Classical K-Theory: Completion of Unimodular Rows

A Thesis

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by

Nishad Devendra Mandlik

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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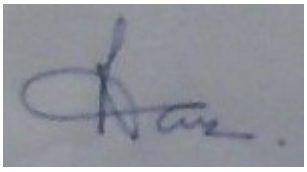
Supervisor: Rabeya Basu

This is to certify that this dissertation entitled Classical K-Theory:
Completion of Unimodular Rows towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents the research carried out by Nishad Devendra Mandlik at Indian Institute of Science Education and Research under the supervision of Rabeya Basu, Assistant Professor, Department of Mathematics , during the academic year 2014-2015.

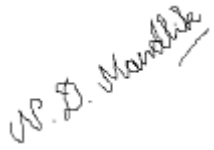
 Rabeya Basu

Committee:
Rabeya Basu
Debargha Banerjee

Dedicated to my parents

Declaration

I hereby declare that the matter embodied in the report entitled Classical K-Theory: Completion of Unimodular Rows are the results of the investigations carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research Pune, under the supervision of Rabeya Basu and the same has not been submitted elsewhere for any other degree.



Nishad Devendra Mandlik

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Abstract

In this thesis, we explore the problem of the completion of unimodular rows which has its historical importance in the solution of Serre's problem. We also study Classical Algebraic K-theory in the context of projective modules and unimodular rows. We end the thesis with the study of a recent theorem on the completion of unimodular row of length 3 and understand its constructive proof using the help of the software *Macaulay2*.

Contents

Chapter 1

Introduction

In 1955, J.P-Serre proposed the following problem: *'Is finitely generated projective modules over polynomial rings free?'* The problem was solved independently by D. Quillen and A. Suslin in 1976. This twenty year period in finding the solution witnessed the generation of spectacular growth in the field of Classical K-Theory. One such endeavour was the notion of completion of unimodular rows. Suslin used the correspondence between the completion of unimodular rows and freeness of finitely generated stably free projective modules to give an elementary proof of Serre's problem.

In this thesis, we will understand the completion of unimodular rows of length 3 by looking at the following theorem by Mohan Kumar and M.P. Murthy:

Theorem 1.0.1. *"If (a, b, c) is a unimodular row in ring A such that $z^2 + bz + ac = 0$ has roots in A, then (a, b, c) is completable." [9]*

It is interesting to understand whether we can find out an explicit completion of a unimodular row to an invertible matrix. V. Kodiyalam succeeded in finding a non-trivial constructive proof for Kumar-Murthy's theorem using the software *Macaulay2*. We will replicate Dr. Kodiyalam's proof using *Macaulay2* and describe the steps used to obtain the result in detail.

To understand the theory related to completion of unimodular rows, we need the preliminaries in Commutative Algebra which we will develop in Chapter 2 of the thesis. We will develop module theory, prove results like the Nakayama Lemma, Prime

Avoidance Lemma and state the properties of localisation of rings, of Noetherian and Artinian rings and Integral Extensions.

We will develop the area of Projective Modules in Chapter 3 and define the Grothendieck group K_0 of a ring. We will also compute the K_0 group for different classes of rings.

We will understand the properties of Elementary matrices and introduce the concept of unimodular rows in Chapter 4. We will prove the equivalence between the completion of unimodular rows and freeness of finitely generated stably free modules and prove basic theorems related to the subject. We will also look at some results in Classical Algebraic K-theory with respect to the Whitehead group K_1, which developed as an offshoot in the process of solving Serre's problem.

In Chapter 5, we will concern ourselves with some non-trivial results in the completion of unimodular rows which are important in the historical pathway of solving Serre's problem. These results include Horrocks' theorem, Quillen's Local-Global Principle and Ravi Rao's proof of the generalised Horrocks' theorem. We also study Swan-Towber's non-trivial constructive result on the completion of the unimodular row of length 3, namely the completion of (a^2, b, c) and its generalisation given by Suslin called as the 'factorial n' theorem.

In Chapter 6, we will focus on the Mohan Kumar and M.P. Murthy's theorem on the completion of unimodular rows and V. Kodiyalam's constructive proof for the same theorem.

Chapter 2

Preliminaries

We will assume our ring A to be commutative ring with identity.

2.1 Modules and Localisation

We are familiar with the notion of vector spaces which naturally developed from the concepts of vectors in a plane. Modules are a generalisation of the concept of vector spaces. We define modules over a general ring. We will use the standard notations and proofs from [1] and [2].

Definition 2.1.1. A *module* M over a ring A is an abelian group under addition and with the operation: $A \times M \to M$ satisfying the following properties: For $a, b \in A$ and $x, y \in M$,

- $(a + b) \cdot x = a \cdot x + b \cdot x.$
- $a \cdot (x + y) = a \cdot x + a \cdot y.$
- $a \cdot (b \cdot x) = ab \cdot x.$
- $1 \cdot x = x.$

Examples:

1. Any vector space over a field.
2. An abelian group G is a $\mathbb{Z}$-module.

3. A is a module over itself.

4. An ideal I of A over A.

5. For A-modules M, N the set $\mathrm{Hom}(M, N)$ is a module over A.

6. $A[X]$, the ring of polynomials is a module over A.

Definition 2.1.2. A subgroup N of the module M over A is a *submodule* if it is closed under scalar multiplication from A. That is, if $n \in N$ then $a \times n \in N$ for all $a \in A$.

Examples:

1. Subspaces of a vector space.

2. An ideal of A is a submodule of A over itself.

3. $B = \{f(X) \in A[X] : \deg(f) \leq n \text{ for fixed } n\}$ is a submodule of $A[X]$ over A.

Definition 2.1.3. If N is a submodule of M then the quotient

$$\frac{M}{N} = \{x + N : x \in M\}$$

is called the *quotient module*. It has an A-module structure given by

$$a \cdot (x + N) = a \cdot x + N.$$

Definition 2.1.4. *The surjective cannonical map: $M \to M/N$ is called projection map.*

Definition 2.1.5. Let M, M' be modules over R. A function $f : M \to M'$ is a *homomorphism* if:

1. *f is a group homomorphism.*

2. *f preserves scalar multiplication* ($f(ax) = af(x)$).

A homomorphism of modules that is bijective is called an *isomorphism*.

Sum and Direct Sum of Modules:

Let N, K be the submodules of M.

The smallest submodule generated by N and K is the module :

$$N + K = \{x + y : x \in N, y \in M\}$$

Similarly, smallest submodules generated by the submodules $N_1, N_2, ..., N_k$ is $N_1 + N_2 + \cdots + N_k$.

A direct sum is when every element of N can be written uniquely as:

$$x = x_1 + x_2 + \cdots + x_k,$$

where $x_i \in N_i$, $1 \leq i \leq k$. That is, $N = N_1 \oplus \cdots \oplus N_k$ if and only if

1. $N = N_1 + N_2 + \cdots + N_k$.
2. $N_i \cap (N_1 + \cdots + N_{i-1} + N_{i+1} + \cdots + N_k) = 0$ for all $i = 1, ..., k$.

The definition of direct sum can be extended to any collection of modules. An A-module M is a direct sum of a collection of submodules $M_{\alpha_{\alpha \in I}}$ if each $x \in M$ can be expressed uniquely as $x = x_{\alpha_1} + x_{\alpha_2} + \cdots + x_{\alpha_n}$.

Isomorphism theorems:

Theorem 2.1.1. ***First Isomorphism Theorem:***
Let M and N be A-modules such that $f : M \to N$ is a homomorphism. Then, Ker $f \subseteq M$ *and Im* $f \subseteq N$. *Moreover,*

$$\frac{M}{\text{Ker}f} \cong \text{Im}f$$

Theorem 2.1.2. ***Second Isomorphism Theorem:***
Let M, N and K be A-modules such that $K \subseteq N \subseteq M$. Then,

$$\frac{M}{N} \cong \frac{(M/K)}{(N/K)}$$

Theorem 2.1.3. ***Third Isomorphism Theorem:***
Let M be a A-module and K, N be A-submodule of M. Then,

$$\frac{(N+K)}{K} \cong \frac{N}{(N \cap K)}$$

An A-module M is free if there exists a basis for M.

We will define the annihilator of a module as Ann $(M) = \{a \in A : aM = 0\}$, Similarly, we define the annihilator of an element as Ann $(x) = \{a \in A : ax = 0\}$. We note Ann (x) is an ideal of A.

Definition 2.1.6. A cyclic A module Ax is *free* if Ann (x) is the zero ideal (0).

An A-module M is called free if it can be expressed as a direct sum $M = \oplus M_\alpha$, where each M_{α_i} is a free cyclic A-module. If $M_{\alpha_i} = Ax_i$, then the collection $\{x_i\}$ is called a basis of the free module M .

A *finitely generated free* module $M \cong M_1 \oplus M_2 \oplus ... \oplus M_k$, where $M_i = Ax_i$. Thus, a finitely generated free module $M \cong A^n$.

Examples:

1. A cyclic module Ax is a free module of rank 1.
2. The A-module A^n is a free module of rank n.
3. The A-module $A[X]$ is a free module of countable rank.
   $\{1, X, ..., X^m, ...\}$.

Split Exact Sequence:

Definition 2.1.7. Consider a sequence of A- modules and homomorphism between them:

$$0 \to N \xrightarrow{\text{f}} M \xrightarrow{\text{g}} K \to 0$$

The given sequence is called a *short exact sequence* if f is an injective homomorphism, g is a surjective homomorphism and Ker g = Im f.

Definition 2.1.8. Given a short exact sequence:

$$0 \to N \xrightarrow{\text{f}} M \xrightarrow{\text{g}} K \to 0$$

if $\exists\ h : K \to M$ such that $gh = 1_K$, then the sequence is called as a *split exact sequence.*

Theorem 2.1.4. *A sequence is split exact if and only if* $M \cong N \oplus K$.

Proof. Let us first assume that $M \cong N \oplus K$. We can define the following maps: firstly the injective map $N \to M$, where $n \mapsto (n, 0)$.

The projective map $M \to K$, where $(n, k) \mapsto k$. Thus, $0 \to N \to M \to K \to 0$ is a split exact sequence as we can construct the map: $K \to M$, where $k \mapsto (0, k)$.

Now, we will assume that $0 \to N \xrightarrow{\mathrm{f}} M \xrightarrow{\mathrm{g}} K \to 0$ is a split exact sequence with $h : K \to M$ such that $gh = 1_K$. We need to prove that there exists an isomorphism from M to $N \oplus K$. We will define the following map- $\theta : M \to N \oplus K$, where $\theta(m) = (f^{-1}(m), g(m))$. □

Localisation

The idea of localisation is a tool in algebra to understand the behaviour of ideals and develops as an analogue of the idea of localising the function say $f(x)$ defined on the real line at a point. The construction of the field of rationals $\mathbb{Q}$ from the ring of integers $\mathbb{Z}$ can be considered as a special case of localisation of a ring.

For further reference on localisation, consider [1] and [2].

Definition 2.1.9. Let A be a ring. A subset S is called *multiplicatively closed set* if $1 \in S$ and $a, b \in S$ implies $ab \in S$.

For any ring A and a multiplicatively closed set S of A, we define the ring $S^{-1}A$ consisting of elements of the form $\frac{a}{s}$, where $a \in A$ and $s \in S$ with ring operations as follows:

$$(a/s) + (b/t) = (at + bs)/st,$$

$$(a/s)(b/t) = ab/st,$$

and with an equivalence relation: $a/s \sim b/t$ if there exists an $r \in S$ such that $r(at - bs) = 0$.

Definition 2.1.10. The ring $S^{-1}A$ is called the *ring of fractions* of A or the localisation of ring A at S.

An important case that we are interested is the localisation of a ring at a prime ideal because we can translate using this technique the concept of local properties of affine algebraic spaces to the corresponding rings. Let A be a ring and $\mathfrak{p}$ be a prime ideal in A. Let $S = A - \mathfrak{p}$. The localisation of A at $\mathfrak{p}$ is donated by $A_{\mathfrak{p}}$. This ring $A_{\mathfrak{p}}$ is a local ring.

Definition 2.1.11. An A-module M is said to be *locally free* if M_p is A_p-free for every prime ideal p of A, where A_p is the localisation of A at the prime ideal p.

Some properties of localisation:

1. If A is a domian and $S = A - 0$ then $S^{-1}A$ is the quotient field of A.

2. The ring homomorphism $f : A \to S^{-1}A$ is defined by $f(x) = x/1$. It is injective if and only if S does not have any zero divisors.

3. The ring homomorphism f defined above satisfies the universal property:

   For every ring homomorphism $g : A \to B$ satisfying the property that $g(s)$ is a unit in B for $s \in S$, there exists a homomorphism $h : S^{-1}A \to B$ such that $g = h \odot f$, where $h(a/s) = g(a)g(s)^{-1}$.

4. If $I \in A$ an ideal, then $S^{-1}I = \left\{ \frac{i}{s} \mid i \in I, s \in S \right\}$ is an ideal of $S^{-1}A$. Moreover, any ideal of $S^{-1}A$ is of the form $S^{-1}I$, where I is an ideal.

5. The prime ideals of $S^{-1}A$ are in a bijective correspondence with the prime ideals $\mathfrak{p}$ of A satisfying $\mathfrak{p} \cap S = \emptyset$.

6. For ideals I, J of A, we have the following isomorphisms
   $S^{-1}(I + J) \cong S^{-1}I + S^{-1}J$,
   $S^{-1}(IJ) \cong S^{-1}(I) \cdot S^{-1}(J)$,
   $S^{-1}(I \cap J) \cong S^{-1}(I) \cap S^{-1}(J)$ and
   $S^{-1}I \cong S^{-1}A$ if and only if $I \cap S \neq \emptyset$.

Localisation of modules:

Suppose M is an A-module and S is a multiplicatively closed set of A. Using the same concept that we used to construct the $S^{-1}A$, we can construct a $S^{-1}A$-module $S^{-1}M$ from M.

We will start with defining an equivalence relation on $S \times M$: For $t \in S$,

$$(m, s) \sim (m', s') \quad \text{if and only if} \quad t(sm' - s'm) = 0.$$

Let $\frac{m}{s}$ denote the set of equivalence class of (m, s) and $S^{-1}M$ denote the set of equivalence classes.

Using the relations:

$$\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + m's}{ss'} \quad ,$$

$$\frac{a}{s} \cdot \frac{m'}{s'} = \frac{am'}{ss'}$$

we construct $S^{-1}A$-module $S^{-1}M$ from M.

Properties of localisation of modules:

1. For $S = A - \mathfrak{p}$, the $A_\mathfrak{p}$-module $S^{-1}M$ is denoted by $M_\mathfrak{p}$ and is called as the *localisation of M at $\mathfrak{p}$*.

2. The localisation $S^{-1}M$ is also an A-module and there exists an A-homomorphism of modules given by

$$f : M \to S^{-1}M,$$

$$f(m) = m/1.$$

3. Let M, N be A-modules and $f \in \mathrm{Hom}_A(M, N)$. There exists a module homomorphism:

$$S^{-1}f : S^{-1}M \to S^{-1}N$$

$$S^{-1}f(m/s) = f(m)/s.$$

$$S^{-1}f \in \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

4. Let

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of A-modules. Then

$$0 \to S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \to 0$$

is an exact sequence of $S^{-1}A$-modules.

