{H}_{n-1} & \mathbf{H}_0 & \dots & \mathbf{H}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_0 \end{pmatrix}. \tag{4.1}$$

The spectrum of the matrix $\mathbf{H}^\lambda$ is made of the union of spectra of the following $n$ matrices of size $2\times 2$:

$$\mathbf{B}_{\rho_k} = \mathbf{H}_0 + \rho_k\mathbf{H}_1 + \rho_k^2\mathbf{H}_2 + \cdots + \rho_k^{n-1}\mathbf{H}_{n-1}. \tag{4.2}$$

where $\rho_k = \exp(ik\theta), k = 0, \dots, n-1$. This follows from [25] and references therein. Let us denote by $\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_{n-1}$ the blocks of the first line in $\mathbf{M}^\lambda$. Then, we have

$$\mathbf{B}_{\rho_k} = \mathbf{M}_0 + \mathbf{M}_1(\rho_k\mathbf{R}_\theta) + \cdots + \mathbf{M}_{n-1}(\rho_k\mathbf{R}_\theta)^{n-1}.$$

where $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ denotes the rotation matrix around the origin with angle $\theta$ in the trigononmetric sense.

Using the expression of $\mathbf{M}^{\lambda}$ (see Theorem 3.1.5), we decompose each one of the blocks $\mathbf{M}_i = \mathbf{M}_i^1 + \mathbf{M}_i^2 + \mathbf{M}_i^3$ with

$$\mathbf{M}_i^1 = -2|\mathbb{P}_n| \begin{bmatrix} a(U_0^1, U_i^1) & a(U_0^1, U_i^2) \\ a(U_0^2, U_i^1) & a(U_0^2, U_i^2) \end{bmatrix},$$

$$\mathbf{M}_i^2 = -\lambda_1 \int_{\mathbb{P}_n} [(\nabla\phi_0 \otimes \nabla\phi_i) - (\nabla\phi_i \otimes \nabla\phi_0)],$$

$$\mathbf{M}_i^3 = 2|\mathbb{P}_n| \int_{\mathbb{P}_n} (\nabla\phi_0 \cdot \nabla\phi_i)(\nabla u_1 \otimes \nabla u_1).$$

where

$$a(U_i^1, U_j^1) := \int_{\mathbb{P}_n} (DU_i^1 DU_j^1 + \lambda_1 U_i^1 U_j^1)$$

for $i, j = 0, \ldots, n-1$ .

In the following, we compute separately the matrices $\mathbf{B}_{\rho_k}^l = \sum_{i=0}^{n-1} \rho_k^i \mathbf{M}_i^l \mathbf{R}_{i\theta}$, for $l = 1, 2, 3$.

**Case** $l = 1$:

We denote by $U_i = (U_i^1, U_i^2)$ the solution of (3.16), with the normalization (3.17). Since the triangulation defining $\phi_i$ is symmetric, we have

$$\mathbf{U}_i(x) = \mathbf{R}_{i\theta}\mathbf{U}_0(\mathbf{R}_{i\theta}^T x).$$

For $0 \le i \le n-1$, we have

$$\mathbf{M}_i^1 \mathbf{R}_{i\theta} = \begin{pmatrix} a(U_0^1, U_i^1) & a(U_0^1, U_i^2) \\ a(U_0^2, U_i^1) & a(U_0^2, U_i^2) \end{pmatrix} \begin{pmatrix} \cos(i\theta) & -\sin(i\theta) \\ \sin(i\theta) & \cos(i\theta) \end{pmatrix} = \begin{pmatrix} a(U_0^1, U_0^1 \circ \mathbf{R}_{i\theta}^T) & a(U_0^1, U_0^2 \circ \mathbf{R}_{i\theta}^T) \\ a(U_0^2, U_0^1 \circ \mathbf{R}_{i\theta}^T) & a(U_0^2, U_0^2 \circ \mathbf{R}_{i\theta}^T) \end{pmatrix}.$$

**Case** $l = 2$:

Since the matrices $\mathbf{M}_i^2$ come from the Hessian of the area, we have

$$\mathbf{M}_1^2 = -\lambda_1 \begin{pmatrix} 0 & 0.5 \\ -0.5 & 0 \end{pmatrix}, \ \mathbf{M}_{n-1}^2 = -\mathbf{M}_1^2 \quad \text{and} \quad \mathbf{M}_i^2 = 0 \quad \text{for} \quad i \notin \{1, n-1\}.$$

Therefore, we get

$$\sum_{i=0}^{n-1} \rho_k^i \mathbf{M}_i^2 \mathbf{R}_{i\theta} = \frac{\lambda_1}{2}(\rho_k \mathbf{R}_{\pi/2+\theta} + \overline{\rho}_k \mathbf{R}_{-\pi/2-\theta}).$$

**Case** $l = 3$:

From the definition of $A_{xx}, A_{yy}, A_{xy}$ (see proof of Theorem 3.1.6), we have

$$\mathbf{M}_0^3 = 4|\mathbb{P}_n| \|\nabla\phi_0\|^2 \begin{pmatrix} A_{xx} & 0 \\ 0 & A_{yy} \end{pmatrix},$$

$$\mathbf{M}_1^3 = 2|\mathbb{P}_n| \int_{T_+} (\nabla\phi_0 \cdot \nabla\phi_1) \begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{pmatrix},$$

$$\mathbf{M}_{n-1}^3 = 2|\mathbb{P}_n| \int_{T_-} (\nabla\phi_0 \cdot \nabla\phi_{n-1}) \begin{pmatrix} A_{xx} & -A_{xy} \\ -A_{xy} & A_{yy} \end{pmatrix}.$$

We observe that $\int_{T_+} (\nabla\phi_0 \cdot \nabla\phi_1)_{T_+} = \int_{T_-} (\nabla\phi_0 \cdot \nabla\phi_{n-1}) = -\cos\theta |\nabla\phi_0|^2$. A direct computation shows that $|\mathbb{P}_n| = 0.5n\sin\theta$, $|\nabla\phi_0|_{T_+} = 1/\sin\theta$ and using the relation (3.23), we find that

$$\sum_{i=0}^{n-1} \rho_k^i (\mathbf{M}_i^2 + \mathbf{M}_i^3) \mathbf{R}_{i\theta} = \frac{2n(1-\cos(k\theta))}{\sin\theta} \begin{pmatrix} A_{xx} & 0 \\ 0 & A_{yy} \end{pmatrix}.$$

**Theorem 4.1.1.** *For* $0 \le k \le n-1$ *we have* $B_{\rho_k} = \begin{pmatrix} \alpha_k & i\gamma_k \\ -i\gamma_k & \beta_k \end{pmatrix}$ *with*

$$\alpha_k = q_k \int_{T_0} (\partial_x u_1)^2 - 2|\mathbb{P}_n| \ a(U_0^1, \sum_{i=0}^{n-1} \cos(ik\theta)(\cos(i\theta)U_i^1 + \sin(i\theta)U_i^2)),$$

$$\beta_k = q_k \int_{T_0} (\partial_y u_1)^2 - 2|\mathbb{P}_n| \ a(U_0^2, \sum_{i=0}^{n-1} \cos(ik\theta)(-\sin(i\theta)U_i^1 + \cos(i\theta)U_i^2)),$$

*where*

$$q_k = \frac{2n(1-\cos(k\theta))}{\sin\theta},$$

*and*

$$\gamma_k = -2|\mathbb{P}_n|\ a(U_0^1, \sum_{i=0}^{n-1}(\sin(ik\theta)(-\sin(i\theta)U_i^1 + \cos(i\theta)U_i^2)).$$

From the expression of $B_{\rho_k}$, we obtain the eigenvalues of $B_{\rho_k}$ in terms of $\alpha_k, \beta_k$ and $\gamma_k$ [5]. Consequently, we know that the eigenvalues of the Hessian matrix $\mathbf{M}^\lambda$ are the union of spectra of eigenvalues of matrices $B_{\rho_k}$, for $0 \leq k \leq n-1$.

The explicit expressions of $\alpha_k, \beta_k$ and $\gamma_k$ in terms of $u_1$ and the couple $(U_0^1, U_0^2)$ are given in [[5], Proposition 4.10].

**Proposition 4.1.2.** *If the Hessian matrix $M^\lambda$ evaluated at $\mathbb{P}_n$, has $2n-4$ eigenvalues that are strictly positive, then $\mathbb{P}_n$ is a local minimum among the family of simple polygons $\mathcal{P}_n$ in $\mathbb{R}^2$.*

*Proof.* To prove that the regular polygon is a local minimum, we need to prove that the Hessian matrix associated with the free variables, denoted by $M'$, is positive definite. This implies that it is enough to prove that the eigenvalues of the matrix $M'$ are strictly positive. If we fix the vertices $a_{n-2}$ and $a_{n-1}$, then the matrix $M'$, corresponding to the free variables, would be obtained by removing the last four rows and columns of $M^\lambda$.

An important property of the eigenvalues of $M'$ is that they are bounded below by the eigenvalues of $M^\lambda$ [37]. This implies that the eigenvalues of $M'$ are non-negative. It is enough to prove that $M'$ doesn't have a zero eigenvalue. Suppose we assume that the matrix $M'$ has a zero eigenvalue and we denote the corresponding eigenvector to be $\xi$. We know that if the last four components of $\xi$ are completed by zeros, we get an eigenvector of $M^\lambda$ corresponding to the zero eigenvalues. We know from Proposition 3.1.1 that $\mathbf{t}_x, \mathbf{t}_y, \mathbf{r}$ and $\mathbf{s}$ are eigenvectors of $M^\lambda$ corresponding to the zero eigenvalues. We note that the last four components of $\mathbf{t}_x, \mathbf{t}_y, \mathbf{r}$ and $\mathbf{s}$ are linearly independent in $\mathbb{R}^4$, implying that the last four components cannot be $(0, 0, 0, 0)$.

This implies that our assumption that the matrix $M'$ has a zero eigenvalue is false. Hence, the matrix $M'$ is positive definite, implying that the regular polygon $\mathbb{P}_n$ is a local minimum among the family of polygons $\mathcal{P}_n$ in $\mathbb{R}^2$.

$\square$

Proposition 4.1.2 essentially states that it is enough to show that the eigenvalues of $\mathbf{M}^\lambda$ are strictly positive to prove that the regular polygon is a local minimum.

## 4.2 Finite element approximation of the eigenvalues of $\mathbf{M}^\lambda$

Since the eigenvalues, $\mu_n$, of $\mathbf{M}^\lambda$ are described analytically in Theorem 4.1.1, they do not allow us to show that the eigenvalues are strictly positive. Therefore, we use finite element approximation of the terms $\alpha_k, \beta_k, \gamma_k$ to numerically certify this fact. We consider certified estimates of approximations for $u_1, \lambda_1$ on the regular polygon $\mathbb{P}_n$ using $\mathbf{P_1}$ finite elements (linear finite element space) in the first step. In the next step, we get certified estimates for the approximation of $\mathbf{U}_i$ and in the last step, we get approximation results for coefficients of the Hessian matrix.

We assume the notations and definitions for this section from Chapter 2, Section 2.9. In this section, we denote the linear finite element space by $\mathcal{V}^h$.

**Step 1: Certified approximation of the first eigenpair**

We denote by $\lambda_1$,$\lambda_{1,h}$ the first Dirichlet eigenvalue satisfying the problems (2.20) and (2.21), respectively.

Results in [41] prove that

$$\lambda_{1,h} > \lambda_1 > \frac{\lambda_{1,h}}{1 + C_1^2 h^2 \lambda_{1,h}^2}.$$

where the constant $C_1, h$ are defined in Chapter 2, Section 2.9.

