Universal Enveloping Algebras and Verma Modules

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by

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Certificate

This is to certify that this dissertation entitled Universal Enveloping Algebras and Verma Modules towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sushant Shivaji Maske at Indian Institute of Science Education and Research under the supervision of Dr. Manish Mishra, Associate Professor, Department of Mathematics IISER Pune and Dr. Volodymyr Mazorchuk, Professor, Department of Mathematics, Uppsala University, during the academic year 2025-2026.

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This thesis is dedicated to my Grandma

Declaration

I hereby declare that the matter embodied in the report entitled Universal Enveloping Algebras and Verma Modules are the results of the work carried out by me at the Department of Mathematics IISER Pune, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Manish Mishra and the same has not been submitted elsewhere for any other degree. I declare that the use of Generative AI in this thesis did not extend beyond what is described in Section 4.6.1 of the 'Guidelines for Generative AI usage at IISER Pune', and accordingly, no attribution is required.

Sushant Shivaji Maske

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Abstract

After Lie algebras were introduced by Sophus Lie, Elie Cartan gave the classification of semi-simple Lie algebras and began studying their representations by the early 20th century. The aim of this thesis is to survey the advances made in the representation theory of semi-simple Lie algebras since then, culminating with the study of the theory of the BGG Category $\mathcal{O}$.

Emmy Noether revolutionized the field of algebra when she developed the modern theory of modules and ideals. She used this language to describe representations of groups and algebras as modules over rings [1]. In the same philosophy, representations of a Lie algebra ($\mathfrak{g}$) are treated as modules over an associative ring called the Universal Enveloping algebra ($U(\mathfrak{g})$) of $\mathfrak{g}$. The first two chapter of this thesis are dedicated to the theory of Enveloping algebras, where this correspondence between representations of $\mathfrak{g}$ and left modules over $U(\mathfrak{g})$ is made concrete.

We use this language of Universal Enveloping algebras to study representations of finite-dimensional complex semi-simple Lie algebras. We do so by first studying finite-dimensional representations. We then gradually broaden the collection of representations that we study. While studying finite-dimensional representations, we realize that these representations come equipped with a particular structure, called the highest weight structure. So, we shift our focus on studying the properties of left $U(\mathfrak{g})$-modules that have this highest weight structure. In doing so we study the work of D. N. Verma, who introduced the theory of Verma modules. Informally, these could be described as universal highest weight modules. Chapter 3 is a detailed account of this theory.

We now realize the importance of highest weight modules to representation theory of semi-simple Lie algebras. So, chapter 4 is dedicated to important techniques in the study of highest weight modules. We introduce two algebraic invariants: central characters (infinitesimal characters) and formal characters. Central characters arise due to the action of the center of $U(\mathfrak{g})$ on highest weight modules whereas formal characters capture the dimension of weight spaces in representations. The work of Harish-Chandra characterizes central characters and provides us with a powerful tool which we use consistently in the rest of the thesis. We know by now Verma modules control the representation theory of highest weight modules. So chapter 5 is dedicated to the structure theory of Verma modules. This chapter makes use of techniques described in the previous chapter.

Finally, we will try to understand the Bernstein-Gelfand-Gelfand Category $\mathcal{O}$, which is a certain sub-category of $U(\mathfrak{g})$ modules for a semi-simple Lie algebra $\mathfrak{g}$. Highest weight modules serve as the building blocks of this category. This category allows us to study in detail the structure and relationships between highest weight modules. We also detail some homological techniques used to study Category $\mathcal{O}$. This comprises of the last two chapters of the thesis.

Contents

Chapter 1

Universal Enveloping Algebras

After Lie algebras were introduced by Sophus Lie, Elie Cartan gave the classification of semi-simple Lie algebras and began studying their representations by the early 20th century. Henri Poincaré (1900), Garrett Birkhoff (1937), and Ernst Witt (1937) independently proved what we now call the PBW theorem. However, the universal enveloping algebra was not explicitly defined until the early 1940s, by Nathan Jacobson and Claude Chevalley (independently). Following the introduction of Category theory the universal property as well as the categorical construction and uniqueness of enveloping algebras were formalized.

1.1 Motivation

Definition: A *Lie Algebra* is a vector space L over a field F , with a bilinear transformation $L \times L \to L$ defined for all elements in L that follows:

1. *Jacobi identity:* $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \qquad \forall x, y, z \in L$
2. $[x,x] = 0 \qquad \forall x \in L$

Given an associative algebra $(A, \cdot)$, we can give it a Lie algebra structure by defining the Lie bracket as the commutator.

$$\begin{aligned} A \times A &\to A \\ [x,y] &\to x \cdot y - y \cdot x \end{aligned}$$

We want to ask the following question:

- It is possible to realize every Lie algebra as a commutator of some associative algebra?

One way to tackle this is to look at representations.

Definition: A *Lie algebra representation* is a linear map of the Lie algebra into an endomorphism ring $\mathrm{End}(V)$ for some vector space V that preserves the Lie bracket. $\rho : \mathfrak{g} \to End(V)$ is a linear map such that $\rho([x, y]) = [\rho(x), \rho(y)] = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x)$ where $\circ$ is the composition in $End(V)$.

In this way one can relate a Lie bracket with the commutator in the endomorphism ring. But this raises further questions. A representation need not always be nice, for example it need not be injective. For finite-dimensional Lie algebras Ado's theorem says that every finite-dimensional Lie algebra has a faithful finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ this solves our problem. But we raised these questions for any Lie algebra, even of an infinite dimension. Also, when using representations, we are faced with another question:

- Given a representation of a Lie algebra $\rho : \mathfrak{g} \to \mathrm{End}(V)$, $\mathrm{Im}(\rho)$ need not be closed under composition. How can we close the image under composition?

Surprisingly both these questions are tackled in the representation theory of Lie algebras by defining an object associated to a Lie algebra $\mathfrak{g}$ called the *Universal enveloping algebra* of $\mathfrak{g}$. Refer to appendix A for the basics of the theory of algebras, Lie algebras and their representations.

1.2 Definition

Consider a Lie algebra $\mathfrak{g}$ and its representation $\rho : \mathfrak{g} \longrightarrow End(V)$. We define an action of $\mathfrak{g}$ on V as $x \cdot v := \rho(x) \cdot v$ $\forall x \in \mathfrak{g}$ and $\forall v \in V$. Now, $x \cdot v$ is a vector in V. Similarly, $(x_1 \circ x_2) \cdot v = (\rho(x_1) \circ \rho(x_2)) \cdot v$ is also a vector in V (Here, $\circ$ is composition in $End(V)$). In this way, any other polynomial in elements of $\mathfrak{g}$ acts on v to give a vector in V. But many such polynomials act in the same way. For example the polynomial $x \circ y - y \circ x$ acts the same as $[x, y]$. We also know that multiplication in $End(V)$ is associative i.e. $x_1 \cdot (x_2 \cdot v) = (x_1 \circ x_2) \cdot v$.

So what we want to do now, is to club together those polynomials in elements of $\mathfrak{g}$ that will necessarily act the same on any $\mathfrak{g}$-module. We have an intuitive idea that we can start with a collection of associative but non-commutative polynomials on elements of $\mathfrak{g}$ and then impose on them the commutator relation.

We now consider the free associative algebra of $\mathfrak{g}$ containing 1. This is analogous to the associative but noncommutative polynomials we described above. We know that a free associative algebra over a vector space exists, is unique, and can be described as the tensor algebra over it. So, we consider $T(\mathfrak{g})$, the tensor algebra over $\mathfrak{g}$. To impose an additional structure, we always quotient. So, we quotient $T(\mathfrak{g})$ by the subalgebra generated by elements of the form $\{x \circ y - y \circ x - [x, y] \mid x, y \in \mathfrak{g}\}$. The resulting vector space is what we call the Universal Enveloping Algebra of $\mathfrak{g}$.

As the name suggests, $U(\mathfrak{g})$ comes with a universal property. That means we can give a different description of $U(\mathfrak{g})$, one that is not constructive (like the one we gave above) but rather involves a universal object.

Definition: For a lie algebra $\mathfrak{g}$ the universal enveloping algebra of $\mathfrak{g}$ is a pair $(U(\mathfrak{g}), \iota)$ where ι is a Lie algebra homomorphism from $\mathfrak{g} \to U(\mathfrak{g})$. The pair satisfies the following universal property: $U(\mathfrak{g})$ is associative algebra such that for any associative algebra A and a Lie algebra homomorphism $\alpha : \mathfrak{g} \to A$, $\exists! \gamma : U(\mathfrak{g}) \to A$ a homomorphism of associative algebras such that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & U_{\mathfrak{g}} \arrow[d, "\gamma"] \\ & A \end{tikzcd}$$

We call ι the *canonical inclusion of* $\mathfrak{g}$ *in* $U(\mathfrak{g})$. We write the product of two elements $x, y \in U(\mathfrak{g})$ as xy while we use the same notation for the action of $U(\mathfrak{g})$ and $\mathfrak{g}$ on V e.g. $x \cdot v$ for any $x \in U(\mathfrak{g})$ or $\mathfrak{g}$.

The broad idea of this definition is that we have an object $(U(\mathfrak{g}))$ such that every Lie algebra homomorphism $\alpha : \mathfrak{g} \to A$ (where A is an associative algebra) factors through it.

Before proving the existence and uniqueness of this object let us look at how it helps us answer the questions we first posed:

- Using the Lie algebra homomorphism ι we can give a natural left action of $\mathfrak{g}$ on $U(\mathfrak{g})$. This gives us a representation of $\mathfrak{g}$ into $End(U(\mathfrak{g}))$. It turns out that this is an injective representation. So, we now have an injective representation of any Lie algebra into an associative algebra. This means we can recognize the Lie bracket as a commutator of that associative algebra.

- The image of an algebra homomorphism into $End(V)$ is closed under composition (for some vector space V). So, for $A = End(V)$, and α a representation and the image of γ closes the image of α under composition.

Henceforward we will try to study this object $(U(\mathfrak{g}), \iota)$

1.3 Existence and Uniqueness

Lemma: The Universal Enveloping Algebra ($U(\mathfrak{g})$) of a Lie algebra $\mathfrak{g}$ exists and is unique upto isomorphism.

Proof: We will first show that if an object with such universal property exists, it is unique upto isomorphism. For the existence part, we will follow the strategy described in the previous section and give a constructive proof. Notice that there is no condition on the underlying field or the dimension of the Lie algebra. Let k be the underlying field and all associative algebras we take have 1.

Uniqueness: Suppose there exist (U_1, ι_1) and (U_2, ι_2) that satisfy the universal property. Then,

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & U_1 \arrow[d, shift left, "\pi_1"] \\ & U_2 \arrow[u, shift left, "\pi_2"] \end{tikzcd} \quad \Rightarrow \quad \pi_1 \circ \pi_2 : U_2 \to U_2 \text{ is a homomorphism of algebras.}$$

Similarly, $\pi_2 \circ \pi_1 : U_1 \to U_1$ is also a homomorphism of associative algebras.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota_2"] \arrow[dr, "\iota_2"'] & U_2 \arrow[d, "\mathrm{Id}"] \\ & U_2 \end{tikzcd}$$ By universal property, Id is the unique homomorphism such that $\mathrm{Id} \circ \iota_2 = \iota_2$.

$$\pi_1 \circ \iota_1 = \iota_2, \quad \pi_2 \circ \iota_2 = \iota_1 \quad \Rightarrow \quad \pi_1 \circ \pi_2 \circ \iota_2 = \iota_2 \quad \Rightarrow \quad \pi_1 \circ \pi_2 = \mathrm{Id}_{U_2}$$

Similarly, $\pi_2 \circ \pi_1 = \mathrm{Id}_{U_1}$. This shows that π_1 and π_2 are inverses of each other. This means that they are an algebra isomorphism. Therefore, if a universal enveloping algebra exists, then it is unique up to isomorphism.