5. Let M, N be A-modules. Then

$$S^{-1}(M \oplus N) \cong S^{-1}(M) \oplus S^{-1}N$$

2.2 Some Theorems in Commutative Algebra

Theorem 2.2.1. ***Prime Avoidance Lemma:***

Let A be a ring and $I \subset A$ an ideal. Suppose for prime ideals P_i, $I \subset \bigcup_{i=1}^{n} P_i$, then $I \subset P_i$ for some i.

Proof. We will use induction on n. The statement is trivially true for $n = 1$. Let us assume the statement to be true for $n-1$, i.e. $I \subset \bigcup_{i=1}^{n-1} P_i \Rightarrow I \subset P_i$ for some i such that$(1 \leq i \leq n-1)$. We assume $I \subset \cup P_i$. If I is contained in union of any $(n-1)$ prime ideals, we can use induction hypothesis. If not, $I \not\subseteq \bigcap_{j \neq i} P_j$, for all i. This means that there exists $a_i \in I$ such that $a_i \notin \bigcup_{j \neq i} P_j$ for all i such that$(1 \leq i \leq n-1)$. If for some $i, a_i \notin P_i$,then $I \not\subseteq \bigcap_{i=1}^{n} P_i$. So we assume that $a_i \in P_i$ for all i. Then the element

$$a = \sum_{i=1}^{n} a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$$

is an element of $I \notin \bigcup_{i=1}^{n-1} P_i$. This leads to a contradiction. Hence, the lemma follows. □

Another version of the Prime Avoidance Lemma is as follows:

Theorem 2.2.2. *Consider a ring A with prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2, \ldots, \mathfrak{p}_t$ and an ideal I generated by the elements $a_1, \ldots, a_n$. such that $I \not\subseteq \mathfrak{p}_i$ for $1 \leq i \leq r$. Then, we can find elements $b_2, ..., b_n \in A$ such that $c = a_1 + a_2 b_2 + \cdots + a_n b_n \notin \bigcup_{i=1}^{r} P_i$*

For details, see [3].

Definition 2.2.1. The intersection of all prime ideals is called the *Jacobson radical* of A, also denoted as $\mathrm{Jac}(A)$

Let $x \in \mathrm{Jac}(A)$, then $1 - ax$ is a unit for $a \in A$.

Theorem 2.2.3. ***First version of Nakayama lemma:***

Let A be a ring and M be a finitely generated A-module. I, an ideal such that $IM = M$. Then there exists an element $x \in I$ such that $(1-x)M = 0$.

Proof. We will assume that $M \neq 0$. As M is finitely generated, let $m_1, \ldots, m_r$ be the generating set of M. Since $IM = M$, we can write $m_i = \sum_{j=1}^{r} \lambda_{ij} m_j$, where $\lambda_{ij} \in A$. Hence,

$$\begin{pmatrix} 1-\lambda_{11} & -\lambda_{12} & \cdots & -\lambda_{1r} \\ -\lambda_{21} & 1-\lambda_{22} & \cdots & -\lambda_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda_{r1} & -\lambda_{r2} & \cdots & 1-\lambda_{rr} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_r \end{pmatrix} = 0.$$

Let

$$\alpha = \begin{pmatrix} 1-\lambda_{11} & -\lambda_{12} & \cdots & -\lambda_{1r} \\ -\lambda_{21} & 1-\lambda_{22} & \cdots & -\lambda_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda_{r1} & -\lambda_{r2} & \cdots & 1-\lambda_{rr} \end{pmatrix}$$

Using the relation $T \cdot \mathrm{adjT} = \det(\mathrm{T})\mathrm{I_r}$, where T is a matrix, we get $\det(\alpha)M = 0$. Since $\alpha = I_r$ modulo I, we get $\det(\alpha) = 1$ modulo I. Hence, there exists an $x \in I$ such that $(1-x)M = 0$. This proves the lemma.

□

Theorem 2.2.4. ***Second Version of Nakayama Lemma:***

Let A be a ring and M be a finitely generated A-module. I is an ideal contained in the Jacobson radical of A such that $IM = M$. Then $M = 0$.

Proof. Using the first version of the lemma, we have $(1-x)M = 0$ for $x \in I$. As $I \subset \mathrm{Jac}(A)$, we have $1-ax$ is a unit in A for $a \in A$. In particular, $1-x$ is a unit in A. □

Corollary 2.2.5. *Let A be a ring and M be a finitely generated A-module. Let N be an A-submodule of M and I, an ideal contained in the Jacobson radical of A. If $M = IM + N$, then $M = N$.*

Proof. The proof follows by applying Nakayama Lemma to the module M/N. □

For further reference, consider [3], [1] and [2].

2.3 Noetherian and Artinian Rings

We will define the concept of Noetherian rings and Noetherian modules which were defined by Emmy Noether and the concept of Artinian rings and Artinian modules defined by Emil Artin.

We will start the section by defining Noetherian modules.

Definition 2.3.1. Let A be a ring. An A-module M is called *Noetherian* if it satisfies the following equivalent conditions:

1. Any non-empty collection of submodules of M has maximal element.
2. Any ascending chain of submodules of M is stationary. We call it as the ascending chain condition, in short ACC.
3. Every submodule of M is finitely generated.

Definition 2.3.2. A ring A is called *Noetherian ring* if A is Noetherian as an A-module. In other words A satisfies the following equivalent conditions:

1. Any non-empty collection of ideals of A has maximal element.
2. Any ascending chain of ideals of A is stationary.
3. Every ideal of A is finitely generated.

Examples:

- The ring $\mathbb{Z}$ is Noetherian as every ideal is principal.
- The ring $k[X]$, where k is a field is Noetherian.
- A Dedekind domain is Noetherian.
- The polynomial ring $k[X_1, X_2, \ldots]$ is not Noetherian.

Some properties of Noetherian rings and modules:

1. Let A be a ring, M an A-module and N an A-submodule of M. Then, M is Noetherian if and only if N and M/N are Noetherian.
2. For A-modules $M_1, \ldots, M_n$, the direct sum $\oplus_{i=1}^{n} M_i$ is Noetherian if and only if each M_i is Noetherian.
3. Let A be a Noetherian ring and M is a finitely generated A-module. The, M is Noetherian.
4. Let A be a Noetherian ring. Then, the polynomial ring $A[X]$ is also Noetherian.

5. Let A be a Noetherian ring and $I \subset A$ is an ideal. Then, the radical of I is a finite intersection of all prime ideals of A.

We will now define Artinian rings and modules

Definition 2.3.3. Let A be a ring. An A-module M is called *Artinian* if it satisfies the following equivalent conditions:

1. Any non-empty collection of submodules of M has minimal element.
2. Any descending chain of submodules of M is stationary. We call it as the descending chain condition, in short DCC.

Definition 2.3.4. A ring A is called *Artinian ring* if A is Artinian as an A-module. In other words A satisfies the following equivalent conditions:

1. Any non-empty collection of ideals of A has minimal element.
2. Any descending chain of ideals of A is stationary.

Examples:

- Any finite abelian group as a $\mathbb{Z}$-module is Artinian.
- The ring $k[X]/(X^n)$, where k is a field and n is a positive integer is Artinian.
- The ring $k[X]$, where k is a field is not Artinian.
- The polynomial ring $k[X_1, X_2, \ldots]$ is not Artinian.

Some properties of Artinian rings and modules:

1. Let A be a ring, M an A-module and N an A-submodule of M. Then, M is Artinian if and only if N and M/N are Artinian.
2. For A-modules $M_1, \ldots, M_n$, the direct sum $\oplus_{i=1}^{n} M_i$ is Artinian if and only if each M_i is Artinian.
3. Let A be a Artinian ring and M is a finitely generated A-module. Then, M is Artinian.
4. An Artinian domain is a field.

5. In an Artinian ring, every prime ideal is maximal.

6. An Artinian ring is semilocal.

7. A ring A is Artinian if and only if it Noetherian and $\dim(A) = 0$.
   Every Artinian module is not necessarily Noetherian.

For further reference, consider [3], [1] and [2].

2.4 Krull Dimension

In this section, we will study the concept of the Krull dimension of a ring A. Moreover, we will understand the relationship between the Krull dimension of a ring with the Krull dimension of the polynomial ring. More specifically, we will prove that for a Noetherian ring A, $\dim(A[X]) = \dim(A) + 1$.

Definition 2.4.1. Let A be a ring. Consider the chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n.$$

The integer n is called the length of the chain. We define the *Krull dimension* of the ring A as the supremum of length of chains of prime ideals in A. We will denote it by $\dim(A)$.

Definition 2.4.2. Let $\mathfrak{p}$ be a prime ideal in the ring A. The *height* of $\mathfrak{p}$ denoted by $\mathrm{ht}(\mathfrak{p})$ is defined as the supremum of length of chains of prime ideals in A which end at $\mathfrak{p}$.

Lemma 2.4.1. *Let A be a ring and $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ be prime ideals is the polynomial ring $A[X]$ such that $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3$. Then, $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A = \mathfrak{p}_3 \cap A$ is false.*

Proof. We will prove by contradiction. Assume that, $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ be prime ideals is the polynomial ring $A[X]$ such that $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3$. Then, $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A = \mathfrak{p}_3 \cap A$. By going modulo $\mathfrak{p}_1 \cap A$, we get $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A = \mathfrak{p}_3 \cap A = 0$. Here, we are considering the ring A to be a domain. We will now consider the localisation of A at $S = A - \{0\}$. We have seen that there exists a one-to-one correspondence between the prime ideals of $S^{-1}A[X]$ and the prime ideals of $A[X]$ that have an empty intersection with S. Hence, we have $S^{-1}\mathfrak{p}_1 \subsetneq S^{-1}\mathfrak{p}_2 \subsetneq S^{-1}\mathfrak{p}_3$. □

Lemma 2.4.2. *Let A be a Noetherian ring and I be an ideal of $A[X]$ with* $\mathrm{ht(I)} = \mathrm{n}$. *Then,* $\mathrm{ht\,(I \cap A)} \geq \mathrm{n} - 1$.

Theorem 2.4.3. *If A is a Noetherian ring, then* $\dim(A[X]) = \dim(A) + 1$.

Proof. Consider the quotient ring $\frac{A[X]}{\mathfrak{m}[X]} \cong \frac{A}{\mathfrak{m}}[X]$ for a maximal ideal $\mathfrak{m}$ of A. As the quotient ring is not a field, we have $\mathfrak{m}[X]$ is not a maximal ideal in $A[X]$. Hence, we have $\dim(A[X]) \geq \dim(A) + 1$. Let us assume that $\dim(A[X]) > \dim(A) + 1$, where $\dim(A) = d$. Let $\mathfrak{M}$ be a maximal ideal of $A[X]$ such that $\mathrm{ht}(\mathfrak{M}) > \mathrm{d}$. Using the Lemma 2.4.2, we have $\mathrm{ht}(\mathfrak{M} \cap \mathrm{A}) > \mathrm{d}$. This leads to a contradiction of the claim that $\dim(A) = d$. Hence, $\dim(A[X]) = \dim(A) + 1$. □

2.5 Integral Extensions

We will define the notion of integral extensions in this section.

Definition 2.5.1. Let A be a subring of B such that $1 \in A$. We call an element $\alpha \in B$ to be *integral* over A if α is the root of a monic polynomial in $A[X]$.

Definition 2.5.2. Let A be a subring of B such that $1 \in A$. The ring B is called an *integral extension* of A if every $b \in A$ is integral over A.

Definition 2.5.3. The *integral closure* of A in B is the set of all elements in B that are integral over A. A ring A is said to be *integrally closed* in B if its integral closure is equal to itself.

Examples:

1. Every element of the ring A is integral over A.
2. Let K be the field extension of F and $\alpha \in K$ be algebraic over the field F. Then, α is integral over F.
3. Let $A = \mathbb{Z}$ and $B = \mathbb{Z}[i]$, the ring of Gaussian integers. Then, $\alpha = 1 + i$ is integral over A.

Theorem 2.5.1. *Let A be a subring of B such that $1 \in A$ and $\alpha \in B$. Then, the following are equivalent:*

1. *α is integral over A.*

2. *$A[\alpha]$ is a finitely generated A-module.*

3. *$A[\alpha]$ is contained in some ring C, that is a finitely generated A-module such that $A \subseteq C \subseteq B$.*

Proof. (1) $\Rightarrow$ (2). If α is integral over A then α is the root of a polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_i \in A$. Therefore,

$$\alpha^n = -a_{n-1}\alpha^n - \cdots - a_0.$$

Thus, all powers of α lie in the A-module generated by $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$. Hence, $A[\alpha]$ is finitely generated by the elements $1, \alpha, \alpha^2, \cdots, \alpha^{n-1}$ as an A-module.

(2) $\Rightarrow$ (3). Take $C = A[\alpha]$.

(3) $\Rightarrow$ (1). Let $c_1, \ldots, c_n$ generate C as an A-module. Let $\alpha c_i = \sum_{j=1}^{r} \lambda_{ij} c_j$, where $\lambda_{ij} \in A$. Let

$$T = \begin{pmatrix} \alpha - \lambda_{11} & -\lambda_{12} & \cdots & -\lambda_{1r} \\ -\lambda_{21} & \alpha - \lambda_{22} & \cdots & -\lambda_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda_{r1} & -\lambda_{r2} & \cdots & \alpha - \lambda_{rr} \end{pmatrix}$$

Using the same steps used in the proof of Nakayama lemma, we get $\det(T)C = 0$. As $1 \in C$, we get $\det(T) = 0$. Calculating the determinant gives us a monic polynomial, leading us to (1). □

Some properties of Integral Extensions:

1. Suppose B is an integral extension of A and $I \subset A$ an ideal. Then, $B/A \cap I$ is an integral extension of A/I.

2. Suppose B is an integral extension of A and S is a multiplicatively closed set in A. Them, $S^{-1}B$ is an integral extension of $S^{-1}A$.

3. Suppose B is an integral extension of A, where A, B are domains. Then, A is a field if and only if B is a field.

4. Suppose B is an integral extension of A; let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{q} \cap A$. Using Properties 2 and 3, we get that $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.

5. If A is a unique factorisation domain, then A is integrally closed.
   An example: the polynomial ring $k[X,Y]$ is integrally closed over its fraction field $k(X,Y)$, where k is a field.