This implies that we have

$$|\lambda_1 - \lambda_{1,h}| \le \lambda_{1,h}^3 C_1^2 h^2/(1 + C_1^2 h^2 \lambda_{1,h}^2). \tag{4.3}$$

**Approximation of** $u_1$:

Let us denote by $p = P_h(u_1)$ ($P_h(u_1)$ is the projection of $u_1$ onto $\mathcal{V}^h$). Then

$$\|\nabla u_1 - \nabla p\|_{L^2} \le C_1 h \|D^2 u_1\|_{L^2} \quad \text{and} \quad \|u_1 - p\|_{L^2} \le C_1 h \|\nabla u - \nabla p\|_{L^2}. \tag{4.4}$$

For $u = u_1 \in H^2$, using $\|D^2 u_1\|_{L^2} = \|\Delta u_1\|_{L^2}$ [[19], Theorem 4.3.1.4], we have

$$\|\nabla u_1 - \nabla p\|_{L^2} \le C_1 h \lambda_1 \quad \text{and} \quad \|u_1 - p\|_{L^2} \le C_1^2 h^2 \lambda_1. \tag{4.5}$$

Let us decompose $p = \alpha u_{1,h} + \widetilde{p}$ where $\int_{\mathbb{P}_n} \widetilde{p} u_{1,h} dx = 0, \alpha \in \mathbb{R}$.

We obtain the error estimate

$$\|\nabla u_1 - \nabla u_{1,h}\|_{L^2} \le \|\nabla u_1 - \nabla p\|_{L^2} + \left(\frac{1 - |\alpha|}{|\alpha|}\right) \|\nabla p\|_{L^2} + \frac{1}{\alpha} \|\nabla \widetilde{p}\|_{L^2}. \tag{4.6}$$

We calculate the following bounds for $\|p\|_{L^2}, \|\nabla p\|_{L^2}$ using the definition of $p, P_h$:

$$\|\nabla p\|_{L^2}^2 = (\nabla u_1, \nabla p)_{\mathbb{P}_n} = \lambda_1 (u_1, p)_{\mathbb{P}_n} \le \lambda_1 \|p\|_{L^2} \le \lambda_1 (\|u_1\|_{L^2} + \|p - u_1\|_{L^2})$$

where $(,)_{\mathbb{P}_n}$ denotes the $L^2$ inner product on $\mathbb{P}_n$. We obtain further quantitative estimates in terms of $h$ from (4.5).

**Bounds for** $\|\widetilde{p}\|_{L^2}, \|\nabla \widetilde{p}\|_{L^2}$ :

Using the definition of $u_1, \widetilde{p}$ and $p$, we have

$$\int_{\mathbb{P}_n} \nabla p \cdot \nabla v_h - \lambda_1 h, \int_{\mathbb{P}_n} p v_h = \int_{\mathbb{P}_n} (\lambda_1 u_1 - \lambda_1 h, p) v_h, \quad \forall v_h \in \mathcal{V}^h.$$

Using the Poincare inequality on the orthogonal of $u_{1,h}$ in $\mathcal{V}^h$, we get

$$\left(1-\frac{\lambda_{1,h}}{\lambda_{2,h}}\right)\int_{\mathbb{P}_n}|\nabla\widetilde{p}|^2dx \leq \frac{\|\lambda_1 u_1-\lambda_{1,h}p\|_{L^2}}{\sqrt{\lambda_{2,h}}}\int_{\mathbb{P}_n}|\nabla\widetilde{p}|^2dx,$$

or

$$\|\overline{p}\|_{L^2} \leq \frac{1}{\lambda_{2,h}^{\frac{1}{2}}}\|\nabla\overline{p}\|_{L^2} \leq \frac{\lambda_{2,h}^{\frac{1}{2}}}{(\lambda_{2,h}-\lambda_{1,h})}(|\lambda_1-\lambda_{1,h}|+\lambda_1\|u_1-p\|_{L^2}). \tag{4.7}$$

where $\lambda_{2,h}$ denotes the second Dirichlet eigenvalue solving the eigenvalue problem (2.21) We obtain further quantitative estimates in terms of $h$ from (4.5) and (4.3).

**Bounds for $\alpha$:**

To conclude, we need bounds for $\alpha$. We have $\int_{\mathbb{P}_n} p^2 = \alpha^2 + \int_{\mathbb{P}_n}\widetilde{p}^2$, which proves that

$$|1-\alpha| \leq |1-\alpha^2| \leq \int_{\mathbb{P}_n}\tilde{p}^2 + \int_{\mathbb{P}_n}(u_1^2-p^2) \leq \int_{\mathbb{P}_n}\overline{p}^2 + \|u_1-p\|_{L^2}(2+\|u_1-p\|_{L^2}).$$

We obtain quantitative estimates from (4.4) and (4.7). Since $\alpha>0$, $\alpha$ can be bounded below, for $h$ small enough.

Similarly, we obtain the $L^2$ error estimate for the first eigenfunction:

$$\|u_1-u_{1,h}\|_{L^2} \leq \|u_1-p\|_{L^2} + \left(\frac{|1-\alpha|}{|\alpha|}\right)\|p\|_{L^2} + \frac{1}{\alpha}\|\widetilde{p}\|_{L^2}. \tag{4.8}$$

**Step 2: Certified approximation of $\mathbf{U}_j$:** We consider the following steps for the approximation:

- Estimates for $\|v-p_v\|_{L^2}$ are considered for $v$ the solution of (3.25) on $\mathbb{P}_n$ with $f\in H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)$, where $\gamma\in(0,\frac{1}{2})$ and $p_v := P_h(v)$ the projection of $v$ onto $\mathcal{V}^h$, in Lemma 4.2.1.

- Using the above estimate for $f\in H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)$, we find estimates for $U$ defined by (4.12) on $\mathbb{P}_n$ with $f=f^{reg}+f^{sing}$ with $f^{reg}\in L^2(\mathbb{P}_n)$ and $f^{sing}\in H^{-\frac{1}{2}-\gamma}(\mathbb{P}_n)$ in Theorem 4.2.1.

- We find estimates for $\mathbf{U}_0 = (U_0^1, U_0^2)$ defined in (3.16) with the normalization condition (3.17) and $\mathbf{f}_0 \in H^{-1}(\mathbb{P}_n, \mathbb{R}^2)$ based on estimates found in the previous step.

**Lemma 4.2.1.** *Let $\gamma \in (0, \frac{1}{2})$ and $v$ the solution of (3.25) on $P_n$ with $f \in H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)$. Then*

$$\|\nabla v - \nabla p_v\|_{L^2} \leq \|f\|_{H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)} (C_1 h)^{\frac{1}{2}-\gamma} (1 + \frac{1}{\lambda_1})^{\frac{1}{2}+\gamma}, \tag{4.9}$$

*and*

$$\|v - p_v\|_{L^2} \leq \|f\|_{H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)} (C_1 h)^{\frac{3}{2}-\gamma} (1 + \frac{1}{\lambda_1})^{\frac{1}{2}+\gamma}. \tag{4.10}$$

*Proof.* From the Aubin-Nitsche argument [5], we have

$$\|v - p_v\|_{L^2} \leq C_1 h \|\nabla v - \nabla p_v\|_{L^2}.$$

Since $f \in H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)$, we get

$$\|\nabla v - \nabla p_v\|_{L^2}^2 = (f, v - p_v)_{H^{-1} \times H_0^1} \leq \|f\|_{H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)} \|v - p_v\|_{H^{\frac{1}{2}+\gamma}(\mathbb{R}^2)}.$$

.

Using the Gagliardo-Nirenberg interpolation inequality [20], we have

$$\|\nabla v - \nabla p_v\|_{L^2}^2 \leq \|f\|_{H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)} \|v - p_v\|_{L^2}^{\frac{1}{2}-\gamma} \|v - p_v\|_{H^1(\mathbb{R}^2)}^{\frac{1}{2}+\gamma}.$$

Further using Poincare inequality, we get

$$\|\nabla v - \nabla p_v\|_{L^2}^2 \leq \|f\|_{H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)} (C_1 h)^{\frac{1}{2}-\gamma} (1 + \frac{1}{\lambda_1})^{\frac{1}{2}+\gamma} \|\nabla v - \nabla p_v\|_{L^2}. \tag{4.11}$$

The estimate (4.11) together with the Aubin-Nitsche argument gives an estimate for $\|v - p_v\|_{L^2}$.

□

**Theorem 4.2.1.** *Let $U \in H_0^1(P_n)$ be the solution of*

$$\begin{cases} -\Delta U - \lambda U = f & in\ \mathbb{P}_n, \\ U = 0 & on\ \partial\mathbb{P}_n, \\ \int_{\mathbb{P}_n} uU dx = 0 \end{cases} \tag{4.12}$$

*where $(f, u_1)_{H^{-1},H_0^1} = 0$. Consider $f = f^{reg} + f^{sing}$ with $f^{reg} \in L^2(P_n)$ and $f^{sing} \in H^{-\frac{1}{2}-\gamma}(P_n)$. We denote by $f_h$ a numerical approximation in $H^{-1}$ of $f$ such that $(f_h, u_{1,h})_{H^{-1},H_0^1} = 0$ and $u_{1,h}$ a numerical approximation of $u_1$, in $H_0^1(P_n)$. Let us denote by $U_h$ the finite element solution in $\mathcal{V}^h$ for*

$$\int_{\mathbb{P}_n} (\nabla U_h \cdot \nabla v - \lambda_{1,h} U_h v) dx = (f_h, v)_{H^{-1}\times H_0^1}, \tag{4.13}$$

*together with the normalization condition,*

$$\int_{\mathbb{P}_n} u_{1,h} U_h dx = 0. \tag{4.14}$$

*Then*

$$\begin{aligned} \|\nabla U - \nabla U_h\|_{L^2} &\leq C_1 h \|\lambda_1 U + f^{reg}\|_{L^2} + \|f^{sing}\|_{H^{-\frac{1}{2}}} (C_1 h)^{\frac{1}{2}-\gamma} \left(1 + \frac{1}{\lambda_1}\right)^{\frac{1}{2}+\gamma} \\ &+ \lambda_{1,h}^{\frac{1}{2}} \left( (C_1 h)^2 \|\lambda_1 U + f^{reg}\|_{L^2} + \|f^{sing}\|_{H^{-\frac{1}{2}-\gamma}(\mathbb{R}^2)} (C_1 h)^{\frac{3}{2}-\gamma} \left(1 + \frac{1}{\lambda_1}\right)^{\frac{1}{2}+\gamma} \right) \\ &+ \lambda_{1,h}^{\frac{1}{2}} (\|V\|_{L^2} \|u_{1,h} - u\|_{L^2}) \\ &+ \frac{\lambda_{2,h}^{\frac{1}{2}}}{\lambda_{2,h} - \lambda_{1,h}} \left( |\lambda_{1,h} - \lambda_1| \|U\|_{L^2} + \lambda_{1,h} \|U - P_h(U)\|_{L^2} + (1 + \lambda_{2,h})^{\frac{1}{2}} \|f - f_h\|_{H^{-1}} \right). \end{aligned}$$

*Proof* We have

$$\|\nabla U - \nabla U_h\|_{L^2} \leq \|\nabla U - \nabla V\|_{L^2} + \|\nabla V - \nabla U_h\|_{L^2},$$

where $V = P_h(U)$ is the projection of $U$ onto $\mathcal{V}^h$.

**Estimate for** $\|\nabla U - \nabla V\|_{L^2}$:

Let us denote $U^{\text{reg}}, U^{\text{sing}}$ to be the solutions of

$$U^{\text{reg}} \in H_0^1(P_n), \quad -\Delta U^{\text{reg}} = \lambda U + f^{\text{reg}}, \quad U^{\text{sing}} \in H_0^1(P_n), \quad -\Delta U^{\text{sing}} = f^{\text{sing}},$$

so that $U = U^{\text{reg}} + U^{\text{sing}}$. We define the functions $V_{reg}, V_{sing}$ such that

$$V^{\text{reg}}, V^{\text{sing}} \in \mathcal{V}^h, \quad V^{\text{reg}} = P_h(U^{\text{reg}}), \quad V^{\text{sing}} = P_h(U^{\text{sing}})$$

and $V^{reg}, V^{sing}$ are the solutions of

$$V^{\text{reg}} \in \mathcal{V}^h, \quad -\Delta V^{\text{reg}} = \lambda U + f^{\text{reg}}, \quad V^{\text{sing}} \in \mathcal{V}^h, \quad -\Delta V^{\text{sing}} = f^{\text{sing}}.$$

Using Lemma 3.2.1, we get,

$$\|\nabla U^{\text{sing}} - \nabla V^{\text{sing}}\|_{L^2} \leq \|f^{\text{sing}}\|_{H^{-\frac{1}{2}-\gamma}} (C_1 h)^{\frac{1}{2}-\gamma} \left(1 + \frac{1}{\lambda_1}\right)^{\frac{1}{2}+\gamma}, \tag{4.15}$$

Similarly, the estimate for $V^{\text{reg}}$ from (4.5) gives

$$\|\nabla U^{\text{reg}} - \nabla V^{\text{reg}}\|_{L^2} \leq C_1 h \|\lambda_1 U + f^{\text{reg}}\|_{L^2}. \tag{4.16}$$

Summing (4.15) and (4.17) gives the estimate for $\|\nabla U - \nabla V\|_{L^2}$.