Existence: We will do this in the following steps:

Step 1: Let $\{x_i\}_{i\in\Lambda}$ be a basis of our Lie algebra $\mathfrak{g}$. Let the structure constants be defined by:

$$[x_\alpha, x_\beta] = \sum_{k\in\Lambda} c^k_{\alpha\beta} x_k$$

Let $\mathcal{F}$ be a free associative algebra of $\mathfrak{g}$. Let $\{a_i\}_{i\in\Lambda}$ be the image of $\{x_i\}_{i\in\Lambda}$ in the inclusion map $\mathfrak{g} \to \mathcal{F}$. The Lie bracket in an associative algebra is defined by:

$$[u, v] = uv - vu$$

Define a two-sided ideal in $U(\mathfrak{g})$ generated by:

$$\mathcal{J} := \left\langle a_\alpha a_\beta - a_\beta a_\alpha - \sum_{k\in\Lambda} c^k_{\alpha\beta} a_k \;\middle|\; \alpha, \beta \in \Lambda \right\rangle$$

This ensures the Lie bracket in the free algebra (i.e, the commutator) coincides with the Lie bracket from the Lie algebra $\mathfrak{g}$ and hence, ensure ι becomes a Lie algebra homomorphism.

Define:

$$U(\mathfrak{g}) := \mathcal{F}/\mathcal{J}$$

We now want to show that the universal enveloping algebra we defined indeed satisfies the definition.

Step 2: $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is a Lie algebra homomorphism

$$\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})$$

Consider the composition:

$$\mathfrak{g} \to \mathcal{F} \to U(\mathfrak{g}) = \mathcal{F}/\mathcal{J}$$

$$x_k \to a_k \to \bar{a}_k \quad \text{(linear map of vector spaces)}$$

Now compute:

$$\iota([x_\alpha, x_\beta]) = \iota\left(\sum_{k\in\Lambda} c^k_{\alpha\beta} x_k\right) = \sum_{k\in\Lambda} c^k_{\alpha\beta}\bar{a}_k = \bar{a}_\alpha\bar{a}_\beta - \bar{a}_\beta\bar{a}_\alpha = [\bar{a}_\alpha, \bar{a}_\beta] = [\iota(x_\alpha), \iota(x_\beta)]$$

Therefore, ι is a Lie algebra homomorphism.

Step 3: To show the given construction satisfies the universal property.

Let A be another associative algebra, and suppose:

$$\varphi : \mathfrak{g} \to A$$

is a Lie algebra homomorphism.

We aim to show that there exists a unique algebra homomorphism $\pi : U(\mathfrak{g}) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & U(\mathfrak{g}) \arrow[d, dashed, "\pi"] \\ & A \end{tikzcd}$$

From the universal property of free algebras:

Let $V = \mathfrak{g}$, and let $\mathcal{F}$ be the free associative algebra on V. Then, for the linear map $\varphi : V \to A$, there exists a unique algebra homomorphism $\tilde{\pi} : \mathcal{F} \to A$ such that: $\tilde{\varphi} = \tilde{\pi} \circ \iota$

$$\begin{tikzcd} V \arrow[r, "\tilde{\iota}"] \arrow[dr, "\tilde{\varphi}"'] & \mathcal{F} \arrow[d, dashed, "\tilde{\pi}"] \\ & A \end{tikzcd}$$

Since φ is a Lie algebra homomorphism, we have:

$$\tilde{\pi}(a_\alpha a_\beta - a_\beta a_\alpha - \sum_k c^k_{\alpha\beta} a_k) = \tilde{\pi}(a_\alpha)\tilde{\pi}(a_\beta) - \tilde{\pi}(a_\beta)\tilde{\pi}(a_\alpha) - \sum_k c^k_{\alpha\beta}\tilde{\pi}(a_k)$$

$$= \varphi(x_\alpha)\varphi(x_\beta) - \varphi(x_\beta)\varphi(x_\alpha) - \sum_k c^k_{\alpha\beta}\varphi(x_k) = [\varphi(x_\alpha), \varphi(x_\beta)] - \varphi([x_\alpha, x_\beta]) = 0$$

Thus, all the generators of $\mathcal{J}$ are in the kernel of $\tilde{\pi}$, so $\mathcal{J} \subseteq \ker\tilde{\pi}$. Hence, $\tilde{\pi}$ descends to a unique algebra homomorphism:

$$\pi : \mathcal{F}/\mathcal{J} = U(\mathfrak{g}) \to A$$

$$\begin{tikzcd}
& \mathcal{F} \arrow[rr, two heads] \arrow[dr, "\tilde{\pi}"] & & U(\mathfrak{g}) \arrow[dl, dashed, "\pi"] \\
\mathfrak{g} \arrow[ur, "\tilde{\imath}"] \arrow[rr, "\varphi"'] & & A &
\end{tikzcd}$$

Both triangles commute $\Rightarrow \quad \varphi = \pi \circ \iota$. This proves the universal property of the universal enveloping algebra $U(\mathfrak{g})$. ■

The universal enveloping algebra of the trivial Lie algebra is just the underlying field. For an abelian Lie algebra, the universal enveloping algebra is its symmetric algebra and for a free Lie algebra generated by a set X, it is the free associative algebra generated by X.

1.4 Poincaré–Birkhoff–Witt theorem

The universal enveloping algebra is an infinite dimensional vector space. This is a trade-off we make while going from a Lie algebra ($\mathfrak{g}$) to an associative algebra ($U(\mathfrak{g})$). So to understand $U(\mathfrak{g})$ better, we will try to find a basis of $U(\mathfrak{g})$. It turns out that we can get a basis of $U(\mathfrak{g})$ in terms of the basis of $\mathfrak{g}$. This result is known as the Poincaré–Birkhoff–Witt theorem. Throughout this section, the underlying field is k and the product of two elements $x, y \in U(\mathfrak{g})$ by xy.

Let $x, y \in \mathfrak{g}$. Let $\bar{x} := \iota(x)$ and $\bar{y} := \iota(y)$ where ι is the canonical inclusion in $U(\mathfrak{g})$. We know that in $U(\mathfrak{g})$, $\bar{x}\bar{y} = \bar{y}\bar{x} + [\bar{x}, \bar{y}]$. Vaguely speaking, we can see that when we want to exchange alphabets of a word in $U(\mathfrak{g})$, we have to add an extra term of length one smaller than the original e.g. when trying to $\bar{x}\bar{y}$ as $\bar{y}\bar{x}$ we have to add the $[\bar{x}, \bar{y}]$ word. This observation will be the key when we try to prove the following theorem.

Theorem 1.4.1. Let $\{x_i\}_{i \in \Lambda}$ be a well-ordered basis of $\mathfrak{g}$. Let $y_i := \iota(x_i)$ be the canonical image of x_i in $U(\mathfrak{g})$. A monomial $y_{i_1} y_{i_2} \cdots y_{i_n}$ is called *standard*, if $i_1 \leq i_2 \leq \cdots \leq i_n$. The theorem says that standard monomials form a basis of $U(\mathfrak{g})$

The proof of the theorem is included in Appendix. The idea is to convert any word as a sum of standard monomials by flipping the terms. To do so, we might have to make use of the commutator relation like we discussed above.

Corollary 1.4.1.1. The map $\iota : \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ is an embedding.

This is an immediate consequence of the PBW theorem. To see this notice that, for any $x \in \mathfrak{g}$ its canonical image in $U(\mathfrak{g})$ is a standard monomial. Recall that in section 1.2 we said that the representation of $\mathfrak{g}$ in $U(\mathfrak{g})$ that is given by the left action is faithful. By the left action, any $x \in \mathfrak{g}$ maps $1 \in U(\mathfrak{g})$ to $\iota(x)$. It is obvious from the PBW theorem that each $x \in \mathfrak{g}$ maps to a distinct element in $End(U(\mathfrak{g}))$. So, the representation is faithful.

Chapter 2

Some Consequences of the PBW theorem

In this chapter we will explore some properties of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ that arise as a consequence of the PBW theorem. From now on, we may sometimes denote $U(\mathfrak{g})$ as $U_{\mathfrak{g}}$.

2.1 $U(\mathfrak{g})$-Mod $\cong$ $\mathfrak{g}$-Mod

Denote by $U(\mathfrak{g})$-Mod and $\mathfrak{g}$-Mod the categories of left $U(\mathfrak{g})$ modules and $\mathfrak{g}$ modules respectively. We know the Lie bracket is non-associative, so a Lie algebra cannot be a viewed as a ring. But, its universal enveloping algebra is a ring and hence representations of Lie algebra can be considered to be modules over its enveloping algebra. So, $U(\mathfrak{g})$ helps us brings module theoretic ideas to representation theory of Lie algebras. This is a motivation for us to study the exact relationship between the two categories, $U(\mathfrak{g})$-Mod and $\mathfrak{g}$-Mod.

It is easy to see from the corollary in 1.4 that every left $U(\mathfrak{g})$ module is a $\mathfrak{g}$ module because $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$. From the universal property of $U(\mathfrak{g})$ we can say that every $\mathfrak{g}$ module is also a left $U(\mathfrak{g})$ module. This hints at an isomorphism of categories. We will now give the functor that will make this equivalence concrete.

Corollary 2: If ρ is a representation of $\mathfrak{g}$, then there exists a unique ρ' a representation of $U(\mathfrak{g})$ such that ρ' is an extension of ρ. In the space of representations of $\mathfrak{g}$ and $U(\mathfrak{g})$, the map $\rho \mapsto \rho'$ is a bijection.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\rho"'] & U(\mathfrak{g}) \arrow[d, "\exists!\rho'"] \\ & \mathfrak{gl}(V) \end{tikzcd}$$

What we mean to say is that every $\mathfrak{g}$-module is a left $U(\mathfrak{g})$-module and vice-versa.

Proof:

Injectivity: By the universal property of $U(\mathfrak{g})$, every $\mathfrak{g}$-module (π, V) is a left $U(\mathfrak{g})$-module; i.e., it can be extended to a unique $\tilde{\pi}$.. If there are two representations (π_1, V) and (π_2, V) such that they have the same extension $\tilde{\pi}$, then we have:

$$\pi_1(x) = \tilde{\pi}(\iota(x)) = \pi_2(x) \quad \forall x \in \mathfrak{g}$$

$$\Rightarrow \pi_1 = \pi_2$$

This shows injectivity.