Theorem 2.5.2. *Suppose B is an integral extension of A and let $\mathfrak{p}$ be a prime ideal of A. Then there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$.*

Proof. We know by Property 2 that $B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$. Hence, the following diagram is commutative:

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A_{\mathfrak{p}} & \longrightarrow & B_{\mathfrak{p}} \end{array}$$

Let $\mathfrak{n}$ be a maximal ideal of $B_{\mathfrak{p}}$. Then, $\mathfrak{m} = \mathfrak{n} \cap A_{\mathfrak{p}}$ is a maximal ideal of $A_{\mathfrak{p}}$. We can now construct $\mathfrak{q}$ as the preimage under the map β of the maximal ideal $\mathfrak{n}$. Thus, $\mathfrak{q} = \beta^{-1}(\mathfrak{n})$ is a prime ideal in A such that $\mathfrak{q} \cap A = \alpha^{-1}(\mathfrak{m}) = \mathfrak{p}$. □

Theorem 2.5.3. ***Going-up theorem:***

Suppose B is an integral extension of A. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_m$ be a chain of prime ideals of B $(m \leq n)$, such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ $(1 \leq i \leq m)$. Then, the chain $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_m$ can be extended to $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_m \subseteq \cdots \subseteq \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ $(1 \leq i \leq n)$.

Proof. We will use induction on m and n. Therefore, we will have to first prove for the case when $m = 1, n = 2$.

Using Property 1, $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. Using the above theorem, there exists a prime ideal, say $\bar{\mathfrak{q}_2}$ of $B/\mathfrak{q}$ such that $\bar{\mathfrak{q}_2} \cap (A/\mathfrak{p}) = \bar{\mathfrak{p}_2}$, the image of $\mathfrak{p}_2$ in $A/\mathfrak{p}$. By lifting $\bar{\mathfrak{q}_2}$ to B, we get a prime ideal $\mathfrak{q}$ with the required properties. □

We will end this section by stating the going-down theorem:

Theorem 2.5.4. ***Going-down theorem:***

Suppose A, B are integral domains such that B is an integral extension of A and A is integrally closed. Let $\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \cdots \supseteq \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{q}_1 \supseteq \mathfrak{q}_2 \supseteq \cdots \supseteq \mathfrak{q}_m$ be a chain of prime ideals of B $(m < n)$, such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ $(1 \leq i \leq m)$. Then, the chain $\mathfrak{q}_1 \supseteq \mathfrak{q}_2 \supseteq \cdots \supseteq \mathfrak{q}_m$ can be extended to $\mathfrak{q}_1 \supseteq \mathfrak{q}_2 \subset \cdots \supseteq \mathfrak{q}_m \supseteq \cdots \supseteq \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ $(1 \leq i \leq n)$.

For further reference, consider [3], [1] and [2].

Chapter 3

Projective Modules and the Grothendieck Group K_0

3.1 Projective Modules

Definition 3.1.1. A module M over a ring A is *stably free* if there exists a finitely generated free module F such that $M \oplus F$ is free.

Thus, when M is finitely generated and stably free, then for some k; $A^k \oplus M$ is finitely generated and free, so $A^k \oplus M = A^n$ for some n.

The concept of a projective module over a ring can be considered as a generalisation of free modules. There are different ways to define projective modules. We will see these definitions and prove their equivalence.

Projective Module as Direct Summand of a Free Module:

Definition 3.1.2. A *projective module* is a direct summand of a free module.

Thus, a module P is called a projective module if for a free module F, there exists a module L such that $P \oplus L = F$.

A module P is called a finitely generated projective module if there exists another module L such that $P \oplus L = A^n$ for some n.

Projective Modules and Split Exact Sequence:

Definition 3.1.3. Given M, N and P are R-modules. Consider the following short exact sequence:

$$0 \to L \xrightarrow{f} F \xrightarrow{g} P \to 0$$

- f, g are R-homomorphisms.
- Ker g = Im f
- f: injection
  g: surjection

The module P is *projective* if the sequence is split exact, that is, there exists a map $h : P \to F$ such that $gh = 1_P$.

This implies that P is a direct summand of F.

$$F = \text{Im(h)} \oplus \text{Ker(g)}$$

We also note that Im $(h) \cong P$ as h is injective.

Projective Modules and Hom Functor:

Given M, N and P are A-modules. We can define the Hom functor as follows:

Consider the set S of all A-homomorphisms of M into N. This set has a A-module structure for the given operations,

$$(f+g)(x) = f(x) + g(x) \quad \text{for} \quad x \in M \quad \text{and} \quad f, g \in S$$

$$(af)(x) = af(x) \quad \text{for} \quad a \in A \quad \text{and} \quad x \in M, f \in S$$

We will denote this module by $\text{Hom}_A(M, N)$.

Let M be a A-module. A homomorphism $f : N \to N$ of A-modules will induce a homomorphism:

$$f^\star : \text{Hom}_A(M, N) \to \text{Hom}_A(M, N')$$

$$f^\star(\alpha) = f\alpha$$

We note that $(gf)^\star = g^\star f^\star$ where $g \in \text{Hom}_A(N', N'')$.

Similarly, for a module N and a homomorphism $g : M \to M'$, there exists an A-module homomorphism:

$$g^\star : \mathrm{Hom}_A(M', N) \to \mathrm{Hom}_A(M, N)$$

$$g^\star(\beta) = \beta g$$

For an A-module M and an exact sequence:

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

we have the induced sequence,

$$0 \to \mathrm{Hom}_A(M, N') \xrightarrow{f^\star} \mathrm{Hom}_A(M, N) \xrightarrow{g^\star} \mathrm{Hom}_A(M, N'') \to 0.$$

This sequence is exact.

Similarly, for an A-module N and an exact sequence:

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

we have the induced sequence,

$$0 \to \mathrm{Hom}_A(M'', N) \xrightarrow{f^\star} \mathrm{Hom}_A(M, N) \xrightarrow{g^\star} \mathrm{Hom}_A(M', N) \to 0.$$

This sequence is exact.

Definition 3.1.4. An A-module P is *projective* if given a surjective homomorphism $g : M \to M''$, then the induced homomorphism:

$$g^\star : \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, M'')$$

is surjective.

Assume that P is free with basis $\{e_i\}$. Given $\alpha \in \mathrm{Hom}_A(P, M'')$, let $\alpha(e_i) = x_i$. Since, g is surjective, choose $y_i \in M$ with $g(y_i) = x_i$. Then the map $\beta : P \to M$, defined by $\beta(e_i) = y_i$ can be extended to an $A-$linear map β by defining $\beta(\Sigma a_i x_i) = \Sigma a_i y_i$. Here, $g^\star(\beta) = \alpha$. Hence, $g^\star$ is surjective. If P is projective (direct summand of a free module), then there exists an A-module Q such that $P \oplus Q = F$ for some free

module F. Let $\pi : F \to P$ be the projection map. Let $\alpha \in \mathrm{Hom}_A(F, M'')$, consider the $\alpha\pi \in \mathrm{Hom}_A(F, M'')$. There exists $\beta \in \mathrm{Hom}_A(F, M)$ with $g^\star(\beta) = \alpha\pi$. Then, $\beta_1 : P \to M$ has the property that $g^\star(\beta_1) = \alpha$. Hence, $g^\star$ is surjective. Conversely, assume that P has the Hom Functor property. Hence there is a surjective map: $g : F \to P$. Then the induced map : $g^\star : \mathrm{Hom}_A(P, F) \to \mathrm{Hom}_A(P, P)$ is surjective. In particular, there exists $\beta \in \mathrm{Hom}_A(P, F)$ with $g^\star(\beta) = g\beta = Id_P$ This shows that the exact sequence

$$0 \to \mathrm{Ker}\ g \to F \xrightarrow{g} P \to 0$$

splits. Hence, P is isomorphic to a direct summand of F.

Projective Modules and Lifting Property:

Definition 3.1.5. A module P is *projective* if, for every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $fh = g$.

We note that the lifting homomorphism h need not be unique.

Idempotents and Projective Modules: (*cf:*[7])

Definition 3.1.6. An element $e \in A$ is called an *idempotent* if $e^2 = e$.

Theorem 3.1.1. *If $e \in A$ is idempotent, then the module $P = eA$ is projective. Conversely, if $P \oplus Q \cong A$, then there exists elements $e \in P$ and $f \in Q$ such that $e + f = 1$ and e, f are idempotents in A.*

Proof. $P = eA$ is an A-module. Consider the set $X = eA \oplus (1 - e)A$. Thus, we get, $1 = e + 1 - e \in A$. Therefore, $X = A$. Hence, P is a projective module.

As $1 \in A$, $1 = e + f$ for unique elements $e \in P$ and $f \in F$. □

As we have assumed our ring A to be commutative the projective module $P = eA$ cannot be free as $P(1 - e) = 0$ but $A(1 - e) \neq 0$.

Theorem 3.1.2. *If A is a local ring, then every finitely generated projective A-module is free.*

Proof. Let $\mathfrak{m}$ be the maximal ideal in A. Consider the element $u \in A - \mathfrak{m}$. It is a unit in A. Since Jac(A) = m, $u \in 1 + \mathfrak{m}$ is also a unit. Hence, I is a radical ideal in A.

Suppose that $P \oplus Q \cong A^n$. Consider $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ as vector spaces over the field $F = A/\mathfrak{m}$. Hence, $P/\mathfrak{m}P \cong F^p$ and $Q/\mathfrak{m}Q \cong F^q$ for some p and q. Since $F^p \oplus F^q \cong F^n$, we have $p+q = n$. Choose a basis of P and Q mapping to the basis of $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$. Let the basis of P and Q be respectively $e_1, \ldots, e_p$ and $f_1, \ldots, f_q$. These bases determine the homomorphism

$$A^p \oplus A^q \to P \oplus Q \cong A^n$$

We thus have a square matrix (a_{ij}) whose reduction over F is invertible. Therefore, using the pullback of the surjection $GL_n(A) \to GL_n(A/I)$, we get that (a_{ij}) is invertible over A. Hence, $(e_1, \ldots, e_p, f_1, \ldots, f_q)$ is a basis for $P \oplus Q$ and hence, P is free on basis $(e_1, \ldots, e_p)$. □

Corollary 3.1.3. *If $\mathfrak{p}$ is a prime ideal in A and P is a finitely generated projective A-module, then the localisation $P_\mathfrak{p} \cong (A_\mathfrak{p})^n$ for some $n \geq 0$.*

Theorem 3.1.4. *Let A be a Dedekind domain. An ideal I of A is finitely generated and projective over A.*

Conversely, every projective module over A can be expressed as a direct sum of ideals in A.

Proof. Let I be an ideal in A. We will choose an ideal generated by $a \in A$ such that $aA \subseteq I$. There exists an ideal J such that $aA = IJ$. Therefore, we can express $a = e_1f_1 + \cdots + e_kf_k$ with $e_i \in I$ and $f_i \in J$. Using this element, we will define the following homomorphisms:

$$I \to A^k$$

$$e \mapsto (e_1f_1/a, ..., e_kf_k/a) \quad \text{and}$$

$$A^k \to I$$

$$(x_1, ..., x_n) \mapsto e_1x_1 + \cdots + e_kx_k$$

The composition of the two maps is the identity map and hence, I is a finitely generated projective module.

Conversely, consider a projective A-module P and the embedding $f : P \to A^n$ for some n with $\mathrm{Ker}(f) \subseteq A^{k-1}$. $\mathrm{Im}(f)$ is an ideal and hence $P \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$. We will now use induction and get the proof that $P \cong I_1 \oplus \cdots \oplus I_k$ for I_j ideals in A. □

Free $\implies$ Stably free $\implies$ Projective:

Free Modules $\implies$ Stably Free Modules: $A^n \cong A^{n-m} \oplus A^m$.

Stably Free Modules $\implies$ Projective Modules: If M is stably free, then $M \oplus A^k \cong A^n$ and A^k is a A module.

Stably free but not Free Module:

Consider the ring $A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$, the ring of polynomial functions over real numbers on the 2-sphere. The map:

$$f : A\frac{\partial}{\partial X} \oplus A\frac{\partial}{\partial Y} \oplus A\frac{\partial}{\partial Z} \to A, \text{where}$$

$$f\left(\frac{\partial}{\partial X}\right) = X; f\left(\frac{\partial}{\partial Y}\right) = Y; f\left(\frac{\partial}{\partial Z}\right) = Z$$

is surjective as $X^2 + Y^2 + Z^2 - 1 = 0$. Here, by $\frac{\partial}{\partial X}$ we mean the partial derivative with respect to X.

Let P be the kernel of this map f. We, therefore, get a short exact sequence:

$$0 \to P \to A\frac{\partial}{\partial X} \oplus A\frac{\partial}{\partial Y} \oplus A\frac{\partial}{\partial Z} \to A.$$

As P is projective, the sequence splits and therefore we have $P \oplus A \cong A^3$. The splitting map is given by

$$A \to A^3, \text{ where}$$
$$1 \mapsto X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}.$$

Hence, P is a projective stably-free module.

But P is not free because the unimodular row is not completable.(See [3])

Projective but not Free Module:

Consider the quotient ring $A = \frac{\mathbb{Z}}{6\mathbb{Z}}$. Let $P = \mathbb{Z}/2\mathbb{Z}$ and $Q = \mathbb{Z}/3\mathbb{Z}$ be an A-module. P is projective because $P \oplus Q \cong A$. However, P is not free as any free module over A will have atleast 6 elements.

3.2 Construction of Projective Modules

We have used ideas from the book by Milnor (*cf:*[6]). Let us consider the ring homomorphisms as follows:

$$\begin{array}{ccc} A & \xrightarrow{i_1} & A_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ A_2 & \xrightarrow{j_2} & A' \end{array}$$

satisfying the following two properties:

1. The ring A is a product of the rings A_1 and A_2 over the ring A'.
   This means, given $a_1 \in A_1$ and $a_2 \in A_2$ with $j_1(a_1) = j_2(a_2)$ in A', there exists a unique element $a \in A$ such that $i_1(a) = a_1$ and $i_2(a) = a_2$.

2. From the two homomorphisms j_1 and j_2 atleast one is surjective.

Construction:

Let P_1 be a projective module over A_1 and P_2 be a projective module over A_2. Let $j_1[P_1]$ and $j_2[P_2]$ be the induced modules over A' induced by the homomorphisms j_1 and j_2 respectively. Let h be an isomorphism: $h : j_1[P_1] \to j_2[P_2]$.

Let $M = (P_1, P_2, h)$ denote the subgroup of $P_1 \times P_2$ consisting of all pairs (p_1, p_2) with $hj_{1\star}(p_1) = j_{2\star}(p_2)$. (*Note*: $j_{1\star}$ denotes the canonical map : $P_1 \to j_1[P_1]$ defined by $j_{1\star}(p_1) = 1 \otimes p_1$.) Thus, we can define the following diagram:

$$\begin{array}{ccc} M & \longrightarrow & P_1 \\ \downarrow & & \downarrow{\scriptstyle hj_{1\star}} \\ P_2 & \xrightarrow{j_{2\star}} & j_2[P_2] \end{array}$$

This diagram satisfies condition 1 and 2.

We define M as follows:
$a \cdot (p_1, p_2) = (i_1(a) \cdot p_1, i_2(a) \cdot p_2)$. This gives us a well defined map for M.

Theorem 3.2.1. *The module $M = M(P_1, P_2, h)$ is projective over A. Moreover, if P_1 and P_2 are finitely generated over A_1 and A_2 respectively, then M is finitely generated over A.*

Theorem 3.2.2. *Every projective A-module is isomorphic to $M(P_1, P_2, h)$ for suitable P_1, P_2 and h.*

Theorem 3.2.3. *The modules P_1 and P_2 are isomorphic to $i_1[M]$ and $i_2[M]$ respectively.*

We will start by proving a special case of the first theorem, when P_1 and P_2 are free modules.