**Estimate for** $\|\nabla V - \nabla U_h\|_{L^2}$:

$$\|\nabla V - \nabla U_h\|_{L^2} \leq \|\nabla V - \nabla \tilde{V}\|_{L^2} + \|\nabla \tilde{V} - U_h\|_{L^2}$$

where $\overline{V} = V - (\int_{\mathbb{P}_n} V u_{1,h} dx) u_{1,h}$ is the component of $V$ orthogonal to $u_{1,h}$

We consider the above definition for $\overline{V}$ since $U$ is orthogonal to $u_{1,h}$, which follows from

(4.14).We have

$$\|\nabla V - \nabla \overline{V}\|_{L^2} = \lambda_{1,h}^{\frac{1}{2}} \left( \int_{P_n} (V u_{1,h} - U u_1) \right) \leq \lambda_{1,h}^{\frac{1}{2}} \left( \|V - U\|_{L^2} + \|V\|_{L^2} \|u_{1,h} - u_1\|_{L^2} \right). \tag{4.17}$$

We immediately notice that $\overline{V}$ is the solution of

$$\overline{V} \in \mathcal{V}^h, \quad -\Delta \overline{V} - \lambda_{1,h} \widetilde{V} = \lambda_{1,h} U - \lambda_{1,h} V$$

Combining with equation (4.12) and using integration by parts, we get

$$\int_{\mathbb{P}_n} |\nabla \overline{V} - \nabla U_h|^2 dx - \lambda_{1,h} \int_{\mathbb{P}_n} |\overline{V} - U_h|^2 dx = (\lambda_1 U + f - \lambda_{1,h} V - f_h, \overline{V} - U_h)_{H^{-1} \times H_0^1}.$$

By the Poincaré inequality in the orthogonal of $u_{1,h}$ we get

$$\left(1 - \frac{\lambda_{1,h}}{\lambda_{2,h}}\right) \|\nabla \overline{V} - \nabla U_h\|_{L^2(\mathbb{P}_n)} \leq \left(\|\lambda_1 U - \lambda_{1,h} V\|_{L^2} \|\overline{V} - U_h\|_{L^2}\right) + \left(\|f - f_h\|_{H^{-1}} \|\overline{V} - U_h\|_{H^1}\right) \tag{4.18}$$

Summing (4.17) and (4.18) gives the estimate for $\|\nabla V - \nabla U_h\|_{L^2}$.

The proof for the estimate $\|U - U_h\|_{L^2}$ follows a similar procedure and is given in ([5], Theorem 5.3).

It can be seen that the estimates become explicit as soon we have estimates for $\|f\|_{H^{-1}}$, $\|f^{reg}\|_{L^2}$, $\|f - f_h\|_{H^{-1}}$, $\|f^{sing}\|_{H^{-1/2-\gamma}}$, and $L^2$ estimates of $U, \nabla U, V, \nabla V$.

**Estimates for** $U, \nabla U, V, \nabla V$:

Since $U$ is orthogonal to $u_1$, we have

$$\sqrt{\lambda_2} \|U\|_{L^2} \leq \|\nabla U\|_{L^2}$$

using the Poincare inequality

We have the following estimate for $U$:

$$\|U\|_{L^2}^2 - \lambda_1\|U\|_{L^2}^2 \le \|f\|_{H^{-1}}\|U\|_{H^1}.$$

which implies that

$$\sqrt{\lambda_2}\|U\|_{L^2} \le \|\nabla U\|_{L^2} \le \frac{\sqrt{\lambda_2(\lambda_2+1)}}{\lambda_2-\lambda_1}\|f\|_{H^{-1}}.$$

We observe that a similar estimate holds for the discrete approximation as well:

$$\sqrt{\lambda_{2,h}}\|U_h\|_{L^2} \le \|\nabla U_h\|_{L^2} \le \frac{\sqrt{\lambda_{2,h}(\lambda_{2,h}+1)}}{\lambda_{2,h}-\lambda_{1,h}}\|f_h\|_{H^{-1}}.$$

To find bounds for $\nabla V$, we use the fact that $V$ is the projection of $U$ onto the space $\mathcal{V}^h$. Furthermore, to find bounds for $V$, we use the Poincare inequality and the bound for $\nabla V$.

**Estimates for** $\|f - f_h\|_{H^{-1}}$:

In our case, we consider the function $\mathbf{U}_0$ defined in (3.16) with the normalization condition (3.17). We assume the definition of $\phi_i$ as given in (3.9). We denote by $\mathbf{f}_0 \in H^{-1}(\mathbb{P}_n, \mathbb{R}^2)$ the right hand side from (3.16) for $j = 0$. We shall drop the index $i = 0$ from $\mathbf{U}_0, \mathbf{f}_0$ and $\phi_0$ since it is fixed. Let us denote by $T_+ = T_0$ and $T_- = T_{n-1}$ the upper and lower triangles in the symmetric triangulation $\mathcal{T}$ of $\mathbb{P}_n$. Then, we have

$$\nabla\phi = 1_{T_+}\left(1, -\frac{1}{\tan\theta}\right) + 1_{T_-}\left(1, \frac{1}{\tan\theta}\right) \quad \text{and} \quad \forall v \in P_n, \quad \|\nabla\phi(x)\| = \frac{1}{\sin(\theta)}1_{T_+\cup T_-}(x).$$

Let us denote by $\mathbf{f} \in H^{-1}(P_n, \mathbb{R}^2)$ the distribution given by

$$\langle f, v\rangle_{H^{-1}\times H_0^1} = \int_{P_n} \left(-(\nabla\phi\otimes\nabla u_1)\nabla v + (\nabla u_1 \otimes \nabla v)\nabla\phi + (\nabla v\otimes\nabla u_1)\nabla\phi\right)\, dx$$

$$+ \int_{\mathbb{P}_n} \mathbf{S}_1^\lambda \nabla\phi \int_{\mathbb{P}_n} u_1 v \; dx + \lambda_1 \int_{\mathbb{P}_n} u_1 v \nabla\phi \; dx \;\forall v \in C_c^\infty(\mathbb{P}_n).$$

We further simplify the expression using integration by parts and the definition of the term $\mathbf{S}_1^\lambda$.

Therefore, we get

$$\int_{\mathbb{P}_n} \mathbf{S}_1^\lambda \nabla\phi = \int_{\mathbb{P}_n} -2(\nabla u_1 \otimes \nabla u_1)\nabla\phi \, dx = -4\int_{T^+} \begin{pmatrix} (\partial_x u_1)^2 - \frac{1}{\tan\theta}(\partial_x u_1)(\partial_y u_1) \\ 0 \end{pmatrix} dx := \begin{pmatrix} s_1^\lambda \\ 0 \end{pmatrix}.$$

Using the fact that the derivative of $\lambda_1(P)|P|$, when evaluated at $\mathbb{P}_n$ is zero , we get

$$\int_{\mathbb{P}_n} -2(\nabla u_1 \otimes \nabla u_1)\nabla\phi \; dx = \begin{pmatrix} \frac{2\lambda_1}{n} \\ 0 \end{pmatrix},$$

implying

$$s_1^\lambda = -2\lambda_1/n.$$

Since the triangulation defined for $\phi_j$ is symmetric, we observe that the above computations hold for the discrete approximation as well.

Hence, let us denote by $f_h$ the distribution given by: $\forall v \in C_c^\infty(\mathbb{R}^2)$

$$(\mathbf{f}_h, v)_{H^{-1}\times H_0^1} = \int_{\mathbb{P}_n} 2(\nabla v \odot \nabla u_{1,h})\nabla\phi \; dx + s_{1,h}^\lambda \int_{\mathbb{P}_n} u_{1,h} v \; dx. \tag{4.19}$$

We get

$$\|f - f_h\|_{H^{-1}} \leq \frac{2\sqrt{2}}{\sqrt{n\sin\theta}} \int_{\mathbb{P}_n} |\nabla u_1 - \nabla u_{1,h}|^2 \; dx + \sqrt{\frac{1}{1+\lambda_1}}(|s_1^\lambda - s_{1,h}^\lambda| + |s_1^\lambda| \|u_1 - u_{1,h}\|_{L^2}).$$

**Estimate for** $\|f^i\|_{H^{-1}}$:

We have

$$(\mathbf{f}, v)_{H^{-1}, H_0^1} = \begin{pmatrix} (f^1, v)_{H^{-1}\times H_0^1} \\ (f^2, v)_{H^{-1}\times H_0^1} \end{pmatrix}$$

$$= \int_{\mathbb{P}_n} (\nabla\phi \cdot \nabla P(v))\nabla u_1 \; dx + \int_{\mathbb{P}_n} (\nabla\phi \cdot \nabla u_1)\nabla P(v) \; dx.$$

where $P(v) = v - (\int_{\mathbb{P}_n} u_1 v)u_1$ is the $L^2$-projection of $v$ on the orthogonal of $u_1$. For the $H^{-1}$ estimate we have

$$(f^1, v)_{H^{-1},H_0^1} = \int_{T_+\cup T_-} (2\partial_x u_1 \partial_x P(v) + \partial_y \phi(\partial_x u_1 \partial_y P(v)) + \partial_y u_1 \partial_x (P(v)))\,.$$

Which implies that

$$\|f^1\|_{H^{-1}} \leq 2\sqrt{2}\left(\int_{T_+} (\partial_x u_1)^2\right)^{\frac{1}{2}} + \frac{1}{\tan\theta}\sqrt{\frac{2\lambda_1}{n}}.$$

We similarly compute the estimate for the term $f^2$.

**Estimates for** $\|f^i_{reg}\|_{L^2}, \|f^i_{sing}\|_{H^{-\frac{1}{2}-\gamma}}$:

We have

$$(f, v)_{H^{-1},H_0^1} = \begin{pmatrix} (f^1, v)_{H^{-1}\times H_0^1} \\ (f^2, v)_{H^{-1}\times H_0^1} \end{pmatrix}$$

$$= \int_{P_n} (\nabla\phi\cdot\nabla v)\nabla u_1 dx + \int_{P_n} (\nabla\phi\cdot\nabla u_1)\nabla v\; dx + \int_{P_n} u_1 v\; dx \begin{pmatrix} s_1^\lambda \\ 0 \end{pmatrix} := \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} + \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix} + \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}.$$

We perform integration by parts on the first integral expression and express the integral over the region $T_i$, where $T_i$ is a triangle in the symmetric triangulation of $\mathbb{P}_n$, as the regular part $P^i_{reg}$. We express the integral over the boundary of $T_i$ as the singular part $P^i_{sing}$. Using the estimate $\|\nabla(\partial_x u_1)\|_{L^2(T_j)} = \sqrt{\lambda_1}\|\partial_x u_1\|_{L^2(T_j)}$ and $\|\nabla(\partial_y u_1)\|_{L^2(T_j)} = \sqrt{\lambda_1}\|\partial_y u_1\|_{L^2(T_j)}$ from [5] , we have

$$\|P^1_{\text{reg}}\|_{L^2} \leq \sqrt{2}\; L_1\|\partial_x u_1\|_{L^2(T_+)} \quad \text{and} \quad \|P^2_{\text{reg}}\|_{L^2} \leq \sqrt{2}\; L_1\|\partial_y u_1\|_{L^2(T_+)}.$$

where $L_1 := \frac{\sqrt{\lambda_1}}{\sin\theta}$.

