Applications of Game Theory in Market Microstructure Modelling

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Certificate

This is to certify that this dissertation entitled 'Applications of Game Theory in Market Microstructure Modelling' towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents work carried out by Bhide Atharva Vivek at the Indian Institute of Technology, Bombay, under the supervision of Dr. Ankur A Kulkarni, Associate Professor, Systems and Control Department, IIT Bombay , during the academic year 2023-2024.

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To my dear *Aai-Baba*, your endless support made this thesis possible !!

Declaration

I hereby declare that the matter embodied in the report entitled 'Applications of Game Theory in Market Microstructure Modelling' are the results of the work carried out by me at the Systems and Control Department, IIT Bombay, under the supervision of Dr. Ankur A Kulkarni and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

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Abstract

In this thesis, we study interactions between the broker and the regulator in the context of front-running in the financial markets. The primary objective is to understand the role of incomplete information and regulators' beliefs in the broker's trading behavior.

We model the front-running using a sequential game with incomplete information and analyze the perfect Bayesian equilibrium of the game. The equilibrium analysis for different market conditions based on the profitability of the price revealed interesting insights into the broker's behavior. We show that, when a client order is trend-reversing, the uninformed broker deliberately trades even when the price is not profitable to shape the regulator's belief away from investigating trading activities. It allows the informed broker to front-run the market and the broker gets higher expected payoff. We also examine the role of information spread and the regulator's willingness to investigate the trading in a broker's optimal strategies. We show that certain equilibria exist only when the information spread is lower or higher compared to the regulator's investigation threshold. We further show that imperfect investigation leads to decreases in the regulator's willingness to investigate.

The study is helpful in gaining behavioral insights about the broker in the context of front-running and deciding regulatory policies against the front-running.

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Chapter 1

Introduction

A financial market serves as an avenue for trading different securities like equities, commodities, bonds, etc. It consists of players like individual traders, institutions, hedge funds, and banks who want to trade in these securities. A broker helps to streamline the trading activities in the market by executing the orders for his clients, and he charges fees for this execution as commission or brokerage. All the trading activity in the financial market is mediated through the broker. The broker is expected to execute the order in the client's best interest so that the client gets the best price. However, the broker has the incentive to deviate from it owing to his superior information about the upcoming client order compared to the remaining market players.

When a client decides to trade a bulk quantity of shares of any security, he places an order with his broker. Hence, a broker knows about the bulk transaction before any other market player. The bulk transaction is likely to move the price of the security or related derivatives in the direction of order flow, i.e., the price is likely to increase when the client buys and decrease when the client sells. The broker has private information about the anticipated price movement that this bulk trade would bring. If the broker trades for himself or for his other preferred client before the bulk client order is placed, he can generate profits from the price movement brought by the bulk client order. This behavior of the broker is known as front-running. [2] When the broker front-runs, he trades in his own interest rather than his client's best interest. Front-running is legally prohibited in several markets across the world. The regulatory authorities impose penalties and occasional prohibitions on the corresponding entities to stop front-running. However, the detection of front-running itself is a tricky issue due to several reasons.

When the broker trades for himself (or his preferred client), it can be based on his judgment of a market trend or based on the private information of bulk client orders. It may also happen that the broker is innocent and trades only based on a market trend and receives a bulk client order afterward. If the broker is trading based on private information of the client order, it is only revealed if a regulator investigates the trading activity further. Due to limited resources and a huge amount of trading activity, the regulator cannot investigate every trade happening in the market. Thus, the regulator's belief about the broker's informativeness drives whether the broker faces an investigation or not. Hence, the broker tries to shape the regulator's belief through his trading activity. In essence, the broker wants to front-run the market whenever it is profitable without getting caught, and the regulator tries to catch front-runners by using limited resources available for investigation. We seek an understanding of the broker's strategy to use his private information and the regulator's front-running detection mechanism.

We formulate a sequential game of incomplete information to model front-running. It helps us incorporate information asymmetry in the model. We also incorporate the role of the regulator's belief in the equilibrium analysis. To the best of our knowledge, the interactions between the broker and the regulator during front-running are not studied using a game theoretic model. Thus, the insights derived from our analysis are novel and helpful in understanding the behavior and devising policies to curb front-running. The interactions between informed traders and market makers in the context of front running have been studied before. [12] [13] The focus of those studies is on analyzing the profitability of informed traders and market dynamics arising from information asymmetry. On the contrary, our focus is on understanding the mechanism of use of information in the presence of a regulatory framework. Thus, we do not consider an interaction among traders but focus on how the broker responds to the investigation while using his information.

Organization of the thesis

In Chapter 2, we explain all the preliminaries required for understanding the problem and model formulation. We start with discussing the legal aspects of front-running and present some recent anecdotes about the front-running cases. In the subsequent section, we present relevant background and concepts from the game theory. We keep it brief by focusing only on the concepts and definitions directly used in the model formulation and analysis. (readers familiar with this material can skip it without losing on continuity) In the last section, we discuss important results from the literature on the front-running. All the results we discuss focus on modeling information asymmetry and the broker's behavior in financial markets. Our model is not based on any of those results, but they are worth noting to set a context for the question at hand.

In Chapter 3, we present a game model of front-running in financial markets. The game consists of a broker and a regulator. We discuss the idea behind the model formulation and the game structure, including the sequence of actions, strategies, and payoffs. We also provide an explanation to relate the game model with details of front-running. Moreover, we also state a perfect Bayesian equilibrium definition as a solution concept in the context of the proposed game model.

In Chapter 4, we compute a general form of the broker's and the regulator's optimal strategy in a perfect Bayesian equilibrium. However, optimal strategies depend on market variables. Thus, we analyze the trending and sideways markets separately. We further distinguish market conditions based on the change in trend brought by the client order. For every market condition, we compute all the possible perfect Bayesian equilibria and present insights about the trading and regulatory behavior in the corresponding equilibrium.

In Chapter 5, we present distinctive behavioral phenomena with the help of perfect Bayesian equilibria in select market conditions. We show that when the client order is trend-reversing, the broker trades even when he is uninformed, and there is no profit to be made from trading. It helps him leverage the information wherever it is available. The existence of such deceptive trading behavior aligns with our intuition that the broker needs to fool the regulator by making some noisy trading. We also analyze the impact of the spread of information. We show a certain spread of information compared to the regulator's investigation threshold allows certain equilibria. The scope and profitability for front-running decrease with the increase in the spread of information, showing that the comparative advantage of the information decreases as it becomes more and more spread out. We also show that the regulator's willingness to investigate decreases with the increase in the cost of investigation and decreases in the benefit from investigation in terms of penalty and reputation. We also analyze the effects of imperfect investigation on the equilibrium strategies.

In Chapter 6, we conclude the thesis by summarizing key results and present possible future directions.

Chapter 2

Preliminaries

In this chapter, we state preliminary concepts and definitions related to financial markets and game theory. In the first section of this chapter, we define the basic concepts related to the front running in financial markets. They are pivotal for setting context and understanding the importance of the problem. Many legalities are associated with the front running, and the official website of the Securities and Exchange Board of India (SEBI) [1] is the primary reference. In the second section of the chapter, we state the basic definitions and important results from game theory required for the model formulation and equilibrium analysis. In the third section, we review the related literature and set a context for the problem.

2.1 Front Running

In this section, we explain the problem of front-running in financial markets and the legal aspects of it.

The financial markets facilitate the trading of different securities such as stocks, bonds, commodities, and derivatives such as futures, options, swaps, etc. The different security exchanges facilitate the trading of these contracts. A broker is an individual or firm that acts as an intermediary between a client and a securities exchange. A client can be an individual investor, hedge fund, or institutional investor, among many other categories. If a client wants to trade a security, he can place the order only through the broker. The broker's responsibility is to execute the order in the client's best interest. The broker charges fees or brokerage for it. Nonetheless, the broker may have an incentive to deviate from his responsibility.

The client order is a bulk order when a client is willing to transect a high volume of contracts of any security, which will likely deviate its price significantly. The broker with whom the client places the bulk order knows about the likely price deviation before the client order is sent to exchanges for execution. Hence, if the broker trades in the same or related security before the client order is executed, he can make a profit from the anticipated price change. Such trading behavior of the broker is categorized as a front running.

The National Association of Securities Dealers Automated Quotations (NASDAQ) defines front-running as

Definition 2.1.1. *(Front Running- NASDAQ) Entering into an equity trade, options, or futures contracts with advanced knowledge of a block transaction that will influence the price of the underlying security to capitalize on the trade.*

It is forbidden by the Securities and Exchange Commission (SEC). [7]

The Securities and Exchange Board of India (SEBI) is India's regulatory body for securities and commodity markets. The front-running is defined by the SEBI as follows in the SEBI Circular dated 25th May 2012 in section 1(iv). [2]

Definition 2.1.2. *(Front Running- SEBI) The usage of non-public information to directly or indirectly buy or sell securities or enter into options or futures contracts, in advance of a substantial order, on an impending transaction, in the same or related securities or futures or options contracts, in anticipation that when the information becomes public; the price of such securities or contracts may change.*

Section 4(2)(q) of SEBI's regulation 'Prohibition of Fraudulent and Unfair Trade Practices Relating to Securities Market (2003),' classifies front running as a manipulative, fraudulent, and unfair trade practice. [3] The SEBI has published several guidelines for the broker, mutual fund managers, and research analysts in regard to front running. [8] [9] In all these documents, SEBI mentions that any kind of front-running is illegal and the corresponding entity involved in front-running faces a disciplinary action.

Even though SEBI has clear regulations and guidelines, several entities such as brokers, security dealers, trading houses, and banks have been found guilty of front-running. The SEBI has taken legal actions against them in terms of penalties and prohibition from participating in trading activities. The recent cases of the front running involve institutions like Axis Mutual Fund [4], LIC [6], IIFL [5].

The front-running is unethical and adversely affects the sanctity of financial markets. Hence, the problem of front-running dynamics and the broker's behavior in such cases is particularly very important from a regulatory perspective.

2.2 Game Theory

In this section, we discuss preliminary concepts of game theory. We confine ourselves to concise definitions and explanations. Moreover, we focus only on concepts directly relevant to the model discussed in the thesis. The primary references for all the definitions and explanations are [18] [19] [21].

2.2.1 Sequential Games

Game theory is a study of mathematical models of strategic interactions among rational individuals. Rational individuals take actions that maximize their payoff, given the information available to them. If there is any uncertainty involved in the game, rational individuals maximize their expected payoff. The game comprises the following primary elements.

1. Players - The individuals in the game who make decisions. We denote the set of players by $N = \{1, 2, 3, .., n\}$

2. Strategies/ Actions - A player's strategy is a function from each of his information sets to the set of actions available at that information set. We denote a strategy for player i by $s_i \in S_i$. A strategy profile comprises of the strategies for all the players and is denoted by $s = (s_1, s_2, ..., s_n)$. The vector of strategies for all other players expect player k is defined as $s_{-k} = (s_1, s_2, ..., s_{k-1}, s_{k+1}, ..., s_n)$. The strategies boil down to

actions in the game of complete information. We denote the set of possible actions for player i by A_i.

3. Payoffs - The payoff is the objective the player tries to maximize. The payoff of player i, denoted by u_i, depends on the strategy profile of all the players. The payoff function is given by

$$u_i : S_1 \times S_2 \times ... \times S_n \to \mathbb{R}$$

If there is external uncertainty in any state of the game, we model it by introducing nature as another player. Nature does not have a payoff and takes an action according to the specified prior distribution. To understand the equilibrium behavior of the players, we seek a solution concept. A Nash equilibrium [20] is one of the most important solution concepts.

Definition 2.2.1. *(Nash equilibrium) A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a Nash equilibrium if*

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*)\ \forall s_i \in S_i,\ \forall i \in N \tag{2.1}$$

In other words, no player has an incentive to deviate from his strategy in a Nash equilibrium when every other player is playing his Nash equilibrium strategy.

In a sequential game, players play one after the other; thus, the action of one player affects the actions of other players. The sequential game in extensive form is represented using a game tree. The players in a particular state of the game are represented by the vertices and actions of players by the edges of the tree. The game starts from the root vertex. The leaf vertex shows the end of the game where the payoff is realized.

2.2.2 Incomplete Information and Beliefs

In a sequential game of incomplete information, players do not know the state of the game (i.e, which vertex of a game tree is reached), or are uncertain about whether the other players know the state of the game, or are uncertain whether the other players know whether the other players know the state of the game, and so on. Thus, the equilibrium of a game depends on the belief of players about the state of a game. Hence, the notion of information and belief is interconnected. The publicly available information is defined using common knowledge. The informal definition of common knowledge is sufficient for our model.

Definition 2.2.2. *(common knowledge) A fact F is common knowledge among the players of a game if all the players know F, all the players know that all the players know F, all the players know that all the players know that all the players know F, and so on (for every finite number of levels).*

In the sequential game of incomplete information, an information I_i is defined for every player $i \in N$ such that $I_i := \{I_i^1, I_i^2, ..., I_i^{k(i)}\}$ is a partition of vertices of a game tree. In other words, set I_i for every player i, is a collection of information sets $I_i^j, j = 1, ..., k(i)$. Every information set I_i^j contains vertices such that the player i is uncertain about which of those vertices the game is at.

In a specific information set, actions available to player i are the same. We denote actions available to player i at the information set I_i^j by $A(I_i^j)$. Thus, the pure strategy s_i of player i is a function

$$s_i : I_i \to \bigcup_{I_i^j \in I_i} A(I_i^j) \tag{2.2}$$

In other words, a strategy of player i should only take actions available to the player in his information set. A mixed strategy of player i is a probability distribution over his pure strategies.

A player wants to maximize his expected payoff, but he does not know which vertex in the information set of the game tree he is at. Thus, his belief about which vertex he is at is crucial in determining an equilibrium. In the sequential game of incomplete information, the belief of player i is a function $\mu_i^j : I_i^j \to [0, 1]$ such that

$$\sum_{x \in I_i^j} \mu_i^j(x) = 1$$

A belief system is denoted by μ_i such that $\mu_i = (\mu_i^j : j \in \{1, 2, ..., k(i)\})$. A player's strategy depends on his belief system. Thus, in addition to strategies, beliefs also influence an equilibrium in the game of incomplete information.

2.2.3 Perfect Bayesian Equilibrium

A perfect Bayesian equilibrium is a refinement of a Nash equilibrium suitable for a game of incomplete information. A perfect Bayesian equilibrium (also referred to as a PBE) includes the belief system of players in the equilibrium profile. To define perfect Bayesian equilibrium, we need to define a few other notions.