Theorem 3.2.4. *For free modules P_1 and P_2, if j_2 is surjective, then the module $M = M(P_1, P_2, h)$ is projective over A.*

Proof. Choose a basis $\{x_\alpha\}$ for P_1 and $\{y_\beta\}$ for P_2. These bases will determine the bases for $j_1[P_1]$ and $j_2[P_2]$ namely $\{j_{1\star}x_\alpha\}$ and $\{j_{2\star}y_\beta\}$ respectively over the ring A'. The isomorphism

$$h : j_1[P_1] \to j_2[P_2]$$

can be described by the following matrix $A = a_{\alpha\beta}$, where

$$h\left(j_{1\star}x\alpha\right) = \sum a_{\alpha\beta}\left\{j_{2\star}y_\beta\right\}$$

We can see that such a matrix can occur in the construction if and only if it is invertible.

If the matrix $A = a_{\alpha\beta}$ is an image of an invertible matrix, then M is free as:

$$y'_\alpha = \sum c_{\alpha\beta}y_\beta, \text{ where}$$

$$a_{\alpha\beta} = j_2 c_{\alpha\beta}$$

The elements $\{y'_\alpha\}$ form a basis for P_2. Now,

$$z_\alpha = (x_\alpha, y'_\alpha) \in M \in P_1 \times P_2$$

Thus M is free with the basis $\{z_\alpha\}$.

Now, we will consider Q_1 and Q_2 with a basis element u_β for each y_β and v_α for each x_α. Let, $g : j_1[Q_1] \to j_2[Q_2]$ be an isomorphism with A^{-1}. Then,

$$M(P_1, P_2, h) \oplus M(Q_1, Q_2, g) \cong M(P_1 \oplus Q_1, P_2 \oplus Q_2, h \oplus g),$$

where $h \oplus g$ corresponds to :

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

over A'. As

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^{-1} & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Since j_2 is surjective, we can lift

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$$

to some invertible matrix of the form

$$\begin{pmatrix} I & \star \\ 0 & I \end{pmatrix}$$

Similarly, we can lift the other factors into invertible matrices which proves our proposition. □

We will now prove the general case of Theorem 3.2.1. We will use the ideas of the proof of the special case.

We first prove the following proposition:

Proposition 3.2.5. *There exist projective modules Q_1 and Q_2 over A_1 and A_2 respectively so that $P_1 \oplus Q_1$ and $P_2 \oplus Q_2$ are free. Moreover $j_1[Q_1] \cong j_2[Q_2]$.*

Proof. $P_1 \oplus N_1$ is free of rank r over A_1 for some module N_1.

$$P_1 \oplus N_1 \cong A_1^r$$

Similarly,

$$P_2 \oplus N_2 \cong A_2^s$$

$$\text{Let} \quad P' = j_1[P_1] \cong j_2[P_2]$$

Therefore,

$$P' \oplus j_1[N_1] \cong (A')^r \quad \text{and} \quad P' \oplus j_2[N_2] \cong (A')^s$$

Thus,

$$j_1[N_1] \oplus (A')^s \cong j_1[N_1] \oplus P' \oplus j_2[N_2] \cong j_2[N_2] \oplus (A')^r$$

By defining

$$Q_1 = N_1 \oplus (A_1)^s \quad \text{and} \quad Q_2 = N_2 \oplus (A_2)^r$$

it follows that

$$j_1[Q_1] \cong j_2[Q_2].$$

□

Proof. Proof of Theorem 3.2.1
We will choose Q_1 and Q_2 as above. Let q be an isomorphism

$$q : j_1[Q_1] \to j_2[Q_2]$$

Now,

$$M(P_1, P_2, h) \oplus M(Q_1, Q_2, k) \cong M(P_1 \oplus Q_1, P_2 \oplus Q_2, h \oplus q)$$

as j_2 is surjective. Hence, $M(P_1, P_2, h)$ is projective.

If P_1 and P_2 are finitely generated, we can choose Q_1 and Q_2 as finitely generated. In the above proof, we have constructed Q_1 and Q_2 and this construction preserves finite generation. Hence, $M(P_1, P_2, h)$ is a direct summand of a finitely generated free module. Therefore, it is finitely generated. □

Proof. Proof of Theorem 3.2.2
If P_1 is projective over A, let

$$P_1 = i_1[P] \quad , \quad P_2 = i_2[P]$$

We can define the isomorphism:

$$h : j_1[P_1] \to j_2[P_2]$$

using the result $j_1 i_1 = j_2 i_2$. Consider the diagram:

$$\begin{array}{ccc} P & \longrightarrow & P_1 \\ \downarrow & & \downarrow{\scriptstyle hj_{1\star}} \\ P_2 & \xrightarrow{j_{2\star}} & j_2[P_2] \end{array}$$

The diagram satisfies Hypothesis 1 and 2 for the analogous case. Hence,

$$P \cong M(P_1, P_2, h).$$

□

Proof. Proof of Theorem 3.2.3
Consider any general $M = M(P_1, P_2, h)$. Using the natural map $M \to P$, we get an A_1-linear map

$$f : i_1[M] \to P_1$$

Our aim is to prove that f is an isomorphism.

As the proof of the theorem 1 consisted in showing that every $M(P_1, P_2, h)$ is a direct summand of some $M(P_1 \oplus Q_1, P_2 \oplus Q_2, h \oplus g)$, the map $f \oplus q$ associated with the direct sum is an isomorphism. This can be true only if f is an isomorphism, which completes the proof.

□

3.3 Grothendieck Group K_0

We will define the Grothendieck group K_0 for a ring A in this section. We have used the books on K-theory by Milnor [6] and Weibel [7] as references.

Definition 3.3.1. Let A be a commutative ring. We say that two projective modules are isomorphic if we can find a one-one and onto homomorphic map between them. Let $\mathcal{P}(A)$ be the set of isomorphism classes of finitely generated projective modules over the ring A. We will define $K_0(A)$ as the quotient group of the free abelian group of the generators $[P]$, where $[P] \in \mathcal{P}(A)$ modulo the group generated by the elements of the form $[P] + [Q]$ - $[P + Q]$.

Definition 3.3.2. The set $\mathcal{P}(A)$ is a monoid. The group $K_0(A)$ can be defined as the group completion of the monoid $\mathcal{P}(A)$. Using the definition of K_0, we get a surjective map

$$f : \mathcal{P}(\mathcal{A}) \to K_0(A)$$

$$f(P) = [P]$$

We say that A-modules M and N are stably isomorphic if given a fixed non-negative t, we have

$$A^t \oplus M \cong A^t \oplus N.$$

K_0 for fields:

In a field, modules are the same as vector spaces. Hence, an isomorphic class of finitely generated projective modules correspond to the class of vector space of a given rank. Thus $\mathcal{P}(A) \cong \mathbb{N}$. Thus, $K_0(F) = \mathbb{Z}$.

K_0 for local rings:

We have seen that every finitely generated projective module is free over a local ring. Therefore, $K_0(A)$, where A is a local ring is generated by $[A]$.

Consider the ring homomorphism

$$\phi : A \to A'$$

We have seen that every finitely generated A-module M has a corresponding A' module $M \otimes A'$.

Therefore, the isomorphism class of finitely generated A-modules has a correspondence with the isomorphism class of finitely generated A'-modules.

Hence, we will get a map

$$\phi_\star : K_0(A) \to K_0(A')$$

Moreover, $(\phi \odot \psi)_\star = \phi_\star \odot \psi_\star$.

The obvious example that comes to mind while understanding such a map would be considering the unique homomorphism i from $\mathbb{Z}$ to any ring A.

$$i_\star : K_0(\mathbb{Z}) \to K_0(A)$$

Definition 3.3.3. The cokernel of the above map $i_\star$ is called as the *projective class group* of a ring A.

$$\text{Projective Class Group} = K_0(A)/\langle A \rangle$$

Another standard example is the homomorphism from a ring to a field, which

induces the following map:

$$j_\star : K_0(A) \to K_0(F)$$

Hence, for this fixed homomorphism, we get the following split exact sequence:

$$0 \to \text{Im } (i_\star) \to K_0(A) \to \text{Ker } (j_\star) \to 0.$$

Hence, we get the relation

$$K_0(A) \cong \text{Im } (i_\star) \oplus \text{Ker } (j_\star)$$

Using the standard notation, we will denote Ker $(j_\star)$ as $\tilde{K_0}(A)$ which is an ideal of the ring A. Moreover, as the image of the map $i_\star$ is a group generated by the free module A, we have:

$$K_0(A) = \mathbb{Z} \oplus \tilde{K_0}(A)$$

Chapter 4

Unimodular Rows and the Whitehead Group K_1

4.1 Elementary Subgroup

We are all familiar with matrices. We know that the simplest matrix is the identity matrix. Using the identity matrix as a base we will try to generate some easy-looking matrices such that we have a non-zero element at *one* non-diagonal position. Note that the determinant of the matrix is 1.

In this section, we have referred to the material in [8].

Definition 4.1.1. Let A be a ring with 1. E_{ij} be a $n \times n$ matrix with 1 in the i,j position and 0 elsewhere. $e_{ij}(\lambda) = I_n + \lambda E_{ij}$, where $\lambda \in A$. The subgroup of $GL_n(A)$ generated by $e_{ij}(\lambda)$; $\lambda \in A$ is called as the *elementary subgroup:* $E_n(A)$.

Examples

1. For $n = 3$, $i = 1$, $j = 2$,
$$e_{13}(\lambda) = \begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. The off-diagonal 2×2 matrix
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

3.

$$\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1-a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= e_{12}(1-a)e_{21}(-1)e_{12}(1)$$

4. All $n \times n$ upper triangular matrices and lower triangular matrices with 1's at the diagonal position. For example,

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = e_{12}(2)e_{13}(-3)e_{23}(-1)$$

4.2 Properties of Elementary Subgroup

We know that the group of n × n matrices with determinant 1 viz. Special Linear group over a ring A is a subgroup of the group of n × n matrices with non-zero determinant viz. General Linear group over the ring A. The determinant of an elementary matrix is always 1. Therefore, the group of the elementary matrices over a ring A is a subgroup of the special linear group.

$$E_n(A) \subseteq SL_n(A) \subseteq GL_n(A).$$

Generator of $E_2(A)$:

$$E(a) = \begin{pmatrix} a & 0 \\ -1 & 0 \end{pmatrix}$$

$$E(a) = e_{12}(1-a)e_{21}(-1)e_{12}(1) \in E_2(A)$$

$E_2(A)$ is generated by $E(a)$; $a \in A$

We can give the following explicit relations which establish the result:

$$e_{12}(a) = E(-a)E(0)^{-1}$$
$$e_{21}(a) = E(0)^{-1}E(a)$$

Inverse in $E_n(A)$:
The inverse of the matrix $e_{ij}(\lambda)$ is $e_{ij}(-\lambda)$.

For example, the inverse of $e_{ij}(-3)$ is $e_{ij}(3)$

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Embedding:
The subgroup $E_n(A)$ can be embedded into the subgroup $E_{n+m}(A)$ for $n \geq 1, m \geq 1$ through the following map:

$$E_n(A) \to E_{n+m}(A) \quad \text{given by} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & I_m \end{pmatrix}$$

Definition 4.2.1. Consider the following sequence of embeddings:

$$GL_n(A) \to GL_{n+1}(A) \to \dots$$

We will define $GL(A)$ as follows:

$$GL(A) = \bigcup_{i=1}^{\infty} GL_n(A)$$

Similarly, we can define

$$SL(A) = \bigcup_{i=1}^{\infty} SL_n(A)$$
$$E(A) = \bigcup_{i=1}^{\infty} E_n(A).$$

$$e_{ij}(\lambda)A_{n\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1}+\lambda a_{j1} & \cdots & a_{in}+\lambda_{jn} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Splitting Property:

$e_{ij}(\lambda)\ e_{ij}(\mu) = e_{ij}(\lambda+\mu)$

For Example,

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Commutator Property:

Consider the elements $\lambda, \mu \in A$,

1. $[e_{ij}(\lambda), e_{kl}(\mu)] = I$ for $j \neq k, i \neq l$

2. $[e_{ij}(\lambda), e_{jk}(\mu)] = e_{ik}(\lambda\mu)$ for $i \neq k$

3. $[e_{ij}(\lambda), e_{ki}(\mu)] = e_{jk}(-\lambda\mu)$ for $j \neq k$

For $\lambda, \mu \in A$, $[e_{ij}(\lambda), e_{ij}(\mu)] = I$:

By Splitting Property, we have

$$e_{ij}(\lambda) \times e_{ij}(\mu) \times e_{ij}(-\lambda) \times e_{ij}(-\mu) = e_{ij}(\lambda+\mu+-\lambda+-\mu) = I.$$

$\lambda, \mu \in A$, $[e_{ij}(\lambda), e_{kl}(\mu)] = I$ for $j \neq k, i \neq l$:

$$\begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mu & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\mu & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mu & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\mu & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $\lambda, \mu \in A$, $[e_{ij}(\lambda), e_{jk}(\mu)] = e_{ik}(\lambda\mu)$ for $i \neq k$:

$$\begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \lambda & \lambda\mu & 0 \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda & \lambda\mu & 0 \\ 0 & 1 & -\mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \lambda\mu & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $\lambda, \mu \in A$, $[e_{ij}(\lambda), e_{ki}(\mu)] = e_{jk}(-\lambda\mu)$ for $j \neq k$:

$$\begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda\mu & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem 4.2.1. Whitehead Lemma:

For $\alpha \in GL_n(A)$, the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in E_{2n}(A)$.

Proof. We have,

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ \alpha^{-1} & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n - \alpha \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n - \alpha^{-1} \\ 0 & I_n \end{pmatrix}$$

Hence,

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in E_{2n}(A).$$

□

Theorem 4.2.2. Perfectness:

$$[GL(A), GL(A)] = E(A) = [E(A), E(A)]$$

Proof. Let $\alpha, \beta \in GL(A)$. Using the splitting properties and commutative properties, we can easily note that:

$$E(A) \subseteq [E(A), E(A)] \subseteq [GL(A), GL(A)] .$$

Consider the matrices α and $\beta \in GL_n(A)$, using the following calculations it is clear that

$$\begin{pmatrix} \alpha\beta\alpha^{-1}\beta^{-1} & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \alpha\beta & 0 \\ 0 & (\alpha\beta)^{-1} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix} \in E_{2n}(A)$$

using the Whitehead Lemma. Thus, $[GL(A), GL(A)] \subseteq E(A)$.