Surjectivity: Now, if A is a $U(\mathfrak{g})$-module, then we can make it a $\mathfrak{g}$-module by defining

$$\tilde{\pi} : \mathfrak{g} \to A, \quad x \mapsto \pi \circ \iota(x)$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\tilde{\pi}"'] & U(\mathfrak{g}) \arrow[d, "\pi"] \\ & A \end{tikzcd}$$

$$\begin{aligned} \tilde{\pi}([x,y]) &= \pi(\iota([x,y])) = \pi(\iota(x)\iota(y) - \iota(y)\iota(x)) \quad \text{since } \iota \text{ is a Lie alg homomorphism} \\ &= \pi(\iota(x))\pi(\iota(y)) - \pi(\iota(y))\pi(\iota(x)) = [\pi(\iota(x)), \pi(\iota(y))]_A = [\tilde{\pi}(x), \tilde{\pi}(y)]_A \\ &\implies \tilde{\pi} \text{ is a Lie alg homomorphism} \end{aligned}$$

$$\therefore A \text{ is a } \mathfrak{g}\text{-module.}$$

$\square$

Note: If (V_1, π_1) and (V_2, π_2) are equivalent representations if and only if $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are equivalent.

Corollary 3: The $\mathcal{U}$-functor: Let $\mathfrak{g}, \mathfrak{g}'$ be Lie algebras, and let $\varphi : \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism of Lie algebras. Then there exists a unique $\Psi : U_{\mathfrak{g}} \to U_{\mathfrak{g}'}$, a homomorphism of associative algebras that extends φ, such that $\Psi(\iota_{\mathfrak{g}}(x)) = \iota_{\mathfrak{g}'}(\varphi(x))$ for all $x \in \mathfrak{g}$ and $\Psi(1) = 1$.

Proof: This follows directly from corollary 1 in 1.4.1.1 by realizing $\mathfrak{g}'$ as embedded in $U(\mathfrak{g}')$. Then, by universal property we have the following diagram that commutes. This completes the proof. Note that $\iota_{\mathfrak{g}}$ and $\iota_{\mathfrak{g}'}$ both are Lie algebra homomorphisms.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota_{\mathfrak{g}}"] \arrow[dr, "\iota_{\mathfrak{g}'} \circ \varphi"'] & U_{\mathfrak{g}} \arrow[d, "\exists!\, \Psi"] \\ & U_{\mathfrak{g}'} \end{tikzcd}$$

□

Definition: We define a map $\mathcal{U}$ as

$$\mathcal{U} : Hom(\mathfrak{g}, \mathfrak{g}') \to Hom(U_{\mathfrak{g}}, U_{\mathfrak{g}'})$$
$$\varphi \in \to \Psi$$

where Ψ is defined as in the proof above. We call this the $\mathcal{U}$-functor as it takes the homomorphisms of the category $\mathfrak{g}$-Mod to homomorphisms of the category $U(\mathfrak{g})$-Mod

Corollary 4: If $\varphi : \mathfrak{g} \to \mathfrak{g}', \quad \varphi' : \mathfrak{g}' \to \mathfrak{g}''$, then $\mathcal{U}(\varphi' \circ \varphi) = \mathcal{U}(\varphi') \circ \mathcal{U}(\varphi)$

Proof :

$$\varphi' \circ \varphi : \mathfrak{g} \to \mathfrak{g}'' \quad \text{is a Lie algebra homomorphism}$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota_{\mathfrak{g}}"] \arrow[dr, "\iota_{\mathfrak{g}'} \circ \varphi' \circ \varphi"'] & U_{\mathfrak{g}} \arrow[d, "\exists! \mathcal{U}(\varphi' \circ \varphi)"] \\ & U_{\mathfrak{g}''} \end{tikzcd}$$

$$\therefore \mathcal{U}(\varphi' \circ \varphi) : U_{\mathfrak{g}} \to U_{\mathfrak{g}''} \quad \text{extends } \varphi' \circ \varphi$$

Now, we consider the diagram given below. We know that the diagram commutes. We now see from this diagram that there is another map from $U_{\mathfrak{g}}$ to $U_{\mathfrak{g}''}$ that lifts

$\iota_{\mathfrak{g}'} \circ \varphi' \circ \varphi$ and is given by $\mathcal{U}(\varphi') \circ \mathcal{U}(\varphi)$. By universal property, $\mathcal{U}(\varphi' \circ \varphi)$ is unique such lift. Hence proved, that $\mathcal{U}(\varphi' \circ \varphi) = \mathcal{U}(\varphi') \circ \mathcal{U}(\varphi)$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\iota_{\mathfrak{g}}"] \arrow[d, "\varphi"'] & U_{\mathfrak{g}} \arrow[d, "\mathcal{U}(\varphi)"] \\
\mathfrak{g}' \arrow[r, "\iota_{\mathfrak{g}'}"] \arrow[d, "\varphi'"'] & U_{\mathfrak{g}'} \arrow[d, "\mathcal{U}(\varphi')"] \\
\mathfrak{g}'' \arrow[r, "\iota_{\mathfrak{g}''}"] & U_{\mathfrak{g}''}
\end{tikzcd}$$

$$\boxed{\mathcal{U}(\varphi' \circ \varphi) = \mathcal{U}(\varphi') \circ \mathcal{U}(\varphi)}$$

This completes the proof. ■

As a result of these corollaries we can see that the category $\mathfrak{g}$-Mod is isomorphic to $U(\mathfrak{g})$-Mod

2.2 Universal enveloping algebras of Lie subalgebras

Take a Lie algebra $\mathfrak{a}$ and let $\mathfrak{b}$ be its Lie subalgebra. We ask the question : How are Universal enveloping algebras of $\mathfrak{a}$ and $\mathfrak{b}$ related? We first break down this question into two Lemmas.

Lemma ($\mathfrak{a}$): Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then the subalgebra of $U(\mathfrak{g})$ generated by $\mathfrak{h}$ is isomorphic to $U(\mathfrak{h})$.

Proof:

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\iota"] \arrow[dr, "\pi"'] & U_{\mathfrak{h}} \arrow[d, "\mathcal{U}(\pi)"] \\
& U_{\mathfrak{g}}
\end{tikzcd}$$

Note that $\mathcal{U}(\pi)$ is the functor we discussed in the previous section 2.1. π is the inclusion of $\mathfrak{h}$ in $U_{\mathfrak{g}}$. $\mathcal{U}(\pi)$ closes the image of π under composition. Therefore, $\mathrm{Im}(\mathcal{U}(\pi))$ is the subalgebra in $U_{\mathfrak{g}}$ generated by $\mathfrak{h}$.

Now, let $(x_1, \ldots, x_p)$ be a basis of $\mathfrak{h}$. PBW theorem says that the set

$\{(\iota(x_1)^{r_1}\cdots\iota(x_p)^{r_p}) \mid r_1 \leq ... \leq r_p, \forall\ r_i \in \mathbb{N}\}$ is a basis of $U(\mathfrak{h})$.

$$\mathcal{U}(\pi)\left(\iota(x_1)^{r_1}\cdots\iota(x_p)^{r_p}\right) = (\mathcal{U}(\pi)\circ\iota(x_1))^{r_1}\cdots(\mathcal{U}(\pi)\circ\iota(x_p))^{r_p}\,, \quad \forall r_i \in \mathbb{N}$$

$$= \pi(x_1)^{r_1}\cdots\pi(x_p)^{r_p} \quad \text{since } \mathcal{U}(\pi)\circ i = \pi$$

By the PBW theorem we can say that the image of the basis of $U_{\mathfrak{h}}$ in $U_{\mathfrak{g}}$ is linearly independent.

$\therefore \mathcal{U}(\pi)$ is an injective map and $U_{\mathfrak{h}}$ is isomorphic to the subalgebra of $U_{\mathfrak{g}}$ generated by $\mathfrak{h}$. We have shown that the map $\mathcal{U}(\pi)$ is injective and its image is generated by $\mathfrak{h}$. □

Lemma ($\mathfrak{b}$): Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ and $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{l}$.

Then

$$f : U(\mathfrak{h}) \otimes_{U(\mathfrak{t})} U(\mathfrak{l}) \longrightarrow U(\mathfrak{g})$$

is a bijective linear map.

Proof: From Lemma $\mathfrak{a}$, $U(\mathfrak{h}), U(\mathfrak{l}), U(\mathfrak{t})$ are embedded inside $U(\mathfrak{g})$.

This means that we can construct a map:

Let

$$\phi : U(\mathfrak{h}) \times U(\mathfrak{l}) \longrightarrow U(\mathfrak{g})$$

defined by $(x, y) \mapsto xy$, a bilinear map.

Now take $z \in U(\mathfrak{t})$. Then

$$\phi(xz, y) = xzy = \phi(x, zy)$$

$$\Rightarrow \exists! f : U(\mathfrak{h}) \otimes_{U(\mathfrak{t})} U(\mathfrak{l}) \longrightarrow U(\mathfrak{g}) \quad \text{a linear map}$$

such that $f \circ \iota = \phi$

$$\begin{tikzcd} U(\mathfrak{h}) \times U(\mathfrak{l}) \arrow[r, "\iota"] \arrow[dr, "\phi"'] & U(\mathfrak{h}) \otimes_{U(\mathfrak{t})} U(\mathfrak{l}) \arrow[d, dashed, "\exists! f"] \\ & U(\mathfrak{g}) \end{tikzcd}$$

Bijectivity:

Take $\{a_i\}$ a basis of $\mathfrak{h}$, $\{c_j\}$ a basis of $\mathfrak{t}$, and $\{b_k\}$ a basis of $\mathfrak{l}/\mathfrak{t}$.

Then the standard monomial

$$a_{i_1}^{u_1} \cdots a_{i_m}^{u_m} \otimes c_{j_1}^{v_1} \cdots c_{j_p}^{v_p} b_{k_1}^{w_1} \cdots b_{k_n}^{w_n}$$

forms a basis of

$$U(\mathfrak{h}) \otimes_{U(\mathfrak{t})} U(\mathfrak{l})$$

Image of these words in $U(\mathfrak{g})$ is

$$a_{i_1}^{u_1} \cdots a_{i_m}^{u_m} c_{j_1}^{v_1} \cdots c_{j_p}^{v_p} b_{k_1}^{w_1} \cdots b_{k_n}^{w_n}$$

which is a basis of $U(\mathfrak{g})$. $\Rightarrow$ The given map is a bijective linear map. $\square$

Using these two lemmas, we can finally answer our question.

Corollary 5: If $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n$ are Lie subalgebras of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$$

then there exists a unique

$$f : U(\mathfrak{g}_1) \otimes \cdots \otimes U(\mathfrak{g}_n) \longrightarrow U(\mathfrak{g})$$

such that

$$(u_1 \otimes \cdots \otimes u_n) \mapsto u_1 \cdots u_n,$$

$$1 \mapsto 1$$

where f is a bijective linear map.

Proof: Follows directly from Lemma $\mathfrak{b}$ as $\mathfrak{g}_i \cap \mathfrak{g}_j = 0$. $\square$

Note: In the above corollary, if $\mathfrak{g}_1, ..., \mathfrak{g}_n$ commute pairwise, then the map f becomes an algebra isomorphism. This motivates the following result.

Corollary 6: Let $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n$ be Lie algebras and $\mathfrak{g}$ be their product. Then

$$f : U(\mathfrak{g}_1) \times \cdots \times U(\mathfrak{g}_n) \longrightarrow U(\mathfrak{g})$$

defined by

$$(u_1, \ldots, u_n) \mapsto u_1 \cdots u_n$$

is an **algebra isomorphism**.