Definition 2.2.3. *(Sequential Rationality) A strategy s_i^* of player i at an information set I_i^j is sequentially rational given s_{-i} and beliefs μ_i if*

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(s_i^*, s_{-i} \mid x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(s_i, s_{-i} \mid x) \ \forall s_i \in S_i \tag{2.3}$$

where $u_i(s_i, s_{-i} \mid x)$ denotes a payoff of player i given the vertex x is reached.

Given the strategies of all other players, a player's expected payoff corresponding to his belief is maximized by playing a sequentially rational strategy.

Definition 2.2.4. *(Bayesian Consistency) A belief $\mu_i = (\mu_i^j : j \in \{1, 2, ..., k(i)\})$ is Bayesian consistent with respect to a mixed strategy profile $s = (s_i : i \in N)$ if it is derived from the strategy profile using Bayes rule, i.e.,*

$$\mu_i^j(x) = \frac{P_s(x)}{\sum_{y \in I_i^j} P_s(y)} \ \forall x \in I_i^j, \ \forall j \in \{1, 2, ..., k(i)\} \tag{2.4}$$

where $P_s(x)$ be a probability of reaching to vertex x under strategy profile s.

Definition 2.2.5. *(Perfect Bayesian Equilibrium) The belief system $\mu^* = (\mu_i^* : i \in N)$ and strategy profile $s^* = (s_i^* : i \in N)$ is a perfect Bayesian equilibrium if for every player i*

1. *s_i^* is sequentially rational given s_{-i}^* and μ_i^* at every information set of i. (sequential rationality)*

2. *μ_i^* is Bayesian consistent with respect to s^*. (belief consistency)*

On game tree paths of zero probability (i.e., impossible to reach those vertices through s_i^*), known as 'off-equilibrium paths,' the beliefs must be specified but can be arbitrary.

A perfect Bayesian equilibrium is a Nash equilibrium given the belief system of every player where the belief is updated according to Bayes rule. The perfect Bayesian equilibrium can be divided into three different categories. [22]

1. Separating equilibrium - the action of players does not give any information, so the beliefs are not updated after observing the actions.

2. Pooling equilibrium - the action of players reveals information about the state of the game, so the beliefs become deterministic after observing the actions.

3. Semi-separating equilibrium - the one that does not fall into these two categories.

The notion is primarily important when we study how much information is revealed from the actions of a player.

2.3 Literature Review

In this section, we discuss the existing literature on the mathematical modeling of trading dynamics in financial markets. We summarize different modeling approaches and key results.

The trading behavior and strategies of the brokers to use private information have been a long question of interest for academicians and market participants alike. Kyle [17] models trading between informed traders, market makers, and noise traders. The objective is to study the mechanism of inclusion of private information in the price and distribution of profits among traders based on their informativeness. He shows that the profit of the informed trader increases as the noise trading increases. He also shows that the information is incorporated in the price when the informed trader trades. Glosten and Milgrom [15] study the the behavior of market makers in terms of bid-ask spread. They model the market dynamics among market makers, informed traders, and noise traders. The study shows that the market makers become cautious (i.e., the bid-ask spread widens) as the fraction of informed traders in the market or the informativeness of the informed trader increases. Moreover, increased trading activity from informed traders leads to a widening of spread in immediate effect but a shrinking of spreads in the long term as informed traders bring information to the market. If the proportion of insiders compared to noise traders is very high, then the spread continues to widen, and no trade is possible. As there is no trade, no new information is brought to the market, and hence, there is a possibility of market breakdown. The standard equilibrium concept used in these studies is based on two premises: the market maker sets the price equal to the expected value of a security given order flow (market efficiency), and an informed trader maximizes his expected profit.

These two studies mainly inspire the literature on modeling the front-running. Thus, the market dynamics in the presence of asymmetric information is at the core of front-running models. Fishman and Longstaff [13] study the brokers' behavior when they can trade both on a personal account and on behalf of their client (called dual trading). They find that the informed client is worse off, and an uninformed client is better off by the broker front-running. The broker has an incentive to trade more aggressively when they can front-run the client, and the broker's expected profit also increases with that front-running. Bernhardt and Taub [11] focus on studying the price dynamics in the presence of front-running. They show that the front-running leads to higher expected profitability even when the front-runner only knows the future order flow and not the intrinsic value of the security. Danthine and Moresi [12] extend Kyle's model to study the conflict between informed traders of two types: informed due to knowledge of the intrinsic value of the asset (fundamentally-informed) and informed due to knowledge of order flow before it hits the market (trade-informed). They show that fundamentally-informed traders place large orders when the market is liquid to ensure that the the price is not much deviated by their trading. On the contrary, the strategy of trade-informed traders does not depend on liquidity, and it is always liquidity absorbing.

All these studies focus on the front-running from the point of view of market dynamics. They focus on the strategic interactions between different types of traders who actually participate in the exchange of securities. The role of the regulator is not incorporated in these models. The model that we propose is related but different from the existing literature on front-running. Our model is similar in the aspects of the broker's informativeness about client orders that are considered in the existing studies. But we model the regulator as a player in the market trying to investigate the front-running. To the best of our knowledge, the interactions between the regulator (like SEBI) and the broker in front-running cases are not studied using a game theoretic framework.

Chapter 3

Model

In this chapter, we motivate and explain the game-theoretic model for the front-running problem. In the first section, we state the problem of front-running and explain our modeling approach. In the second section, we present the model details, such as game flow, strategies, and payoffs of the players. We also discuss the solution concept and compute the optimal strategies of the players in different conditions. We build on these strategies to understand the equilibrium behavior in different conditions in the next chapter.

3.1 Model Framework

In this section, we present the front-running situation in detail and how we formulate a game model to capture it. The front-running involves conflicting interactions between two players: a broker and a regulator. The broker can trade for himself and for his client. When a client places an order with the broker, the client expects the broker to execute the order in the client's best interest. When there is no client order, there is a prevalent market trend, and we assume that the broker knows the trend direction. Thus, he also knows the profit that he can generate by trading with a trend-following strategy. When a client order is placed, it can be trend-following or trend-reversing. We also assume that the broker knows the average profit that he can generate in the presence of the client order as well.

At any instance, a client can place a bulk order with his broker. The corresponding broker is informed about the presence or absence of the client order. Any other broker in the market is uninformed about the client order status. An informed broker can decide his strategy based on client order status to generate a non-negative profit. On the contrary, an uninformed broker does not know whether there is a client order or not. Thus, he does not know whether the prevalent market trend is sustained or altered by the client order. Hence, he can not decide whether to trade according to the strategy with the client order or without the client order. In turn, an uninformed broker needs to decide his trading strategy without the information of client order status. In this situation, he follows a strategy that gives him better profitability on average. We formalize it in the broker's payoffs. We model the broker's trading decision only focusing on whether he trades or not. If the broker trades, the exact trading strategy (i.e., whether to buy or sell) is irrelevant from the perspective of detecting front-running.

When the regulator observes that the client order is executed by the broker, there are two possibilities about the informativeness of the corresponding broker. If the broker knows the client order status before his own trading activity, he is guilty of front-running. In another case, the broker might have traded only based on the average profit achievable, and he might have received the client order after his own trading. In this case, the broker is essentially uninformed at the time of making the trading decision. If the regulator investigates, the truth is revealed, and the broker in the former category is punished for front-running, but the latter is not. When the client order hits the market, the information advantage disappears quickly. Hence, the front-running opportunity is alive only between the client order placed to the broker and the broker executing the client order. We model the impact of the client order only through its presence or absence. The impact depends on several other factors, such as its size, liquidity of a security, etc. We incorporate the effect of all other parameters in the payoff function.

When a regulator decides whether to investigate the broker or not, he already knows whether the client order status and the broker has traded for himself or not. Based on this information, the regulator is interested to know whether the broker was informed of the client order when he took his own trading decision. The regulator is rewarded when it catches the front-runner rightly. The reward comes both from the penalty received from the broker and reputational benefits for keeping the sanctity of the markets. The regulator's benefit from the investigation depends on the investigation results in terms of the revelation of the type of broker. However, the fixed cost of the investigation is faced irrespective of its results.

Hence, the regulator faces a trade-off between investigating frequently, which might increase the chance of catching front-runners but carrying heavy fixed costs, and investigating rarely, which decreases the cost but also decreases the chance of catching front-runners.

The payoff of the broker depends on the average profit that he achieves by trading and the penalty that he incurs when the regulator catches him front-running. Thus, only an informed broker in the presence of a client order is concerned about the penalty. However, the informed broker also wishes to use his private information about the client order to achieve front-running profits. But, if the regulator investigates his trading activity, he needs to pay a penalty that might result in heavy losses. Hence, the informed broker wants to drive the regulator away from investigating his trading. To achieve that, the regulator should believe that the broker is uninformed. Hence, the regulator's belief about the broker's informativeness is crucial from the broker's point of view, and the broker tries to shape that belief in his best interest.

In turn, there is a trade-off for both the broker and the regulator. The broker wants to front-run without getting caught, and the regulator wants to catch front-running without investigating too often. The broker's profit depends on how successfully he can drive the regulator away from investigation. We are particularly interested in modeling how the broker drives the regulator's belief and how the regulator devices his investigation policy.

3.2 Game Details

In this section, we explain the game theoretic model and equilibrium concept. As mentioned before, the sequence of actions and each player's information when they take an action is crucial in analyzing outcomes of front-running. Thus, the suitable choice is a sequential game with incomplete information. In the game of incomplete information, the belief of an information-deficient player serves a central role in determining equilibrium strategies. We incorporate the importance of belief in the solution concept.

3.2.1 Sequence of Play

The gameplay happens in the following sequence.

Nature's action

The nature chooses client order status $\Theta \in$ {presence of client order (O), absence of client order ($\bar{O}$)} according to prior distribution α such that $\mathbb{P}(O) = \alpha$ and the type of the broker $T \in$ {informed (I), uninformed (U)} according to prior β such that $\mathbb{P}(I) = \beta$. The market price $p \in [0, \infty)$ is realized according to prior f that may depend on past prices. All the random variables are mutually independent, and all the prior distributions are common knowledge.

The informed broker knows client order status (the realized value of Θ) at the time of deciding whether to trade for himself or not. On the contrary, an uninformed broker does not know the client order status at the time of his own trading decision. If the broker knows about the client order status but has no opportunity to trade for himself before the client order hits the market, he is considered uninformed. In short, an informed broker is one who has an opportunity to front-run.

Broker's action and client order execution

The broker's action is the trading decision based on his information. As the front-running opportunity is alive only before the client order hits the market, the broker's trading denotes whether the broker trades for himself before the client order (if any) hits the market. A trading decision is denoted by Q, where $Q = q$ denotes trading and $Q = \bar{q}$ denotes no trading. When a broker decides to trade, he trades a unit quantity. All profits and penalties are for unit quantity. Even when the quantity varies, the payoff scales with the quantity and the broker's behavior is unchanged.

The trading direction (i.e., whether the broker buys or sells) depends on the market trend. But, once the broker trades for himself, his trading direction is unimportant from the front-running perspective. Thus, both these actions are included in $Q = q$.

The client order (if any) hits the market after the broker takes his action.

Regulator's action

The regulator decides whether to investigate the broker's trading that happens before the client order is executed in the market or not. The regulator takes this decision only after the client order (if any) is executed. Hence, it knows the client order status and the broker's trading decision. The action of regulator A is either to investigate a or not investigate $\bar{a}$. If the regulator decides to investigate, the true type of the broker is revealed. When the regulator investigates an informed broker ($T = I$) when he trades ($Q = q$), and there is a client order ($\Theta = O$), it rightly catches the front running happening in the market and penalizes the corresponding broker.

To understand the important variables of the game, we enlist them again.

1. The client order status $(\Theta) \in \{O, \bar{O}\}$ realized according to prior α; $\mathbb{P}(\Theta = O) = \alpha$.
2. Type of broker $(T) \in \{I, U\}$ realized according to prior β; $\mathbb{P}(T = I) = \beta$.
3. Price $(p) \in [0, \infty)$ realized according to prior f.
4. Trading decision (or broker's action) $(Q) \in \{\bar{q}, q\}$.
5. Investigation decision (or regulator's action) $(A) \in \{a, \bar{a}\}$.

The extensive form of the game is shown in the figure (3.1).

3.2.2 Strategies

The strategies of the broker and the regulator are the functions from their information set to their actions. The broker knows his type and a client order status if he is informed. We denote

$$I_B = \{(I, O), (I, \bar{O}), (U)\} \tag{3.1}$$

The information of the broker comprises his type, possibly a client order status, and the price. Hence, a strategy of the broker π is given by

$$\pi : I_B \times [0, \infty) \rightarrow \Delta(Q) \tag{3.2}$$

where $\Delta(S)$ denotes the space of probability distributions over set S. In other words, given $i_B \in I_B, p \in [0, \infty)$, $\pi(\cdot \mid i_B, p)$ is a probability distributions over Q.

The regulator knows the client order status and the trading decision of the broker.

$$I_R = \{(O, \bar{q}), (O, q), (\bar{O}, \bar{q}), (\bar{O}, q)\} \tag{3.3}$$

Thus, the information of the regulator comprises the broker's trading decision, the client order status, and the price. Hence, a strategy of the regulator σ is given by

$$\sigma : I_R \times [0, \infty) \rightarrow \Delta(A) \tag{3.4}$$

Thus, given $i_R \in I_R, p \in [0, \infty)$, $\sigma(\cdot \mid i_R, p)$ is a probability distributions over A.

3.2.3 Payoffs

Broker's payoff

The broker's payoff comprises a profit and a penalty. The broker's average profit is a function of the client order status and price. The profit is achieved only when the broker trades, and we signify it using an indicator function. Hence, we consider the general form of the broker's profit as

$$\text{Average profit of the broker} = \zeta(\Theta, p)\mathbb{1}_{\{Q=q_1\}} \tag{3.5}$$

Moreover, we get some constraints on the broker's average profit.

Remark 3.2.1. *1. An informed broker knows the client order status and the market trend in the presence and absence of a client order. Thus, he can employ a trend-following strategy, i.e., decide whether to buy or sell based on the trend direction, to ensure a* *non-negative profit.*

$$\zeta(\Theta, p) \geq 0 \text{ if } T = I$$

2. *An uninformed broker knows the trend direction and how much profit he can generate by following the trend. But, the client order can alter the trend, and the average profitability, and the uninformed broker does not know whether there is a client order. Hence, if the client order reverses the trend, he needs to decide his strategy to suit either the absence or presence of client order. He chooses it in such a way that the expected profit is non-negative. He can always do it because flipping buy and sell makes negative profits positive. Define,* $\zeta^U(p) := \mathbb{E}_{\Theta\sim\alpha}(\zeta(\Theta, p))$.

$$\zeta^U(p) = \mathbb{E}_{\Theta\sim\alpha}(\zeta(\Theta, p)) = \Big[\alpha\zeta(O, p) + (1-\alpha)\zeta(\bar{O}, p)\Big] \geq 0 \text{ if } T = U$$

The penalty faced by the broker comes from the regulator's investigation when the broker trades. The penalty depends on the client order status, broker's type, and price. The broker is penalized only when he trades and the regulator investigates. We consider the general form of the penalty as

$$\text{Average penalty of the broker} = \rho(\Theta, T, p)\mathbb{1}_{\{(Q=q)\cap(A=a)\}} \tag{3.6}$$

We have some constraints on the broker's penalty based on the conditions of front-running.