Therefore, $[GL(A), GL(A)] \subseteq E(A)$. □

Theorem 4.2.3. *For any ring A and any ideal $I \subset A$, the following map is commutative:*

$$\begin{array}{ccc} M_n(A) \times A^n & \longrightarrow & M_n(A/I) \times (A/I) \\ \downarrow & & \downarrow \\ A^n & \longrightarrow & (A/I)^n \end{array}$$

Proof. For the proof we refer [3]. □

Theorem 4.2.4. *For an ideal I in the ring A, the map $E_n(A) \to E_n(A/I)$ is surjective for any $n \geq 1$.*

Proof. Let $e_{ij}(\bar{\lambda})$ be the image of $e_{ij}\lambda$ under the map $E_n(A) \to E_n(A/I)$. Consider the generators $E_{ij}(\bar{\lambda})$ of the the group $E_n(A/I)$. □

We note that the map $SL_n(A) \to SL_n(A/I)$ is not surjective in general. (*cf:*[3], Appendix.)

4.3 Unimodular Rows

The concept of unimodular rows and their completion was developed mainly during the process of solving Serre's conjecture. We will develop this concept and understand various results on completion of unimodular rows.

We refer to the material on unimodular rows in [3] , [4] and [5].

Definition 4.3.1. Let A be a ring. A row $(a_1, a_2, ..., a_n) \in A^n$ is said to be *unimodular* (of length n) if the ideal $\langle a_1, a_2, ..., a_n \rangle = A$. The set of unimodular rows of length n is denoted by $Um_n(A)$.

Definition 4.3.2. A unimodular row $(a_1, a_2, ..., a_n)$ is said to be *completable* if there is a matrix in $GL_n(A)$ whose first row is $(a_1, a_2, ..., a_n)$.

A unimodular row is completable if and only if $(a_1, a_2, ..., a_n)$ is $GL_n(A)$ equivalent to $(1, 0, ..., 0)$.

We will state some examples which we will discuss with proofs:

1. Any unimodular row of length $n \geq 1$ is completable for the ring $\mathbb{Z}$.
2. Any unimodular row of length $n \geq 1$ is completable for a PID.
3. Any unimodular row of length $n \geq 1$ is completable for $k[X]$, where k is a field.

Equivalence between Completion of Unimodular Rows and Freeness of Stably Free Projective Modules:

One way to define the equivalence between the completion of unimodular rows and the freeness of finitely generated stably free projective modules is by defining the this module as the cokernel of the inclusion map that specifies the matrix.

$$\sigma : A \xrightarrow{(a_1 ... a_n)^t} A^n$$

is an injective map between free modules. Let P be the module defined by this unimodular row $(a_1, \ldots, a_n)$. Then, $P \oplus A \cong A^n$. The stably free module is free it means that $P \cong A^{n-1}$ and therefore $P \oplus A \cong A^n$ will generate a new basis for A^n. This means that there exists an invertible matrix with the first column as $(a_1, \ldots, a_n)^t$ that provides a map between the basis. In other words, the unimodular row is complete.

A second way to look at the same concept albeit in a more general setting is described as follows:

Consider a finitely generated stably free A-module P such that $P \oplus A^m \cong A^n$. We can say that every such stably free module of rank $n - m$ is the kernel of a surjective $m \times n$ matrix say

$$\sigma : A^n \to A^m$$

This is true again because a lift of the basis for A^m gives us a decomposition $P \oplus A^m \cong A^n$.

A special case of such a map is when σ is a row, also called as the unimodular row. Thus, we have $\sigma : A^n \to A$ as the surjective map. If P is free, we have a new basis for A^n in the form of a matrix say g whose first row is the row σ.

Therefore, there lies a correspondence between the freeness of finitely generated stably free projective modules and completion of unimodular rows to invertible matrices.

4.4 On Completion of Unimodular Rows

We state some specific cases when the unimodular row is completable. ([3])

Theorem 4.4.1. *Let A be a ring and $(a_1, ..., a_n)$ be a unimodular row of length n which contains a unimodular row of shorter length. Then the row $(a_1, ..., a_n)$ is completable.*

Proof. The unimodular row $(a_1, ..., a_n)$ contains a unimodular row of shorter length. Without loss of generality, let us assume that $(a_1, ..., a_{n-1})$ is unimodular. We can find $b_1, ..., b_{n-1} \in A$ such that

$$1 - a_n = a_1 b_1 + ... + a_{n-1} b_{n-1}.$$

$$1 = a_1 b_1 + ... + a_{n-1} b_{n-1} + a_n$$

$(a_1, ..., a_n)$ is completable as:

$$\begin{aligned}(a_1, a_2, ..., a_{n-1}, a_n) &\overset{E_n(A)}{\equiv} (a_1, a_2, ..., a_{n-1}, 1)\\ &\overset{E_n(A)}{\equiv} (0, 0, ..., 0, 1)\\ &\overset{E_n(A)}{\equiv} (1, 0, ..., 0, 1)\\ &\overset{E_n(A)}{\equiv} (1, 0, ..., 0, 0).\end{aligned}$$

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & -a_2 & \cdots & -a_{n-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$$

□

Proposition 4.4.2. *Any diagonal matrix in $SL_n(A)$ belongs to $E_n(A)$.*

Proof. Let us consider the diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

Let us denote $D = diag\,(d_1, d_2, \cdots, d_n)$, where each d_i is a unit, as $D \in SL_n(A)$. Using the following relation,

$$D = diag\left(d_1, d_1^{-1}, 1, \cdots, 1\right) \cdot diag\,(d_1, d_2, \cdots, d_n)$$

If we prove that $diag\,(d, d^{-1}) \in E_n(A)$, then we can prove our result using induction on n. We will prove our claim through the following elementary transformations:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} = \begin{pmatrix} 0 & d^{-1} \\ -d & 0 \end{pmatrix} \to \begin{pmatrix} 1 & d^{-1} \\ -d & 0 \end{pmatrix} \to \begin{pmatrix} 1 & d^{-1} \\ 0 & 1 \end{pmatrix}$$

This can be summarised explicity as:

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e_{21}(-d)e_{12}(d^{-1})e_{21}(-d)$$

□

The above result is a special case of Whitehead's lemma.

Theorem 4.4.3. *In a Euclidean domain A, every unimodular row is elementary completable. Consequently, $SL_n(A) = E_n(A)$.*

Proof. Consider the Euclidean norm, $\delta : A \to \mathbb{N}$. Let $(a_1, \ldots, a_n)$ be a unimodular row in the Euclidean domain and consider the set of $(a'_1, \ldots, a'_n)$ elementary equivalent to $(a_1, \ldots, a_n)$. Choose the element of this set with minimal $\delta(a'_1)$. If $a'_1 = 0$, then we are done by Theorem.. Else, we claim that $a'_1 \mid a'_j$ for $j \geq 2$. If not, we get that $a'_2 = ma'_1 + r$ such that $\delta(r) \leq \delta(a'_1)$. Thus we have a unimodular row $(r, -a'_1, ..., a'_n)$ which contradicts the minimality of a'_1. Therefore, we get

$$(a'_1, ..., a'_n) \overset{GL_n(A)}{\equiv} (a'_1, 0, ..., 0)$$

Hence, a'_1 is a unit. Let $M \in SL_n(A)$. Let $\alpha = e_1M$, then e_1M is a unimodular row of length n Thus, using elementary transformations as above, we have

$$M \overset{GL_n(A)}{\equiv} \begin{pmatrix} a'_1 & 0 & \ldots & 0 \\ \star & M' & & \\ \vdots & & & \end{pmatrix}$$

We argue that:

$$M \overset{E_n(A)}{\equiv} \begin{pmatrix} a'_1 & 0 & \ldots & 0 \\ 0 & M' & & \\ \vdots & & & \end{pmatrix} \overset{E_n(A)}{\equiv} \begin{pmatrix} a'_1 & 0 & \ldots & 0 \\ 0 & a'^{-1}_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \end{pmatrix}$$

Hence, using Whitehead lemma, $M' \in SL_n(A)$. Thus, by induction, $SL_n(A) \subset E_n(A)$. It is clear that $E_n(A) \subset SL_n(A)$. Therefore, $SL_n(A) = E_n(A)$ for Euclidean domains. □

Theorem 4.4.4. *If A is a local ring, then* any *unimodular row is completable.*

Proof. Consider the unimodular row $(a_1, ..., a_n)$.

Case 1: a_i is a unit for some i. This means that the row $(a_1, ..., a_n)$ contain a unimodular row of shorter length.

Case 2: None of the a_i's are units. As A is a local ring, $\langle a_1, a_2, ..., a_n\rangle \subset \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal. This is a contradiction. □

Theorem 4.4.5. *If A is a semilocal ring, then* any *unimodular row is completable for $n \geq 2$.*

Proof. Let $m_1, ..., m_k$ be the ideals of A. Consider the unimodular row $(a_1, ..., a_n)$.

Case 1: a_i is a unit for some i. This means that $(a_1, ..., a_n)$ contain a unimodular row of shorter length.

Case 2: None of the a_i's are units.

Using the Prime Avoidance Lemma, we can find $b_2, ..., b_n$ such that $d = a_1 + a_2b_2 + ... + a_nb_n \notin \cup\mathfrak{m}_i$. Hence, it is a unit.

$$\begin{aligned}(a_1, a_2, ..., a_n) &\overset{E_n(A)}{\equiv} (d, a_2, ..., a_n)\\ &\overset{E_n(A)}{\equiv} (d, 0, ..., 0)\\ &\overset{E_n(A)}{\equiv} (1, 0, ..., 0).\end{aligned}$$

□

Theorem 4.4.6. *If A is a* Noetherian ring *with* $\dim(A) = d$ *then* any *unimodular row of length is completable for $n \geq d + 2$.*

Proof. Let $(a_1, ..., a_n)$ be a unimodular row of length n.

$$(a_1, ..., a_n) \overset{E_n(A)}{\equiv} (b_1, ..., b_n)$$

where ht $\langle \mathrm{b_1}, ..., \mathrm{b_i}\rangle \geq \mathrm{i}\ \forall\ 1 \leq i \leq n$. Therefore ht $\langle \mathrm{b_1}, ..., \mathrm{b_{d+1}}\rangle \geq \mathrm{d}+1$. But $\dim(A) = d$. Hence, $(b_1, ..., b_{d+1})$ is unimodular. That is, $(b_1, ..., b_n)$ contains a unimodular row of shorter length. Hence, $(b_1, ..., b_n)$ is completable.

Infact,

$$(b_1, ..., b_n) \overset{E_n(A)}{\equiv} (1, 0, ..., 0).$$

Thus,

$$(a_1, ..., a_n) \overset{E_n(A)}{\equiv} (b_1, ..., b_n) \overset{E_n(A)}{\equiv} (1, 0, ..., 0).$$

□

Examples:

Consider the Noetherian ring $\mathbb{Z}$. $\dim(\mathbb{Z}) = 1$. Hence, any unimodular row of length ≥ 3 is completable.

Consider the ring $\mathbb{C}[X, Y]$. Here $\dim(\mathbb{C}[X, Y]) = 2$. Hence, $n \geq 4$, every unimodular row is completable. As $\mathbb{C}[X, Y]$ is a UFD, the result is also true for $n \geq 3$.

Theorem 4.4.7. *If A is a* Noetherian ring *and $n \geq \dim(A/Jac(A)) + 2$, then* any *unimodular row of length n is completable.*

Proof. Let $(a_1, ..., a_n)$ be a unimodular row of length n. Then, $(\bar{a_1}, ..., \bar{a_n})$ is a unimodular row in $\bar{A} = A/Jac(A)$. Given the conditions on n, $(\bar{a_1}, ..., \bar{a_n})$ is completable. We have,

$$(\bar{a_1}, ..., \bar{a_n}) \overset{E_n(\bar{A})}{\equiv} (\bar{1}, \bar{0}, ..., \bar{0}).$$

As the map $E_n(A) \to E_n(\bar{A})$ is surjective we get,

$$(a_1, ..., a_n) \overset{E_n(A)}{\equiv} (1 + c_1, c_2, ..., c_n)$$

for $c_1, ..., c_n \in Jac(A)$. As $c_1 \in Jac(A)$, $1 + c_1$ is a unit. Therefore,

$$(1 + c_1, c_2, ..., c_n) \overset{E_n(A)}{\equiv} (1, 0, ..., 0).$$

□

4.5 Whitehead group K_1

We have studied the structure of projective modules and the properties and structure of the elementary subgroup. Hence, we have the machinery to develop the theory of K_1. We will use the books by Milnor [6] and Weibel [7] for developing the theory of K_1.

Definition of K_1: For the ring A, we define the Whitehead group K_1 as follows

$$K_1(A) = \frac{GL(A)}{[GL(A), GL(A)]}$$

It follows from Whitehead lemma that $[GL(A), GL(A)] = E(A)$.
$K_1(A) = \frac{GL(A)}{E(A)}$.

Defining the group operation in K_1:

The first way is to define the multiplication in $GL(A)$ with identity I.

Another equivalent way is:

$$M \oplus N = \begin{pmatrix} MN & 0 \\ 0 & N^{-1}M^{-1} \end{pmatrix}$$

As,

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} MN & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix}$$

the two definitions are equivalent.

We have

$$[A \oplus B] = [AB \oplus I] = [AB]$$

If $\phi : A \to B$ is a ring homomorphism then by taking the induced map $GL(A) \to GL(B)$ and composing it with the quotient map we get a map $K_1(A) \to K_1(B)$. Hence, K_1 is a functor.

4.5.1 The subgroup SK_1

Let us consider the following map for a fixed n:

$$\det : GL_n(A) \to A^\times$$

$$\det(\alpha) = \det \alpha.$$

This induces the map:

$$GL_n(A)/E_n(A) \to R^\times$$

We know,

$$\frac{GL_n(A)}{E_n(A)} \to \frac{GL_{n+1}(A)}{E_{n+1}(A)} \to ...$$

Hence,

$$\sigma : K_1(A) \to R^\times$$

$$\sigma(\alpha E(A)) = \det \alpha.$$

We will define $SK_1(A) = \ker(\sigma)$.

Using this definition, we get the exact sequence:

$$1 \to SK_1(A) \to K_1(A) \to A^\times \to 1$$

This sequence splits given by the map:

$$A^\times \to K_1(A),$$

where a unit maps to the diagonal matrix:

$$\begin{pmatrix} u & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Hence,$K_1(A) = SK_1(A) \oplus A^\times$ and

$$SK_1(A) = SL(A)/E(A).$$

4.5.2 Computation of K_1

Theorem 4.5.1. *If F is a field, then $K_1 \cong F^\times$.*

Proof. The statement is equivalent to proving that $SK_1(F)$ is trivial. That is, $SL(F) = E(F)$. It is trivial to note that $E(F) \subseteq SL(F)$. We need to only prove that $SL(F) \subseteq E(F)$.

Let $M \in SL_n(F)$. Let a_{i1} be the first non-zero entry in the first column of M. Using elementary transformations, we can move it to a_{11}. The transformation is given by $e_{1i}(1)e_{i1}(-1)e_{1i}(1)$. Thus, we can assume that $a_{11} \neq 0$.
Now, using the transformations: $e_{n1}(a_{n1}a_{11}^{-1})...e_{21}(a_{21}a_{11}^{-1})$, we get the column $(1, 0, ..., 0)^t$.