For the singular part, using ([5],Remark 5.5) we have

$$\|P^1_{\text{sing}}\|_{H^{-\frac{1}{2}-\gamma}} \leq \frac{1+\cos\theta}{\sin\theta}\sqrt{\lambda_1}\|u_1\|_{L^2(S_0)}C_\gamma, \quad \|P^2_{\text{sing}}\|_{H^{-\frac{1}{2}-\gamma}} \leq 2\sqrt{\lambda_1}\|u_1\|_{L^2(S_0)}C_\gamma.$$

where $S_0 = [0, 1]$.

We again perform integration by parts on the second integral expression denoted by $Q^1, Q^2$ and we again denote the regular parts by $Q^1_{reg}, Q^2_{reg}$ and the singular parts by $Q^1_{sing}, Q^2_{sing}$.

A direct computation gives the following estimates for the regular parts:

$$\|Q^1_{\text{reg}}\|_{L^2} \le \sqrt{2}\; L_1 \|\partial_x u_1\|_{L^2(T_+)} \quad \text{and} \quad \|Q^2_{\text{reg}}\|_{L^2} \le \sqrt{2} L_1 \|\partial_y u_1\|_{L^2(T_-)}.$$

We also get $R^2 = 0$ and $\|R^1\|_{L^2} = |s_1^\lambda| = 2\lambda_1/n$.

Finally, we have

$$\|f^1_{\text{reg}}\|_{L^2} \le 2\sqrt{2}\; L_1 \|\partial_x u_1\|_{L^2(T_+)} + \frac{2\lambda_1}{n} \quad \text{and} \quad \|f^2_{\text{reg}}\|_{L^2} \le 2\sqrt{2} L_1 \|\partial_y u_1\|_{L^2(T_-)}.$$

$$\|f^1_{\text{sing}}\|_{H^{-\frac{1}{2}-\gamma}} \le 2\frac{1+\cos\theta}{\sin\theta}\sqrt{\lambda_1}\|u_1\|_{L^2(S_0)} C_\gamma, \quad \|f^2_{\text{sing}}\|_{H^{-\frac{1}{2}-\gamma}} \le 2\sqrt{\lambda_1}\|u_1\|_{L^2(S_0)} C_\gamma.$$

**Remark 4.2.1.** *We use results from [21] to obtain bounds for terms of the form $f_{sing}$. The result uses trace inequalities on fractional Sobolev spaces.*

**Step 3: Certified approximation of eigenvalues of $M^\lambda$**

We use the notations from Theorem 4.2.1 for two generic problems with solutions $U_a, U_b$ corresponding to the right-hand sides $f^a, f^b$. We denote the bilinear forms

$$a : H^1_0(P_n) \times H^1_0(P_n) \to \mathbb{R}, \quad a(u,v) = \int_{P_n} \nabla u \cdot \nabla v - \lambda_1 \int_{P_n} uv,$$

$$a_h : \mathcal{V}^h \times \mathcal{V}^h \to \mathbb{R}, \quad a_h(u,v) = \int_{P_n} \nabla u \cdot \nabla v - \lambda_{1,h} \int_{P_n} uv.$$

Our objective is to find error estimates of the form

$$|a(U^a, U^b) - a_h(U^a_h, U^b_h)|.$$

in order to get an estimate of order $h^{1-2\gamma}$ for $\alpha_k$, $\beta_k$, $\gamma_k$ in Theorem 4.1.1. By triangle inequality, we have

$$|a(U^a, U^b) - a_h(U^a_h, U^b_h)| \le$$

$$|a(U^a, U^b) - a(V^a, V^b)| + |a(V^a, V^b) - a_h(\widetilde{V}^a, \widetilde{V}^b)| + |a_h(\widetilde{V}^a, \widetilde{V}^b) - a_h(U_h^a, U_h^b)|. \tag{4.20}$$

**First term**

The explicit expression for the first term is given by

$$a(U^a, U^b) - a(V^a, V^b) = \int_{P_n} \nabla(U^a - V^a) \cdot \nabla(U^b - V^b) - \lambda_1 \int_{P_n} [(U^a - V^a)V^b + (U^b - V^b)U^a],$$

so that

$$|a(U^a, U^b) - a(V^a, V^b)| \leq$$

$$\|\nabla(U^a - V^a)\|_{L^2} \|\nabla(U^b - V^b)\|_{L^2} + \lambda_1 \|U^a - V^a\|_{L^2} \|V^b\|_{L^2} + \lambda_1 \|U^a\|_{L^2} \|U^b - V^b\|_{L^2}.$$

Lemma 4.2.1 applied to the $L^2$-norms of both the functions and their gradients give control in $h^{1-2\gamma}$.

We also find estimates for the second term and third term in (4.20) similar to the estimation of the first term. We use Lemma 4.2.1 again to find estimates of terms of the form $\|\nabla(U^a - V^a)\|_{L^2}$, where $V^a$ represents the projection of $U^a$ onto $\mathcal{V}^h$.

We denote by

$$W^{\alpha_k} = \sum_{i=0}^{n-1} \cos(ik\theta) U_0^1 \circ \mathbf{R}_{i\theta}^T$$

$$W^{\beta_k} = \sum_{i=0}^{n-1} \cos(ik\theta) U_0^2 \circ \mathbf{R}_{i\theta}^T$$

$$W^{\gamma_k,1} = \sum_{i=0}^{n-1} \sin(ik\theta) U_0^2 \circ \mathbf{R}_{i\theta}^T$$

$$W^{\gamma_k,2} = \sum_{i=0}^{n-1} \sin(ik\theta) U_0^1 \circ \mathbf{R}_{i\theta}^T$$

so that $a(U_0^1, W^{\alpha_k}), a(U_0^1, W^{\beta_k}), a(U_0^1, W^{\gamma_k,1}), a(U_0^1, W^{\gamma_k,2})$ allow us to express $\alpha_k, \beta_k$ and $\gamma_k$ respectively.

Denote by $f^{\alpha_k} \in H^{-1}$ the distribution

$$\langle f^{\alpha_k}, v\rangle_{H^{-1},H_0^1} = \sum_{i=0}^{n-1}(\cos(i+1)k\theta + \cos(ik\theta))\lambda_1 \int_{T_i} u_1 v$$

$$+\sum_{i=0}^{n-1} \frac{\cos(i+1)k\theta - \cos(ik\theta)}{\sin\theta} \int_{T_i} \begin{pmatrix} -\sin(2i+1)\theta & \cos(2i+1)\theta \\ \cos(2i+1)\theta & \sin(2i+1)\theta \end{pmatrix} \nabla u_1 \cdot \nabla v.$$

We note that $\int_{P_n} W^{\alpha_k} u_1 = 0$ due to the orthogonality of $U_0^1$ on $u_1$. We observe that $(f^{\alpha_k}, u_1) = 0$ by the symmetry of the first eigenfunction $u_1$. These observations imply that $W^{\alpha_k}$ is the unique solution of the problem

$$a(W,v) = \langle f^{\alpha_k}, v\rangle_{H^{-1},H_0^1}, \quad \forall v \in H_0^1(P_n), \int_{P_n} W u_1 = 0.$$

Corresponding to the terms $\beta_k, \gamma_k$, we obtain the distributions $f^{\beta_k}, f^{\gamma_k}$. The explicit expression of the distributions $f^{\beta_k}, f^{\gamma_k}$ are given in [5].

In our case, we consider $U_a = U_0^1$ and $U_b = W^{\alpha_k}, W^{\beta_k}, W^{\gamma_k}$.

In order to find estimates for $\|\nabla U_b - \nabla U_b^h\|_{L^2}$, we use Theorem 4.2.1 where $f$ in the right hand side is replaced by $f^{\alpha_k}, f^{\beta_k}, f^{\gamma_k}$ for $\alpha_k, \beta_k, \gamma_k$ respectively. This implies that we require the following estimates: $\|f^{\alpha_k}\|_{H^{-1}}, \|f^{\alpha_k}_{reg}\|_{L^2}, \|f^{\alpha_k} - f^{\alpha_k}_h\|_{H^{-1}}, \|f^{\alpha_k}_{sing}\|_{H^{-1/2-\gamma}}$, corresponding to the term $\alpha_k$ and similar estimates corresponding to the terms $\beta_k$ and $\gamma_k$.

We find the above estimates in terms of the estimates $|\lambda_1 - \lambda_{1,h}|$ , $\|u_1 - u_{1,h}\|_{L^2}$, which are expressed clearly in terms of the mesh size $h$. We also know from Theorem 4.2.1 that we obtain estimates of order $O(h^{1-2\gamma})$. Hence, we obtain better approximations to the eigenvalues of $\mathbf{M}^\lambda$ as we lower the mesh size $h$.

We find the explicit estimate for the term $\alpha_k$ and the estimates for the terms $\beta_k, \gamma_k$ can be computed similarly.

**The term $\alpha_k$:**

Using Poincare inequality, we get

$$\|v\|_{L^2} \leq \frac{1}{\sqrt{1+\lambda_1}}\|v\|_{H^1}$$

Using the computations in [5] and the above inequality, we get

$$\|f^{\alpha_k}\|_{H^{-1}} \leq \lambda_1\sqrt{\frac{1+\cos(k\theta)}{1+\lambda_1}} + \frac{\sqrt{\lambda_1(1-\cos(k\theta))}}{\sin\theta}.$$

Denote $K_i^{\alpha_k}$ as

$$K_i^{\alpha_k} = \frac{\cos((i+1)k\theta) - \cos(ik\theta)}{\sin\theta}\begin{pmatrix} -\sin(2i+1)\theta & \cos(2i+1)\theta \\ \cos(2i+1)\theta & \sin(2i+1)\theta \end{pmatrix}$$

Then we have

$$\sum_{i=0}^{n-1}\int_{T_i} K_i^{\alpha_k}\nabla u_1 \cdot \nabla v = -\sum_{i=0}^{n-1}\int_{T_i} \operatorname{div}(K_i^{\alpha_k}\nabla u_1)v + \sum_{i=1}^{n-1}\int_{\partial T_i} K_i^{\alpha_k}\nabla u_1 \cdot nv$$

Since $u_1 \in H^2(P_n)$, the first term in the expression is regular. We refer to the second term on the right-hand side of the expression to be the singular part.

For the regular part, using the fact that $\|D^2u_1\|_{L^2} = \lambda_1$ we obtain

$$\|f_{\text{reg}}^{\alpha_k}\|_{L^2} \leq \sqrt{1+\cos(k\theta)}\lambda_1 + \frac{\sqrt{2}}{\sin\theta}\sqrt{1-\cos(k\theta)}\lambda_1.$$

For the singular part, we consider the following. We obtain for $v \in H_0^1(\mathbb{P}_n)$

$$\sum_{i=0}^{n-1}\int_{\partial T_i}(K_i^{\alpha_k}\nabla u_1 \cdot n)v = -\sum_{i=0}^{n-1}\int_{S_i}\frac{\cos\theta}{\sin\theta}2\cos ik\theta(1-\cos k\theta)(\partial_r u_1)v.$$

Using [[5],Remark 5.5], we get

$$\|f_{\text{sing}}^{\alpha_k}\|_{H^{-\frac{1}{2}-\gamma}} \leq 2\sum_{i=0}^{n-1}\frac{\cos\theta}{\sin\theta}|\cos ik\theta(1-\cos k\theta)|\sqrt{\lambda_1}\|u_1\|_{L^2(S_0)}C_\gamma.$$

We consider the discrete version of $f^{\alpha_k}$, replacing $u_1$ and $\lambda_1$ by their discrete approximation

$$\langle f_h^{\alpha_k}, v\rangle_{H^{-1},H_0^1} = \sum_{i=0}^{n-1} \left(\cos((i+1)k\theta) + \cos(ik\theta)\right) \lambda_{1,h} \int_{T_i} u_{1,h} v$$

$$+ \sum_{i=0}^{n-1} \frac{\cos((i+1)k\theta) - \cos(ik\theta)}{\sin\theta} \int_{T_i} \begin{pmatrix} -\sin(2i+1)\theta & \cos(2i+1)\theta \\ \cos(2i+1)\theta & \sin(2i+1)\theta \end{pmatrix} \nabla u_{1,h} \cdot \nabla v.$$