Remark 3.2.2. 1. *The front-running is only possible when there is a client order. Hence, the regulator does not penalize trading activity when there is no client order. Thus,*

$$\rho(\Theta = \bar{O}, T, p) = 0$$

2. *Once the regulator investigates, if the broker is revealed to be uninformed, he does not face any penalty because there is no instance of front-running. Thus,*

$$\rho(\Theta, T = U, p) = 0$$

The payoff of the broker, Φ_B, depends on the profit and penalty. It is given by

$$\begin{aligned}\Phi_B(\Theta, T, p, Q, A) &= \text{Average profit of the broker - Average penalty of the broker} \\ &= \zeta(\Theta, p)\mathbb{1}_{\{Q=q\}} - \rho(\Theta, T, p)\mathbb{1}_{\{(Q=q)\cap(A=a)\}} \\ &= \Big[\zeta(\Theta, p) - \rho(\Theta, T, p)\mathbb{1}_{\{A=a\}}\Big]\mathbb{1}_{\{Q=q\}} \qquad (3.7)\end{aligned}$$

Regulator's payoff

The payoff of the regulator has two terms: a fixed cost, C, corresponding to the investigation such that $C > 0$, and a benefit from the investigation, B, as a function of the revealed type of the broker, order flow, and price. As the investigation is assumed to be completely accurate, the revealed type is the same as the true type of the broker. The regulator gets benefits only when the broker trades. Moreover, if the regulator ignores the trading and decides not to investigate, its benefit and cost are both zero. An indicator function signifies these characteristics. Hence, the payoff of the regulator, Φ_R, is given by

$$\Phi_R(\Theta, T, p, Q, A) = \left[B(\Theta, T, p)\mathbb{1}_{\{Q=q\}} - C\right]\mathbb{1}_{\{A=a\}} \qquad (3.8)$$

The regulator's benefit is in terms of penalty received and reputational benefits. Hence, the benefit is at least as much as the penalty.

$$B(\Theta, T, p) \geq \rho(\Theta, T, p) \forall \Theta \in \{O, \bar{O}\}, T \in \{I, U\}, p \in [0, \infty).$$

We do not impose any more conditions on the penalty and payoff functions ζ, ρ, and B to maintain generality. The equilibrium depends on the interdependence between these functions.

3.3 Equilibrium Definition

In the context of the proposed game, the belief of the regulator about the type of broker is crucial in the regulator's investigation policy. Thus, a perfect Bayesian equilibrium is a suitable choice for the solution concept. In this section, we define the perfect Bayesian equilibrium and the broker's and the regulator's optimal strategies in the perfect Bayesian equilibrium. The framework established in this section help us compute PBEs for different market conditions and analyze broker's and regulator's behavior in the next chapter.

Belief system

In the perfect Bayesian equilibrium, the regulator's optimal strategy depends on the belief system that it holds. Based on the regulator's optimal strategy, the broker chooses his optimal strategy. If the broker's optimal strategy is Bayes' consistent with the regulator's belief, then these strategies and the belief system give a PBE. We first define the belief system $\tilde{\beta}$ of the regulator about the type of broker.

$$\tilde{\beta} := \left\{\tilde{\beta}(\cdot \mid (O,\bar{q}),p), \tilde{\beta}(\cdot \mid (O,q),p), \tilde{\beta}(\cdot \mid (\bar{O},\bar{q}),p), \tilde{\beta}(\cdot \mid (\bar{O},q),p)\right\}_{p\in\mathbb{R}^+}$$

To illustrate, $\tilde{\beta}(I \mid (O,q),p)$ is the probability that the regulator assigns to the informed broker when there is a client order and the broker's trading activity at a price p. The belief system incorporates the regulator's perception of the broker's informativeness.

Regulator's strategy

The regulator's optimal strategy corresponding to its information (Θ, Q, p) depends on the belief system $\tilde{\beta}$. The regulator's optimal strategy (possibly mixed) is denoted by $\sigma^*(\cdot \mid (\Theta, Q, p), \tilde{\beta})$. It maximizes the regulator's expected payoff, given the belief system. Thus,

$$\sigma^*(\cdot \mid (\Theta, Q, p), \tilde{\beta}) \in \underset{\sigma(\cdot|(\Theta,Q),p)}{\operatorname{argmax}} \mathbb{E}_{A\sim\sigma(\cdot|(\Theta,Q),p)}\mathbb{E}_{T\sim\tilde{\beta}(\cdot|(\Theta,Q),p)}\Phi_S\left(\Theta, T, p, Q, A\right) \tag{3.9}$$

where $\mathbb{E}_{X\sim f_X}$ denotes the expectation with respected to random variable X where X is distributed according to pdf f_X. Similarly, we can compute an optimal strategy of the regulator for all possible combinations of (Θ, Q, p) dependent on the belief system. Thus, we get the optimal strategy of the regulator, denoted by $\sigma^*(\cdot \mid \tilde{\beta})$, given his belief system.

$$\sigma^*(\cdot \mid \tilde{\beta}) = \left\{\sigma^*\left(\cdot \mid (\Theta, Q, p), \tilde{\beta}\right)\right\}_{\Theta\in\{O,\bar{O}\}, Q\in\{\bar{q},q\}, p\in\mathbb{R}^+} \tag{3.10}$$

In essence, the regulator's optimal strategy, depending on the belief system (eq. 3.10), is an array of optimal strategies for different information sets.

Broker's strategy

The broker's optimal strategy in PBE is the one that maximizes his expected payoff, given that the regulator plays its optimal strategy. In turn, the optimal strategy of the broker also depends on the belief system that the regulator holds. The optimal strategy of the broker is denoted by $\pi^*(\cdot \mid \tilde{\beta})$. Define,

$$\pi := \left\{\pi(\cdot \mid i_B, p)\right\}_{i_B \in I_B, p \in \mathbb{R}^+}$$

Thus,

$$\pi^*(\cdot \mid \tilde{\beta}) \in \underset{\pi}{\operatorname{argmax}} \mathbb{E}_{p\sim f}\mathbb{E}_{\Theta\sim\alpha}\mathbb{E}_{T\sim\beta}\mathbb{E}_{Q\sim\pi(\cdot|i_B,p)}\mathbb{E}_{A\sim\sigma^*\left(\cdot|(\Theta,Q,p),\tilde{\beta}\right)}\Phi_B\left(\Theta, T, p, Q, A\right) \tag{3.11}$$

Belief consistency

By definition of PBE, the broker's optimal strategy $\pi^*(\cdot \mid \tilde{\beta})$ should be consistent with the belief system. Hence, it should satisfy,

$$\tilde{\beta}\left(I \mid (\Theta, Q), p\right) = \frac{\beta\pi^*\left(Q \mid (\Theta, I), p\right)}{\beta\pi^*\left(Q \mid (\Theta, I), p\right) + (1-\beta)\pi^*\left(Q \mid (U), p\right)}$$
$$, \forall\ \Theta \in \{O, \bar{O}\}, Q \in \{\bar{q}, q\}, p \in \mathbb{R}^+ \tag{3.12}$$

If the belief consistency is satisfied, $\left[\pi^*(\cdot \mid \tilde{\beta}), \sigma^*(\cdot \mid \tilde{\beta}), \tilde{\beta}\right]$ is PBE.

The computation of the PBE starts with fixing a belief system for the regulator. The belief system leads to the corresponding optimal strategies according to (eq. 3.10) and (eq. 3.11). The broker's optimal strategy is used to check the consistency of the belief system according to (eq. 3.12). If the belief system is consistent, we get a PBE; otherwise, we repeat the process with a different belief system.

We compute the broker's and regulator's optimal strategies for different market conditions in the next chapter.

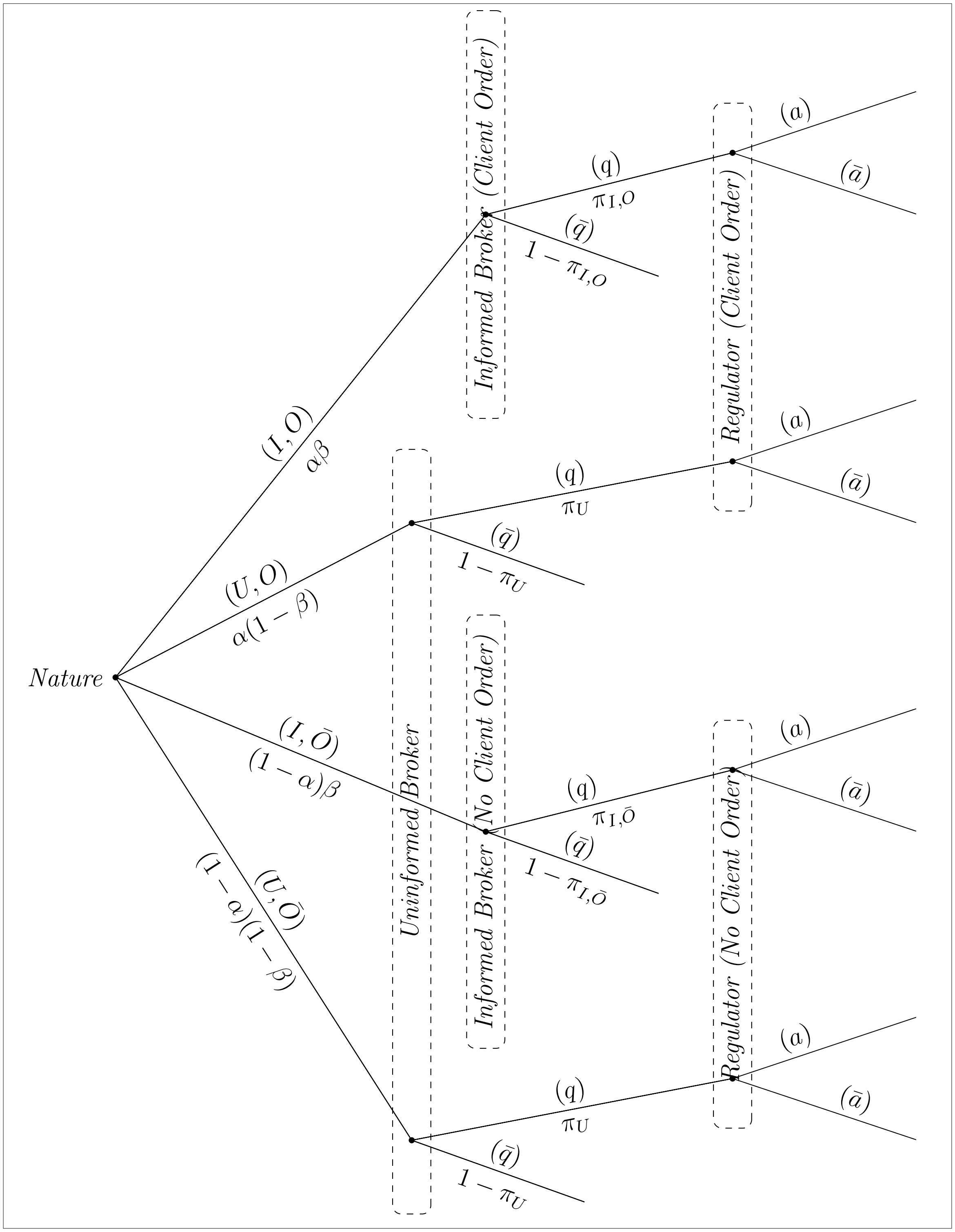

Figure 3.1: Extensive form of the game for fixed price p. (Payoffs and regulator's strategies are mentioned in the details)

Chapter 4

Equilibrium Analysis

In this chapter, we compute perfect Bayesian equilibrium for different market conditions. In the first section, we compute the broker's and regulator's optimal strategies for different market variables. It simplifies the PBE computation. In the next section, we consider different market conditions and analyze the suitable optimal strategies that lead to PBEs. We specifically consider trending and sideways markets and further study the impact of the client order on the prevalent market trend. This analysis reveals interesting insights about the broker's deceptive behavior, the regulator's investigation policy, and the role of information spread. We analyze these details in the next chapter.

4.1 Equilibrium Strategies and Beliefs

In this section, we compute the broker's and regulator's optimal strategies depending on profit, penalty and other relevant market variables. This section simplifies the computation that helps in analyzing different market conditions.

4.1.1 Regulator's Optimal Strategy

We first find the regulator's optimal strategy as a function of its belief system.

Proposition 4.1.1. *When there is no trading (i.e., $Q = \bar{q}$), the optimal strategy of the* *regulator is always not to investigate ($\bar{a}$).*

Proof. The regulator's payoff (eq. 3.8) is given by

$$\Phi_R(\Theta, T, p, Q, A) = \left[B(\Theta, T, p)\mathbb{1}_{\{Q=q\}} - C\right]\mathbb{1}_{\{A=a\}}$$

When there is no trading (i.e., $Q = \bar{q}$),

$$\Phi_R(\Theta, T, p, Q = \bar{q}, A) = [-C]\,\mathbb{1}_{\{A=a\}} = \begin{cases} -C & A = a \\ 0 & A = \bar{a} \end{cases} \tag{4.1}$$

Thus, the regulator's payoff is independent of the client order, broker's type, and price when there is no trading. As $C > 0$, we get the regulator's optimal strategy given by (eq. 3.9) when there is no trading.

$$\sigma^*(\cdot \mid (O, \bar{q}, p), \tilde{\beta}) = \sigma^*(\cdot \mid (\bar{O}, \bar{q}, p), \tilde{\beta}) = \bar{a},\ \forall p \in [0, \infty) \tag{4.2}$$

□

We can observe that the regulator's optimal strategy when there is no trading is independent of the belief system. It is aligned with our intuition that the investigation can be beneficial only when there is trading.

When there is trading activity (i.e., $Q = q$), we define components of the belief system for simplification.

$$\tilde{\beta}_O(p) := \tilde{\beta}(I \mid (O, q), p) \Rightarrow 1 - \tilde{\beta}_O(p) := \tilde{\beta}(U \mid (O, q), p)$$
$$\tilde{\beta}_{\bar{O}}(p) := \tilde{\beta}(I \mid (\bar{O}, q), p) \Rightarrow 1 - \tilde{\beta}_{\bar{O}}(p) := \tilde{\beta}(U \mid (\bar{O}, q), p)$$

To illustrate, $\tilde{\beta}_O(p)$ denotes the probability that the regulator assigns its belief to an informed broker when it observes the client order and trading activity at a price p.