Proof: Follows from corollary 5. $\square$

2.3 Universal enveloping algebras of ideals and quotients

In this section, we ask the question: How are universal enveloping algebras of a Lie algebra and its ideals and quotients related? To answer this question, we will make use of the following Lemma.

Lemma ($\mathfrak{c}$): Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and $\{x_\lambda\}_{\lambda \in \Lambda}$ be a basis of a complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then $\{x_1^{v_1} \cdots x_n^{v_n} \mid v_1 \leq ... \leq v_n, \; \forall \; v_i \in \mathbb{N}\}$ forms a basis of $U(\mathfrak{g})$ as a left and right $U(\mathfrak{h})$-module.

Proof: Extend the given basis to a basis of $U(\mathfrak{g})$. Then any $u \in U(\mathfrak{g})$ can be written uniquely as:

$$u = \sum_{n,m} x_1^{v_1} \cdots x_n^{v_n} y_1 \cdots y_m$$

$$u = \sum x_1^{v_1} \cdots x_n^{v_n} y_{v_1,\ldots,v_m} \quad \text{where } y_{v_1,\ldots,v_m} \in U(\mathfrak{h})$$

because of corollary 5 $U(\mathfrak{h}) \hookrightarrow U(\mathfrak{g})$is an embedding

$$\therefore x_1^{v_1} \cdots x_n^{v_n} \text{ is a basis for } U(\mathfrak{g}) \text{ as a right } U(\mathfrak{h})\text{-module.}$$

Similarly, they are a basis for $U(\mathfrak{g})$ as a left $U(\mathfrak{h})$-module. $\square$

Corollary 7: Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$.

(i) The left and right ideal of $U(\mathfrak{g})$ generated by $\mathfrak{h}$ coincide. Denote the ideal by R

(ii) Let $j : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ be the natural projection. Then

$$\mathcal{U}(j) : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}/\mathfrak{h})$$

is a surjective homomorphism (of unital associative algebras) with kernel R. Here $\mathcal{U}$ is the $\mathcal{U}$-functor from section 2.1

Proof: For $x \in \mathfrak{g}$ let $\tilde{x} := \iota(x) \in U(\mathfrak{g})$. For any $u \in U(\mathfrak{g})$, Lemma $\mathfrak{c}$ says that:

$$u = \sum_n \tilde{x}_1^{v_1} \cdots \tilde{x}_n^{v_n} h_x \quad \text{for some } h_x \in U(\mathfrak{h})$$

$$u = \sum_m h_y \, \tilde{y}_1^{v_1} \cdots \tilde{y}_m^{v_m} \quad \text{for some } h_y \in U(\mathfrak{h})$$

where $\{x_i\}_{i \in \alpha}$ and $\{y_j\}_{j \in \beta}$ are two basis of a complement of $\mathfrak{h}$ in $\mathfrak{g}$. Now,

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "j"'] & U(\mathfrak{g}) \arrow[d, "\mathcal{U}(j)"] \\ & U(\mathfrak{g}/\mathfrak{h}) \end{tikzcd}$$

Let ε_x, ε_y be the constant terms of h_x and h_y.

$$\Rightarrow \mathcal{U}(j)(u) = \sum j(x)_1^{v_1} \cdots j(x)_n^{v_n} \varepsilon_x = \sum \varepsilon_y j(y)_1^{v_1} \cdots j(y)_m^{v_m}$$

$$\therefore \ker(\mathcal{U}(j)) = \left\{ \sum_n \tilde{x}_1^{v_1} \cdots \tilde{x}_n^{v_n} U_+(\mathfrak{h}) \,\middle|\, n, v_1, \ldots, v_n \in \mathbb{N} \right\} \quad = \text{left ideal generated by } \mathfrak{h}$$

$$= \left\{ \sum_m U_+(\mathfrak{h}) \, \tilde{y}_1^{v_1} \cdots \tilde{y}_m^{v_m} \,\middle|\, m, v_1, \ldots, v_m \in \mathbb{N} \right\} \quad = \text{right ideal generated by } \mathfrak{h}$$

where $U(\mathfrak{h})_+$ denotes non-scalar elements in $U(\mathfrak{h})$. This shows part (i) and we only need to prove surjectivity of $U(i)$ to complete the proof of part (ii). Let $\{x_i\}_{i \in \alpha}$ be the basis of a complement of $\mathfrak{h}$ in $\mathfrak{g}$ and let $\{\tilde{x}_i\}_{i\in\alpha}$ be the canonical image of this basis in $U(\mathfrak{g})$. These x_i's also act as a basis of $\mathfrak{g}/\mathfrak{h}$. So, let $\{\overline{x}_i\}_{i\in\alpha}$ be the canonical image of this basis in $U(\mathfrak{g}/\mathfrak{h})$. Then, by the above commutative diagram we get

$$\mathcal{U}(j)\left(\{\tilde{x}_1^{v_1}\cdots\tilde{x}_n^{v_n} \mid n, v_1, \ldots, v_n \in \mathbb{N}\}\right) = \{\overline{x_1}^{v_1}\cdots\overline{x_n}^{v_n} \mid n, v_1, \ldots, v_n \in \mathbb{N}\}$$

which forms a basis of $U(\mathfrak{g}/\mathfrak{h})$.

$$\therefore \mathcal{U}(j) \text{ is a surjection } \blacksquare$$

2.4 Filtration of $U(\mathfrak{g})$

When we construct $U_{\mathfrak{g}}$ from tensor algebra of $\mathfrak{g}$. We know that $T(\mathfrak{g})$ comes with a filtration on it. This filtration structure carries over to $U_{\mathfrak{g}}$ which we will explore in this section.

Definition: $U_n(\mathfrak{g}) :=$ the vector subspace of $U(\mathfrak{g})$ generated by all words of the form $x_1x_2\cdots x_p, \quad p \leq n, \quad x_i \in \mathfrak{g}$.

For example: $U_0(\mathfrak{g}) = k\cdot 1$, $U_1(\mathfrak{g}) = k\cdot 1 \oplus \mathfrak{g}$. Where k is the underlying field. One has to be careful here $U_n(\mathfrak{g}) \neq k\cdot 1 \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^n$ because the map $\pi : T(\mathfrak{g}) \to U(\mathfrak{g})$ is not an isomorphism.

Alternatively, a "coordinate-free" definition of the filtration of $U(\mathfrak{g})$ can be given as:

$$U_n(\mathfrak{g}) := \pi\left(k \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots \oplus \mathfrak{g}^{\otimes n}\right) = \pi(T_n(\mathfrak{g})),$$

where $\pi : T(\mathfrak{g}) \to U(\mathfrak{g})$ is the quotient map we defined in 1.3

The sequence $\{U_n(\mathfrak{g})\}_{n\in\mathbb{N}_0}$ is called the **canonical filtration** of $U(\mathfrak{g})$.

Moreover,

$$U(\mathfrak{g}) = \bigcup_{n\in\mathbb{N}_0} U_n(\mathfrak{g})$$

2.4.1 Properties

1. $U_m(\mathfrak{g}) \cdot U_n(\mathfrak{g}) \subseteq U_{m+n}(\mathfrak{g})$

2. $U_n(\mathfrak{g}) \subsetneq U_{n+1}(\mathfrak{g})$

3. If $(e_1, \ldots, e_r)$ is a basis of $\mathfrak{g}$, then a basis of $U_n(\mathfrak{g})$ consists of words $e_1^{v_1} \cdots e_r^{v_r}$ where $v_1 + \cdots + v_r \leq n$, by the PBW theorem.

Thus, if $\mathfrak{g}' \subseteq \mathfrak{g}$ is a Lie subalgebra, then

$$U_n(\mathfrak{g}') = U_n(\mathfrak{g}) \cap U(\mathfrak{g}').$$

2.4.2 Graded structure

Given a filtered algebra $U(\mathfrak{g})$, we can associate a graded algebra G^n. The idea is to convert the union into a direct sum, i.e., $\mathrm{Gr}(\mathrm{U}_{\mathfrak{g}}) = \bigoplus_{i \in \mathbb{N}_0} \mathrm{G}_i$, where each G_i consists of homogeneous elements of degree i.

Definition: A *grading* on an algebra structure X is a direct sum decomposition:

$$X = \bigoplus_{i \in \mathbb{N}_0} X_i$$

such that $X_i X_j \subseteq X_{i+j}$ $\forall i, j \in \mathbb{N}$ where elements of X_i are called *homogeneous of degree* i.

For example, a *graded algebra* A over a ring R is an associative algebra A that can be written as a direct sum of subalgebras:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_i \oplus \cdots$$

with the property that multiplication respects the grading:

$$A_i \cdot A_j \subseteq A_{i+j}$$

Graded algebra associated to $U(\mathfrak{g})$

Every filtered ring has an associated graded ring while every graded ring has a canonical filtration. We will show this for $U(\mathfrak{g})$, but the same method can be generalized very easily.

$\{U_n(\mathfrak{g})\}_{n\in\mathbb{N}_0}$ be the filtration of $U(\mathfrak{g})$. Define inductively $G_n := U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ while setting $U_{-1}(\mathfrak{g}) = \{0\}$. Then $G := \bigoplus_i G_i$ is our associated graded algebra. Similarly, a graded algebra $G := \bigoplus_i G_i$ has a filtered structure where the n^{th} filtration is the direct sum of first n terms of $\bigoplus_i G_i$. Denote by $Gr(U_\mathfrak{g})$ the graded algebra associated to $U_\mathfrak{g}$

Note: An algebra is also a ring. So the terms filtered ring, filtered algebra, graded ring, graded algebra are used interchangeably in the context of $U(\mathfrak{g})$.

Properties of $Gr(U_\mathfrak{g})$

Here we use G to denote $Gr(U_\mathfrak{g})$

- The multiplication map for graded components is k-bilinear $G_n \times G_m \to G_{n+m}$. This can be seen by the simple computation:

  $$U_n/U_{n-1} \times U_m/U_{m-1} \to U_{n+m}/U_{n+m-1}$$

  $$(x + U_{n-1}) \cdot (y + U_{m-1}) = xy + U_{n+m-1}$$

  Similarly by passage to quotient we can see that if U is unital and associative, then G is also unital and associative

- The graded algebra G is commutative:

  If $x \in G_n$, $y \in G_m$ then:

  $$(x + U_{n-1})(y + U_{m-1}) = xy + U_{n+m-1}$$

  $$(y + U_{m-1})(x + U_{n-1}) = yx + U_{n+m-1}$$

  $$\Rightarrow xy - yx + U_{n+m-1} \in U_{n+m-1}$$

Since,

$$xy - yx \equiv 0 \text{ in } G_{n+m}$$

This means, $xy = yx$ in G_{n+m}. Hence proved that G is commutative.

2.4.3 Symmetrization

The associated graded algebra of $U(\mathfrak{g})$ being commutative suggests that there is a relation between $Gr(U_\mathfrak{g})$ and $S(\mathfrak{g})$ the symmetric algebra over $\mathfrak{g}$.