By direct computation, we obtain

$$\langle f_h^{\alpha_k} - f^{\alpha_k}, v\rangle = \sum_{i=0}^{n-1} \left(\cos((i+1)k\theta) + \cos(ik\theta)\right) \int_{T_i} (\lambda_1 u_1 - \lambda_{1,h} u_{1,h}) v$$

$$+ \sum_{i=0}^{n-1} \int_{T_i} K_i^{\alpha_k} (\nabla u - \nabla u_h) \cdot \nabla v,$$

which implies

$$\|f^{\alpha_k} - f_h^{\alpha_k}\|_{H^{-1}} \le \sqrt{\frac{1+\cos k\theta}{1+\lambda_1}} (|\lambda_1 - \lambda_{1,h}| + \lambda_{1,h} \|u_1 - u_{1,h}\|_{L^2}) + \frac{\sqrt{1-\cos k\theta}}{\sin\theta} \|\nabla u - \nabla u_h\|_{L^2}.$$

**The term $\beta_k$:**

Similar computations as before for $\alpha_k$ gives

$$\|f^{\beta_k}\|_{H^{-1}} \le \lambda_1 \frac{\cos\theta}{\sin\theta} \sqrt{\frac{1-\cos(k\theta)}{1+\lambda_1}} + \frac{\sqrt{\lambda_1(1-\cos(k\theta))}}{\sin\theta}.$$

$$\|f_{\text{sing}}^{\beta_k}\|_{H^{-\frac{1}{2}-\gamma}} \le 2 \sum_{i=0}^{n-1} |\cos ik\theta(1-\cos k\theta)| \sqrt{\lambda_1} \|u_1\|_{L^2(S_0)} C_\gamma.$$

$$\|f_{\text{reg}}^{\beta_k}\|_{L^2} \le \frac{\cos\theta}{\sin\theta} \sqrt{1-\cos(k\theta)}\lambda_1 + \frac{\sqrt{2}}{\sin\theta} \sqrt{1-\cos(k\theta)}\lambda_1.$$

$$\|f^{\beta_k} - f_h^{\beta_k}\|_{H^{-1}} \le \frac{\cos\theta}{\sin\theta} \sqrt{\frac{1-\cos k\theta}{1+\lambda_1}} \left(|\lambda_1 - \lambda_{1,h}| + \lambda_{1,h} \|u_1 - u_{1,h}\|_{L^2}\right)$$

$$+ \frac{\sqrt{1-\cos k\theta}}{\sin\theta} \|\nabla u - \nabla u_h\|_{L^2}.$$

**The term $\gamma_k$**

We perform computations similar to $\alpha_k$ for $\gamma_k^1$, which gives

$$\|f^{\gamma_k,1}\|_{H^{-1}} \leq \lambda_1 \frac{\cos\theta}{\sin\theta}\sqrt{\frac{1-\cos(k\theta)}{1+\lambda_1}} + \frac{\sqrt{\lambda_1(1-\cos(k\theta))}}{\sin\theta}.$$

$$\|f^{\gamma_k,1}_{\text{sing}}\|_{H^{-\frac{1}{2}-\gamma}} \leq 2\sum_{i=0}^{n-1} |\sin ik\theta(1-\cos k\theta)| \sqrt{\lambda_1}\|u_1\|_{L^2(S_0)}C_\gamma.$$

$$\|f^{\gamma_k,1}_{\text{reg}}\|_{L^2} \leq \frac{\cos\theta}{\sin\theta}\sqrt{1-\cos(k\theta)}\lambda_1 + \frac{\sqrt{2}}{\sin\theta}\sqrt{1-\cos(k\theta)}\lambda_1.$$

$$\begin{aligned}\|f^{\gamma_k,1} - f_h^{\gamma_k,1}\|_{H^{-1}} \leq{}& \frac{\cos\theta}{\sin\theta}\sqrt{\frac{1-\cos k\theta}{1+\lambda_1}}\left(|\lambda_1-\lambda_{1,h}| + \lambda_{1,h}\|u_1-u_{1,h}\|_{L^2}\right)\\ &+\frac{\sqrt{1-\cos k\theta}}{\sin\theta}\|\nabla u - \nabla u_h\|_{L^2}.\end{aligned}$$

Similar computations as before for $\alpha_k$ gives the following estimates for $\gamma_k^2$:

$$\|f^{\gamma_k,2}\|_{H^{-1}} \leq \lambda_1\sqrt{\frac{1+\cos(k\theta)}{1+\lambda_1}} + \frac{\sqrt{\lambda_1(1-\cos(k\theta))}}{\sin\theta}.$$

$$\|f^{\gamma_k,2}_{\text{sing}}\|_{H^{-\frac{1}{2}-\gamma}} \leq 2\sum_{i=0}^{n-1} \frac{\cos\theta}{\sin\theta}|\sin ik\theta(1-\cos k\theta)| \sqrt{\lambda_1}\|u_1\|_{L^2(S_0)}C_\gamma.$$

$$\|f^{\gamma_k,2}_{\text{reg}}\|_{L^2} \leq \sqrt{1+\cos(k\theta)}\lambda_1 + \frac{\sqrt{2}}{\sin\theta}\sqrt{1-\cos(k\theta)}\lambda_1.$$

$$\|f_h^{\gamma_k,2} - f^{\gamma_k,2}\|_{H^{-1}} \leq \sqrt{\frac{1+\cos k\theta}{1+\lambda_1}}\left(|\lambda_1-\lambda_{1,h}| + \|u_1-u_{1,h}\|_{L^2}\right) + \frac{\sqrt{1-\cos k\theta}}{\sin\theta}\|\nabla u - \nabla u_h\|_{L^2}.$$

For detailed calculations, see ([5], section 5.3).

We conclude with the following theorem:

**Theorem 4.2.2.** *The terms $\alpha_k, \beta_k, \gamma_k$, given by Theorem 4.1.1 admit an error estimate of order $O(h^{1-2\gamma})$ for every $\gamma \in (0, 1/2)$,* We prove in the next chapter that these error estimates are essential to finding intervals in which the eigenvalues belong. We decrease the mesh size $h$ until we find an interval containing the eigenvalues that don't contain zero.

# Chapter 5

# Proof of the conjecture

The primary reference for this chapter is [5], supplemented by [7]. This chapter explains the numerical computations required to approximate the eigenvalues of $\mathbf{M}^{\lambda}$ and the strategy to prove the conjecture.

## 5.1 Numerical simulations

We denote by $\mathbb{P}_n$ the regular polygon with $n$ sides inscribed in the unit circle with a vertex at $(1,0)$. We consider the symmetric triangulation of $\mathbb{P}_n$, $\mathcal{T}^h$, that we constructed earlier for the definition of $\phi_i$ as in Figure 3.1. We divide each triangle $T_j, j = 0, \ldots, n-1$ in the symmetric triangulation into a mesh of triangles similar to $\frac{1}{m}T_j$, given an integer $m \geq 1$. In this way, we construct a mesh $T^h$ of median length $h = 1/m$. Some examples of meshes constructed for $n = 5, 6, 7$ are given in Figure 5.1. The figure is taken from [5].

Using the mesh $T^h$, we compute using $\mathcal{P}_1$ finite elements (linear finite element space):

- The first two eigenvalues $\lambda_{1,h}, \lambda_{2,h}$ of the discrete Dirichlet eigenvalue problem (2.21).
- The first eigenfunction $u_{1,h}$ of the discrete Dirichlet eigenvalue problem (2.21)

- The solutions $\mathbf{U}_h^j$ of

$$\int_{\mathbb{P}_n} D\mathbf{U}_h \nabla v - \lambda_{1,h} \int_{\mathbb{P}_n} \mathbf{U}_h v = (\mathbf{f}_h, v)_{H^{-1},H_0^1}$$

  where the distributions $\mathbf{f}_h^j, j = 0, \ldots, n-1$ are given by (4.19). The normalization condition is given by $\int_{\mathbb{P}_n} \mathbf{U}_h^j u_{1,h} = 0$.

- The approximations of $W^{\alpha_k}, W^{\beta_k}, W^{\gamma_k,1}, W^{\gamma_k,2}$ for $1 \leq k \leq n-1$ are obtained using $(U_{j,h}^1, U_{j,h}^2)$. We obtain the discrete approximations of $\alpha_k, \beta_k, \gamma_k$ that are of order $O(h^{1-2\gamma})$ for $\gamma \in (0, 0.5)$.

The procedure described above provides for each interval $I_{\alpha_k}, I_{\beta_k}, I_{\gamma_k}$ and for each $k = 1, \ldots, n-1$ for which $\alpha_k \in I_{\alpha_k}, \beta_k \in I_{\beta_k}$ and $\gamma_k \in I_{\gamma_k}$. Using the tool Intlab [[39]], we find intervals $I_j(h, \gamma)$ that contain the eigenvalues of $\mathbf{M}^\lambda$, $\mu_j, 0 \leq j \leq 2n-1$ described in Theorem 4.2.1. The exponent $\gamma$ appears only in the choice of constants and exponents. We choose $\gamma$ appropriately using a grid search. The intervals $I_j(h, \gamma)$, for $\gamma \in (0, 0.5)$, depend on the value of mesh size $h$ chosen to construct the mesh $T^h$. If among the intervals $I_j(h, \gamma)$ we obtain only two that contain zero, this implies that the $2n-4$ eigenvalues of $\mathbf{M}^\lambda$ are strictly positive. This shows that the regular polygon $\mathbb{P}_n$ is a local minimum for $P \mapsto |P|\lambda_1(P)$. If this case fails, we decrease the mesh size $h$ and repeat the procedure on a refined mesh constructed on the triangulation $\mathcal{T}^h$.

We require intervals around the eigenvalues small enough such that they do not contain zero. This implies we require rather small values of $h$, which in turn requires large computational power. Therefore, we use the software FreeFEM [13] in its parallel version together with the libraries PETSc[3], SLEPc[40], and Hypre[36]. The computations done are explained in detail in [5]. The results shown in Figure 5.2 suggest that the regular polygon is a local minimizer for $n \in \{5, 6, 7, 8\}$.

In Figure 5.3, we present the non-zero eigenvalues of the Hessian matrix for $9 \leq n \leq 15$. The figure shows that eigenvalues are indeed positive. This implies that the regular polygon is a local minimizer in these cases. We note that Figures 5.2 and 5.3 are taken from [5].