Proposition 4.1.2. *When there is a client order and trading activity (i.e., $\Theta = O, Q = q$),* *the optimal strategy of the regulator is given by*

$$\sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \begin{cases} a & \tilde{\beta}_O(p) > \lambda_O \\ a \ w.p. \ \delta_O, \bar{a} \ w.p. \ 1-\delta_O & \tilde{\beta}_O(p) = \lambda_O \\ \bar{a} & \tilde{\beta}_O(p) < \lambda_O \end{cases} \tag{4.3}$$

where $\lambda_O = \frac{C-B(O,U,p)}{B(O,I,p)-B(O,U,p)}$ *and* δ_O *signifies the mixed strategy such that* $\delta_O \in [0,1]$.

Proof. When $Q = q$, the regulator's payoff (eq. 3.8) is given by

$$\Phi_R(\Theta, T, p, Q = q, A) = [B(\Theta, T, p) - C] \, \mathbb{1}_{\{A=a\}}$$

Hence, the regulator's optimal strategy (eq. 3.9) is given by

$$\sigma^*(\cdot \mid (\Theta = O, Q = q, p), \tilde{\beta}) \in \underset{\sigma(\cdot|(O,q),p)}{\operatorname{argmax}} \mathbb{E}_{A\sim\sigma(\cdot|O,q,p)} \mathbb{E}_{T\sim\tilde{\beta}(\cdot|O,q,p)} [B(O,T,p) - C] \, \mathbb{1}_{\{A=a\}} \tag{4.4}$$

Using the notations defined earlier,

$$\begin{aligned} \mathbb{E}_{T\sim\tilde{\beta}(\cdot|O,q,p)} [B(O,T,p) - C] \, \mathbb{1}_{\{A=a\}} &= \{(\tilde{\beta}_O(p)) [B(O,I,p) - C] + (1-\tilde{\beta}_O(p)) [B(O,U,p) - C]\} \mathbb{1}_{\{A=a\}} \\ &= \{(\tilde{\beta}_O(p)) [B(O,I,p) - B(O,U,p)] + [B(O,U,p) - C]\} \mathbb{1}_{\{A=a\}} \\ &= \begin{cases} (\tilde{\beta}_O(p)) [B(O,I,p) - B(O,U,p)] + [B(O,U,p) - C] & A = a \\ 0 & A = \bar{a} \end{cases} \end{aligned}$$

Define $\lambda_O = \frac{C-B(O,U,p)}{B(O,I,p)-B(O,U,p)}$, then

$$(\tilde{\beta}_O(p)) [B(O,I,p) - B(O,U,p)] + [B(O,U,p) - C] > 0 \Leftrightarrow \tilde{\beta}_O(p) > \lambda_O$$

Hence, the proposition holds. $\square$

Remark 4.1.1. *We define the regulator's investigation threshold (λ_O) when there is a client order as*

$$\lambda_O = \frac{C - B(O,U,p)}{B(O,I,p) - B(O,U,p)}$$

.

When the regulator is indifferent between its actions (i.e., when $\tilde{\beta}_O(p) = \lambda_O$), it chooses its mixed strategy (i.e., δ_O) in such a way that the broker becomes indifferent between his actions. The following lemma gives the regulator's optimal strategy when there is no client order.

Lemma 4.1.3. *When there is no client order but the trading activity (i.e., $\Theta = \bar{O}, Q = q$), the optimal strategy of the regulator is given by*

$$\sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = \begin{cases} a & \tilde{\beta}_{\bar{O}}(p) > \lambda_{\bar{O}} \\ a \text{ } w.p. \text{ } \delta_{\bar{O}}, \bar{a} \text{ } w.p. \text{ } 1 - \delta_{\bar{O}} & \tilde{\beta}_{\bar{O}}(p) = \lambda_{\bar{O}} \\ \bar{a} & \tilde{\beta}_{\bar{O}}(p) < \lambda_{\bar{O}} \end{cases} \tag{4.5}$$

where $\lambda_{\bar{O}} = \frac{C - B(\bar{O},U,p)}{B(\bar{O},I,p) - B(\bar{O},U,p)}$ and $\delta_{\bar{O}}$ signifies the mixed strategy such that $\delta_{\bar{O}} \in [0, 1]$.

The regulator's investigation threshold ($\lambda_{\bar{O}}$) when there is no client order is defined as

$$\lambda_{\bar{O}} = \frac{C - B(\bar{O}, U, p)}{B(\bar{O}, I, p) - B(\bar{O}, U, p)}$$

Even though we have found the optimal strategy of the regulator when there is no client order, it does not affect the broker's optimal strategy because the penalty is zero in the absence of the client order, i.e., $\rho(\Theta = \bar{O}, T, p) = 0$.

4.1.2 Broker's Optimal Strategy

The broker's optimal strategy can be simplified by defining components of the strategy. The tuple $[\pi_{I,O}(p), \pi_{I,\bar{O}}(p), \pi_U(p)]$ defines the broker's strategy profile where

$$\pi_{I,O}(p) := \pi(q \mid (I, O), p) \tag{4.6}$$

is a probability with which an informed (I) broker trades at a price p when there is a client order (O).

$$\pi_{I,\bar{O}}(p) := \pi(q \mid (I, \bar{O}), p) \tag{4.7}$$

is a probability with which an informed (I) broker trades at a price p when there is no client order ($\bar{O}$).

$$\pi_U(p) := \pi(q \mid (U), p) \tag{4.8}$$

is a probability with which an uninformed (U) broker trades at a price p. We define the probability of investigation ($\mathbb{P}_O(\{A = a\})$) as the probability with which the regulator investigates trading in his optimal strategy in the presence of a client order.

$$\mathbb{P}_O(\{A = a\}) := \mathbb{E}_{A \sim \sigma^*(\cdot|(O,q),p)} \mathbb{1}_{\{A=a\}}$$

Proposition 4.1.4. *The dependence of the broker's optimal strategy on the SEBI's belief system is only through* $\mathbb{P}_O(\{A = a\})$*. Moreover, the broker's optimal strategy in the presence of client order is*

$$\pi^*_{I,O}(p) = \begin{cases} 1 & \zeta(O,p) - \rho(O,I,p)\mathbb{P}_O(\{A = a\}) > 0 \\ \textit{any value in [0,1]} & \zeta(O,p) - \rho(O,I,p)\mathbb{P}_O(\{A = a\}) = 0 \\ 0 & \zeta(O,p) - \rho(O,I,p)\mathbb{P}_O(\{A = a\}) < 0 \end{cases} \tag{4.9}$$

Proof. For the fixed market price p, the broker's optimization equation (eq. 3.11) is given by

$$\pi^*(\cdot \mid p, \tilde{\beta}) \in \underset{\pi(\cdot|i_B,p)}{\operatorname{argmax}} \mathbb{E}_{\Theta\sim\alpha}\mathbb{E}_{T\sim\beta}\mathbb{E}_{Q\sim\pi(\cdot|i_B,p)}\mathbb{E}_{A\sim\sigma^*\left(\cdot|(\Theta,Q,p),\tilde{\beta}\right)}\Phi_B\left(\Theta,T,p,Q,A\right) \tag{4.10}$$

Define the expected payoff of the broker at price p

$$\mathbb{E}(\Phi_B \mid p) := \mathbb{E}_{\Theta\sim\alpha}\mathbb{E}_{T\sim\beta}\mathbb{E}_{Q\sim\pi(\cdot|i_B,p)}\mathbb{E}_{A\sim\sigma^*\left(\cdot|(\Theta,Q,p),\tilde{\beta}\right)}\Phi_B\left(\Theta,T,p,Q,A\right) \tag{4.11}$$

Hence,

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) = \alpha\beta &\left[\pi_{I,O}(p)\mathbb{E}_{A\sim\sigma^*(\cdot|O,q,p)}\left\{\zeta(O,p) - \rho(O,I,p)\mathbb{1}_{\{A=a\}}\right\}\right] + \\ &\alpha(1-\beta)\left[\pi_U(p)\mathbb{E}_{A\sim\sigma^*(\cdot|O,q,p)}\left\{\zeta(O,p) - \rho(O,U,p)\mathbb{1}_{\{A=a\}}\right\}\right] + \\ &(1-\alpha)\beta\left[\pi_{I,\bar{O}}(p)\mathbb{E}_{A\sim\sigma^*(\cdot|\bar{O},q,p)}\left\{\zeta(\bar{O},p) - \rho(\bar{O},I,p)\mathbb{1}_{\{A=a\}}\right\}\right] + \\ &(1-\alpha)(1-\beta)\left[\pi_U(p)\mathbb{E}_{A\sim\sigma^*(\cdot|\bar{O},q,p)}\left\{\zeta(\bar{O},p) - \rho(\bar{O},U,p)\mathbb{1}_{\{A=a\}}\right\}\right]\end{aligned} \tag{4.12}$$

Using the constraints on the penalty ρ under different client order statuses,

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) = \alpha\beta &\left[\pi_{I,O}(p)\mathbb{E}_{A\sim\sigma^*(\cdot|O,q,p)}\left\{\zeta(O,p) - \rho(O,I,p)\mathbb{1}_{\{A=a\}}\right\}\right] \\ &+ \alpha(1-\beta)\left[\pi_U(p)\mathbb{E}_{A\sim\sigma^*(\cdot|O,q,p)}\left\{\zeta(O,p)\right\}\right] \\ &+ (1-\alpha)\beta\left[\pi_{I,\bar{O}}(p)\mathbb{E}_{A\sim\sigma^*(\cdot|\bar{O},q,p)}\left\{\zeta(\bar{O},p)\right\}\right] \\ &+ (1-\alpha)(1-\beta)\left[\pi_U(p)\mathbb{E}_{A\sim\sigma^*(\cdot|\bar{O},q,p)}\left\{\zeta(\bar{O},p)\right\}\right] \end{aligned} \tag{4.13}$$

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) = \alpha\beta &\left[\pi_{I,O}(p)\mathbb{E}_{A\sim\sigma^*(\cdot|O,q,p)}\left\{\zeta(O,p) - \rho(O,I,p)\mathbb{1}_{\{A=a\}}\right\}\right] \\ &+ \alpha(1-\beta)\left[\pi_U(p)\left\{\zeta(O,p)\right\}\right] + (1-\alpha)\beta\left[\pi_{I,\bar{O}}(p)\left\{\zeta(\bar{O},p)\right\}\right] \\ &+ (1-\alpha)(1-\beta)\left[\pi_U(p)\left\{\zeta(\bar{O},p)\right\}\right] \end{aligned} \tag{4.14}$$

Moreover, we have $\mathbb{P}_O(\{A=a\}) = \mathbb{E}_{A\sim\sigma^*(\cdot|O,q,p)}\mathbb{1}_{\{A=a\}}$

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) = \alpha\beta &\left[\pi_{I,O}(p)\left\{\zeta(O,p) - \rho(O,I,p)\mathbb{P}_O(\{A=a\})\right\}\right] \\ &+ \alpha(1-\beta)\left[\pi_U(p)\left\{\zeta(O,p)\right\}\right] + (1-\alpha)\beta\left[\pi_{I,\bar{O}}(p)\left\{\zeta(\bar{O},p)\right\}\right] \\ &+ (1-\alpha)(1-\beta)\left[\pi_U(p)\left\{\zeta(\bar{O},p)\right\}\right] \end{aligned} \tag{4.15}$$

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) = \alpha\beta &\left[\pi_{I,O}(p)\left\{\zeta(O,p) - \rho(O,I,p)\mathbb{P}_O(\{A=a\})\right\}\right] \\ &+ (1-\alpha)\beta\left[\pi_{I,\bar{O}}(p)\left\{\zeta(\bar{O},p)\right\}\right] \\ &+ (1-\beta)\left[\pi_U(p)\left\{\alpha\zeta(O,p) + (1-\alpha)\zeta(\bar{O},p)\right\}\right] \end{aligned} \tag{4.16}$$

Hence, the broker's optimal strategy depends on the SEBI's belief only through $\mathbb{P}_O(\{A=a\})$. Moreover, we can independently maximize the broker's expected payoff corresponding to different components of his strategy. It gives us the required form of $\pi^*_{I,O}(p)$. □

The broker's expected payoff given by (eq. 4.16) can be used to compute the remaining components of his optimal strategy.

Remark 4.1.2. *The broker's optimal strategy whenever he is uninformed or there is no client order is independent of the SEBI's belief and is only a function of market variables.*

The broker's optimal strategy in these cases is given by

$$\pi^*_{I,\bar{O}}(p) = \begin{cases} 1 & \zeta(\bar{O},p) > 0 \\ \textit{any value in [0,1]} & \zeta(\bar{O},p) = 0 \end{cases} \tag{4.17}$$

$$\pi^*_{U}(p) = \begin{cases} 1 & \zeta^U(p) > 0 \\ \textit{any value in [0,1]} & \zeta^U(p) = 0 \end{cases} \tag{4.18}$$

4.1.3 Belief Consistency

The broker's and SEBI's optimal strategies contribute to the PBE if the belief consistency is satisfied. The belief system is signified using $\tilde{\beta}_O(p)$ and $\tilde{\beta}_{\bar{O}}(p)$ and we only need to check the consistency for them. Using notations defined before, the belief consistency is given by

$$\tilde{\beta}_O(p) = \frac{\alpha\beta\pi^*_{I,O}(p)}{\alpha\beta\pi^*_{I,O}(p) + \alpha(1-\beta)\pi^*_U(p)} = \frac{\beta\pi^*_{I,O}(p)}{\beta\pi^*_{I,O}(p) + (1-\beta)\pi^*_U(p)} \tag{4.19}$$

$$\tilde{\beta}_{\bar{O}}(p) = \frac{(1-\alpha)\beta\pi^*_{I,\bar{O}}(p)}{(1-\alpha)\beta\pi^*_{I,\bar{O}}(p) + (1-\alpha)(1-\beta)\pi^*_U(p)} = \frac{\beta\pi^*_{I,\bar{O}}(p)}{\beta\pi^*_{I,\bar{O}}(p) + (1-\beta)\pi^*_U(p)} \tag{4.20}$$

Whenever the RHS of any of these equations is not well-defined, the respective belief is 'off-equilibrium paths,' and we need not check consistency for it by definition.