The universal property of symmetric algebra $S(\mathfrak{g})$ says that for any commutative algebra A and a linear map $f\colon \mathfrak{g} \to A$, there exists a unique algebra homomorphism $\pi\colon S(\mathfrak{g}) \to A$ making the following diagram commute:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & S(\mathfrak{g}) \arrow[d, "\exists!\pi"] \\ & A \end{tikzcd}$$

For $A = Gr(U_\mathfrak{g})$ we get a canonical homomorphism $\pi : S(\mathfrak{g}) \to Gr(U_\mathfrak{g})$. By making use of the PBW theorem, we can show that this homomorphism is indeed an isomorphism of symmetric algebras i.e, $S(\mathfrak{g}) \cong Gr(U_\mathfrak{g})$

Theorem 2.4.1. $Gr(U_\mathfrak{g}) \cong S(\mathfrak{g})$ as commutative algebras.

Proof: Basis of $S_n(\mathfrak{g})$ are elements of the form $x_{i_1}x_{i_2}\cdots x_{i_n}$ where $i_1 \leq i_2 \leq \cdots \leq i_n$, $\{x_i\}$ are basis of $\mathfrak{g}$. Here $S(\mathfrak{g})$ is a graded ring with n^{th} gradation $S_n(\mathfrak{g})$. We denote with π the canonical algebra homomorphism $\pi : S(\mathfrak{g}) \to U(\mathfrak{g})$ that we get from the universal property of $S(\mathfrak{g})$

$$\therefore \pi(x_{i_1}\cdots x_{i_n}) = \pi(x_{i_1})\cdots\pi(x_{i_n}) = x_{i_1}\cdots x_{i_n}, \quad \text{since } \mathfrak{g} \hookrightarrow U(\mathfrak{g})$$

$$\Rightarrow \pi(x_{i_1}\cdots x_{i_n}) \text{ is a basis element of } U_n(\mathfrak{g}) \setminus U_{n-1}(\mathfrak{g})$$

$$\Rightarrow S_n(\mathfrak{g}) \cong \frac{U_n(\mathfrak{g})}{U_{n-1}(\mathfrak{g})} = G_n, \quad \forall n \in \mathbb{N}_0 \Rightarrow S(\mathfrak{g}) \cong Gr(U_\mathfrak{g})$$

$$\begin{tikzcd} & S(\mathfrak{g}) \arrow[d, "\exists!\pi"] \\ \mathfrak{g} \arrow[ur, "i"] \arrow[r, hook, "f=\iota"] & U(\mathfrak{g}) \end{tikzcd}$$

■

We can now say that the graded algebra associated with $U(\mathfrak{g})$ is $S(\mathfrak{g})$.

Definition: A ring is said to be *Noetherian* if it satisfies the ascending chain condition on right and left ideals.

Let $\mathfrak{g}$ be a finite-dimensional lie algebra with basis $\{e_1, ...e_n\}$. Then, $S(\mathfrak{g}) \cong k[x_1, .., x_n]$. The Hilbert basis theorem, a standard result from algebraic geometry, says that $k[x_1, ..., x_n]$ is Noetherian. We also know that a filtered ring is Noetherian if its associated graded ring is Noetherian. Hence, we have the following result.

Theorem 2.4.2. For finite-dimensional Lie algebra $\mathfrak{g}$. $U(\mathfrak{g})$ is a Noetherian ring.

This result is very useful because finitely-generated modules of Noetherian rings are Noetherian. This will be used in the upcoming chapters. For example, simple modules of Lie algebras are singly-generated. Results like these show the importance of $U(\mathfrak{g})$ as we can use module-theoretic ideas to study $\mathfrak{g}$-modules.

Chapter 3

Highest weight theory

We now shift our focus to the study of finite-dimensional representations of semi-simple Lie algebras. The highest weight theory was introduced by Elie Cartan and Hermann Weyl. The theorem of highest weight is also known as the Cartan–Weyl theorem and introduced by Weyl in his paper in 1925 and his 1939 book "The Classical Groups"[17]. Although their work was rooted in analytic and topological methods. The thesis is confined to the algebraic treatment of this theory introduced by Claude Chevalley and Harish-Chandra in the 1940s–1950s. In his 1966 thesis and subsequent papers [13], Daya-Nand Verma introduced Verma modules. This laid foundation for the work of Bernstein–Gelfand–Gelfand (1970s). This thesis follows the books of Jacques Dixmier[8] and Humphreys[9, 12] which is a unified treatment the works of Verma and BGG.

We will show that finite-dimensional representations of semi-simple Lie algebras come with a highest weight structure. So, the study of highest weight modules plays a crucial role in representation theory of semi-simple Lie algebras. Verma modules play the role of universal highest weight modules. In this chapter, we first provide the classification of finite-dimensional representations of $sl_2(\mathbb{C})$. This will be useful as Verma modules of $sl_2(\mathbb{C})$ will serve as our prime example. Then we introduce the language of highest weight theory and then go on to define Verma modules. At the end we will be able to classify the finite-dimensional representations of a semi-simple Lie algebra.

From this section onward, the role of Cartan decomposition of semi-simple Lie algebras takes center stage. The decomposition is true over algebraically closed fields with characteristic 0. In the previous chapters, we worked with arbitrary fields, but henceforward, we will be working with the complex field.

To study representations of semi-simple Lie algebras, we have the following two results at our disposal. Refer to appendix **??** to recall them.

- Weyl's complete reducibility theorem says that any finite-dimensional representation of $\mathfrak{g}$ is semi-simple i.e. completely reducible; where $\mathfrak{g}$ is a semi-simple Lie algebra over a characteristic field 0.
- Shur's lemma says that any nonzero homomorphism between irreducible representations of Lie algebras is an isomorphism.

3.1 Modules of $sl_2(\mathbb{C})$

$sl_2(\mathbb{C})$ is the smallest simple lie algebra. Cartan decomposition tells us that every semi-simple lie algebra consists of copies of sl(2,$\mathbb{C}$). Therefore, it is expected that representation theory of $sl_2(\mathbb{C})$ will play a crucial role in the representation theory of semi-simple Lie algebras. In this section, we will focus on finite-dimensional representations of $sl_2(\mathbb{C})$.

Let us start with some known representations of $sl_2(\mathbb{C})$. Because $sl_2(\mathbb{C})$ is simple, the only one-dimensional representation it has is the trivial representation.
$sl_2(\mathbb{C}) :=$ vector subspace of $M_2(\mathbb{C})$ generated by the vectors $\{e, f, h\}$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h = [e, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We call this the natural matrix representation. It is a 2-dimensional representation. $sl_2(\mathbb{C})$ is a 3-dimensional vector space so the adjoint representation will give a 3-dimensional representation of $sl_2(\mathbb{C})$.

Notice that the above given examples of $sl_2(\mathbb{C})$ representations are all irreducible. In fact, they are the only irreducible representations of their respective dimensions. So naturally we would ask the questions:

- Is there an irreducible representation of $sl_2(\mathbb{C})$ of dimension n, where $n \in \mathbb{N}$?
- Is the irreducible representation of dimension n unique?

It turns out to much delight that the answer to these questions is yes. To show this, we will construct a collection of irreducible representations $\{V^{(n)}\}$ of $sl_2(\mathbb{C})$ where $V^{(n)}$

is of dimension n and show that any irreducible representation is isomorphic to one of these. By Schur's Lemma, we can say that each $V^{(n)}$ is unique upto isomorphism.

Theorem 3.1.1. For any $n \in \mathbb{N} \cup \{0\}$, there exists an irreducible representation $V^{(n)}$ of $sl_2(\mathbb{C})$ of dimension $n+1$. Any finite-dimensional irreducible representation V is isomorphic to $V^{(d)}$ where $d+1$ is the dimension of V.

To prove the theorem, we will rely on the following lemma.

Lemma: V be an $sl_2(\mathbb{C})$ module. Let $v \in V$ be an eigenvector of $h \in sl_2(\mathbb{C})$ with eigenvalue λ. Then, $e \cdot v$ and $f \cdot v$ are eigenvectors of h with eigenvalue $\lambda + 2$ and $\lambda - 2$ respectively.

Proof of Lemma: We know that $[h, e] = h \circ e - e \circ h = 2e$. So,

$$\begin{aligned} h \cdot (e \cdot v) = (h \circ e) \cdot v &= ([h, e]) \cdot v + (e \circ h) \cdot v \\ &= (2e) \cdot v + e \cdot (h \cdot v) = 2(e \cdot v) + e \cdot (\lambda v) \\ &= (\lambda + 2) e \cdot v \end{aligned}$$

Therefore, $e \cdot v$ is an eigenvector of h with eigenvalue $\lambda + 2$. Similarly, using $[h, f] = -2f$ we can show that $f \cdot v$ is an eigenvector of h with eigenvalue $\lambda - 2$. $\square$

Proof of Theorem:

Step 1: Let $V^{(0)}$ be the 1-dimensional representation. We will now give a construction of $V^{(n)}$.

Definition: $V^{(n)}$ be the complex vector space generated by the symbols $\{v_{-n}, v_{-n+2}, ...v_{n-2}, v_n\}$ We define the action of $e, f, h \in sl_2(\mathbb{C})$ on $V^{(n)}$ as follows:

- $f \cdot v_i = v_{i-2} \;\; \forall i \neq -n$ and $f \cdot v_{-n} = 0$
- $h \cdot v_i = i v_i \;\; \forall i$
- $e \cdot v_i = \frac{(n+i+2)(n-i)}{4} v_{i+2} \;\; \forall i \neq n$ and $e \cdot v_n = 0$

The way we have defined the action is such that $V^{(n)}$ becomes an $sl_2(\mathbb{C})$ module of dimension $n+1$. All we have to show is that it is an irreducible representation. Assume that $W \subsetneq V^{(n)}$ is a non-trivial proper subspace of $V^{(n)}$ closed under the action of $sl_2(\mathbb{C})$. Then it must contain an eigenvector of the map $\Phi_h : V^{(n)} \to V^{(n)}, w \mapsto h \cdot w$. That means it must contain v_i for some $i \in \{-n, -n+2, ..., n-2, n\}$. Using the action of e and f on v_i we can get $\{v_{-n}, v_{-n+2}, ...v_{n-2}, v_n\}$. Hence, $W = V^{(n)}$. But this is in contradiction to our assumption that W was a proper subspace. Hence, it can be concluded that $V^{(n)}$ is an irreducible $sl_2(\mathbb{C})$ module of dimension $n+1$.

Step 2: Assume V is a finite-dimensional irreducible representation. Consider the action of h, $\Phi_h : V \to V, w \mapsto h \cdot w$. If $\Phi_h = 0$ then V is the trivial representation. This is due to that fact that for any semi-simple Lie algebra, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In this case, V is isomorphic to V^0.

Assume, $\Phi_h \neq 0$. Since, we have assumed the underlying field to be algebraically closed, Φ_h has a non-trivial eigenvector. Denote by V_λ is the eigenspace of Φ_h with eigenvalue λ. Since, V is finite dimensional, there are only finitely many eigenvalues of Φ_h whose corresponding eigenvectors are non-zero. Let λ' be such an eigenvalue with maximal real part. Then using the lemma we can conclude that $e \cdot V_{\lambda'} = 0$. Take $v \in V_{\lambda'}$. Consider the vector subspace M generated by $\{v, f \cdot v, f^2 \cdot v, ...\}$. All these vectors are eigenvectors of Φ_h with different eigenvalues. Any two non-zero vectors in this collection are linearly independent.