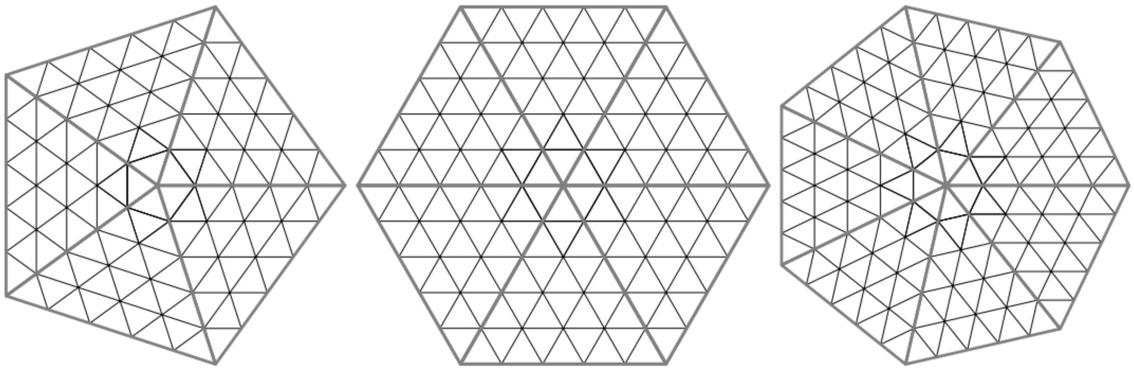

Figure 5.1: Symmetric meshes obtained for regular polygons with $n$ sides, where $n = 5, 6, 7$

| Pentagon | | | |
|---|---|---|---|
| Eig. | l.b. | u.b. | mult. |
| 2.568803 | 2.359297 | 2.784816 | 2 |
| 8.015038 | 7.558395 | 8.460722 | 2 |
| 13.458443 | 13.012758 | 13.915086 | 2 |

| Hexagon | | | |
|---|---|---|---|
| Eig. | l.b. | u.b. | mult. |
| 1.323826 | 1.040291 | 1.629895 | 2 |
| 3.916803 | 3.112218 | 4.719205 | 2 |
| 12.990672 | 12.188270 | 13.795257 | 2 |
| 7.566593 | 6.326083 | 8.803012 | 1 |
| 11.540733 | 10.304314 | 12.781243 | 1 |

| Heptagon | | | |
|---|---|---|---|
| Eig. | l.b. | u.b. | mult. |
| 0.747352 | 0.446026 | 1.096876 | 2 |
| 2.056766 | 0.963449 | 3.148214 | 2 |
| 4.655979 | 3.078862 | 6.228621 | 2 |
| 12.292485 | 10.719843 | 13.869602 | 2 |
| 12.582047 | 11.490599 | 13.675364 | 2 |

| Octagon | | | |
|---|---|---|---|
| Eig. | l.b. | u.b. | mult. |
| 0.452095 | 0.182855 | 0.774247 | 2 |
| 1.171933 | 0.309482 | 2.034382 | 2 |
| 2.772135 | 1.273803 | 4.268064 | 2 |
| 12.049631 | 11.187182 | 12.912082 | 2 |
| 13.037208 | 11.541279 | 14.535540 | 2 |
| 3.999568 | 1.460555 | 6.536411 | 1 |
| 11.740713 | 9.203870 | 14.279726 | 1 |

Figure 5.2: Numerical approximation of the $2n-4$ non-zero eigenvalues of $\mathbf{M}^\lambda$ for $n \in \{5,6,7,8\}$ together with intervals

| $n = 9$ | mult. |
|---|---|
| 0.2888 | 2 |
| 0.7145 | 2 |
| 1.7104 | 2 |
| 2.8667 | 2 |
| 11.4506 | 2 |
| 12.1695 | 2 |
| 13.4392 | 2 |

| $n = 10$ | mult. |
|---|---|
| 0.1927 | 2 |
| 0.4601 | 2 |
| 1.1017 | 2 |
| 1.9625 | 2 |
| 2.4640 | 1 |
| 10.8361 | 2 |
| 11.9253 | 1 |
| 12.7814 | 2 |
| 13.5487 | 2 |

| $n = 11$ | mult. |
|---|---|
| 0.1334 | 2 |
| 0.3096 | 2 |
| 0.7386 | 2 |
| 1.3501 | 2 |
| 1.9129 | 2 |
| 10.2373 | 2 |
| 12.1968 | 2 |
| 13.2741 | 2 |
| 13.4475 | 2 |

| $n = 12$ | mult. |
|---|---|
| 0.0952 | 2 |
| 0.2160 | 2 |
| 0.5128 | 2 |
| 0.9473 | 2 |
| 1.4287 | 2 |
| 1.6659 | 1 |
| 9.6701 | 2 |
| 12.0620 | 1 |
| 12.6398 | 2 |
| 13.2059 | 2 |
| 13.5861 | 2 |

| $n = 13$ | mult. |
|---|---|
| 0.0697 | 2 |
| 0.1554 | 2 |
| 0.3669 | 2 |
| 0.6801 | 2 |
| 1.0598 | 2 |
| 1.3586 | 2 |
| 9.1413 | 2 |
| 12.2461 | 2 |
| 12.8768 | 2 |
| 13.0664 | 2 |
| 13.7288 | 2 |

| $n = 14$ | mult. |
|---|---|
| 0.0521 | 2 |
| 0.1146 | 2 |
| 0.2694 | 2 |
| 0.4994 | 2 |
| 0.7918 | 2 |
| 1.0742 | 2 |
| 1.1995 | 1 |
| 8.6527 | 2 |
| 12.1611 | 1 |
| 12.4975 | 2 |
| 12.5693 | 2 |
| 13.4024 | 2 |
| 13.7331 | 2 |

| $n = 15$ | mult. |
|---|---|
| 0.0397 | 2 |
| 0.0864 | 2 |
| 0.2022 | 2 |
| 0.3744 | 2 |
| 0.5989 | 2 |
| 0.8406 | 2 |
| 1.0115 | 2 |
| 8.2033 | 2 |
| 12.0933 | 2 |
| 12.2933 | 2 |
| 12.9147 | 2 |
| 13.6292 | 2 |
| 13.6320 | 2 |

Figure 5.3: Numerical approximation of the non-zero eigenvalues of the shape Hessian matrix for $9 \leq n \leq 15$ on meshes of size $h = 10^{-3}$.

## 5.2 Proof of the conjecture

In this section, we prove that the proof of the conjecture can be reduced to a finite number of numerical computations under the implicit assumption that the conjecture is true! Assuming that the area of a polygon with $n$ sides is fixed, we devise a procedure to find a value $D_{max}$ such that if the diameter of the polygon exceeds $D_{max}$, then the polygon cannot be optimal corresponding to our problem. Similarly, we shall also find the minimal edge length, $e_{min}$ and minimum inradius, $r_{min}$ of an optimal polygon. All these results gives a compact set of polygons in $\mathbb{R}^{2n-4}$ outside which any polygon cannot be optimal for $|P|\lambda_1(P)$.

Let us denote by $\overline{\mathcal{P}}_n$ the closure of the class of simple polygons with at most $n$ edges with the Hausdorff distance of the complements. We denote by $Q_n$ the optimal polygon.

We denote

$$l_n^* = \min\{|P|\lambda_1 P : P \in \mathcal{P}_n\},$$

for every $n \geq 3$.

**Theorem 5.2.1.** *For $n \geq 3$, let $P \in \mathcal{P}_n$. There exists a value $D_{max} > 0$ such that if $diam(P) > D_{max}$, then*

$$|P|\lambda_1(P) > l_n^*,$$

*where $D_{max} := 2d^* + (k + n - 2)8C_0$.*

To find an explicit bound for the diameter of an optimal polygon, $D_{max}$, we require the properties of the torsion function $w_\Omega$ of a domain $\Omega$ which we consider below without proofs from [7].

**(0)** : The torsion energy of $\Omega$, $E(\Omega)$, is defined by

$$E(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega u dx.$$

We denote by $w_\Omega$, the solution of (3.25) for $f = 1$ and we call $w_\Omega$ to be the torsion function of $\Omega$.

**(1)** : We get
$$\int_{Q_n} w_{Q_n}\, dx \leq \frac{\pi}{8}.$$
from the *Saint-Venant inequality*. The inequality states that the $L^1$ norm of the torsion function of a domain is bounded by the $L^1$ norm of the torsion function of a ball.

**(2)**: We state the following result from [7] without proof:

$$\text{if } w_\Omega(x_0) \geq \eta > 0, \text{ then } \int_{B_\delta(x_0)} w_\Omega\, dx \geq \frac{\pi\eta}{2}\delta^2,$$

where $\delta = 2\sqrt{\pi}$.

This essentially states that the value of the torsion function of a domain at a point gives information about the integral of the torsion function in a neighbourhood of the point.

**(3)**: We consider the following property stated in [7]:

For all $c > 0$, suppose there exist $C_0, r_0 > 0$, $C_0 r_0 < \min\left\{\frac{c}{2}, \frac{1}{2K}\right\}$ such that if for some $r \leq r_0$ and $x_1, \ldots, x_n \in \mathbb{R}$ such that $S_{2r}(x_i) \cap S_{2r}(x_j) = \emptyset$ ($S_{2r}(x_i)$ defined in property (4)) for all $i \neq j$, it holds
$$\max\{w_\Omega(x) : x \in \cup_i S_{2r}(x_i)\} \leq C_0 r_0,$$

then we have that

$$E(\Omega \setminus \cup_i S_{2r}(x_i)) + c|\Omega \setminus \cup_i S_{2r}(x_i)| \leq E(\Omega) + c|\Omega|.$$

The property would be used essentially later to find the maximum diameter for an optimal domain.

**(4)** : Let us consider a strip $S_{2r_0}(a) := [a - 2r_0, a + 2r_0] \times \mathbb{R}$ such that

$$\max_{S_{2r_0}(a)} w_\Omega > C_0^2,$$

Suppose that $C_0 = r_0$. Then we have

$$\int_{B_{2C_0}(x_0)} w_\Omega \, dx \geq 2\pi C_0^4,$$

where $x_0$ is a maximum point of $w$ in $S_{2r_0}(a)$. In particular

$$\int_{S_{4r_0}(a)} w_\Omega \, dx \geq 2\pi C_0^4.$$

We introduce $k \in \mathbb{N}$ such that

$$k = \left\lfloor \frac{\pi/8}{2\pi C_0^4} \right\rfloor + 1 = \left\lfloor \frac{1}{16 C_0^4} \right\rfloor + 1.$$

where $\pi/8$ denotes the upper bound of $L^1$ norm of $w$ from the Saint Venant inequality mentioned in Property 1. Suppose the diameter of $Q_n$ is larger than $8C_0k$. If we take the $x$-axis along the diameter, there will be at least one strip of width $4C_0$, denoted by $S'_{4C_0}$, such that

$$\max_{S'_{4C_0}} w_\Omega < C_0^2.$$

**(5)**: We obtain the following bound for $w_\Omega(x)$ [6]:

$$|w_\Omega(x)| \leq 2\sqrt{2}\; d(x)^{1/2} \left( \frac{|\Omega|^{3/4}}{\pi^{3/4}} \right). \tag{5.1}$$

We use this inequality for $\Omega = Q_n$ so that using $|Q_n| = \pi$, we have the bound

$$w_{Q_n}(x) \leq 2\sqrt{2}\; d(x)^{1/2},$$

where $d(x)$ is the distance of the point $x$ from $\partial\Omega$.

We introduce $e_1$ and $d_1$ such that

$$2\sqrt{2}(e_1)^{1/2} < C_0^2 \quad \text{and} \quad d_1 = \frac{\pi}{e_1}. \tag{5.2}$$

Suppose, for contradiction, we assume that the diameter of $Q_n$ is the segment $[a_0, a_m]$

and it is greater than $D_{max}$. We construct $k+n-2$ adjacent strips of width $8C_0$ around the midpoint of the segment $[a_0, a_m]$. Suppose we remove at most $n-2$ strips that contain a vertex in their interior. From property (4), we see that there exists a strip of width $S_{4C_0}$ such that

$$\max_{S_{4C_0}(a)} w < C_0^2.$$

. From property (3), this implies

$$E(Q_n \setminus S_r(a)) + c|Q_n \setminus S_r(a)| < E(Q_n) + c|Q_n|. \tag{5.3}$$

The intersection of the strip $S_r(a)$ with $Q_n$ is a union of trapezes $\{T_j\}_{j\in J}$. When removing any of these trapezes, one splits the polygon $Q_n$ into two polygons. In particular, if a vertex is on the boundary of the strip, one splits the polygon into more than two polygons.

Hence, from the arguments of [[7], Corollary 4.3], we get

$$E(Q_n \setminus T_j) + c|Q_n \setminus T_j| < E(Q_n) + c|Q_n|. \tag{5.4}$$

From [[7], Lemma 3.1], we get

$$|Q_n \setminus \overline{T}_j|\lambda_1(Q_n \setminus \overline{T}_j) < |Q_n|\lambda_1(Q_n) \text{ for every } j \in J. \tag{5.5}$$

Suppose we remove a trapeze, say $T_j$, from $Q_n \cap S_{C_0}(a)$. The polygon $Q_n$ splits into two polygons $P^1_{T_j}$ and $P^2_{T_j}$, which have $n+4$ edges in total. We have two possibilities:

1. Both $P^1_{T_j}$ and $P^2_{T_j}$ are polygons with less than $n$ sides.
2. One of $P^1_{T_j}$ and $P^2_{T_j}$ is a polygon with $n+1$ edges and the other one is a triangle.