This is the matrix:

$$\begin{pmatrix} a_{11} & \star \\ 0 & M' \end{pmatrix}$$

Repeating the same procedure for the matrix M', we get a diagonal matrix by induction.

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}.$$

Let

$$E_a^k = \begin{pmatrix} I_k & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in E(A).$$

$$E^0_{d_{nn}\ldots d_{22}} \ldots E^{n-2}_{d_{22}}.$$

$$D = \begin{pmatrix} d_{11}d_{22}\ldots d_{nn} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = D'$$

Hence, $D' \in E(A)$.

$\det(D') = \det(D) = \det(M)$. Hence, $\det(D') = 1$. That is $d_{11}d_{22}\ldots d_{nn} = 1$. Hence, $D = I_n$. Thus, $M \in E(A)$.

Thus, we have proved,

$$SL(F) \subseteq E(F).$$

□

4.5.3 K_1 and Completion of Unimodular Rows

Theorem 4.5.2. *Let A be a ring in which every unimodular row is completable. Then, $SL_n(A) = E_n(A)$. In other words, $SK_1(A)$ is trivial.*

Proof. Let $M \in SL_n(A)$. Using elementary transformations,

$$M \overset{E_n(A)}{\equiv} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \star & & & \\ \vdots & & & \\ \star & & & \end{pmatrix}$$

Using the result that a unimodular row containing a shorter row is completable, we have

$$M \overset{E_n(A)}{\equiv} \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$$

where N is a $(n-1) \times (n-1)$ matrix.
Applying induction on N, we have: $M \overset{E_n(A)}{\equiv} I_n$. Thus, $SL_n(A) = E_n(A)$ □

Corollary 4.5.3. *SK_1 is trivial for the following class of rings:*

- *Fields*
- *Euclidean domains*
- *Local rings*
- *Semi-local rings*

Proof. For the given class of rings, we have proved that every unimodular row is completable. □

Examples:

- $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{-1, 1\}$
- $K_1(\mathbb{Z}[i]) = \{-1, 1, -i, i\}$

Chapter 5

Results on Unimodular Rows

We will prove some important theorems on the completion of unimodular rows which are historically important in the context of the proof of Serre's problem. More information about Serre's problem and related results can be found in [3] and [5] which serve as a reference for this chapter.

5.1 Horrocks' Theorem

Theorem 5.1.1. *Let $(A, \mathfrak{m})$ be a local ring and $(f_1(X), ..., f_n(X))$ be a unimodular row in $A[X]$ with one entry monic. Then $(f_1(X), ..., f_n(X))$ is completable.*

Lemma 5.1.2. *Let k be a field and B a ring containing k such that B is finite dimensional k-vector space having dimension l say. Then, the number of maximal ideals of $B \leq l$.*

Proof. Let us assume that there are $l+1$ maximal ideals in B. They are:

$$\mathfrak{m}_1, ..., \mathfrak{m}_{l+1}.$$

We can easily check the following:

$$\mathfrak{m}_i + \mathfrak{m}_1\mathfrak{m}_2...\mathfrak{m}_{i-1}\mathfrak{m}_{i+1}...\mathfrak{m}_{l+1} = B.$$

As, sum of two ideals is an ideals and $\mathfrak{m}_i \subset \mathfrak{m}_i + \mathfrak{m}_1\mathfrak{m}_2...\mathfrak{m}_{i-1}\mathfrak{m}_{i+1}...\mathfrak{m}_{l+1}$, this ideal sum is an ideal which contains a maximal ideal. Hence, it is the whole ring B.

Hence, we can choose $c_i \in \mathfrak{m}_1\mathfrak{m}_2...\mathfrak{m}_{i-1}\mathfrak{m}_{i+1}...\mathfrak{m}_{l+1}$ such that $c_i - 1 \in \mathfrak{m}_i$. We claim that $c_1, ..., c_{l+1}$ are linearly independent over k.

Assume, to the contrary that c_i are linearly dependent i.e. $\sum_{i=1}^{l+1} a_i c_i = 0$ in B , where $a_i \in k$ and not all a_i are zero.

Without loss of generality, let us assume that $a_1 \neq 0$, so that $c_1 = -\sum_{i=2}^{l+1} a_1^{-1} a_i c_i$ showing that $c_1 \in \mathfrak{m}_1$. Since, $c_1 - 1 \in \mathfrak{m}_1$, we have $1 \in \mathfrak{m}_1$, a contradiction.

Thus, the claim $c_1, ..., c_{l+1}$ are linearly independent over k.

This claim leads to a contradiction. Hence, the lemma. □

Proof. **Proof of Theorem 5.1.1**:
Any unimodular row of length 2 is completable. Consider $n \geq 3$.

Without loss of generality, we assume that $f_1(X)$ is monic. Let $\deg(f_1(X)) = \mathrm{t}$ and $B = A[X]/\langle f_1(X)\rangle$. Thus, B is a finitely generated $A-$module generated by the images of $1, X, ..., X^{t-1}$. Hence, $A \to B$ is an integral extension. Hence, $B/\mathfrak{m}B$ is a finite dimensional $A/\mathfrak{m}$- vector space. $\mathfrak{m}B \neq B$ from Nakayama lemma. As $A \to B$ is an integral extension and $B/\mathfrak{m}B$ is a finite dimensional $A/\mathfrak{m}$- vector space, using the properties of integral extensions and module homomorphisms, we get that the maximal ideals of $B/\mathfrak{m}B$ are in one-to-one correspondence with the maximal ideals of B. Using the lemma, we know that the number of maximal ideals of $B/\mathfrak{m}B \leq l$, where $l = \dim B/\mathfrak{m}B$ and hence finite. Thus, B is semi-local.

Let us now consider the elements modulo $\langle f_1(X)\rangle$. As, B is semi-local,

$$\big(f_2(\bar{X}), ..., f_n(\bar{X})\big) \overset{E_{n-1}(B)}{\equiv} (\bar{1}, \bar{0}, ..., \bar{0}).$$

So, there exists $\alpha \in E_{n-1}(B)$ such that:

$$\alpha(f_2(\bar{X}), ..., f_n(\bar{X}))^t = (\bar{1}, \bar{0}, ..., \bar{0})$$

As the map $A[X] \to A[X]/\langle f_1(X)\rangle$ is surjective, we can lift α to $\sigma \in E_{n-1}(A[X])$. Thus,

$$\alpha(f_2(X), ..., f_n(X))^t = (1 + f_1(X)h_1(X), f_1(X)h_2(X), ..., f_1(X)h_{n-1}(X))^t$$

Here, $h_i(X) \in A[X]$ for $1 \leq i \leq n-1$.

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \\ \vdots \\ f_n(X) \end{pmatrix} = \begin{pmatrix} (f_1(X) \\ 1 + f_1(X)h_1(X) \\ f_1(X)h_2(X) \\ \vdots \\ f_1(X)h_{n-1}(X) \end{pmatrix} \overset{E_n(A[X])}{\equiv} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

□

5.2 Local-Global Principle

Theorem 5.2.1. *Suppose A is a ring and $(f_1(X), ..., f_n(X))$ is a unimodular row. If,*

$$(f_1(X), ..., f_n(X)) \overset{GL_n(A_{\mathfrak{m}}[X])}{\equiv} (f_1(0), ..., f_n(0))$$

where $A_{\mathfrak{m}}$ is the localisation at maximal ideal $\mathfrak{m}$, for every maximal ideal in A; then

$$f_1(X), ..., f_n(X)) \overset{GL_n(A[X])}{\equiv} (f_1(0), ..., f_n(0).$$

Remark 5.2.2. *In order to check that $(f_1(X), ..., f_n(X)) \overset{GL_n(A_{\mathfrak{m}}[X])}{\equiv} (f_1(0), ..., f_n(0))$, we have to check $(f_1(X), ..., f_n(X)) \overset{GL_n(A_{\mathfrak{m}}[X])}{\equiv} (1, ..., 0)$ as $(f_1(0), ..., f_n(0))$ is unimodular row in the local ring $A_{\mathfrak{m}}$ and therefore elementary equivalent to $(1, 0, ..., 0)$.*

Lemma 5.2.3. *Let A be an integral domain, S is a multiplicatively closed set in A. If*

$$(f_1(X), ..., f_n(X)) \overset{GL_n(S^{-1}A[X])}{\equiv} (f_1(0), ..., f_n(0))$$

then there exists $c \in S$ such that,

$$(f_1(X + cY), ..., f_n(X + cY)) \overset{GL_n(A[X,Y])}{\equiv} ((f_1(X), ..., f_n(X)).$$

Proof. Let $f(X) = (f_1(X), ..., f_n(X))$ and $f(0) = (f_1(0), ..., f_n(0))$.

Using the given statement, there exists $\sigma \in GL_n(S^{-1}A[X])$ such that $f(X) = \sigma(X)f(0)$. i.e. $f(0) = \sigma(X)^{-1}f(X)$, a constant. This is thus an invariant under the

translation

$$X \to Y$$

$$\sigma(X)^{-1}f(X) = \sigma(X+Y)^{-1}f(X+Y) = f(0)$$

We will define the following maps:

$$\tau(X,Y) = \sigma(X)\sigma(X+Y)^{-1}$$

$$\tau(X,Y)f(X+Y) = \sigma(X)\sigma(X+Y)^{-1}f(X+Y) = \sigma(X)f(0) = f(X)$$

Since $\tau(X,0) = I_n$, we can find $c \in S$ such that $\tau(X,cY) \in M_n(A[X,Y])$. As, $\det(\sigma(X))$ is a unit in S,

$$\det(\sigma(X)) = \det(\sigma(X+cY)^{-1})$$

Hence, $\det(\tau(X)) = \det(\sigma(X))\det(\sigma(X+cY)^{-1}) = 1$.

$$\tau(X,cY)f(X+cY) = \sigma(X)\sigma(X+cY)^{-1}f(X+cY) = \sigma(X)f(0) = f(X)$$

and $\tau(X,cY) \in SL_n(A[X,Y])$. □

Proof. Proof of Theorem 5.2.1:
Let $f(X)$ be as above. Let J be an ideal consisting of all the elements c as in the lemma. We claim that $J = A$. Suppose not. $J \subset \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

Using the hypothesis,

$$(f_1(X), ..., f_n(X)) \overset{GL_n(A_{\mathfrak{m}}[X])}{\equiv} (f_1(0), ..., f_n(0)).$$

Using the lemma,

$$(f_1(X+cY), ..., f_n(X+cY)) \overset{GL_n(A[X,Y])}{\equiv} (f_1(X), ..., f_n(X)) \text{ for some } c \in A - \mathfrak{m}.$$

Thus, $c \in J$, but $c \notin \mathfrak{m}$. Hence, we arrive at a contradiction. As $1 \in J$,

$$\sigma(X,Y)\begin{pmatrix} f_1(X+Y) \\ \vdots \\ f_n(X+Y) \end{pmatrix} = \begin{pmatrix} f_1(X) \\ \vdots \\ f_n(X) \end{pmatrix}$$

Let ϕ be the homomorphism: $A[X,Y] \to A[Y]$.
Thus, we obtain a matrix $\sigma(0,Y) \in GL_n(A[Y])$ such that

$$\sigma(X,Y)\begin{pmatrix} f_1(Y) \\ \vdots \\ f_n(Y) \end{pmatrix} = \begin{pmatrix} f_1(0) \\ \vdots \\ f_n(0) \end{pmatrix}$$

Replacing Y by X, we arrive at the proof. □

5.3 Generalisation of Horrocks' Theorem

In this section, we give a proof for a generalisation of Horrocks' theorem given by Ravi Rao.

Theorem 5.3.1. *Let A be a domain. Let $(f_1(X), ..., f_n(X))$ be a unimodular row in $A[X]$ with one element monic. Then, $(f_1(X), ..., f_n(X))$ is completable.*

Proof. We will use Horrocks' theorem and Quillen's Local-Global principle to prove the theorem.

Let

$$\begin{aligned} f_1(X) &= X^{r_1} + \cdots + \quad a_{11}X + \quad a_{10} \\ &\vdots \\ f_n(X) &= a_{nr_n}X^{r_n} + \cdots + \quad a_{n1}X + \quad a_{n0} \end{aligned}$$

We define,

$$\begin{aligned} g_1(X) &= a_{10}X^{r_1} + \cdots + \quad 1 \\ &\vdots \\ g_n(X) &= a_{n0}X^{r_n} + \cdots + \quad a_{nr_n} \end{aligned}$$

We defined $g(X)$ in order to prove that it is unimodular completable row and we will use it prove the completion of $f(X)$. We will start by proving that $(g_1(X), ..., g_n(X))$ is unimodular.

We know that the homomorphism: $\phi : A[X] \to A[Y]$ maps identity in $A[X]$ to the identity in $A[Y]$.

As $(f_1(X), ..., f_n(X)) \in Um_n(A[X])$, we have

$$(f_1(X^{-1}), ..., f_n(X^{-1})) \in Um_n(A[X^{-1}]).$$

Hence, there exists $h_i(X) \in A[X]$ such that

$$\sum_{i=1}^{n} f_i(X^{-1})h_i(X^{-1}) = 1.$$

$$\text{That is,} \quad \sum_{i=1}^{n} X^{-r_i} g_i(X^{-1})h_i(X^{-1}) = 1.$$

For a sufficiently large d, we have $X^d \in \langle g_1(X), ..., g_n(X)\rangle$.

We claim that

$$\langle g_1(X), ..., g_n(X)\rangle = A[X].$$

Assume to the contrary that $\langle g_1(X), ..., g_n(X)\rangle \subset \mathfrak{M}$ for maximal ideal $\mathfrak{M} \in A[X]$. $X^d \in \mathfrak{M}$.

As, M is prime ideal, $X \in \mathfrak{M}$. As $g_i(X) \in \mathfrak{M}$, $g_1(X) \in \mathfrak{M}$. As, $g_1(0) = 1, 1 \in \mathfrak{M}$. Contradiction.

Thus,

$$(g_1(X), ..., g_n(X)) \in Um_n(A[X]).$$

Now, we will prove that there exists a monic polynomial in this row in the localisation of ring A. As $(f_1(X), ..., f_n(X)) \in Um_n(A[X])$, $(f_1(0), ..., f_n(0)) \in Um_n(A)$. Hence, $(a_{10}, ..., a_{n0})$ is a unimodular row. In the localisation of A, atleast one element of a_{i0} is a unit. Hence, there exists a monic polynomial in $(g_1(X), ..., g_n(X))$ in the localisation of the ring A. Therefore, we can apply Horrocks' theorem,

$$(g_1(X), ..., g_n(X)) \overset{GL_n(A_{\mathfrak{m}}[X])}{\equiv} (1, 0, ..., 0)$$

for all maximal ideals $\mathfrak{m}$ of A.

Applying Quillen's Local-Global Principle, we have

$$(g_1(X), ..., g_n(X)) \overset{GL_n(A[X])}{\equiv} (g_1(0), ..., g_n(0)).$$

Put X=1,

$$(g_1(1), ..., g_n(1)) \overset{GL_n(A)}{\equiv} (g_1(0), ..., g_n(0)).$$

As, $g_1(0) = 1$, the unimodular row $(g_1(0), ..., g_n(0))$ is completable.