It is thus clear that M is closed under the action of f and h. Using $[e, f] = e \circ f - f \circ e = h$ we can see that $e \cdot (f \cdot v) \in M$. Using induction we can say that $e \cdot (f^k \cdot v) \in M$. Hence, M is closed under the action of $sl_2(\mathbb{C})$. Simplicity of V means that $M = V$. Let $n \in \mathbb{N}$ be the smallest natural number such that $f^{n+1} \cdot v = 0$. As a result $V \cong V^{(n)}$ ■

This result now gives us a complete classification of irreducible modules of $sl_2(\mathbb{C})$. Along with Weyl's complete reducibility theorem, this gives us the classification of all finite dimensional representations of $sl_2(\mathbb{C})$.

3.2 Highest weight modules

Consider W a finite-dimensional irreducible representation of $sl_2(\mathbb{C})$. We know that the linear map $\Phi_h : W \to W, w \mapsto h \cdot w$ has non-trivial eigenvectors (as the complex field is algebraically closed). $M := \oplus W_\lambda$ the direct sum of all non-trivial eigenspaces. Using the lemma in 3.1 we can say that M is closed under the action of $sl_2(\mathbb{C})$. Using simplicity of W we can see that $M = W = \oplus W_\lambda$.

In the above result we will now try to replace $sl_2(\mathbb{C})$ with a complex semi-simple Lie algebra $\mathfrak{g}$. Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. We will see that just as in the $sl_2(\mathbb{C})$ case, we can give a direct sum decomposition of V that consists of eigenvectors of the action of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We first fix the following conventions for the rest of this chapter:

Conventions:

$\mathfrak{g}$ is a finite-dimensional complex semi-simple Lie algebra. We fix $\mathfrak{h}$ a Cartan subalgebra. Let $\mathfrak{g} = \mathfrak{h} \underset{\alpha\in\Phi}{\oplus} \mathfrak{g}_\alpha$ be the Cartan decomposition and Φ be the corresponding set of roots. Fix a basis Δ and let Φ^+ and Φ^- be the set of positive and negative roots respectively. $\mathfrak{n}_+ := \{v \in \mathfrak{g}_\alpha \mid \alpha \in \Phi^+\}$ and $\mathfrak{n}_- := \{v \in \mathfrak{g}_{-\alpha} \mid \alpha \in \Phi^+\}$. Let $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$ be the Borel subalgebra.

Definitions: For a $\mathfrak{g}$-module V

- A vector $\lambda \in \mathfrak{h}^*$ is called a *weight*
- The vector subspace $V_\lambda := \{v \in V \mid h \cdot v = \lambda(h)v \ , \forall h \in \mathfrak{h}\}$ is called the *weight space* corresponding to the weight λ
- The set $\{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}$ is called the *support of V*
- V is said to be a *weight module* if $V = \underset{\lambda\in\mathfrak{h}^*}{\oplus} V_\lambda$

Now, take $\rho : \mathfrak{g} \to End(V)$ a finite-dimensional irreducible representation of $\mathfrak{g}$. Since $\mathfrak{h}$ is an abelian subalgebra, its elements commute and so do the representations of these elements. We know that over complex numbers, any collection of commuting endomorphisms of a finite-dimensional vector space has a common eigenvector. Therefore, *there exists a non-trivial weight space* V_λ.

Lemma: If $\alpha \in \Phi$ then , $\mathfrak{g}_\alpha(V_\lambda) \subseteq V_{\lambda+\alpha}$

Proof: Let $x \in \mathfrak{g}_\alpha$, $h \in \mathfrak{h}$ and $v \in V_\lambda$ then,

$$\begin{aligned} h \cdot (x \cdot v) &= x \cdot (h \cdot v) + [h, x] \cdot v \\ &= x \cdot (\lambda(h)v) + \alpha(h)x \cdot v \\ &= (\lambda + \alpha)(h)(x \cdot v) \end{aligned}$$

Hence proved that $x \cdot v \in V_{\lambda+\alpha}$ ■

From the lemma, it is easy to see that $\bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ is a $\mathfrak{g}$-submodule of V. We have also shown it is non-trivial. Since, V is irreducible this implies $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. Hence, we have shown that a finite-dimensional irreducible $\mathfrak{g}$-module is a weight module.

Remark: A general representation need not have a weight space decomposition. But finite-dimensional representations always decompose into weight spaces. We have seen that irreducible finite-dimensional representations have a weight decomposition. This together with Weyl's complete reducibility theorem ensure that all finite-dimensional representations of have a weight space decomposition.

In the case of irreducible $sl_2(\mathbb{C})$ modules, we used the fact that there was a weight λ with a maximal real part. We then showed that we can generate the entire module using a vector from the weight space V_λ. We can do something similar here.

Definition: For a finite-dimensional $\mathfrak{g}$-module V,

- $\lambda \in \mathfrak{h}^*$ is a *highest weight* if $V_{\lambda+\alpha} = 0 \;\; \forall \; \alpha \in \Phi_+$
- $v \in V_\lambda$ is called a *highest weight vector* or a *maximal vector*
- If V generated by a highest weight vector, it is called a *highest weight module*

A finite-dimensional $\mathfrak{g}$-module V has a highest weight because otherwise its support will be infinite-dimensional. Let λ' be a highest weight and let $v \in V_{\lambda'}$. If V is also irreducible, then it is equal to the $\mathfrak{g}$-module generated by v. This is similar to what we did in the proof of the theorem in 3.1. We can now conclude the following theorem:

Theorem 3.2.1. All simple finite-dimensional modules of $\mathfrak{g}$ are highest weight modules.

We will now try to further explore this highest weight structure. To do so, we will now consider highest weight modules of $\mathfrak{g}$ of arbitrary dimension. The following theorem details some properties of such representations.

Proposition 3.2.2. M be a highest weight $\mathfrak{g}$-module generated by maximal vector v with highest weight $\lambda \in \mathfrak{h}^*$. Then,

1. $M = U(\mathfrak{g}) \cdot v = U(\mathfrak{n}_-) \cdot v$
2. $M = \bigoplus\limits_{\mu \in \mathfrak{h}^*} M_\mu$ (i.e., M is a weight module)
3. Every weight of M is of the form $\lambda - \sum_i n_i \alpha_i$ where $\alpha_i \in \Delta^+$ and $n_i \in \mathbb{Z}_{\geq 0}$
4. Each weight space M_μ is finite dimensional and $\dim M_\lambda = 1$
5. Every endomorphism of M is scalar

Proof:

(1) By definition M is a $\mathfrak{g}$-module generated by a maximal vector v of weight λ.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\rho"'] & U(\mathfrak{g}) \arrow[d, "\exists! \rho'"] \\ & \mathfrak{gl}(M) \end{tikzcd}$$

From the universal property of $U(\mathfrak{g})$ it is obvious that $x \cdot v = \iota(x) \cdot v$. Hence, $M = \mathfrak{g} \cdot v = U(\mathfrak{g}) \cdot v$. We know that, $\mathfrak{n}_-$ annihilates the maximal vector v and $\mathfrak{h}$ scales it. Using Corollary 5 in 2.2 we can say, $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$ and hence, $U(\mathfrak{g}) \cdot v = U(\mathfrak{n}_-) \cdot v$ $\square$

(2) The maximal vector $v \in M_\lambda$ and from the lemma in 3.2 we can say that

$$\bigoplus_{\mu \in \mathfrak{h}^*} M_\mu \subseteq M$$

But $v \in \bigoplus\limits_{\mu \in \mathfrak{h}^*} M_\mu$ and it generates M. So, the inclusion is an equality. Therefore, we can say that $M = \bigoplus\limits_{\mu \in \mathfrak{h}^*} M_\mu$ i.e. M is a weight module. $\square$

(3) From Corollary 5 in 2.2 we get

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \Rightarrow U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$$

We know that $M = U(\mathfrak{g}) \cdot v$ where v is a maximal vector with weight λ. This means that $U(\mathfrak{n}_+)$ annihilates v whereas $U(\mathfrak{h})$ just scales v. So, $M = U(\mathfrak{n}_-) \cdot v$. Let,

$$\mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}, \quad \{x_1, \ldots, x_k\} \text{ such that } x_i \in \mathfrak{g}_{-\alpha_i},\ \alpha_i \in \Phi^+$$

$$\implies \text{Basis of } U(\mathfrak{n}^-) \text{ is } \{x_1^{n_1} \cdots x_k^{n_k} | n_i \in \mathbb{N} \cup \{0\}\}$$

$$\implies (x_1^{n_1} \cdots x_k^{n_k}) \cdot v \in M_\mu \text{ where } \mu = \lambda - \sum_{i=1}^{k} n_i \alpha_i$$

Combining this with the result from (2), $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu = U(\mathfrak{n}_-) \cdot v$ we can say that any weight is of the form $\lambda - \sum_i n_i \alpha_i$ where $\alpha_i \in \Delta^+$ and $n_i \in \mathbb{Z}_{\geq 0}$ $\square$

(4) From (3) we have $M = U(\mathfrak{n}_-) \cdot v$ and for any weight μ of M, $\lambda - \mu$ is a $\mathbb{Z}_{\geq 0}$ sum of positive roots. We can rewrite this sum using different positive roots only finitely many times. This means that for only finitely many different words $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$

$$(x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}) \cdot v \in M_\mu \text{ such that } \lambda - \mu = \sum_{i=1}^{k} n_i \alpha_i$$

where $\{x_1, x_2, \cdots, x_k\}$ is basis of $\mathfrak{n}_-$. So, we can say that the weight space M_μ is finite-dimensional. Similarly, M_λ is 1-dimensional because there is only one way to write

$$x_{i_1}^{n_1} \cdots x_{i_k}^{n_k}(v) \in M_\lambda \Rightarrow v \in M_\lambda \qquad \square$$

(5) Let $T \in \mathrm{Hom}_{\mathfrak{g}}(M, M)$, then

$$T(x \cdot w) = x \cdot T(w), \quad \forall x \in \mathfrak{g}, w \in M$$

For $h \in \mathfrak{h}$ and maximal vector v

$$T(h \cdot v) = h \cdot T(v) = \lambda(h) T(v)$$

$\implies T(v) \in M_\lambda$ and hence $T(v) = z \cdot v$ for some $z \in \mathbb{C}$. Using (3) we can say that for any $w \in M$

$$T(w) = T(x_1^{n_1} \cdots x_r^{n_r} \cdot v) = x_1^{n_1} \cdots x_r^{n_r} \cdot T(v) = z \cdot w$$

Hence, proved that T is just zId_M for some $z \in \mathbb{C}$. ■

Definition: Let $\mu, \lambda \in \mathfrak{h}^*$. We say that $\mu \leq \lambda$ if $\lambda - \mu$ can be expressed as a $\mathbb{Z}_{\geq 0}$-linear combination of positive roots.

In this way we are able to define a partial order on $\mathfrak{h}^*$. We can now reinterpret the property (3) from the proposition above as: $\mu \in Supp(M)$ then $\mu \leq \lambda$.