Once we check the belief consistency, the perfect Bayesian equilibrium is given by the strategy and belief profile.

$$\left[\left(\sigma^*(\cdot \mid (O,q,p),\tilde{\beta}), \sigma^*(\cdot \mid (\bar{O},q,p),\tilde{\beta}))\right), \left(\pi^*_{I,O}(p), \pi^*_{I,\bar{O}}(p), \pi^*_U(p)\right), \left(\tilde{\beta}_O(p), \tilde{\beta}_{\bar{O}}(p)\right)\right]$$

We can observe that components $\pi^*_{I,\bar{O}}(p)$ and $\pi^*_U(p)$ of the broker's optimal strategy are independent of the regulator's strategy and only depend on market conditions. Moreover, $\pi^*_{I,O}(p)$ only depends on $\sigma^*(\cdot \mid (O,q,p),\tilde{\beta})$. Hence, while analyzing equilibrium for different market trends, we can compute $\pi^*_{I,\bar{O}}(p)$ and $\pi^*_{I,O}(p)$ independently. The belief consistency equations also allow us to do that. We get different PBEs by combining $\pi^*_{I,\bar{O}}(p)$ and $\pi^*_{I,O}(p)$ with corresponding belief systems and regulator's optimal strategy.

4.1.4 Separating and Pooling Equilibrium

In this game setting, we need to define separating and pooling equilibria for both client order statuses separately.

Definition 4.1.1. *When there is a client order ($\Theta = O$) at price p,*

1. *Separating Equilibria - if* $\pi^*_{I,O}(p) = 0, \pi^*_U(p) = 1$ *or* $\pi^*_{I,O}(p) = 1, \pi^*_U(p) = 0$
2. *Pooling Equilibria - if* $\pi^*_{I,O}(p) = \pi^*_U(p) = 1$ *or* $\pi^*_{I,O}(p) = \pi^*_U(p) = 0$
3. *Semi-separating Equilibria - the ones which are neither separating nor pooling.*

Definition 4.1.2. *When there is no client order ($\Theta = \bar{O}$) at price p,*

1. *Separating Equilibria - if* $\pi^*_{I,\bar{O}}(p) = 0, \pi^*_U(p) = 1$ *or* $\pi^*_{I,\bar{O}}(p) = 1, \pi^*_U(p) = 0$
2. *Pooling Equilibria - if* $\pi^*_{I,\bar{O}}(p) = \pi^*_U(p) = 1$ *or* $\pi^*_{I,\bar{O}}(p) = \pi^*_U(p) = 0$
3. *Semi-separating Equilibria - the ones which are neither separating nor pooling.*

These definitions of separating and pooling equilibria help us categorize PBEs based on how much information about the broker's type is revealed to the regulator through his trading activity.

4.2 Market Trends

In this section, we compute and interpret PBEs for different market conditions. We have computed the regulator's and the broker's optimal strategy. As they depend on the market conditions (through market variables such as profit and penalty functions), we need to compute the PBEs for different market conditions separately. When there is no client order, the market can be trending or sideways.

1. Trending Market ($\zeta(\bar{O}, p) > 0$) - In the trending market, trading with an appropriate strategy (buy/sell) is always profitable. Thus, an informed broker is willing to trade in this market condition.

2. Sideways Market ($\zeta(\bar{O}, p) = 0$) - In the sideways market, trading with any strategy leads to zero profit. Thus, an informed broker is indifferent between trading and not trading in this market condition.

We analyze these two market conditions independently. We further make different cases based on the change in the trend brought by the client order.

4.2.1 Trending Markets

When the market is trending (i.e., $\zeta(\bar{O}, p) > 0$) in the absence of the client order, an informed broker can employ a trend-following strategy to generate profits and, thus, always trade in the trading market. Hence, we get the optimal strategies and the corresponding beliefs in the absence of client order.

1. $\pi^*_{I,\bar{O}}(p) = 1$ if $\beta < \lambda_{\bar{O}}$. [Belief system: $\tilde{\beta}_{\bar{O}}(p) < \lambda_{\bar{O}} \Rightarrow \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = \bar{a}$]

2. $\pi^*_{I,\bar{O}}(p) = 1$ if $\beta = \lambda_{\bar{O}}$. [Belief system: $\tilde{\beta}_{\bar{O}}(p) = \lambda_{\bar{O}}(p) \Rightarrow \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = a$ w.p. $\delta_{\bar{O}}, \bar{a}$ w.p. $1 - \delta_{\bar{O}}$]

3. $\pi^*_{I,\bar{O}}(p) = 1$ if $\beta > \lambda_{\bar{O}}$. [Belief system: $\tilde{\beta}_{\bar{O}}(p) > \lambda_{\bar{O}} \Rightarrow \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = a$]

The optimal strategy of the informed broker is the absence of client order is the same, but the belief depends on the spread on information (β) in the market.

Remark 4.2.1. *When there is no client order ($\Theta = \bar{O}$), all the equilibria are pooling equilibria (because $\pi^*_{I,\bar{O}}(p) = \pi^*_U(p) = 1$). Hence, the regulator gets no more information about the broker's type from his trading decision.*

In the trending market, the client order can be trend-following or trend-reversing. If the client order is trend-following, an uninformed broker has to follow the trend that is independent of the client order status. Thus, he can ensure a positive expected profit. If the client order is trend-reversing, an uninformed broker needs to choose whether to follow the trend in the presence or absence of the client order. In this case, he follows the trend that gives him higher average profitability. When the profitability in both the trend directions is equal and opposite, the uninformed broker essentially gets zero expected profit by trading. If the client order makes the market sideways, the uninformed broker follows the trend in the absence of the client order to ensure a positive expected profit.

The informed broker's average profitability in the presence of client order lies in different ranges compared to the penalty. Thus, we get different cases of market conditions based on the price profitability. We present different possibilities of $\pi^*_{I,O}(p)$ and $\pi^*_U(p)$ along with corresponding belief systems and regulator's optimal strategy in the table (4.1). A perfect Bayesian equilibrium is a combination of optimal strategies and beliefs in the presence and absence of a client order as mentioned before. We also explain every case in detail below.

Neutral Client Order ($\zeta(O, p) = 0$)

When the client order is neutral, the client order essentially stops the trend and makes the market sideways. the uninformed broker can align his strategy with the trend in the absence of a client order. In that case, he can always get a positive expected profit. Hence,

$$\zeta^U(p) = (1 - \alpha)\zeta(\bar{O}, p) > 0$$

We get the following sub-cases of optimal strategies.

1. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) < \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

   The regulator's belief is against investigation. Hence, the informed broker is indifferent between trading and not trading when there is a client order and trades with the probability that is consistent with the regulator's belief. the uninformed broker always trades using a suitable strategy.

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) &= (1 - \alpha)\beta\zeta(\bar{O}, p) + (1 - \beta)(1 - \alpha)\zeta(\bar{O}, p) \\ &= (1 - \alpha)\zeta(\bar{O}, p)\end{aligned}$$

	$\zeta^U(p) > 0$ $(\pi^*_U(p) = 1)$ (Profitable on average for uninformed broker)	$\zeta^U(p) = 0$ (any $\pi^*_U(p)$ such that $\pi^*_U(p) \in [0,1]$) (Non-profitable on average for uninformed broker)
$\zeta(O,p) = 0$ (Neutral Client Order)	1. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) < \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = 0$ and $\beta \geq \lambda_O$ [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$]	Not Possible
$0 < \zeta(O,p) < \rho(O,I,p)$ (Moderately profitable Client Order)	1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = \frac{\zeta(O,p)}{\rho(O,I,p)}$ and $\beta \geq \lambda_O$ [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$]	1. $\pi^*_{I,O}(p) = 1$ if $\frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta} < \pi^*_U(p)$ and $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$ with $\delta_O = \frac{\zeta(O,p)}{\rho(O,I,p)}$. [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$]
$\zeta(O,p) = \rho(O,I,p)$ (Marginally profitable Client Order)	1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = 1$ and $\beta \geq \lambda_O$ [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$] 3. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ and $\beta > \lambda_O$ [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$]	1. $\pi^*_{I,O}(p) = 1$ if $\frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta} < \pi^*_U(p)$ and $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$ with $\delta_O = 1$. [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$] 3. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$. [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$]
$\zeta(O,p) > \rho(O,I,p)$ (Highly profitable Client Order)	1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = 1$ if $\beta > \lambda_O$. [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$]	1. $\pi^*_{I,O}(p) = 1$ if $\frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta} < \pi^*_U(p)$ and $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = 1$ if $\frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta} > \pi^*_U(p)$. [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$]

Table 4.1: Optimal strategies and beliefs for different price regions in the 'trending market'

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = 0$ and $\beta \geq \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

   The regulator's belief is such that it is indifferent between its actions. Hence, the regulator chooses not to investigate to make the informed broker indifferent between trading and not trading when there is a client order. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief. The equilibrium exists only if the spread of the information is equal or higher compared to the regulator's investigation threshold (i.e., $\beta \geq \lambda_O$).

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) &= (1-\alpha)\beta\zeta(\bar{O}, p) + (1-\beta)(1-\alpha)\zeta(\bar{O}, p)\\ &= (1-\alpha)\zeta(\bar{O}, p)\end{aligned}$$

3. The belief $\tilde{\beta}_O(p) > \lambda_O$ does not lead to an optimal strategy consistent with the belief.

We have computed $\pi^*_{I,O}(p)$ and $\pi^*_{I,\bar{O}}(p)$ separately. Hence, we get a total of 6 PBE classes by combining three different sub-cases of $\pi^*_{I,\bar{O}}(p)$ and two different sub-cases of $\pi^*_{I,O}(p)$. For example, one of the PBE classes is

$$\Bigg[\Big(\sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}, \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = \bar{a})\Big), \Big(\text{any } \pi^*_{I,O}(p) \text{ such that}$$

$$\pi^*_{I,O}(p) < \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}, \pi^*_{I,\bar{O}}(p) = 1, \pi^*_U(p) = 1\Big), \Big(\tilde{\beta}_O(p) < \lambda_O, \tilde{\beta}_{\bar{O}}(p) < \lambda_{\bar{O}}\Big)\Bigg] \text{ if } \beta < \lambda_{\bar{O}}$$

This class also contains several PBEs depending on the exact belief system and consistent strategies. For example,

$$\Bigg[\Big(\sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}, \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = \bar{a})\Big),$$

$$\Big(\pi^*_{I,O}(p) = \frac{1-\beta}{\beta}\frac{\lambda_O}{2-\lambda_O}, \pi^*_{I,\bar{O}}(p) = 1, \pi^*_U(p) = 1\Big), \Big(\tilde{\beta}_O(p) = \frac{\lambda_O}{2}, \tilde{\beta}_{\bar{O}}(p) = \beta\Big)\Bigg] \text{ if } \beta < \lambda_{\bar{O}}$$

Similarly, we can enlist all the possible PBEs. However, we analyze the strategies independently for ease of understanding.

Moderately profitable Client Order ($\rho(O, I, p) > \zeta(O, p) > 0$)

When the client order is moderately profitable, the trading in the presence of the client order is profitable only when the regulator does not investigate the trade. In this case, the client order can be either trend-following or trend-reversing. If the client order is trend-following, the uninformed broker can follow the same trend to ensure a positive expected profit. If the client order is trend-reversing, the expected profit of the uninformed broker can be positive or zero, depending on the degree of trend-reversal. Hence,

$$\zeta^U(p) > 0 \text{ or } \zeta^U(p) = 0$$

Hence, we get two sub-cases of the overall market trend.

Profitable on average for the uninformed broker ($\zeta^U(p) > 0$)

If the market is profitable on average for the uninformed broker, the uninformed broker has a strictly positive expected profit. The uninformed broker's optimal strategy is always to trade. (i.e., $\pi^*_U(p) = 1$). We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$ with $\beta < \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

   The regulator's belief is against investigation. Hence, the informed broker trades always owing to the profitability of the price when there is a client order. The equilibrium exists only if the spread of the information is lower compared to the regulator's investigation threshold (i.e., $\beta < \lambda_O$).

   $$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O, p) + (1 - \alpha)\beta\zeta(\bar{O}, p) + (1 - \beta)\zeta^U(p)$$

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = \frac{\zeta(O,p)}{\rho(O,I,p)} \in (0, 1)$ and $\beta \geq \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$ w.p. δ_O, X w.p. $1 - \delta_O$]

   The regulator's belief is such that it is indifferent between its actions. Hence, the regulator chooses a suitable mixed strategy to make the broker indifferent between trading and not trading. Thus, the broker chooses a mixed strategy consistent with the regulator's belief. The equilibrium exists only if the spread of the information is equal or higher compared to the regulator's investigation threshold (i.e., $\beta \geq \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p) + (1-\beta)\zeta^U(p)$$

This sub-case resembles the second sub-case in a neutral client order case.

3. The belief $\tilde{\beta}_O(p) > \lambda_O$ does not lead to an optimal strategy consistent with the belief.

Remark 4.2.2. *When $\rho(O, I, p) > \zeta(O, p) > 0$, $\zeta^U(p) > 0$, and the regulator's optimal strategy is not to investigate, we get a pooling equilibrium (because $\pi^*_{I,O}(p) = \pi^*_{I,\bar{O}}(p) = \pi^*_U(p) = 1$) irrespective of the client order.*

Non-profitable on average for the uninformed broker ($\zeta^U(p) = 0$)

The client order is trend-reversing. Moreover, the trend reversal is such that both directions are equally profitable for the uninformed broker. Hence, the uninformed broker gets zero expected profit by following any of the trend directions. Thus, the uninformed broker is indifferent between trading and not trading. Hence, any mixed strategy is optimal for him. We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$, any $\pi^*_U(p) > \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$ with $\beta < \lambda_O$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

   The regulator's belief is against investigation. Hence, the informed broker always trades owing to the profitability of the price when there is a client order. Even though the uninformed broker has no profitability, he trades with a non-zero probability of satisfying belief consistency. It allows the informed broker to take full advantage by driving the regulator away from the investigation. The equilibrium exists only if the spread of the information is lower compared to the regulator's investigation threshold (i.e., $\beta < \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O, p) + (1-\alpha)\beta\zeta(\bar{O}, p)$$

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$ with $\delta_O = \frac{\zeta(O,p)}{\rho(O,I,p)} \in (0, 1)$, any $\pi^*_U(p) \in [0, 1]$.
   [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$ w.p. δ_O, X w.p. $1 - \delta_O$]

The regulator's belief is such that it is indifferent between its actions. Hence, the regulator chooses a suitable mixed strategy to make the broker indifferent between trading and not trading. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief. It also depends on the strategy of the uninformed broker.

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p)$$

3. The belief $\tilde{\beta}_O(p) > \lambda_O$ does not lead to an optimal strategy consistent with the belief.

We can observe the deceptive trading behavior as anticipated while formulating the game. We explore this behavior in detail in the next chapter.