If the first situation occurs, it contradicts optimality by (5.5). Hence, We suppose that the second situation above occurs for each trapeze $T_j$. We claim that there are only triangles on one side of the strip [5]. Hence, assuming that there are triangles only on the left side of the strip, we choose the triangle with the vertex $\mathbf{a}_0$. We know that the vertex is at a distance of at least $d_1$ from the strip, by our previous computations. We continuously move the strip $S_{4C_0}(a)$ to the right until it intersects its neighbouring vertex or a different vertex $\mathbf{a}_k$. In both cases, removing the trapeze splits the polygon into two or three polygons with less than $n$ edges. This has been demonstrated in Figure 5.4. The figure has been taken from [5].

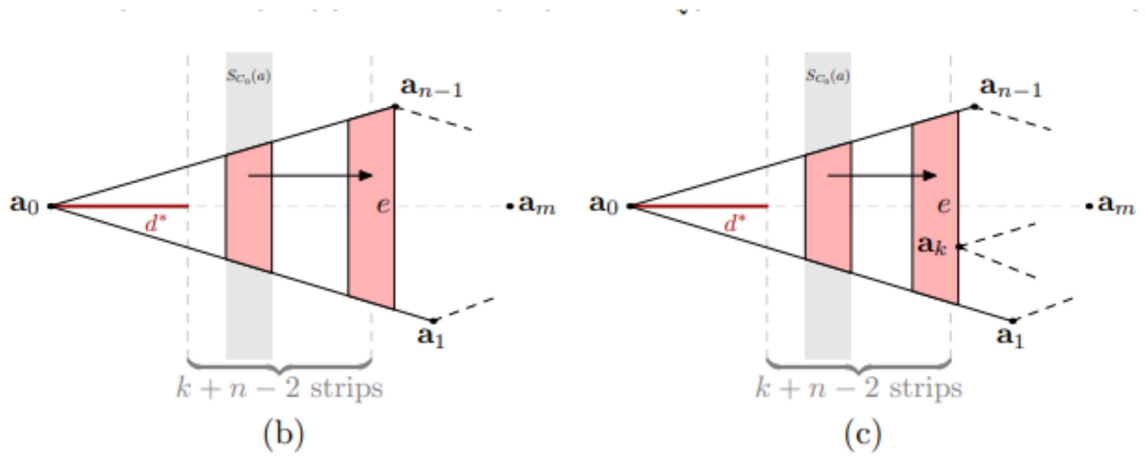


Figure 5.4: (b) The strip $S_{4C_0}(a)$ meets a neighbour of $a_0$ (c) or another vertex $a_k$ .

If we assume that the length of the longest side of the trapeze is $l$, then the area of the triangle on the left together with the trapeze is at least $d_1 l/2$. We know that the area is less than $\pi$ since the set is fully contained in the polygon. This implies that $l/2 \leq e_1$ and we get that the maximum of $w_{Q_n}$ is less than $C_0^2$ from (5.2), leading to a contradiction.

**Theorem 5.2.2.** *Suppose $P = [\boldsymbol{a}_0 \dots \boldsymbol{a}_{n-1}] \in \overline{\mathcal{P}}_n$ and $diam(P) \leq D_{max}$. There exists $\delta_0 > 0$ such that if $|\boldsymbol{a}_0 \boldsymbol{a}_1| \leq \delta \leq \delta_0$, then*

$$|P|\lambda_1(P) \geq l^*_{n-1} - C\delta^{\frac{1}{2}}, \tag{5.6}$$

*where $C$ depends only on $n$. We choose $\delta_0$ such that*

$$l^*_{n-1} - C\delta_0^{\frac{1}{2}} > l^*_n. \tag{5.7}$$

This essentially implies that the length of an edge of an optimal polygon, with a fixed area in $\overline{\mathcal{P}}_n$, cannot be smaller than a certain threshold.

We use the following lemma in the proof:

**Lemma 5.2.1.** *Let $P = [\boldsymbol{a}_0 \dots \boldsymbol{a}_{n-1}] \in \overline{\mathcal{P}}_n$ be such that $|P| = \pi$ and $diam(P) \leq D_{max}$. Let $P' \in \overline{\mathcal{P}}_n, P' = [\boldsymbol{b}_0 \dots \boldsymbol{b}_{n-1}]$ such that we have $|a_i b_i| \leq \delta$ for every $i = 0, \dots, n-1$ . Then*

$$|\lambda_1(P') - \lambda_1(P)| \leq 4\sqrt{2\pi}e^{\frac{1}{4\pi}} \left(\max\{\lambda_1(P), \lambda_1(P')\}\right)^2 \lambda_1(P \cap P')\delta^{\frac{1}{2}}.$$

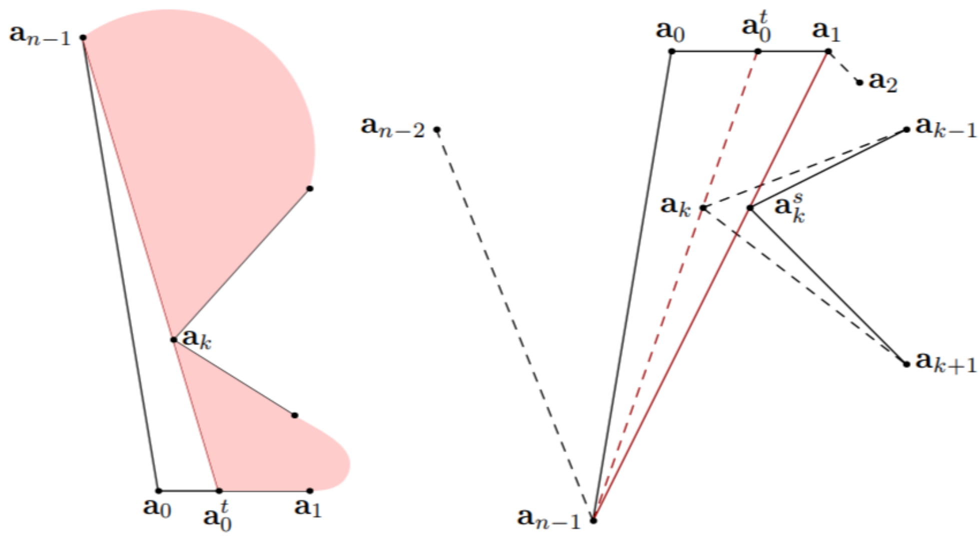


Figure 5.5: The segment $\mathbf{a}_{n-1}\mathbf{a}_0^t$ intersecting a vertex $\mathbf{a}_k$ in case the angle $\hat{a}_0$ is convex (left), in case the angles $\hat{a}_0$, $\hat{a}_1$ are concave (right)

The proof of the lemma follows from [[7], Inequality 2.6].

*Proof* (of Theorem 5.2.2) Let $P = [\mathbf{a}_0 \dots \mathbf{a}_{n-1}] \in \overline{\mathcal{P}}_n$ is such that $|P| = \pi, \mathrm{diam}(P) \leq D_{max}$ and $|\mathbf{a}_0\mathbf{a}_1| \leq \delta$. Assume that $\pi\lambda_1(P) < l^*_{n-1}$. Assume at least one of the angles $\mathbf{a}_0, \mathbf{a}_1$ is convex. We move the point $\mathbf{a}_0$ continuously towards $\mathbf{a}_1$, denoting it by $\mathbf{a}_0^t = [1-t]\mathbf{a}_0 + t\mathbf{a}_1$. If it doesn't intersect any vertex of $P$ for $t \in (0,1)$, we denote the case $t = 1$ to be the polygon $P'$.

Using Makai's inequality [12] and the fact $\rho_{P'} \geq \rho_P - \delta$, where $\rho_P$ denotes the inradius of $P$, we get

$$\lambda_1(P') \leq \frac{1}{\rho^2}\lambda_1(B_1) \leq \frac{1}{\left(\pi/4l^*_{n-1}\right)^{\frac{1}{2}} - 2\delta}\lambda_1(B_1). \tag{5.8}$$

Finally using Lemma 4.2.1 and the equation (5.8), we get

$$\frac{\lambda^*_{n-1}}{\pi - D_{\max}\delta} - \lambda_1(P) \leq \frac{\lambda^*_{n-1}}{|P'|} - \lambda_1(P) \leq \lambda_1(P') - \lambda_1(P)$$

$$\leq 4\sqrt{2\pi}e^{\frac{1}{4\pi}} \left( \frac{1}{\left(\pi/(\lambda^*_{n-1})^{1/2} - 2\delta\right)^3} - \lambda_1(B_1) \right)^{\frac{3}{2}} \delta^{\frac{1}{2}}.$$

In case the edge $[\mathbf{a}_{n-1}\mathbf{a}_0^t]$ meets a vertex, the polygon $P^t$ is split into two polygons, as shown in Figure 5.5. We choose the one with the smallest eigenvalue and repeat the argument. The proof for the case that both the angles are concave also follows a similar argument as given in [[5], Theorem 7.3]. Figure 5.5 is taken from [5].

**Theorem 5.2.3.** *For every $n \geq 5$, the proof of the Pólya - Szegö conjecture can be reduced to a finite number of numerical computations, assuming the conjecture is true.*

*Proof* The proof follows from induction. From Chapter 2, we know that the Pólya - Szegö conjecture has been solved for $n = 3, 4$. Suppose the inequality is true for polygons with up to $n-1$ edges. We denote by $\mathbb{P}_n = [\mathbf{a}_0^* \dots \mathbf{a}_{n-1}^*]$ the regular polygon inscribed in the unit circle with vertex $a_0^* = (1, 0)$. According to our assumption, we have $l_{n-1}^* = \lambda_1(\mathbb{P}_{n-1})|\mathbb{P}_{n-1}|$. To prove the conjecture for $n$ edges, we shall analyze the problem (3.1) in the following steps.

**Step 1**: From Section 4.1, we compute the $2n-4$ eigenvalues of the shape Hessian matrix $\mathbf{M}^\lambda$ and certify its positivity using the finite element approximation methods discussed in Section 4.2. This proves that $\mathbb{P}_n$ is a local minimum.

We denote by $\mathcal{P} \subset \overline{\mathcal{P}}_n$ the family of polygons $P = [\mathbf{a}_0\mathbf{a}_1 \dots \mathbf{a}_{n-1}]$ with $\mathbf{a}_i = (x_i, y_i)$. We consider the two points fixed $\mathbf{a}_0 = \mathbf{a}_0^*, \mathbf{a}_1 = \mathbf{a}_1^*$ and we assume that the edge $[\mathbf{a}_0\mathbf{a}_1]$ is the longest edge. We identify the polygon $P$ as a point in $\mathbb{R}^{2n-4}$, with the coordinates $(x_2, y_2, \dots, x_{n-1}, y_{n-1})$.

**Step 2**: Using Theorem 3.2.1, we compute a neighbourhood of $\mathbb{P}_n$ where the local minimality holds i.e

$$|P|\lambda_1(P) \geq |\mathbb{P}_n|\lambda_1(\mathbb{P}_n).$$

This implies that for a value $\varepsilon_0$, we have $|P|\lambda_1(P) \geq |\mathbb{P}_n|\lambda_1(\mathbb{P}_n)$ for every $P = [\mathbf{a}_0 \dots \mathbf{a}_{n-1}]$ with $\mathbf{a}_0 = \mathbf{a}_0^*, \mathbf{a}_1 = \mathbf{a}_1^*$, such that $\forall i = 2, \dots, n-1, |\mathbf{a}_i\mathbf{a}_i^*| \leq \varepsilon_0$. We denote this neighbourhood by $\mathcal{L}_n$, a closed set.

**Step 3**: Since $[\mathbf{a}_0\mathbf{a}_1]$ is the maximal length of an edge by our assumption, we get the bound

$$\sqrt{\frac{|\mathbb{P}_n|}{|P|}}|\mathbf{a}_0\mathbf{a}_1| \leq D_{max}.$$

Using Theorem 5.2.2, we get a lower bound for $e_{min}$. Similarly, using Makai's inequality [12], we get a lower bound for the inradius (denoted as $\rho_{min}$) of an optimal $n-$gon. All these geometric constraints: inradius, shortest edge length and measure determine a smaller compact set $\widetilde{\mathcal{P}}$ such that $\widetilde{\mathcal{P}} \subseteq \mathcal{P}$ in $\mathbb{R}^{2n-4}$.