3.3 Verma modules

We now have a good idea about the structure of highest weight modules of a semi-simple Lie algebra. It is now natural to try to tackle the problem of classifying the finite-dimensional irreducible representations of semi-simple Lie algebras.

For each weight $\lambda \in \mathfrak{h}^*$ there could be any number of highest weight modules with highest weight λ. The idea we have is to define a universal object with the highest weight structure. What we want to do is create a family of highest weight modules corresponding to a particular weight.

We will use a method used often in representation theory, the method of induction. We start with a family of modules of the Borel subalgbera $\mathfrak{b}$ and then induce to $\mathfrak{g}$.

We notice that the action on $\mathfrak{h}$ on any highest weight module is semi-simple while the action of $\mathfrak{n}_+$ is locally finite. So we consider the $\mathfrak{b}$-module $\mathbb{C}_\lambda$; a trivial $\mathfrak{n}_+$-module and a one-dimensional $\mathfrak{h}$ module with action of $h \in \mathfrak{h}$ given as $h \cdot z = \lambda(h)z$. By 2.1 we can also say $\mathbb{C}_\lambda$ is a $U(\mathfrak{b})$-module. From Corollary 5 in 2.2 we know that $U(\mathfrak{b})$ is a subalgebra of $U(\mathfrak{g})$. Using extension of scalars we get $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ a $U(\mathfrak{g})$-module.

Definition: A *Verma module* of weight λ (denoted as $M(\lambda)$) is defined as

$$M(\lambda) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

The vector $1 \otimes 1$ is non-zero in $M(\lambda)$. $U(\mathfrak{n}_+)$ annihilates $1 \otimes 1$ whereas any $h \in \mathfrak{h}$ scales it by $\lambda(h)$. Hence, $1 \otimes 1$ is a maximal vector in $M(\lambda)$. Also, $M(\lambda)$ is generated by $1 \otimes 1$ as a $U(\mathfrak{g})$-module. Hence, $M(\lambda)$ is a highest weight module with highest weight λ. $1 \otimes 1$ is called the *canonical generator* of $M(\lambda)$. Since $M(\lambda)$ is a highest weight module, it follows all the results from the previous theorem.

Remark: The Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is a maximal solvable subalgebra of $\mathfrak{g}$. We know that finite-dimensional irreducible modules of solvable Lie algebras are one-dimensional (when working with algebraically closed fields). So, $\mathbb{C}_\lambda$ is an irreducible $U(\mathfrak{b})$-module. And if instead of $\mathbb{C}_\lambda$ we had used an arbitrary finite-dimensional $\mathfrak{b}$-module N, then the induced module $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} N$ need not have been a highest weight module.

We wanted Verma modules to be universal objects. We want to relate an arbitrary highest weight module of highest weight λ to the Verma module $M(\lambda)$. The following observation will help us state the desired result.

Theorem 3.3.1. The map $U(\mathfrak{n}_-) \to M(\lambda)$ where $u \mapsto u(1 \otimes 1)$ is an $\mathfrak{n}_-$ module isomorphism when $U(\mathfrak{n}_-)$ is provided with left regular representation. i.e. $M(\lambda)$ is a free left $U(\mathfrak{n}_-)$ module of rank 1.

Proof: We know that $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}$. From Corollary 5 in 2.2

$$U(\mathfrak{g}) \cong U(\mathfrak{n}_-) \underset{\mathbb{C}}{\otimes} U(\mathfrak{b})$$

is an isomorphism of left $U(\mathfrak{n}_-)$ modules. Now,

$$M(\lambda) = U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \cong \left((U(\mathfrak{n}_-) \underset{\mathbb{C}}{\otimes} U(\mathfrak{b}) \right) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda$$

By property of tensor products,

$$M(\lambda) \cong U(\mathfrak{n}_-) \underset{\mathbb{C}}{\otimes} \left((U(\mathfrak{b}) \underset{\mathfrak{U}(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \right) \cong U(\mathfrak{n}_-) \underset{\mathbb{C}}{\otimes} \mathbb{C}$$

$$\therefore M(\lambda) \cong U(\mathfrak{n}_-) \quad \blacksquare$$

The theorem hints at a description of Verma modules in terms of $U(\mathfrak{g})$. In fact, we can give an alternative description of $M(\lambda)$ using generators and relations. This result is given as the following corollary.

Corollary: $M(\lambda) \cong U(\mathfrak{g})/I$ as left $U(\mathfrak{g})$-modules. Here, I is a left ideal generated by $\mathfrak{n}_+$ and elements of the form $\{h - \lambda(h) \cdot 1| \;\; h \in \mathfrak{h},\; \lambda \in \mathfrak{h}^*\}$

Proof: Give $U(\mathfrak{g})$ left regular representation and consider the map,

$$\begin{aligned} \varphi : U(\mathfrak{g}) &\to M(\lambda) \\ u &\mapsto u \cdot (1 \otimes 1) \end{aligned}$$

It is easy to see that φ is a surjective homomorphism of left $U(\mathfrak{g})$-modules. From Corollary 5 in 2.2 we have $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$
From the theorem we know that $U(\mathfrak{n}_-)$ acts freely on $1 \otimes 1$. The non-empty words in $U(\mathfrak{n}_+)$ lie in the kernel of φ as they annihilate $1 \otimes 1$. Similarly, the set $\{h - \lambda(h)| \; h \in \mathfrak{h}\}$ also annihilates the maximal vector of M. Hence, $ker(\varphi) = I$ the left ideal generated by $U(\mathfrak{n}_+)$ and $\{h - \lambda(h)| \; h \in \mathfrak{h}\}$ $\therefore$ $M(\lambda) \cong U(\mathfrak{g})/I$ □

Now, observe that the action of $U(\mathfrak{n}_-)$ on all highest weight modules is not free e.g, finite-dimensional modules. But, highest weight modules are generated by a single element as a $U(\mathfrak{n}_-)$-module as shown in the proposition in 3.2. This hints us at the following universal property of Verma modules.

Theorem 3.3.2. (Universal property of Verma modules) Let V be any highest weight $\mathfrak{g}$-module generated by a vector v of highest weight λ. Then,

(a) there exists a left $U(\mathfrak{g})$-module homomorphism, unique upto scalars

$$\varphi : M(\lambda) \twoheadrightarrow V$$

and this homomorphism is surjective.

(b) Furthermore, φ is bijective if and only if $u \cdot v \neq 0$ for all nonzero $u \in U(\mathfrak{n}_-)$.

Proof:

(a) Since $1 \otimes 1$ generates $M(\lambda)$, where it is mapped determines the homomorphisms from $M(\lambda)$. Let v be the maximal vector of V. Consider the map,

$$\psi : M(\lambda) \to V$$
$$u \cdot (1 \otimes 1) \mapsto u \cdot v$$

It is easy to see that this is a homomorphism of left $U(\mathfrak{g})$-modules. This shows the existence of non-trivial elements in $Hom_{U(\mathfrak{g})}(M(\lambda), V) \neq 0$.

Claim: $\phi \in Hom_{\mathfrak{g}}(V, W)$ where V, W are $\mathfrak{g}$-modules. Let $x \in V_\mu$ then $\phi(x) \in W_\mu$.

Proof: Take any $h \in \mathfrak{h}$. Then, $h \cdot \phi(x) = \phi(h \cdot x) = \phi(\lambda(h)x) = \lambda(h)\phi(x)$

We know that, $1 \otimes 1 \in M(\lambda)_\lambda$. Then using the claim above, we can say that if we take any non-trivial $\Phi \in Hom_{U(\mathfrak{g})}(M(\lambda), V) \implies \Phi(1 \otimes 1) \in V_\lambda$. Since, V is a highest weight module with highest weight λ, the weight space V_λ is 1-dimensional as was proved in the proposition in 3.2.

If $v \in V_\lambda$ is the generator of V then $\Phi(1 \otimes 1) = cv$ for some $c \in \mathbb{C}$. This is also consistent with the fact that endomorphisms of highest weight modules are unique upto scalars. Since v generates V, Φ is a surjection. □

(b) From the previous theorem we know that $M(\lambda)$ is a free $U(\mathfrak{n}_-)$ module of rank 1. This means that if we take $u \neq 0 \in U(\mathfrak{n}_-)$, then $u \cdot (1 \otimes 1) \neq 0$.

From the proposition in 3.2 we know that $V = U(\mathfrak{n}_-) \cdot v$. Using this we can see easily that if $\Phi \in Hom_{U(\mathfrak{g})}(M(\lambda), V)$ is non-trivial then Φ is injective if and only if $\forall\ u \neq 0 \in U(\mathfrak{n}_-), \quad \Phi(u \cdot (1 \otimes 1)) = u \cdot v \neq 0$ ■

We can also interpret (b) as follows: A highest weight module M with highest weight λ is a Verma module if and only if the action of $U(\mathfrak{n}_-)$ on M is free.

Example: As an example, let us look at $sl_2(\mathbb{C})$. The Cartan-subalgebra of $sl_2(\mathbb{C})$ is one-dimensional and spanned by $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence, $h^* = \{\lambda : h \to \mathbb{C}\}$ i.e. $h^* \cong \mathbb{C}$. For a fixed $\lambda \in \mathbb{C}$,

$$M(\lambda) := \langle v_i \mid i \in \mathbb{N}_0 \rangle, \quad \text{such that}$$

- $f \cdot v_i = v_{i+1}$
- $h \cdot v_i = (\lambda - 2i)v_i \ \ \forall i$
- $e \cdot v_i = a_i v_{i-1} \ \ \forall i \neq 0$ and $e \cdot v_0 = 0$

where $a_i := i(\lambda - (i-1))$. The way action of $sl_2(\mathbb{C})$ is defined is deliberate to make $M(\lambda)$ an $sl_2(\mathbb{C})$-module. $M(\lambda)$ is an infinite-dimensional module generated by v_0 which is a maximal vector of weight λ. $M(\lambda)$ is a highest weight module of highest weight λ. We can easily see that action of f on $M(\lambda)$ is free. Therefore, using the corollary we can say $M(\lambda)$ the Verma module of $sl_2(\mathbb{C})$ corresponding to weight λ.

3.4 Simple highest weight modules

Consider the Verma modules of $sl_2(\mathbb{C})$ given in 3.3. They look similar to the irreducible finite-dimensional $sl_2(\mathbb{C})$-modules $V^{(n)}$ defined in 3.1. Could it be possible that the Verma modules of $sl_2(\mathbb{C})$ are also irreducible? We explore this in the following result.

Proposition 3.4.1. *$M(\lambda)$ be the Verma module of $sl_2(\mathbb{C})$ as defined previously. Then,*

(a) *$M(\lambda)$ is simple $\iff \lambda \notin \mathbb{N} \cup \{0\}$*

(b) *If $\lambda \in \mathbb{N} \cup \{0\}$, $M(\lambda)$ is indecomposable and it has a unique simple submodule $M(-\lambda-2)$ which is itself simple, and:*

$$M(\lambda)/M(-\lambda-2) \cong V^{(\lambda)}$$

Proof: Let $\lambda \neq \mathbb{N} \cup \{0\}$ and V be a proper $\mathfrak{g}$-submodule of $M(\lambda)$. Any non-zero $v \in V$ can be written as a finite sum $v = \sum_{i=0}^{k} \alpha_i v_i$. We know that the action of e raises the weight of a weight vector by 2. Since, λ is not an integer, we can act e enough times on v so that we are left with only a multiple of v_0. Thus, $v_0 \in V$. But v_0 generates $M(\lambda)$ and this contradicts V being proper. Hence, $M(\lambda)$ is simple.