Marginally profitable Client Order ($\rho(O, I, p) = \zeta(O, p) > 0$)

When the client order is marginally profitable, the trading profits match exactly with the penalty that the informed broker faces for the front running when the regulator investigates the trade. In this case, the client order can be either trend-following or trend-reversing. Based on the degree of trend reversal, the uninformed broker can have strictly positive or zero expected profit. Hence,

$$\zeta^U(p) > 0 \text{ or } \zeta^U(p) = 0$$

Hence, we get two sub-cases of the overall market trend.

Profitable on average for the uninformed broker ($\zeta^U(p) > 0$)

The uninformed broker's optimal strategy is always to trade. (i.e., $\pi_U^*(p) = 1$). We get the following sub-cases of optimal strategies.

1. $\pi_{I,O}^*(p) = 1$ with $\beta < \lambda_O$, $\pi_U^*(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O, p) + (1-\alpha)\beta\zeta(\bar{O}, p) + (1-\beta)\zeta^U(p)$$

This sub-case is the same as the first sub-case in the case of moderately profitable client order when the market is profitable on average for the uninformed broker.

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = 1$ and $\beta \geq \lambda_O$, $\pi^*_U(p) = 1$.
[Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

   The regulator's belief is such that it is indifferent between its actions. Hence, the regulator chooses to investigate to make the informed broker indifferent between trading and not trading. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief. The equilibrium exists only if the spread of the information is equal or higher compared to the regulator's investigation threshold (i.e., $\beta \geq \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p) + (1-\beta)\zeta^U(p)$$

   This sub-case resembles the second sub-case in the case of moderately profitable client order when the market is profitable on average for the uninformed broker.

3. any $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ and $\beta > \lambda_O$, $\pi^*_U(p) = 1$.
[Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

   The regulator's belief is towards investigation; hence, the regulator always investigates. Thus, the informed broker is indifferent between trading and not trading when there is a client order. He chooses a mixed strategy consistent with the regulator's belief. The equilibrium exists only if the spread of the information is higher compared to the regulator's investigation threshold (i.e., $\beta > \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p) + (1-\beta)\zeta^U(p)$$

Non-profitable on average for the uninformed broker ($\zeta^U(p) = 0$)

The uninformed broker is indifferent between trading and not trading. Hence, any mixed strategy is optimal for him. We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$, any $\pi^*_U(p) > \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$ when $\beta < \lambda_O$.
[Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O, p) + (1-\alpha)\beta\zeta(\bar{O}, p)$$

This sub-case is the same as the first sub-case in the case of moderately profitable client order when the market is non-profitable on average for the uninformed broker.

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$ with $\delta_O = 1$, any $\pi^*_U(p) \in [0, 1]$.
[Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

   The regulator's belief is such that it is indifferent between its actions. Hence, the regulator always chooses to investigate to make the informed broker indifferent between trading and not trading. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief. It also depends on the strategy of the uninformed broker.

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p)$$

   This sub-case resembles the second sub-case in the case of moderately profitable client order when the market is non-profitable on average for the uninformed broker.

3. any $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$, any $\pi^*_U(p) \in [0, 1]$.
[Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

   The regulator's belief is towards investigation; hence, the regulator always investigates. Hence, the informed broker is indifferent between trading and not trading. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief. It also depends on the strategy of the uninformed broker.

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p)$$

In this case, Similar to the moderately profitable client order, we can observe deceptive trading behavior.

Highly profitable Client Order ($\zeta(O, p) > \rho(O, I, p)$)

When the client order is highly profitable, the trading profits exceed the penalty, and hence, trading is lucrative even when the regulator investigates. Thus, the investigation becomes futile, and the broker always wants to trade. In this case, the client order can be either trend-following or trend-reversing. Based on the degree of trend reversal, the uninformed broker can have strictly positive or zero expected profit. Hence,

$$\zeta^U(p) > 0 \text{ or } \zeta^U(p) = 0$$

Hence, we get two sub-cases of the overall market trend.

Profitable on average for the uninformed broker ($\zeta^U(p) > 0$)

The uninformed broker's optimal strategy is always to trade. (i.e., $\pi^*_U(p) = 1$). We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$ with $\beta < \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

   $$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O, p) + (1-\alpha)\beta\zeta(\bar{O}, p) + (1-\beta)\zeta^U(p)$$

   This sub-case is the same as the first sub-case in the case of moderately profitable client order when the market is profitable on average for the uninformed broker.

2. The belief $\tilde{\beta}_O(p) = \lambda_O$ does not lead to an optimal strategy consistent with the belief.

3. $\pi^*_{I,O}(p) = 1$ if $\beta > \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

   The regulator's belief is towards the investigation. But, the price is extremely profitable, so the broker always trades, even by accepting the penalty. The equilibrium exists only if the spread of the information is sufficiently high compared to the regulator's investigation threshold (i.e., $\beta > \lambda_O$).

   $$\mathbb{E}(\Phi_B \mid p) = \alpha\beta(\zeta(O, p) - \rho(O, I, p)) + (1-\alpha)\beta\zeta(\bar{O}, p) + (1-\beta)\zeta^U(p)$$

Remark 4.2.3. *When the client order is highly profitable, and the price is profitable on average for the uninformed broker, all PBEs are pooling equilibria. It is because trading is profitable for all types of brokers in all scenarios of investigation, and hence, there is always trading.*

Non-profitable on average for the uninformed broker ($\zeta^U(p) = 0$)

The uninformed broker is indifferent between trading and not trading. Hence, any mixed strategy is optimal for him. We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$, any $\pi^*_U(p) > \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$ and $\beta < \lambda_O$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

   $$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O, p) + (1-\alpha)\beta\zeta(\bar{O}, p)$$

   This sub-case is the same as the first sub-case in the case of moderately profitable client order when the market is non-profitable on average for the uninformed broker.

2. The belief $\tilde{\beta}_O(p) = \lambda_O$ does not lead to an optimal strategy consistent with the belief.

3. $\pi^*_{I,O}(p) = 1$, any $\pi^*_U(p) < \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$.
   [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

   The regulator's belief is towards investigation; hence, the regulator always investigates. The informed broker still trades owing to the high profitability of the price when there is a client order. The strategy of the uninformed broker should be consistent with the belief.

   $$\mathbb{E}(\Phi_B \mid p) = \alpha\beta(\zeta(O, p) - \rho(O, I, p)) + (1-\alpha)\beta\zeta(\bar{O}, p)$$

We present general characteristics of the broker's expected payoff across all the cases of trending markets in the following remark.

Remark 4.2.4. *In the trending market,*

1. *the broker's expected payoff is maximum when the regulator's belief is against the investigation, i.e., $\tilde{\beta}_O(p) < \lambda_O$. It is jointly maximum when the client order is neutral.*

2. *when the client order is marginally profitable, the broker's expected payoff is equal when $\tilde{\beta}_O(p) = \lambda_O$ or $\tilde{\beta}_O(p) > \lambda_O$.*

Thus, the broker is better off by driving the regulator away from the investigation in the trending market.

4.2.2 Sideways Markets

When the market is sideways (i.e., $\zeta(\bar{O}, p) = 0$) in the absence of the client order, the informed broker is indifferent between trading and not trading. Hence, we get the optimal strategies and the corresponding beliefs in the absence of client order.

1. any $\pi^*_{I,\bar{O}}(p)$ such that $\pi^*_{I,\bar{O}}(p) < \frac{\lambda_{\bar{O}}}{1-\lambda_{\bar{O}}} \frac{(1-\beta)\pi^*_U(p)}{\beta}$ [Belief system: $\tilde{\beta}_{\bar{O}}(p) < \lambda_{\bar{O}} \Rightarrow \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = \bar{a}$]
2. $\pi^*_{I,\bar{O}}(p) = \frac{\lambda_{\bar{O}}}{1-\lambda_{\bar{O}}} \frac{(1-\beta)\pi^*_U(p)}{\beta}$ and any $\delta_{\bar{O}} \in [0, 1]$ [Belief system: $\tilde{\beta}_{\bar{O}}(p) = \lambda_{\bar{O}} \Rightarrow \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = a$ w.p. $\delta_{\bar{O}}, \bar{a}$ w.p. $1 - \delta_{\bar{O}}$]
3. any $\pi^*_{I,\bar{O}}(p)$ such that $\pi^*_{I,\bar{O}}(p) > \frac{\lambda_{\bar{O}}}{1-\lambda_{\bar{O}}} \frac{(1-\beta)\pi^*_U(p)}{\beta}$ [Belief system: $\tilde{\beta}_{\bar{O}}(p) > \lambda_{\bar{O}} \Rightarrow \sigma^*(\cdot \mid (\bar{O}, q, p), \tilde{\beta}) = a$]

The regulator's belief shapes the broker's trading probability. As a result, the broker trades with the probability that is consistent with the belief system. The belief consistency also depends on the uninformed broker's trading probability $\pi^*_U(p)$.

In the sideways market, the client order may bring a trend to the market. If the client order brings the trend, the uninformed broker can follow that trend to ensure positive expected profit. If the client order fails to bring a trend to the market, an uninformed broker gets zero expected profit. Thus, similar to the informed broker, the uninformed broker remains indifferent between trading and not trading.

Similar to the trending market, we get different cases of market conditions based on price profitability. We present different possibilities of $\pi^*_{I,O}$ and π^*_U along with corresponding belief systems and regulator's optimal strategy in the table (4.2). We also explain every case in detail below.

Neutral Client Order ($\zeta(O, p) = 0$)

When the client order is neutral, the market remains sideways even in the presence of the client order. In this scenario, any type of broker is indifferent between trading and not trading. Hence, any mixed strategy is optimal for the informed broker. Thus, the optimal

	$\zeta^U(p) > 0$ $(\pi^*_U(p) = 1)$ (Profitable on average for uninformed broker)	$\zeta^U(p) = 0$ (any $\pi^*_U(p)$ such that $\pi^*_U(p) \in [0,1]$) (Non-profitable on average for uninformed broker)
$\zeta(O,p) = 0$ (Neutral Client Order)	Not Possible	1. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) < \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$ with $\delta_O = 0$. [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$]
$0 < \zeta(O,p) < \rho(O,I,p)$ (Moderately profitable Client Order)	1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = \frac{\zeta(O,p)}{\rho(O,I,p)}$ and $\beta \geq \lambda_O$ [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$]	Not Possible
$0 < \zeta(O,p) = \rho(O,I,p)$ (Marginally profitable Client Order)	1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = 1$ and $\beta \geq \lambda_O$ [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1-\delta_O$] 3. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ and $\beta > \lambda_O$ [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$]	Not Possible
$\zeta(O,p) > \rho(O,I,p)$ (Highly profitable Client Order)	1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$] 2. $\pi^*_{I,O}(p) = 1$ if $\beta > \lambda_O$. [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$]	Not Possible

Table 4.2: Optimal strategies and beliefs for different price regions in the 'sideways market'

strategy is chosen to be consistent with the belief. As the market is sideways irrespective of client order, an uninformed broker also has zero expected profit. Hence,

$$\zeta^U(p) = 0$$

We get the following sub-cases of optimal strategies.

1. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) < \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$, any $\pi^*_U(p) \in [0,1]$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$]

   The regulator's belief is against investigation. Hence, the informed broker is indifferent between trading and not trading when there is a client order and, hence, trades with the probability that is consistent with the regulator's belief. An uninformed broker is also indifferent between trading and not trading.

   $$\mathbb{E}(\Phi_B \mid p) = 0$$

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$ with $\delta_O = 0$, any $\pi^*_U(p) \in [0,1]$.
   [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$]

   The regulator's belief is such that it is indifferent between its actions. Hence, the regulator chooses not to investigate to make the informed broker indifferent between trading and not trading when there is a client order. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief.

   $$\mathbb{E}(\Phi_B \mid p) = 0$$

3. The belief $\tilde{\beta}_O(p) > \lambda_O$ does not lead to an optimal strategy consistent with the belief.

In this case, the market is sideways in every possible scenario. There is nothing much to gain, and the broker is indifferent between his actions. Similar to the trending market, we get a total of 6 PBE classes by combining three different sub-cases of $\pi^*_{I,\bar{O}}(p)$ and two different sub-cases of $\pi^*_{I,O}(p)$. For example, one of them is

$$\Bigg[\Big(\sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}, \sigma^*(\cdot \mid (\bar{O},q,p),\tilde{\beta}) = \bar{a}\Big), \Big(\text{any } \pi^*_{I,O}(p) \text{ such that}$$

$$\pi^*_{I,O}(p) < \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}, \text{any } \pi^*_{I,\bar{O}}(p) \text{ such that } \pi^*_{I,\bar{O}}(p) < \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta},$$

$$\text{any } \pi^*_U(p) \in [0,1]\Big), \Big(\tilde{\beta}_O(p) < \lambda_O, \tilde{\beta}_{\bar{O}}(p) < \lambda_{\bar{O}}\Big)\Bigg]$$

Moreover, each of these PBE classes can lead to several possible PBEs similar to the one described in the trending market scenario.

Moderately profitable Client Order ($\rho(O, I, p) > \zeta(O, p) > 0$)

When the client order is moderately profitable, the client order essentially brings the trend to the sideways market. In this scenario, an uninformed broker can align his strategy with the trend in the presence of the client order. Thus, he always gets a positive expected profit. Hence,

$$\zeta^U(p) = \alpha\zeta(O,p) > 0$$

We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$ with $\beta < \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$]

   The regulator's belief is against investigation. Hence, an informed broker trades always owing to the profitability of the price when there is a client order. The uninformed broker also always trades. The equilibrium exists only if the spread of the information is lower compared to the regulator's investigation threshold (i.e., $\beta < \lambda_O$).

   $$\begin{aligned}\mathbb{E}(\Phi_B \mid p) &= \alpha\beta\zeta(O,p) + (1-\beta)\alpha\zeta(O,p)\\ &= \alpha\zeta(O,p)\end{aligned}$$

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = \frac{\zeta(O,p)}{\rho(O,I,p)}$ and $\beta \geq \lambda_O$, $\pi^*_U(p) = 1$.

[Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$ w.p. $\delta_O, \bar{a}$ w.p. $1 - \delta_O$]

The regulator's belief is such that it is indifferent between its actions. Hence, the regulator chooses a mixed strategy to make the informed broker indifferent between trading and not trading. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief. The equilibrium exists only if the spread of the information is equal or higher compared to the regulator's investigation threshold (i.e., $\beta \geq \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = (1 - \beta)\alpha\zeta(O, p)$$

3. The belief $\tilde{\beta}_O(p) > \lambda_O$ does not lead to an optimal strategy consistent with the belief.

Remark 4.2.5. *Similar to the trending market, when the regulator's belief is against investigation, we get a pooling equilibrium (i.e., $\pi^*_{I,O}(p) = \pi^*_U(p) = 1$) when there is a client order.*

Marginally Profitable Client Order ($\rho(O, I, p) = \zeta(\bar{O}, p) > 0$)

When the client order is marginally profitable, the trading profits match exactly with the penalty that the informed broker faces for the front running when the regulator investigates the trade. By aligning the strategy similar to the moderately profitable client order, an uninformed broker can always get a positive expected profit. Hence,

$$\zeta^U(p) = \alpha\zeta(O, p) > 0$$

We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$, $\pi^*_U(p) = 1$.
[Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) &= \alpha\beta\zeta(O, p) + (1 - \beta)\alpha\zeta(O, p) \\ &= \alpha\zeta(O, p)\end{aligned}$$

This sub-case is the same as the first sub-case in a moderately profitable client order case.