We note that there exist universal upper bounds for eigenvalue and for the area in $\widetilde{\mathcal{P}}$. Below we work with $\delta \leq \frac{\rho_{min}}{4} < 1$. From Lemma 5.2.1, we know that for two polygons $P, P' \in \widetilde{\mathcal{P}}$ such that if the distance between two vertices is at most $\delta$, then there exists K such that $|\lambda_1(P) - \lambda_1(P')| \leq K\delta^{\frac{1}{2}}$. We also have

$$||P| - |P'|| \leq K'\delta$$

where $K' = nD_{max} + n\pi$. This implies that there exists a constant $K''$ such that

$$|\lambda_1(P)|P| - \lambda_1(P')|P'|| \leq K''\delta^{\frac{1}{2}}.$$

holds for $P, P'$.

**Step 4**: We assume that $\mathbb{P}_n$ is the only minimizer for $P \to |P|\lambda_1(P)$ in $\widetilde{\mathcal{P}}$. From Step 2, we know that there exists $\varepsilon_1 > 0$ such that outside the neighbourhood $\mathcal{L}_n$ of $\mathbb{P}_n$ in $\mathbb{R}^{2n-4}$ we have $|P|\lambda_1(P) > l_n^* + \varepsilon_1$. We consider $\delta > 0$ such that $K''(2\delta)^{\frac{1}{2}} < \frac{\varepsilon_1}{4}$ and $2\delta \leq \delta_0$ with $\delta_0$ from (5.7).

Since the set $\widetilde{\mathcal{P}} \setminus \mathcal{L}_n$ is compact, we cover the set with at most $c_{2n-4}\left(\frac{D_{max}}{\delta}\right)^{2n-4}$ balls $(B_j)_{j\in J}$ of radius $\delta$, where $c_{2n-4}$ is a dimensional constant.

Choose one of the balls $B_j$ enumerated above. We consider an admissible polygon $P \in \widetilde{\mathcal{P}} \setminus \mathcal{L}_n$ contained in the ball $B_j$. We obtain a certified numerical estimate of $|P|\lambda_1(P)$, obtaining a certified estimate interval of length $\varepsilon/4$. If the computation gives

$$|P|\lambda_1(P) \geq l_n^* + \frac{\varepsilon}{2}, \tag{5.9}$$

then the polygon $P$ is not optimal and by the choice of $\delta$, there doesn't exist an optimal polygon in the ball $B_j$. If the condition (5.9) holds for every admissible polygon in every ball $B_j$, then the conjecture is solved. We consider $\varepsilon_1 = 1$ and verify that (5.9) holds every time there exists an admissible polygon in one of the balls by repeating the above procedure. If it holds, we stop. if it doesn't hold for any polygon, we divide $\varepsilon_1$ by 2 and repeat the procedure. The procedure we described stops in a finite number of steps.

# Chapter 6

# Conclusion

We conclude with a brief mention of a related problem to the Dirichlet eigenvalue problem that we have seen until now.

Let us consider the quantity

$$T(\Omega) = \int_{\Omega} w_{\Omega} dx,$$

where $w$ is the torsion function given by (3.25) where $f = 1$. The quantity $T(\Omega)$ is called torsional rigidity.

We know that the Saint-Venant inequality implies that the ball maximizes the torsional rigidity among all sets of area $\pi$. Polya and Szego conjectured the following [15]:

**Conjecture**: The unique solution to the problem

$$\max_{P \in \mathcal{P}_n, |P| = \pi} T(P)$$

is the regular polygon with $n$ sides and area $\pi$.

The results we obtained for the Dirichlet eigenvalue problem can be used to solve the above conjecture as well. The proof of local maximality again involves computing the Hessian matrix for the regular polygon. The expression of the Hessian matrix is obtained by Laurain in [[29],Proposition 14].

# Bibliography


[1] M.Bourlard et al. “Coefficients of the singularities for elliptic boundary value problems on domains with conical points. III - Finite Element methods on Polygonal domains”. In: *SIAM Journal of Numerical Analysis* 29.1 (1992), pp. 136–155.

[2] Michel Pierre Antoine Henrot. *Shape Optimization and Variation*. EMS Tracts in Mathematics. European Mathematical Society, Zurich, 2018. ISBN: 978-3-03719-178-1.

[3] S. Balay W.D. Gropp L.C. McInnes B.F.Smith. *Efficient management of parallelism in object oriented numerical software libraries.* Modern Software Tools in Scientific Computing. Birkhauser Press, 1997. ISBN: 978-1-4612-1986-6.

[4] Julie Clutterbuck Ben Andrews. “Proof of the Fundamental Gap Conjecture”. In: *Journal of the American Mathematical Society* 24.3 (2011), pp. 899–916.

[5] Dorin Bucur Beniamin Bogosel. “On the Polygonal Faber - Krahn Inequality”. In: *Journal de l'École polytechnique - Mathématiques* 11 (2024), pp. 19–105. DOI: 10.5802/jep.250.

[6] M.van den Berg and E.Bolthausen. “Estimates for Dirichlet eigenfunctions.” In: *Journal of the London Mathematical Society* 59.2 (1999), pp. 607–619. DOI: 10.1112/S0024610799007267.

[7] Dorin Bucur and Dario Mazzoleni. “A Surgery Result for the Spectrum of the Dirichlet Laplacian”. In: *SIAM Journal on Mathematical Analysis* 47.6 (2015), pp. 891–921. DOI: 10.1137/140992448.

[8] Lawrence C.Evans. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, 2010.

[9] D.Bucur and I.Fragala. “Proof of the honeycomb asymptotics for optimal Cheeger clusters,” in: *Advances in Mathematics* 350 (2019), pp. 97–129.

[10] Monique Dauge. *Elliptic Boundary Value Problems on corner domains*. Lecture Notes in Mathematics. Springer Verlag Berlin, 1988.

[11] E.M.Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series. Princeton Unviersity Press, Princeton, N.J, 1970.

[12] E.Makai. "A lower estimation of the principal frequencies of simply connected domains". In: *Acta Mathematica Academiae Scientiarum Hungarica* 16 (1965), pp. 319–323. DOI: 10.1007/BF01904840.

[13] F.Hecht. "New development in FreeFem++." In: *Journal of Numerical Mathematics* 20.3-4 (2012), pp. 251–265. DOI: https://doi.org/10.1515/jnum-2012-0013.

[14] Pedro Freitas. "Upper and Lower bounds for the first Dirichlet eigenvalue of a triangle". In: *Proceedings of the American Mathematical Society* 134.7 (2006), pp. 2083–2089. DOI: 10.1090/S0002-9939-06-08339-0.

[15] G.Polya and G.Szego. *Isoperimetric Inequalities in Mathematical Physics*. Annals of Mathematical studies. Princeton university Press,Princeton,N.J, 1951. ISBN: 9780198520115.

[16] George J.Fix Gilbert Strang. *An Analysis of the Finite Element Method*. Prentice-Hall series in Automatic computation. Wellesley-Cambridge Press, 1997.

[17] Giulio Schimperna Giuseppe Savare. "Domain pertrubations and estimates for the solutions of second order elliptic equations". In: *Journal de l'École polytechnique - Mathématiques* 81.10 (2002), pp. 1071–1112. DOI: http://dx.doi.org/10.1002/andp.19053221004.

[18] D.S. Grebenkov and B.T.Nguyen. "Geometrical structure of Laplacian eigenfunctions". In: *SIAM Review* 55.4 (2013), pp. 601–667. DOI: https://doi.org/10.1137/120880173.

[19] Pierre Grisvard. *Elliptic Problems in Nonsmooth domains*. Monographs and studies in mathematics. Pitman Publishing, 1985. ISBN: 0-273-08647-2.

[20] Petru Mironescu Haïm Brezis. "Gagliardo-Nirenberg inequalities and non-inequalities: the full story". In: *Annales de l'Institut Henri Poincaré C, Analyse non linéaire,* 35.5 (2018), pp. 1355–1376.

[21] Young Ja Pak Hee Chul Pak. "Sharp trace inequalities on fractional Sobolev spaces". In: *Mathematische Nachrichten* 284.5 (2011), pp. 761–763. DOI: https://doi.org/10.1002/mana.200810206.

[22] Antoine Henrot. *Extremum problems of eigenvalues of elliptic operator*. Frontiers in mathematics. Birkhauser Verlag, 2006.

[23] Antoine Henrot. *Shape Optimization and spectral theory*. DeGruyter Open Poland, 2017. ISBN: 9783110550856.

[24] I.Fragala and B.Velichkov. "Serrin-type theorem for triangles". In: *Proceedings of the American Mathematical Society* 147.4 (2019), pp. 1615–1626. DOI: `10.1090/PROC/14352`.

[25] Garry J.Tee. "Eigenvectors of block circulant and alternating circulant matrices". In: *Research Letters in the Information and Mathematical Sciences* 8 (2005), pp. 123–142. DOI: `http://dx.doi.org/10.1002/andp.19053221004`.

[26] J.W.S.Rayleigh. *The Theory of Sound*. Dover, New York, 1945. ISBN: 9780198520115.

[27] Bernhard Kawohl. *Rearrangements and Convexity of level sets in PDE*. Lecture notes in mathematics. Springer Verlag Heidelberg, 1985.

[28] L.Tartar. *An introduction to Sobolev spaces and interpolation spaces,* Lecture Notes of the Unione Matematica Italiana. Springer Berlin, Heidelberg, 2007.

[29] Antoine Laurain. "Distributed and boundary expressions of first and second order shape derivatives in nonsmooth domains." In: *Journal de Mathématiques Pures et Appliquées* 9.134 (2020), pp. 328–368.

[30] M.H.Protter. "Lower Bounds for the First Eigenvalue of Elliptic Equations". In: *Annals of Mathematics* 71.3 (1960), pp. 423–444.

[31] William Mclean. *Strongly Elliptic systems and Boundary Integral Equations*. Cambridge University Press, 2000.

[32] Shubhalaxmi Mukherjee. *An Eigenvalue Optimisation Problem for Triangles and Quadrilaterals*. MS thesis. http://dr.iiserpune.ac.in:8080/xmlui/handle/123456789/6069, 2021.

[33] Alessio Porretta. "A note on the Sobolev and Gagliardo-Nirenberg inequality when p ¿ N". In: *Advanced Nonlinear studies* 20.2 (2020), pp. 361–371. DOI: `10.1515/ans-2020-2086`.

[34] R.A.Adams and J.F.Fournier. *Sobolev spaces*. Academic Press, 1978. ISBN: 9780080541297.

[35] R.Courant and D.Hilbert. *Methods of Mathematical Physics*. Wiley classics library. Interscience Publishers, 1953.

[36] J.E.Jones R.D.Falgout and U.M.Yang. *The Design and Implementation of hypre, a Library of Parallel High Performance Preconditioners*. Lecture Notes in Computational Science and Engineering. Springer Berlin, 2006.

[37] Charles R.Johnson Roger A.Horn. *Matrix Analysis*. International series of monographs on physics. Cambridge University Press,Cambridge, Second edition, 2013. ISBN: 978-0-521-83940-2.

[38] S.Kesavam. *Functional Analysis*. Texts and readings in mathematics. Hindustan Book agency, 2009. ISBN: 978-81-85931-87-6.

[39] S.M.Rump. “Verification methods: Rigorous results using floating-point arithmetic”. In: *Acta Numerica* 19 (2010), pp. 287–449. DOI: https://doi.org/10.1017/S096249291000005X.

[40] J.E.Roman V.Hernandez and V.Vidal. “. SLEPc: a scalable and flexible toolkit for the solution of eigenvalue problems”. In: *ACM Transactions on Mathematical Software* 31.3 (2005), pp. 351–362. DOI: https://doi.org/10.1145/1089014.1089019.

[41] Shin’ichi Oishi Xuefeng Liu. “Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape”. In: *SIAM Journal on Numerical Analysis* 51.3 (2013). DOI: https://doi.org/10.1137/120878446.