Thus,

$$(g_1(1),...,g_n(1)) \overset{GL_n(A)}{\equiv} (g_1(0),...,g_n(0)) \overset{GL_n(A)}{\equiv} (1,0,...,0)$$

By hypothesis, $(f_1(X),...,f_n(X))$ is a unimodular row with one element monic.

Therefore, we can apply Horrocks' theorem,

$$(f_1(X),...,f_n(X)) \overset{GL_n(A_{\mathfrak{m}}[X])}{\equiv} (1,0,...,0)$$

for all maximal ideals $\mathfrak{m}$ of A.

Applying Quillen's Local-Global principle, we have

$$(f_1(X),...,f_n(X)) \overset{GL_n(A[X])}{\equiv} (f_1(0),...,f_n(0)).$$

Put X=1,

$$(f_1(1),...,f_n(1)) \overset{GL_n(A)}{\equiv} (f_1(0),...,f_n(0)).$$

But,

$$(f_1(1),...,f_n(1)) = (g_1(1),...,g_n(1))$$

Thus,

$$(f_1(X),...,f_n(X)) \overset{GL_n(A[X])}{\equiv} (1,0,...,0)$$

□

5.4 Suslin's Factorial n Theorem

We will prove the famous generalisation of the Swan-Towber theorem given by Suslin. ([5])

Theorem 5.4.1. *Let A be a ring, $(a_0,a_1,...,a_n) \in Um_{n+1}(A)$. Then, the row $(a_0^{r_0},...,a_n^{r_n})$ is completable if $n! \mid r_0...r_n$*

For $n = 2$, the theorem is the statement that (a^2,b,c) is completable for a unimodular row (a,b,c). Murthy-Swan-Towber proved the $n = 2$ using an explicit construction.

If $ap + bq + cr = 1$ in the ring A then,

$$\begin{pmatrix} a^2 & b & c \\ b+ar & -r^2+bpr & -p+qr-bpq \\ c-aq & p+qr+cpr & -q^2-cpq \end{pmatrix} \in SL_3(A).$$

Another completion is proved by using the determinantal identity provided by Krusmeyer and Suslin in $Z[a,b,c,p,q,r]$ as follows:

$$\det \begin{pmatrix} a^2 & b & c \\ -b-2ar & r^2 & p-qr \\ -c+2aq & -p-qr & q^2 \end{pmatrix} \in SL_3(A).$$

Let us understand the construction of this identity: ([12])

Consider the following alternating matrix:

$$V = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & r & -q \\ -b & -r & 0 & p \\ -c & q & -p & 0 \end{pmatrix}$$

For alternating or skew-symmetric matrices, we can express the square root of the determinant of the matrix as a polynomial function in terms of the entries of the matrix. This polynomial is called the Pfaffian of the matrix.

Let $(1,2,...,2n)$ denote the Pfaffian of an alternating matrix. A recursive definition to calculate the Pfaffian is:

$$(1,2,...,2n) = \sum_{i=1}^{2n} (-1)^j (1,j)(2,...,\hat{j},...,2n)$$

where $\hat{j}$ denotes that index j is omitted.

Using the above result,

$$\text{Pf}(V) = (1,2)(3,4) - (1,3)(2,4) + (1,4)(2,3) = ap - b(-q) + cr.$$

Hence, $\det(V) = (ap+bq+cr)^2$ and $V \in SL_n(A)$.

Consider the following elementary transformations:

$$e_{41}(c)e_{31}(b)e_{21}(a)e_{14}(-r)e_{13}(-q)e_{12}(-p).$$

We have,

$$e_{41}(c)e_{31}(b)e_{21}(a)e_{14}(-r)e_{13}(-q)e_{12}(-p)V = \begin{pmatrix} 1 & a & b & c \\ 0 & a^2 & r+ab & -q+ac \\ 0 & -r+ab & b^2 & p+ac \\ 0 & q+ac & -r+bc & p^2 \end{pmatrix}$$

We thus get:

$$\begin{pmatrix} a^2 & r+ab & -q+ac \\ -r+ab & b^2 & p+ac \\ q+ac & -r+bc & p^2 \end{pmatrix} \in SL_3(A).$$

Replace V by $V^\star$, where

$$V^\star = \begin{pmatrix} 0 & a & -r & q \\ -a & 0 & b+ar & c-aq \\ r & -b-ar & 0 & p \\ -q & -c+aq & -p & 0 \end{pmatrix}$$

Repeating the same procedure for $V^\star$, we get Krusmeyer's identity!

To prove Suslin's theorem, we will first prove the following proposition:

Proposition 5.4.2. *Let $(a_0, ..., a_n) \in Um_{n+1}(A)$ be such that $(\bar{a}_0, ..., \bar{a}_{n-1})$ is completable in $\bar{A} = A/Aa_n$. Then, $(a_0, ..., a_{n-1}, a_n^n) \in Um_{n+1}(A)$ is completable in A.*

Proof. Let $\alpha \in M_n(A)$ with first row $(a_0, ..., a_{n-1})$ such that $\bar{\alpha} \in GL_n(\bar{A})$. Let $\beta \in M_n(A)$ be such that $\bar{\alpha}\bar{\beta} = I_n$. Then, $\alpha\beta = I_n + a_n\gamma$, $\beta\alpha = I_n + a_n\delta$ for suitable γ and $\delta \in M_n(A)$. We therefore get,

$$\begin{pmatrix} \alpha & a_nI_n \\ \delta & \beta \end{pmatrix} \begin{pmatrix} \beta & -a_nI_n \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ \star & I_n \end{pmatrix} \in GL_{2n}(A)$$

Thus,

$$\begin{pmatrix} \alpha & a_nI_n \\ \delta & \beta \end{pmatrix} \in GL_{2n}(A).$$

Now we will transform a_nI_n to a diagonal matrix $\text{diag}(a_n^n, 1, ..., 1)$. Let $d = \det(\alpha) \in A$. Since, d is a unit in $\bar{A}$, a_n is a unit in A/dA, we apply Whitehead's lemma over the ring A/dA.

Moreover, the map $E_n(A) \to E_n(A/dA)$ is surjective. Hence,

$$\begin{pmatrix} {a_n}^n & 0 \\ 0 & I_{n-1} \end{pmatrix} = (a_nI_n)\sigma + d\tau$$

for some $\sigma \in E_n(A)$ and $\tau \in M_n(A)$. Let $\alpha' = adj(A)$.

$$\phi = \begin{pmatrix} \alpha & a_nI_n \\ \delta & \beta \end{pmatrix} \begin{pmatrix} I_n & \alpha'\tau \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} \alpha & \lambda \\ \delta & \star \end{pmatrix} \in GL_{2n}(A)$$

Here,

$$\lambda = \alpha\alpha'\tau + a_n\sigma = d\tau + a_n\sigma = \begin{pmatrix} a_n^n & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

Rewriting ϕ in block form: $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$, where α_1 is $n \times (n+1)$ matrix- α adjoined with $(a_n^n, 0, ..., 0)^t$ and $\alpha_2 = \begin{pmatrix} 0 \\ I_n \end{pmatrix}$.

After performing elementary transformations on the last row of ϕ, we get an invertible matrix:

$$\phi' = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3' & 0 \end{pmatrix}$$

Since, $\alpha_2 = \begin{pmatrix} 0 \\ I_n \end{pmatrix}$, the submatrix of ϕ' complimentary to I_{n-1} is an invertible $(n+1) \times (n+1)$ matrix with top row $(a_0, a_1, ..., a_n^n)$. Hence, we arrive at the proof. □

Corollary 5.4.3. *For any unimodular row $(a_0, a_1, ..., a_n)$ in A, the factorial unimodular row : $(a_0, a_1, a_2^2, ..., a_n^n)$ is completable.*

Proof. Using induction on n in Proposition 5.4.2, we get the result. □

Lemma 5.4.4. *Let A be a local domain. $a_0, ..., a_n$ is a unimodular row in A, then*

$$(a_0^r, a_1 + a_0X, a_2, ..., a_n) \overset{GL_{n+1}(A[X])}{\equiv} (1, 0, ..., 0).$$

Proof. Let $\mathfrak{m}$ be the unique maximal ideal in A. If $a_i \notin \mathfrak{m}$ for some $i \in \{0, 2, ..., n\}$, then the row $(a_0^r, a_1 + a_0X, a_2, ..., a_n)$ contains a unimodular row of shorter length. Hence, it is completable. Thus, we will assume that $a_i \in \mathfrak{m}$ for $i \in \{0, 2, ..., n\}$. As $a_0, ..., a_n$ is a unimodular row in A, a_1 is a unit in A.

Claim: $(a_0^r, a_1 + a_0X)$ is a unimodular row. If not, $\langle a_0^r, a_1 + a_0X\rangle \in M$ for a maximal ideal M of $A[X]$. Thus, $a_0 \in M$ and therefore, $a_1 \in M$. This leads to a Contradiction. Hence, $(a_0^r, a_1 + a_0X, a_2, ..., a_n)$ contains a unimodular row of shorter length. Hence, it is completable. □

Lemma 5.4.5. *Let A be a domain. For $r \in N$, and $(a_0, a_1, ..., a_n)$ unimodular row in A;*

$$(a_0^r, a_1, ..., a_n) \overset{GL_{n+1}(A)}{\equiv} (a_0, a_1^r, ..., a_n).$$

Proof. Using Quillen's Local-Global principle and Lemma 5.4.4,

$$(a_0^r, a_1 + a_0X, a_2, ..., a_n) \overset{GL_{n+1}(A[X])}{\equiv} (a_0^r, a_1, a_2, ..., a_n)$$

Put $X = -1$,

$$(a_0^r, a_1 - a_0, a_2, ..., a_n) \overset{GL_{n+1}(A)}{\equiv} (a_0^r, a_1, a_2, ..., a_n)$$

$a_1^r - a_0^r = \lambda(a_1 - a_0)$. So,

$$(a_0^r, a_1 - a_0, a_2, ..., a_n) \overset{GL_{n+1}(A)}{\equiv} (a_1^r, a_1 - a_0, a_2, ..., a_n)$$

From,

$$(a_0^r, a_1 - a_0, a_2, ..., a_n) \overset{GL_{n+1}(A)}{\equiv} (a_0^r, a_1, a_2, ..., a_n)$$

let $a_0 \to a_1$ and $a_1 \to a_1 - a_0$,

$$(a_1^r, a_1 - a_0, a_2, ..., a_n) \overset{GL_{n+1}(A)}{\equiv} (a_1^r, -a_0, a_2, ..., a_n)$$

$$(a_1^r, -a_0, a_2, ..., a_n) \overset{GL_{n+1}(A)}{\equiv} (a_0, a_1^r, a_2, ..., a_n)$$

Hence,

$$(a_0^r, a_1, ..., a_n) \overset{GL_{n+1}(A)}{\equiv} (a_0, a_1^r, ..., a_n)$$

□

Theorem 5.4.6. *Let A be a domain and $(a_0, a_1, ..., a_n)$ be a unimodular row in A. Then, $(a_0, a_1, ..., a_n^{n!})$ is completable.*

Proof. Follows from Corollary 5.4.3 and Lemma 5.4.5. □

Proof. Proof of Suslin's n! Theorem:
Follows from Lemma 5.4.5 and Theorem 5.4.6. □

5.5 Monic Polynomial Theorem

Lemma 5.5.1. *Let A be a Noetherian ring with Krull dimension d. If $\mathfrak{M}$ is a maximal ideal of $A[X]$ with $\mathrm{ht}(\mathfrak{M}) = \mathrm{d} + 1$, then $\mathfrak{M} \cap A$ is a maximal ideal of A.*

Proof. Using the Lemma of 2.4.2 we have the result $\mathrm{ht}(\mathfrak{M} \cap \mathrm{A}) \geq \mathrm{d} + 1$. Therefore, we arrive at the lemma. □

Lemma 5.5.2. *Let A be a Noetherian ring with Krull dimension d. If $\mathfrak{M}$ is a maximal ideal of $A[X]$ with $\mathrm{ht}(\mathfrak{M}) = \mathrm{d} + 1$, then $\mathfrak{M}$ contains a monic polynomial.*

Proof. Using the Lemma 2.4.1 and Lemma 2.4.2 we can prove that $\mathfrak{M} \cap A$ is a maximal ideal of A. As we know that $\mathfrak{M} \cap A$ is not maximal and $\mathfrak{M} \cap A \subset \mathfrak{M}$. Hence, we will look at the set $\mathfrak{M} - (\mathfrak{M} \cap A)A[X]$. Let $f(X)$ be the lowest degree polynomial in this set. Let $f(X) = a_n X^n + \cdots + a_0$.

We claim that $a_n \notin \mathfrak{M} \cap A$. If not, $a_n \in \mathfrak{M} \cap A$. Hence,

$$a_{n-1}X^{n-1} + \cdots + a_0 \in \mathfrak{M} - (\mathfrak{M} \cap A)A[X].$$

This is a contradiction because of our choice of $f(X)$.

Hence, $a_n \notin \mathfrak{M} \cap A$ implies that there exists an element say $b_n \in A$ such that $a_n b_n - 1 \in \mathfrak{M} \cap A$. Therefore, the following construction

$$g(X) = b_n f(X) - (a_n b_n - 1)X^n$$

is our required monic polynomial in $\mathfrak{M}$. □

Corollary 5.5.3. *Let A be a Noetherian ring with Krull dimension d. If I is a maximal ideal of $A[X]$ with $\mathrm{ht}(\mathrm{I}) = \mathrm{d} + 1$, then I contains a monic polynomial.*

Proof. As A is Noetherian, the radical of the ideal I is an intersection of prime ideals in A which are minimal over I. As $\mathrm{ht(I)} = \mathrm{d}+1$, we can say that $\mathrm{ht}(\mathfrak{p}_\mathrm{i}) \geq \mathrm{d}+1$. As A is Noetherian, $\dim(A[X]) = d+1$, we have $\mathrm{ht}(\mathfrak{p}_\mathrm{i}) = \mathrm{d}+1$. Hence, the prime ideals are maximal. Therefore, using the above lemma, we can say that each $\mathfrak{p}_\mathrm{i}$ contains a monic polynomial. As $\sqrt{I} = \cap_{i=1}^{n}\mathfrak{p}_\mathrm{i}$, the product of all the monic polynomials in $\mathfrak{p}_\mathrm{i}$ will give us a monic polynomial in $\sqrt{I}$. A large power of such a polynomial gives us a monic polynomial over I. □

Theorem 5.5.4. *Let A be a Noetherian ring with dimension d. Consider the unimodular row $(f_1(X), f_2(X), ..., f_n(X))$ in $A[X]$, where $n \geq d+2$. Then, we can find a matrix $\beta(X) \in GL_n(X)$ such that $\beta(X)$ can be connected to the identity matrix and*

$$\beta(X)\begin{pmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Proof. We know from the theory of elementary matrices that such a matrix can be connected to the identity matrix. Therefore, the result follows for $n \geq d+3$ as the dimension of the ring $A[X]$ is $d+1$.
Hence, we need to check for the case $n = d+2$. By a lemma, we can find $\sigma \in E_n(A[X])$ such that:

$$\sigma(X)\begin{pmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{pmatrix} = \begin{pmatrix} g_1(X) \\ g_2(X) \\ \vdots \\ g_n(X) \end{pmatrix}$$

and the height $\mathrm{ht}\,\langle \mathrm{g_1(X)}, ..., \mathrm{g_i(X)}\rangle \geq \mathrm{i}$ for $1 \leq i \leq n$. Using the fact that the height, $\mathrm{ht}\,\langle \mathrm{g_1(X)}, ..., \mathrm{g_{n-1}(X)}\rangle \geq \mathrm{n}-1 = \mathrm{d}+1$, it follows from Corollary 5.5.3 that the ideal $\langle g_1(X), ..., g_{n-1}(X)\rangle$ contains a monic polynomial say $h(X)$. Adding a large power of $h(X)$ to $g_n(X)$ we will get that $g_n(X)$ is monic. Adding a large power of $g_n(X)$ to $g_1(X)$ we will get that $g_1(X)$ is monic. Hence, by theorem, there exists $\delta(X,T)$ such that $\delta(X,0) = I_n$ and

$$\delta(X,1) = \begin{pmatrix} g_1(X) \\ g_2(X) \\ \vdots \\ g_n(X) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\sigma(X)$ is an elementary matrix which we can connected to the identity matrix.
Hence, there exists $\delta'(X,T)$ such that $\delta'(X,0) = I_n$ and $\delta'(X,1) = \sigma(X)$
Let $\alpha = \delta\delta'$. Then $\alpha(X,0) = I_n$ and

$$\alpha(X,1)\begin{pmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Set $\beta(X) = \alpha(X,1)$. □

Chapter 6

Unimodular Rows of Length 3

6.1 A Theorem by Mohan Kumar and M.P. Murthy

A modern development in our treatment of completion of unimodular rows is a theorem proved by Mohan Kumar and MP Murthy. [9] This theorem talks about a particular unimodular row of length 3.