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = 1$ and $\beta \geq \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

   The regulator's belief is such that it is indifferent between its actions. Hence, the regulator chooses to investigate to make the informed broker indifferent between trading and not trading. Thus, the informed broker chooses a mixed strategy consistent with the regulator's belief. The equilibrium exists only if the spread of the information is equal or higher compared to the regulator's investigation threshold (i.e., $\beta \geq \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = (1-\beta)\alpha\zeta(O, p)$$

3. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ and $\beta > \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

   The regulator's belief is towards the investigation. Thus, the informed broker becomes indifferent between trading and not trading and chooses a mixed strategy consistent with the regulator's belief. The equilibrium exists only if the spread of the information is higher compared to the regulator's investigation threshold (i.e., $\beta > \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = (1-\beta)\alpha\zeta(O, p)$$

Highly profitable Client Order ($\zeta(\bar{O}, p) > \rho(O, I, p)$)

When the client order is highly profitable, the trading profits exceed the penalty, and hence, trading is lucrative even when the regulator investigates. Hence, the investigation becomes futile, and the broker always wants to trade. An uninformed broker can always get a positive expected profit. Hence,

$$\zeta^U(p) = \alpha\zeta(O, p) > 0$$

We get the following sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$, $\pi^*_U(p) = 1$.

[Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) &= \alpha\beta\zeta(O, p) + (1 - \beta)\alpha\zeta(O, p)\\ &= \alpha\zeta(O, p)\end{aligned}$$

This sub-case is the same as the first sub-case in a moderately profitable client order case.

2. The belief $\tilde{\beta}_O(p) = \lambda_O$ does not lead to an optimal strategy consistent with the belief.

3. $\pi^*_{I,O}(p) = 1$ if $\beta > \lambda_O$, $\pi^*_U(p) = 1$.
[Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

The regulator's belief is towards investigation; hence, the regulator always investigates. The informed broker still trades owing to the high profitability of the price when there is a client order. The uninformed broker also always trades. The equilibrium exists only if the spread of the information is higher compared to the regulator's investigation threshold (i.e., $\beta > \lambda_O$).

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta(\zeta(O, p) - \rho(O, I, p)) + (1 - \beta)\alpha\zeta(O, p)$$

We can observe that the broker's optimal strategies in the presence of the client order in corresponding market conditions are the same in the trending and sideways markets. Even though this similarity exists, the broker's strategy in the absence of client order is different, and hence, the complete PBE is different in these two market conditions.

Chapter 5

Results and Discussion

In this chapter, we present insights about the broker's trading behavior, the impact of information spread in the market, and the regulator's investigation policy. In the first section, we present our key result about the broker's deceptive trading behavior. In the next section, we analyze the impact of the spread of information (β). The next section is focused on the regulator's behavior and possible policy implications. In the last section, we discuss a generalization where the regulator's investigation is not fool-proof.

5.1 Deceptive Trading

The important objective of the game model is to understand the conditions required for the front running to be successful. The analysis reveals a fascinating insight into the role of uninformed brokers in the front running. In this section, we analyze the deceptive trading behavior of the uninformed broker and its implications for front-running.

5.1.1 Trend Reversal and Moderately profitable Client Order

If the client order is moderately profitable in the trending market (4.2.1), i.e., $\zeta(\bar{O}, p) > 0$ and $0 < \zeta(O, p) < \rho(O, I, p)$ and the client order is trend-reversing such that $\zeta^U(p) = 0$, then we two sub-cases of the informed broker's optimal strategies in presence of client order depending on the belief $\tilde{\beta}_O(p)$.

Belief against investigation

In the first sub-case, $\tilde{\beta}_O(p) < \lambda_O$,

$$\pi^*_{I,O}(p) = 1, \pi^*_U(p) \in \left(\frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}, 1\right] \;\&\; \beta < \lambda_O$$

Here, the uninformed broker trades with a positive probability even though he has no incentive to trade. It enables the informed broker to exploit all the profit arising from information advantage. The trading from the uninformed broker makes a semi-separating equilibrium, which would have been a separating equilibrium without his trading. Hence, the regulator's belief remains against the investigation. Thus, the uninformed broker acts as a deterrent for the regulator in the process of identification of the informed broker. However, this equilibrium exists only if the spread of information is sufficiently low (i.e., $\beta < \lambda_O$). It shows that the regulator's belief can be shaped only when the information spread is within a certain limit.

The lower bound on the trading probability of the uninformed broker ($\frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$) signifies the degree of effort required to deceive the regulator. The bound increases with the spread of information (increase in β) and increases in the regulator's willingness to investigate (decrease in λ_O). Hence, the uninformed broker needs to create more noise by trading with a higher probability when the regulator is either convinced about the information spread or is willing to investigate aggressively.

Indifferent belief

In the second sub-case, $\tilde{\beta}_O(p) = \lambda_O$,

$$\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta} \;\&\; \pi^*_U(p) \in [0,1]$$

Here, the trading probability of the informed broker is directly proportional to the trading probability of the uninformed broker. However, the regulator's mixed strategy eliminates possible profit from the front-running for the informed broker, and he is indifferent between his actions.

Comparison of expected payoffs

The expected payoff in the first sub-case is

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O,p) + (1-\alpha)\beta\zeta(\bar{O},p)$$

On the other hand, in the second sub-case

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O},p)$$

Hence, the expected payoff of the broker is higher by $\alpha\beta\zeta(O,p)$ when the uninformed broker successfully deceives the regulator. This additional gain can be attributed to the front-running. The additional gain from the front running is proportional to the probability of the client order (α) and the information spread (β). As $\beta < \lambda_O$, the gain from front-running is bounded above by $\alpha\lambda_O\zeta(O,p)$. Thus, the increase in the regulator's willingness to investigate (decrease in λ_O) decreases the possible gains from the front-running.

Hence, we can conclude that the uninformed broker's trading helps to increase the expected payoff of the broker if he is able to shape the regulator's belief against the investigation. This behavior can also be observed when the client order is marginally profitable, and the market is non-profitable on average for the uninformed broker (4.2.1).

5.1.2 Highly profitable Client Order

If the client order is highly profitable in a trending market (4.2.1), i.e., $\zeta(\bar{O},p) > 0$ and $\zeta(O,p) > \rho(O,I,p)$ such that it reverses the trend to make $\zeta^U(p) = 0$, the informed broker always trades ($\pi^*_{I,O}(p) = 1$) in all the PBEs. This is because the profit achievable from the trading exceeds the penalty. The effect of the uninformed broker's trading is on the expected payoff. We have two sub-cases.

1. $\pi^*_{I,O}(p) = 1$, any $\pi^*_U(p) > \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$ and $\beta < \lambda_O$.

[Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = \bar{a}$]

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O,p) + (1-\alpha)\beta\zeta(\bar{O},p)$$

2. $\pi^*_{I,O}(p) = 1$, any $\pi^*_U(p) < \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$.
[Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O,q,p),\tilde{\beta}) = a$]

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta(\zeta(O,p) - \rho(O,I,p)) + (1-\alpha)\beta\zeta(\bar{O},p)$$

In these two sub-cases, when the uninformed broker trades with a higher probability (i.e., $\pi^*_U(p) > \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$), it successfully drives the regulator's belief away from the investigation. Hence, the penalty is avoided, and the expected payoff of the broker is higher. Thus, the uninformed broker's trading strategy decides the expected payoff of the broker when the client order is highly profitable. In essence, the uninformed broker's trading helps the informed broker to front-run and hence increases the expected payoff of the broker.

5.2 Impact of Information Spread

The spread of information in the markets is crucial in deciding the broker's behavior. The lower information spread gives him a competitive advantage. As the information is more and more spread out, its usefulness for trading decreases. In the model, the spread of information is signified by the prior of the type of broker. Higher values of β correspond to more spread-out information as the broker is more likely to be informed.

The degree of information spread impacts the equilibrium in almost every market condition. We analyze this impact in two ways: on the existence of equilibrium and the nature of optimal strategies.

5.2.1 Existence of Equilibria

In several market conditions, some equilibrium strategies exist only when there is a certain amount of information spread in the market. The role of information spread in these cases is primarily in the regulator's belief shaping, as certain belief systems lead to an equilibrium strategy only for a certain degree of information spread. We present an example to illustrate the role of information spread in the existence of optimal strategies.

In the marginally profitable client order in the sideways market (4.2.2), i.e., $\zeta(\bar{O}, p) = 0$ and $0 < \zeta(O, p) = \rho(O, I, p)$, $\zeta^U(p) > 0$. We get three sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$ if $\beta < \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

$$\begin{aligned}\mathbb{E}(\Phi_B \mid p) &= \alpha\beta\zeta(O, p) + (1 - \beta)\alpha\zeta(O, p) \\ &= \alpha\zeta(O, p)\end{aligned}$$

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ with $\delta_O = 1$ and $\beta \geq \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

$$\mathbb{E}(\Phi_B \mid p) = (1 - \beta)\alpha\zeta(O, p)$$

3. any $\pi^*_{I,O}(p)$ such that $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}$ and $\beta > \lambda_O$, $\pi^*_U(p) = 1$.
   [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

$$\mathbb{E}(\Phi_B \mid p) = (1 - \beta)\alpha\zeta(O, p)$$

Among these sub-cases, the regulator's belief is against the investigation in the first sub-case. To support this belief, the information spread should be sufficiently low compared to the regulator's threshold for investigation. (i.e., $\beta < \lambda_O$). If this fails to hold, the equilibrium will not exist. Moreover, we can observe that this particular equilibrium gives the highest expected payoff to the broker among all the equilibrium classes. Thus, we conclude that the higher spread of information actually kills the broker's most favorable equilibrium. It also matches our insight that the lower information spread gives the broker a comparative advantage in terms of higher expected profit.

On the other hand, the higher information spread compared to the regulator's threshold for investigation is the requirement for the other two sub-cases. In those sub-cases, the regulator always investigates the trade. Hence, the higher spread of information makes a broker prone to investigation, and the front-running remains no more lucrative for him.

To understand the mechanism behind it, if the spread of information is high, the broker is likely to be informed. Hence, the regulator is more likely to investigate the informed broker than the uninformed broker. As a result, the regulator is willing to investigate. We present the general observation across all the sub-cases of the sideways market through the following remark.

Remark 5.2.1. *In the sideways market, when the client order brings the trend to the market,*

1. *When the regulator's belief is strictly against the investigation (i.e., $\tilde{\beta}_O(p) < \lambda_O$), the PBE exists only when there is low spread of information than the regulator's threshold for investigation in the presence of client order (i.e., $\beta < \lambda_O$).*

2. *When the regulator's is indifferent between his actions (i.e., $\tilde{\beta}_O(p) = \lambda_O$), the PBE exists only when there is marginally high or more spread of information than the regulator's threshold for investigation in the presence of client order (i.e., $\beta \geq \lambda_O$).*

3. *When the regulator's belief is strictly towards the investigation (i.e., $\tilde{\beta}_O(p) > \lambda_O$), the PBE exists only when there is higher spread of information than the regulator's threshold for investigation in the presence of client order (i.e., $\beta > \lambda_O$).*

The requirement of a certain level of information spread for the existence of optimal strategies is observed for several market conditions. We can observe the general trend of a higher spread of information leading to the investigation, whereas a lower spread of information helps the broker shield from the investigation. Even though this is the case, there is no restriction on the information spread for some equilibrium strategies.

5.2.2 Nature of Optimal Strategies

In some market conditions, a broker's trading probability (or the range of trading probabilities) depends on the information spread. To illustrate it, we consider the marginally profitable client order in the trending market (4.2.1), i.e., $\zeta(\bar{O}, p) > 0$ and $0 < \zeta(O, p) = \rho(O, I, p)$. If the market is non-profitable on average for the uninformed broker, i.e., $\zeta^U(p) = 0$, we get three sub-cases of optimal strategies.

1. $\pi^*_{I,O}(p) = 1$, any $\pi^*_U(p) > \frac{1-\lambda_O}{\lambda_O}\frac{\beta}{1-\beta}$ when $\beta < \lambda_O$. [Belief system: $\tilde{\beta}_O(p) < \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \bar{a}$]

$$\mathbb{E}(\Phi_B \mid p) = \alpha\beta\zeta(O, p) + (1-\alpha)\beta\zeta(\bar{O}, p)$$

2. $\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$ with $\delta_O = 1$, any $\pi^*_U(p) \in [0, 1]$. [Belief system: $\tilde{\beta}_O(p) = \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p)$$

3. any $\pi^*_{I,O}(p) > \frac{\lambda_O}{1-\lambda_O}\frac{(1-\beta)\pi^*_U(p)}{\beta}$, any $\pi^*_U(p) \in [0, 1]$. [Belief system: $\tilde{\beta}_O(p) > \lambda_O \Rightarrow \sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = a$]

$$\mathbb{E}(\Phi_B \mid p) = (1-\alpha)\beta\zeta(\bar{O}, p)$$

In the first sub-case, the trading probability of the uninformed broker has a lower bound. The lower bound increases with the information spread (β). Thus, the uninformed broker needs to trade aggressively as mentioned in the explanation of the deceptive trading activity.

In the second sub-case, the trading probability of the informed broker decreases with the increase in the information spread (β). It shows that the use of information decreases when it is more spread out. It is also insightful to examine the extreme cases of information spread. We summarize them in the following lemma.

Lemma 5.2.1. *When the regulator's belief is not strictly away from the investigation, i.e., $\tilde{\beta}_O \geq \lambda_O$, and the informed broker plays a mixed strategy in the presence of a client order,*

1. *When the information is highly spread out, the informed broker's trading probability tends to zero, i.e.,*

$$\lim_{\beta \to 1} \pi^*_{I,O}(p) = 0$$

2. *As the degree of secrecy of the information increases, the informed broker trades with a higher probability.*

Proof. When the regulator's belief is not strictly inclined towards investigation, and the informed broker plays a mixed strategy in the presence of a client order, the informed broker either plays a fixed mixed strategy or plays a range of mixed strategies (table 4.1, table 4.2). We handle these cases separately.