Let us first consider a unimodular row of length 2 over any general commutative ring A. Let (a_1, a_2) be the unimodular row such that $a_1b_1 + a_2b_2 = 1$. On inspection, we get the following completion :

$$\begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}$$

Thus, a length 2 unimodular row is always completable.

Our next question is to look at unimodular rows of length 3. However, length 3 unimodular rows are not completable over any general ring A.

Hence, it is imperative that we understand the result of Kumar and Murthy on the completion of unimodular rows of length 3 over a particular ring A.

We will start our journey by stating a result from Algebraic geometry which was given by Swan.

Definition 6.1.1. A *line bundle* M over a commutative ring A is a finitely generated projective A-module of constant rank 1. (*cf:*[7], Page 15)

Lemma 6.1.1. ***Swan's Lemma:***
If M is an 2-generated line bundle over the ring A. Then, M^n is 2-generated for $n \geq 0$. Moreover, $M^n \oplus M^{-n} \cong A^2$ and hence free.

Theorem 6.1.2. *Let A be a commutative ring with identity. If (a, b, c) is a unimodular row in A and if the polynomial $z^2 + bz + ac = 0$ has roots in A, then (a, b, c) is completable. (cf: [9])*

Proof. (a, b, c) is a unimodular row in A.
Therefore, there exists elements a', b' and $c' \in A$ such that $aa' + bb' + cc' = 1$.
We now construct a quotient ring in indeterminates a, a', b, b', c, c', z modulo the ideals generated by $aa' + bb' + cc' - 1$ and $z^2 + bz + ac$.

$$R = \frac{\mathbb{Z}\left[a, a', b, b', c, c', z\right]}{(aa' + bb' + cc' - 1, z^2 + bz + ac)}$$

Consider the ideal $I = (z, a)$. $I^2 = (z^2, az, a^2)$.

We know that $aa' + bb' + cc' = 1$. Multiplying this equation by az, we get $az = az(aa' + bb' + cc')$. That is, $az = a^2(a'z - cb' - c^2c'b^{-1}) + z^2(-ab' - acc'b^{-1})$. Hence, $I^2 = (z^2, a^2)$. Thus I^2 is a 2-generated ideal.

We will look at the following surjective map:

$$\phi : R^3 \to I^2$$

given by

$$\phi\left(e_1\right) = z^2; \phi\left(e_2\right) = az; \phi\left(e_3\right) = a^2.$$

e_i form the basis of the module R^3.

$$\phi\left(ae_1 + be_2 + ce_3\right) = az^2 + baz + ca^2 = a\left(z^2 + bz + ac\right) = 0$$

Thus, ϕ factors through $P = R^3/\left(ae_1 + be_2 + ce_3\right)$.
Therefore, $P \cong I^2 \oplus I^{-2}$.

We know that I is 2-generated. We can use Swan's lemma to see that P is free. Let α be a root of the polynomial $z^2 + bz + ac = 0$. By assumption, $\alpha \in A$. Consider

the homomorphism $\psi : R \to A$.

$$\psi(z) = \alpha$$

The projective module related to this homomorphism, $P \otimes_R A$ is free as P is free. $\square$

6.2 Algorithmic Approach for Murthy-Kumar's Theorem

Using the software *Macaulay2*, V. Kodiyalam gave a constructive proof for the theorem by Mohan Kumar and MP Murthy.

I replicated this construction using the software *Macaulay2*, using the ideas of Kumar-Murthy in [9] and V. Kodiyalam in [10].

The construction of the proof is as follows:

Proposition 6.2.1. *Consider the matrix*

$$P = \begin{pmatrix} gj & g^2j^2x + h^2j^2z - hjv & g^2k^2x + h^2k^2z + hju \\ gk + hj & gjt - hkv & hku - gjs \\ hk & g^2j^2w + h^2j^2y + gkt & g^2k^2w + h^2k^2y - gks \end{pmatrix}$$

in the indeterminates - $g, h, i, j, k, s, t, u, v, w, x, y, z$. Then,

$$\det(X) = (g^2j^2x + g^2k^2t + h^2j^2u + h^2k^2v)(g^2j^2w + g^2k^2x + h^2j^2y + h^2k^2z)$$

Theorem 6.2.2. *Let A be a commutative ring with identity. If (a, b, c) is a unimodular row in A and if the polynomial $z^2 + bz + ac = 0$ has roots in A, then (a, b, c) is completable.*

Proof. Let the roots of the polynomial $z^2 + bz + ac = 0$ be α and β. We know that, $\alpha + \beta = -b$ and $\alpha\beta = ac$ for the given quadratic polynomial. We will consider the following ring homomorphism:

$$\phi : \mathbb{Z}[gj, gk, hj, hk] \to A$$

$$\phi(gj) = a, \;\; \phi(gk) = -\alpha, \;\; \phi(hj) = -\beta, \;\; \phi(hk) = c$$

As (a, b, c) is unimodular, so is (a, α, β, c) and correspondingly is $(a^2, \alpha^2, \beta^2, c^2)$.

Therefore, there exists $s, t, u, v, w, x, y, z \in A$ such that:

$$a^2s + \alpha^2t + \beta^2u + c^2v = 1 = a^2w + \alpha^2x + \beta^2y + c^2z$$

Therefore, the homomorphic image of the matrix X (in the above proposition) in A has a determinant of 1. Moreover, the homomorphic image of P has as its first column: (a, b, c) , giving us the following determinantal identity for the completion of unimodular row (a, b, c).

$$\begin{pmatrix} a & a^2x + \beta^2z + \beta v & \alpha^2x + c^2z - \beta u \\ b & at - cv & cu - as \\ c & a^2w + \beta^2y - \alpha t & \alpha^2w + c^2y + \alpha s \end{pmatrix}$$

□

Proof. **Constructive Proof using the software *Macaulay2*:**

Using the proof given by Mohan Kumar and MP Murthy:

Consider the ring

$$A = \frac{\mathbb{Z}[a, b, c, d, p, q, r]}{(ad - bc, pa + q(b + c) + rd - 1)}$$

Let us look at the unimodular row $(a, b + c, d)$ The associated projective module of the unimodular row be P.

$$0 \to A \to A^3 \to P \to 0$$

Using the software *Macaulay2*, we check that A is an integral domain.

Consider the ideal generated by a and b $((a, b))$. As $pa + q(b + c) + rd = 1$ and $ad = bc$, we have $ab = a^2(pb + qd) + b^2(qa + rc)$. Therefore:

$$(a, b)^2 = (a^2, b^2)$$

$$(a, b) \cdot (a, c) = (a^2, ab, ac, bc) = (a^2, ab, ac, ad) = (a \cdot 1).$$

Thus, (a, b) is invertible ideal. Similarly, the ideal $I = (a^2, b^2)$ is invertible with the inverse $J = (a^2, c^2)$.

Thus, we can map P onto the 2-generated ideal I. As I is invertible, the kernel of this map is the inverse J. Hence, $P \cong I \oplus I^{-1}$.

Using Swan's lemma, P is free. Now, we want a construction of the determinantal identity that shows that P is free.

Therefore, we have:

$$0 \to J \xrightarrow{i} A^2 \xrightarrow{\pi} I \to 0$$

The explicit surjection is:

$$\pi : A^2 \to I$$

$$\pi(e_1) = a^2 \quad , \quad \pi(e_2) = b^2$$

Let $K = \ker\pi$,

Using *Macaulay2*, K is generated by the columns of the matrix

$$\begin{pmatrix} -b^2 & -d^2 \\ a^2 & c^2 \end{pmatrix}$$

as a submodule of A^2.

Let us define the map

$$J \to K$$

$$x \mapsto ye_1 + xe_2$$

where $d^2x + c^2y = 0$.

Using the relation $ad = bc$ and the fact that J is generated by a and c, we can check this map is an isomorphism.

We can now define the injection:

$$i : J \to A^2$$

$$i(x) = ye_1 + xe_2$$

where $d^2x + c^2y = 0$.

We thus have the explicit maps i and π for the short exact sequence defined above.

Our aim is to find a splitting map, say $\rho : A^2 \to J$ explicity.

$$\rho(e_1) = -(ta^2 + vc^2) \quad , \quad \rho(e_2) = sa^2 + uc^2$$

For $\rho \cdot i$ to be the identity map at J, we need the condition $sa^2 + tb^2 + uc^2 + vd^2$. Using the unimodularity condition, we can find such $s, t, u, v \in A$. Therefore, the

short exact sequence splits. Thus:

$$A^2 \cong J \oplus I$$

If we construct an isomorphism from P to $J \oplus I$, we are done.

Using the definition of P as the associated projective module of the unimodular row $(a, b+c, d)$, P is the cokernel of the map: $A \to A^3$. Hence,

$$P = \frac{A^3}{(ae_1 + (b+c)e_2 + de_3)}$$

Consider the surjection map:

$$A^3 \to I$$

$$e_1 \mapsto b^2 \quad , \quad e_2 \mapsto -ab \quad , \quad e_3 \mapsto a^2.$$

It is clear that the map factors through P as

$$\tilde{\pi} : P \to I$$

Let L be the kernel of the map $A^3 \to I$.

Using *Macaulay2*, L is generated by the columns of the following matrix

$$\begin{pmatrix} c & -a & 0 & 0 \\ d & -b & -c & a \\ 0 & 0 & -d & b \end{pmatrix}$$

We can clearly see that L contains the vector $ae_1 + (b+c)e_2 + de_3$.

$$\text{Thus,} \quad \ker(\tilde{\pi}) = \frac{L}{(ae_1 + (b+c)e_2 + de_3)}$$

$$\text{Therefore,} \quad \ker(\tilde{\pi}) = (c\bar{e_1} + d\bar{e_2}, -a\bar{e_1} - b\bar{e_2}, -c\bar{e_2} - d\bar{e_3}, a\bar{e_2} + b\bar{e_3})$$

As $ae_1 + be_2 = -ce_1 - de_3$, $\ker(\tilde{\pi})$ has three generators.

We will define the following map $\tilde{i} : J \to P$

$$\tilde{i}(c^2) = c\bar{e_1} + d\bar{e_2} \quad , \quad \tilde{i}(a^2) = -a\bar{e_2} - b\bar{e_3}$$

Now, $\tilde{i}(ac) = \tilde{i}\left\{a^2(pc+qd) + c^2(qa+rb)\right\} = a\bar{e_1} + b\bar{e_2}$

Hence, $\tilde{i}(J) = \ker(\tilde{\pi})$. Therefore, $\ker(\tilde{\pi})$ J giving us our required short exact sequence:

$$0 \to J \xrightarrow{\tilde{i}} P \xrightarrow{\tilde{\pi}} I \to 0$$

We want this short exact sequence to split. We instead consider the following tensored sequence:

$$0 \to JJ \to JP \to JI \to 0$$

$$0 \to J^2 \xrightarrow{\hat{i}} JP \xrightarrow{\hat{\pi}} A \to 0$$

where $\hat{i} = \tilde{i}$ on J^2, $\hat{\pi} = a_2\tilde{\pi}$ on JP.

The map

$$\hat{\pi} : JP \to A$$

$$a^2\bar{e_1} \mapsto b^2 \quad , \quad a^2\bar{e_3} \mapsto a^2 \quad , \quad c^2\bar{e_1} \mapsto d^2 \quad , \quad c^2\bar{e_3} \mapsto c^2.$$

We now construct the splitting map $\hat{\sigma} : A \to JP$

$$\hat{\sigma}(1) = xa^2\bar{e_1} + wa^2\bar{e_3} + zc^2\bar{e_1} + yc^2\bar{e_3}$$

and

$$wa^2 + xb^2 + yc^2 + zd^2$$

Thus, we have a split exact sequence. However, we want a splitting map for the original short exact sequence. We will, therefore, tensor with I to get the following splitting map $\tilde{\sigma} : I \to P$

$$\tilde{\sigma}(a^2) = a^2(x\bar{e_1} + w\bar{e_3}) + c^2(z\bar{e_1} + y\bar{e_3}) \quad , \quad \tilde{\sigma}(b^2) = b^2(x\bar{e_1} + w\bar{e_3}) + d^2(z\bar{e_1} + y\bar{e_3})$$

Therefore,

$$J \oplus I \cong P$$

Thus, we will define the isomorphism between $A^2 \to P$ by composing the isomorphism maps between $A^2 \to J \oplus I$ and $J \oplus I \cong P$. This map is :

$$e_1 \mapsto (a^2x + c^2z - cv)\bar{e_1} + (at - dv)\bar{e_2} + (a^2w + c^2y + bt)\bar{e_3}$$

$$e_2 \mapsto (b^2x + d^2z + cu)\bar{e_1} + (du - as)\bar{e_2} + (b^2w + d^2y - bs)\bar{e_3}$$

We thus get the following determinantal identity for the completion of the uni-

modular row $(a, b+c, d)$

$$\begin{pmatrix} a & a^2x + c^2z - cv & b^2x + d^2z + cu \\ b+c & at - dv & du - as \\ d & a^2w + c^2y + bt & b^2w + d^2y - bs \end{pmatrix}$$

Hence, the proof is complete. □

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