When the informed broker plays a fixed mixed strategy,

$$\pi^*_{I,O}(p) = \frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta} \Rightarrow \pi^*_{I,O}(p) \propto \frac{1-\beta}{\beta}$$

Thus, as $\beta \to 1$, $\pi^*_{I,O}(p) \to 0$. Moreover, $\pi^*_{I,O}(p)$ increases with decrease in the β. Thus, the secrecy increases the trading probability of the informed broker. When the informed broker plays a range of mixed strategies,

$$\pi^*_{I,O}(p) = \left(\frac{\lambda_O}{1-\lambda_O}\frac{1-\beta}{\beta}, 1\right]$$

Thus, $\pi^*_{I,O}(p)$ it is bounded below by a constant proportional to $\frac{1-\beta}{\beta}$. Hence, using a similar analysis, the results hold. $\square$

5.3 Investigation Mechanisms

In this section, we present insights into the regulator's investigation. We study the role of the investigation threshold in determining the broker's trading. It also helps us understand the possible policy implications and incentives to the regulator.

5.3.1 Impact of Investigation Threshold

The investigation threshold in the regulator's optimal strategy (i.e., $\lambda_O, \lambda_{\bar{O}}$) works as a proxy for the regulator's conviction in the investigation. It is given by

$$\lambda_O = \frac{C - B(O,U,p)}{B(O,I,p) - B(O,U,p)} \tag{5.1}$$

To keep $\lambda_O \in [0, 1]$,

$$B(O, I, p) \geq C \geq B(O, U, p)$$

Thus, the benefit from the investigation of an informed broker should be higher than the cost of the investigation, and the benefit from the investigation of an uninformed broker should be lower than the cost of the investigation. If the benefit from the investigation of the uninformed broker exceeds the cost, the regulator will investigate every trade. Whereas, if the cost of investigation exceeds the benefit of investigating an informed broker, the regulator will not investigate any trade at all.

The lower values of λ_O show the higher interest in the investigation as the regulator investigates if the belief $\tilde{\beta}_O(p)$ exceeds λ_O. The following lemma summarizes the factors affecting the regulator's willingness to investigate.

Lemma 5.3.1. *The regulator's willingness to investigate when there is a client order increases with*

1. *increase in the benefit acquired from the investigation of an informed broker.*
2. *decrease in the cost of the investigation.*
3. *increase in the benefit acquired from the investigation of an uninformed broker.*

Proof. The regulator's willingness to investigate in the presence of a client order increases as λ_O decreases.

$$\lambda_O = \frac{C - B(O, U, p)}{B(O, I, p) - B(O, U, p)}$$

Hence, λ_O decreases with decrease in C, increase in $B(O, I, p)$ and increase in $B(O, U, p)$. □

A similar result holds in the case of the absence of a client order. However, the benefit from the investigation is likely to be less than the cost of the investigation in that case, and hence, the regulator never investigates.

The impact of the investigation threshold is similar to the impact of the information spread. It can affect both the existence and the nature of equilibria.

Existence of equilibria

The existence of certain equilibria depends on the relation between β, λ_O and $\lambda_{\bar{O}}$. This impact is already examined as the impact of information spread. In that case, we have compared the information spread with the regulator's threshold for investigation. Thus, we have implicitly studied the relevance of the investigation threshold in the existence of equilibria.

Nature of equilibria

The impact of the investigation threshold is also similar to the impact of the information spread. We present a lemma to understand when the informed broker plays a mixed strategy.

Lemma 5.3.2. *When the regulator's belief is not strictly away from the investigation, i.e.,* $\tilde{\beta}_O \geq \lambda_O$*, and the informed broker plays a mixed strategy in the presence of a client order,*

1. *when the regulator decides always to investigate, the informed broker's trading probability tends to zero.*

2. *the increase in the regulator's interest in investigation decreases the trading probability (or range of probabilities) of an informed broker.*

Proof. The increased interest is signified by the decrease in λ_O. Moreover, we have

$$\pi^*_{I,O}(p) \propto \frac{\lambda_O}{1 - \lambda_O}$$

Hence, the proof follows similarly to 5.2.1 $\square$

5.3.2 Policy Implications

The equilibrium analysis gives helpful insights into the regulator's investigation behavior. We discuss the ways to nudge the regulator towards investigation and discuss possible issues due to which the front-running is not disincentivized enough.

Remark 5.3.1. *In any market condition, when the client order is highly profitable (i.e., $\zeta(O,p) > \rho(O,I,p)$), the informed broker always trades, i.e., $\pi^*_{I,O}(p) = 1$. Hence, the front-running can not be stopped or restricted at all.*

Thus, the regulator needs to set penalty levels at least as high as the average profit achievable at a price p when there is a client order. If the penalty is lesser than that, the front-running can not be prevented at all.

As the regulator's willingness to investigate increases, an informed broker might lose some opportunities for front-running. Hence, the policy should focus on increasing the regulator's willingness to investigate. The comparison of the cost and benefit of the regulator is done in (5.3.1). There are two possible ways to increase the regulator's willingness to investigate.

The regulator's willingness to investigate can be increased by decreasing the cost of the investigation. This cost can be lowered by bringing functional efficiencies in the investigation mechanism. The updating of the technology and increasing organizational governance are justified based on lowering the fixed cost of the investigation. In all, the fixed cost mostly depends on the process of investigation rather than the public perception of it.

Another way is to increase the benefit acquired from the investigation. If the regulator investigates an uninformed broker, it gets no benefit in terms of penalty. But, there can be benefits in terms of an increase in the efficiency and faith in the markets. If these kinds of positive externalities are factored into the regulator's benefit, the regulator's benefit will increase. The reputational benefits can even be amplified when the investigation results reveal the front-running incident.

5.4 Imperfect Investigation

In the game model, we have assumed that once the regulator decides to investigate a trade, it correctly discovers the true type of the broker. Now, we deviate from this assumption and study the optimal strategy of the regulator when the investigation is not foolproof. Thus, the investigation results can be wrong with certain fixed, known probabilities. The broker's optimal strategy (4.1.4) depends only on the regulator's investigation probability. Hence, once we incorporate the impact of the imperfect investigation on the regulator's optimal strategy, the broker's optimal strategy follows similarly to before.

5.4.1 Additions in the Game Flow

When the regulator investigates the trade, it can misidentify informed brokers as uninformed and vice versa. The table (5.1) depicts the probabilities with which identification happens.

	Revealed Type	
True Type	informed (I_R)	uninformed (U_R)
informed (I)	p_i	$1 - p_i$
uninformed (U)	$1 - p_u$	p_u

Table 5.1: Probability of Identification

Thus, the regulator's expected payoff when it investigates changes and its optimal strategy also changes accordingly. Moreover, we assume that the regulator's payoff depends on the revealed type of the broker. The only relevant scenario is when there is a client order and the broker trades. When the broker does not trade, the regulator's optimal strategy remains unchanged.

Proposition 5.4.1. *When there is a client order, and the broker trades (i.e., $\Theta = O, Q = q$), the optimal strategy of the regulator is given by*

$$\sigma^*(\cdot \mid (O, q, p), \tilde{\beta}) = \begin{cases} a & \tilde{\beta}_O(p) > \lambda_O^i \\ a \text{ w.p. } \delta_O, \bar{a} \text{ w.p. } 1 - \delta_O & \tilde{\beta}_O(p) = \lambda_O^i \\ \bar{a} & \tilde{\beta}_O(p) < \lambda_O^i \end{cases} \tag{5.2}$$

where $\lambda_O^i = \frac{C - [p_u B(O,U,p) + (1-p_u) B(O,I,p)]}{(p_i + p_u - 1)[B(O,I,p) - B(O,U,p)]}$ *and* δ_O *signifies the mixed strategy such that* $\delta_O \in [0, 1]$.

Proof. When $Q = q$, regulator's payoff (eq. 3.8) is given by

$$\Phi_S(\Theta, T, p, Q = q, A) = [B(\Theta, T, p) - C] \, \mathbb{1}_{\{A=a\}}$$

Hence, the regulator's optimal strategy (eq. 3.9) is given by

$$\sigma^*(\cdot \mid (\Theta = O, Q = q, p), \tilde{\beta}_O(p)) \in \underset{\sigma(\cdot|O,q,p)}{\operatorname{argmax}} \mathbb{E}_{A\sim\sigma(\cdot|O,q,p)} \mathbb{E}_{T\sim\tilde{\beta}(\cdot|O,q,p)} \left[B(O,T,p) - C\right] \mathbb{1}_{\{A=a\}} \tag{5.3}$$

Using the notations defined earlier and considering the imperfect investigation,

$$\begin{aligned}
&\mathbb{E}_{T\sim\tilde{\beta}(\cdot|O,q,p)} \left[B(O,T,p) - C\right] \mathbb{1}_{\{A=a\}} \\
&\qquad = \{(\tilde{\beta}_O(p)) \left[p_i[(B(O,I,p) - C] + (1-p_i)[(B(O,U,p) - C]\right] \\
&\qquad + (1-\tilde{\beta}_O(p)) \left[(1-p_u)[B(O,I,p) - C] + p_u[B(O,U,p) - C]\right]\}\mathbb{1}_{\{A=a\}} \\
&\qquad = \{(\tilde{\beta}_O(p)) \left[p_i[(B(O,I,p) - B(O,U,p)] + [B(O,U,p) - C]\right] \\
&\qquad + (1-\tilde{\beta}_O(p)) \left[p_u[B(O,U,p) - B(O,I,p)] + [B(O,I,p) - C]\right]\}\mathbb{1}_{\{A=a\}} \\
&\qquad = \{(\tilde{\beta}_O(p)) \left[(p_i + p_u - 1)[(B(O,I,p) - B(O,U,p)]\right] \\
&\qquad\qquad + p_u B(O,U,p) + (1-p_u)B(O,I,p) - C\}\mathbb{1}_{\{A=a\}}
\end{aligned}$$

Define, $\lambda_O^i = \frac{C-[p_u B(O,U,p)+(1-p_u)B(O,I,p)]}{(p_i+p_u-1)[B(O,I,p)-B(O,U,p)]}$. Thus, the proposition holds. □

We have shown that the form of the regulator's optimal strategy is the same. We only need to make changes in the regulator's investigation threshold. A similar result holds when there is no client order, but it is of little interest because the broker is not penalized.

5.4.2 Interpretation of Equilibrium

To understand the consequences of the imperfect investigation, we focus on the investigation threshold

$$\lambda_O^i = \frac{C - [p_u B(O,U,p) + (1-p_u)B(O,I,p)]}{(p_i + p_u - 1)[B(O,I,p) - B(O,U,p)]}$$

We first examine the impact of detection accuracy.

Lemma 5.4.2. *When the regulator has full detection accuracy for uninformed brokers, i.e., $p_u = 1$, the regulator's willingness to investigate (λ_O^i) is inversely proportional to its accuracy, given by the probability of detecting an informed broker correctly (p_i).*

Proof. We have,

$$\lambda_O^i = \frac{C - [p_u B(O,U,p) + (1-p_u)B(O,I,p)]}{(p_i + p_u - 1)[B(O,I,p) - B(O,U,p)]}$$

Moreover, the regulator has full detection accuracy for uninformed brokers, i.e., $p_u = 1$. Hence,

$$\lambda_O^i = \frac{C - [B(O,U,p)]}{p_i[B(O,I,p) - B(O,U,p)]}$$

Hence, $\lambda_O^i \propto \frac{1}{p_i}$ and the result follows. $\square$

Remark 5.4.1. *The regulator's willingness to investigate decreases with the decrease in its detection accuracy for informed brokers.*

Thus, the lower detection accuracy for an informed broker drives the regulator away from the investigation. According to the analysis of the regulator's behavior in the previous section, the lower willingness to investigate leads to a better environment for front-running. Thus, the investigation accuracy directly affects front-running in certain market conditions.

The impact of p_u is more complex and depends on the exact values of benefit and cost. But, the uninformed broker is better off by revealing his true type to the regulator. Thus, the interests of the broker and the regulator match when the broker is uninformed. Hence, p_u is likely to be close to one.

Chapter 6

Conclusion and Future Directions

In this thesis, we proposed a game theoretic model for front-running in financial markets. During front-running, the information asymmetry between the broker and the regulator is at the core of the broker's trading strategy and the regulator's policy to detect and penalize front-running. A game theoretic model that we proposed captures these behavioral nuances and also gives valuable insights into the role of information and the regulator's willingness to investigate the activity in different market conditions. We formulated a model in the framework of a sequential game with incomplete information and used perfect Bayesian equilibrium as a solution concept. We computed equilibria in different market conditions based on the profit achievable for an informed and an uninformed broker.

The analysis of equilibria gives insights into the broker's trading behavior. Our most notable result is about the deceptive trading behavior of the broker. In a trending market, when the client order causes trend reversal, the broker trades with a positive probability even when he gets zero profit and is uninformed about the client order. The uninformed broker's trading helps deceive the regulator as the regulator can not distinguish between normal trading based on the trend and front-running. It helps the broker to front-run when he is informed about the client order. When the uninformed broker shows such deceptive behavior, the regulator does not investigate the trading, and the expected payoff of the broker is higher. Thus, the deceptive trading behavior not only makes front-running possible but also helps exploit gains from the front-running and achieve a higher expected payoff.

We also studied the impact of information spread in the market on the broker's behavior.

The impact of information spread exists in certain equilibria based on the regulator's belief and market conditions. Wherever there is an impact of information, the higher the information spread, the lower the trading probability of the informed broker. Thus, we showed that competitive advantage decreases as the market becomes more and more aware of the information. If the information spread is higher than the regulator's threshold for investigation, it becomes impossible for the broker to avoid investigation in some market conditions. Thus, the front-running can not go undetected.

The regulator's investigation threshold drives its optimal strategy. The investigation threshold depends on the cost and benefit structure of the regulator. We showed that increasing the benefit from the investigation in terms of both penalty and reputation increases the willingness to investigate. Interestingly, the willingness is positively correlated with the benefit of investigating both informed and uninformed brokers. Moreover, the willingness decreases with an increase in the cost of the investigation. Thus, the policy should focus on increasing the benefits and reducing the cost of investigation. We also studied the case when the regulator's investigation was not perfect. The imperfection in the investigation mostly comes from the inability to detect the informed broker. In that case, the regulator's willingness to investigate decreases as its accuracy decreases.

Even though our game model made few strong assumptions, the equilibrium analysis revealed interesting insights about the front-running situation. We assumed that there were no transaction costs or taxes. Our preliminary calculations showed that the inclusion of such frictions leads to a market condition where trading profits are positive but lower than transaction costs. Hence, it may reveal some more equilibria and thus be an interesting future direction. We also assumed that the broker knows the average profitability of price and that only uncertainty in the trend direction comes from the presence of the client order. An introduction to uncertainty in the broker's knowledge of average price profitability would be another good future direction to explore.

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