Now, if we show that when $\lambda \in \mathbb{N} \cup \{0\}$, $M(\lambda)$ is indecomposable and it has a unique simple submodule $M(-\lambda-2)$, we are essentially done. Since, $-n-2 \notin \mathbb{N}$ and $M(-n-2)$ will be simple by our earlier analysis.

Let $\lambda = n \in \mathbb{N} \cup \{0\}$. This means that $a_{n+1} = 0$ which makes v_{n+1} a maximal vector of weight $-n-2$. Since the action of f on $M(n)$ is free, this means that $M(-n-2) \hookrightarrow M(n)$.

Let V be any submodule of $M(n)$. We know that action of f is free on $M(n)$ so for any vector $v \in V$ the vector, $f^{n+1} \cdot v$ definitely lies in $M(-n-2)$. That means every submodule of $M(n)$ coincides with $M(-n-2)$. Hence, $M(-n-2)$ is the unique simple submodule of $M(n)$ and $M(n)$ is indecomposable. The quotient $M(n)/M(-n-2)$ has basis $\{v_0, ..., v_n\}$ and hence the quotient is isomorphic to $V^{(n)}$
■

Remark: Take a submodule V of $M(n)$. We know that V intersects $M(-n-2)$. If $M(-n-2) \subsetneq V \subsetneq M(n)$, then there exists $v \in V$ such that v is spanned by $\{v_0, ..., v_n\}$. But that means that by acting e enough number of times on v, we can get a scalar multiple of v_0. And v_0 will generate $M(n)$. So, the only proper non-trivial submodule of $M(n)$ is $M(-n-2)$.

Now if we are able to determine how many simple highest weight modules exists for a given highest weight, we will essentially be able to classify all simple highest weight modules of $sl_2(\mathbb{C})$. We can expect Schur's lemma to plays a key role in such a result.

Theorem 3.4.2. **(Classification of simple highest weight modules of $sl_2(\mathbb{C})$)** Let $L(\lambda)$ denote the simple highest weight module of $sl_2(\mathbb{C})$ of highest weight $\lambda \in \mathbb{C}$. Then, $L(\lambda)$ is unique and it is of the following form

$$L(\lambda) = \begin{cases} M(\lambda), & \lambda \notin \{\mathbb{N} \cup 0\}, \\ \mathbf{V}^{(n+1)}, & \lambda = n \in \{\mathbb{N} \cup 0\}. \end{cases}$$

Proof: We have shown in the proof of theorem 3.1.1 that $V^{(n)}$ is simple highest weight module with highest weight $n \in \{\mathbb{N} \cup 0\}$. And proposition 3.4.1 shows $M(\lambda)$ is simple when $\lambda \notin \{\mathbb{N} \cup 0\}$. All we have to show is that for a given weight, a simple highest weight module of that weight is unique.

Let V be a simple highest weight module of $sl_2(\mathbb{C})$ of highest weight λ. From universal property of Verma modules 3.3.2 we have a surjective map $\varphi \in Hom_{sl_2(\mathbb{C})}(M(\lambda), V)$

and $\varphi \neq 0$. If $M(\lambda)$ is simple, then by Schur's lemma $M(\lambda) \cong V$. If it is not simple, then using 3.4.1 we know that it has a unique simple submodule say K. So, $\tilde{\varphi} : M(\lambda)/K \to V$ be the quotient map. Again, by Schur's lemma, this is an isomorphism since $\varphi \neq 0$ ■

We can see that for the case of $sl_2(\mathbb{C})$, Verma modules are either simple, or they have a unique simple submodule. A similar result is true for Verma modules of semi-simple Lie algebras. Verma modules are either simple or they have a unique maximal submodule. It also will be shown in the next chapters that there exists a smallest simple submodule of $M(\lambda)$.

Proposition 3.4.3. Let $\lambda \in \mathfrak{h}^*$. $M(\lambda)_+ := \sum\limits_{\mu \neq \lambda} M(\lambda)_\mu$. Every proper $\mathfrak{g}$-submodule of $M(\lambda)$ is contained in $M(\lambda)_+$

Proof: F be a proper $\mathfrak{g}$-submodule of $M(\lambda)$. Since, $M(\lambda)$ is a weight module, we can write $F = \sum\limits_{\mu \in \mathfrak{h}^*} F \cap M(\lambda)_\mu$. According to proposition 3.2.2, a vector in $M(\lambda)_\lambda$ generates $M(\lambda)$ and since F is proper, $F \cap M(\lambda)_\lambda = 0$. Therefore, $F = \sum\limits_{\mu \neq \lambda} F \cap M(\lambda)_\mu \subset M(\lambda)_+$ ■

Remark: Let K be the sum of all proper $\mathfrak{g}$-submodules of $M(\lambda)$. The proposition implies $K \subset M(\lambda)_+$ i.e. if K is itself a proper $\mathfrak{g}$-submodule. Hence, K will be the unique maximal submodule of $M(\lambda)$. By $L(\lambda)$ we denote quotient $M(\lambda)/K$. Hence, $L(\lambda)$ is a simple highest weight module with highest weight λ.

Theorem 3.4.2 gives the classification of simple highest weight modules of $sl_2(\mathbb{C})$. It said that corresponding to every weight, there is a unique simple highest weight module. We now want to show this is also true for $\mathfrak{g}$.

Proposition 3.4.4. V be a simple highest weight $\mathfrak{g}$-module of highest weight λ. Then, V is isomorphic to $L(\lambda)$.

Proof: Using the universal property of Verma modules 3.3.2 we have a surjective map $\varphi \in Hom_{\mathfrak{g}}(M(\lambda), V)$ and $\varphi \neq 0$. K be the unique maximal submodule of $M(\lambda)$. Since V is simple, $K \subset ker(\varphi)$. Now, $\tilde{\varphi} : M(\lambda)/K \to V$ be the quotient map. Since

$L(\lambda) \cong M(\lambda)/K$ and V both are simple $\mathfrak{g}$-modules, and $\varphi \neq 0$, using Schur's Lemma we can conclude that $L(\lambda) \cong V$ ■

3.5 Highest Weight theorem

In the previous sections, we have shown the following results for semi-simple Lie algebras,

- Every simple finite-dimensional module is a highest weight module (theorem 3.2.1).
- For each weight, the corresponding simple highest weight module is unique (proposition 3.4.4).

From these two results we can conclude that for some weights, $L(\lambda)$ is finite-dimensional. The natural question to ask now is for what weights $\mu \in \mathfrak{h}^*$, is $L(\mu)$ finite-dimensional? In the case of $sl_2(\mathbb{C})$ we saw that the highest weights of simple finite-dimensional modules were $\{\mathbb{N} \cup 0\}$ (theorem 3.1.1). In this section, we arrive at a similar result for $\mathfrak{g}$.

Cartan decomposition tells that a semi-simple Lie algebra is a sum of subalgebras which are isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. So it makes sense to expect that representation theory of $\mathfrak{sl}_2(\mathbb{C})$ ought to help us understand the representations of $\mathfrak{g}$.

Definition: Let $\lambda \in \mathfrak{h}^*$ be a weight

- λ is called integral if $\lambda(\alpha) \in \mathbb{Z} \quad \forall\, \alpha \in \Phi$
- λ is called dominant if $\lambda(\gamma) \geq 0 \quad \forall\, \gamma \in \Phi_+$

For $\alpha \in \Delta$, let $\mathfrak{sl}_\alpha := \{x_\alpha,\ h_\alpha,\ y_\alpha\}$ be the the subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$. Here, $x_\alpha \in \mathfrak{g}_\alpha,\ h_\alpha \in \mathfrak{h},\ y_\alpha \in \mathfrak{g}_{-\alpha}$. Since, V is a $\mathfrak{g}$-module, restricting ρ to the $\mathfrak{sl}_\alpha$ submodule of $\mathfrak{g}$, V can be viewed as a finite-dimensional $\mathfrak{sl}_\alpha$-module. Since $\mathfrak{sl}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$, we will use representation theory of $sl_2(\mathbb{C})$ to prove the following theorem :

Theorem 3.5.1. *V is a finite-dimensional $\mathfrak{g}$-module. For $w \in W$,*

(a) If $\mu \in$ *support of* V implies $w \cdot \mu \in$ *support of* V with the same multiplicity.

(b) If V is simple then the highest weight of V is dominant integral.

Proof: Let $\alpha \in \Delta$, if $\mu \in$ *support of* V then, $\mu(\alpha) = n \in \mathbb{Z}$. This is because all weights of h_α in an $\mathfrak{sl}_2(\mathbb{C})$-module are integers and occur along with their negatives. This also means that V contains a simple $\mathfrak{sl}_2(\mathbb{C})$-module of the form $V^{(k)}$ where $k \geq |n|$.

- Case (1) $n \geq 0$: Then, x_α^n maps V_μ injectively into $V_{\mu - n\alpha}$
- Case (2) $n \leq 0$: Then, $x_{-\alpha}^{-n}$ maps V_μ injectively into $V_{\mu - n\alpha}$

In any case, multiplicity of $s_\alpha(\mu) = \mu - n\alpha$ is at least n. Since, $\mathfrak{s}_\alpha \cdot \mathfrak{s}_\alpha(\mu) = \mu$, This means that $s_\alpha(\mu)$ is a weight with the same multiplicity as μ. We know that $\{s_\alpha \mid \alpha \in \Delta\}$ generates the Weyl group W. This proves the first part of the theorem.

If V is simple, we know that it will be a highest weight module. Let λ be the highest weight of V. Let μ be any other weight in the *support of* V then, $\mu \leq \lambda$. Again using the fact that all weights of h_α are integers, we can say that $\lambda(h_\alpha) = m \in \mathbb{Z}$. From the first part of the theorem we can say that $\mathfrak{s}_\alpha(\lambda)$ is also a weight in *support of* V. Note if $\lambda(h_\alpha) = m$ then, $\mathfrak{s}_\alpha(\lambda)(h_\alpha) = -m$. But $\mathfrak{s}_\alpha(\lambda) \leq \lambda$. This means that $(\lambda - \mathfrak{s}_\alpha(\lambda))(h_\alpha) \geq 0$. So, $\lambda(h_\alpha) \in \mathbb{Z}^{\geq 0}$ and since this analysis is true for all simple roots, we have showed that λ is dominant integral. ■

We now know that a simple finite-dimensional $\mathfrak{g}$-module V is isomorphic to $L(\lambda)$ for a dominant integral weight λ. It is natural to question: for any dominant integral weight μ, is $L(\mu)$ finite-dimensional? The answer is yes and the result is known as the theorem of highest weight. This theorem gives a characterization of simple-finite dimensional $\mathfrak{g}$-modules in terms of dominant integral weights.

Theorem 3.5.2. $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$. Then, the following are equivalent

(a) $L(\lambda)$ is finite-dimensional

(b) $\lambda \in \mathfrak{h}^*$ is a dominant integral weight

(c) if μ is a weight of $L(\lambda)$ then for any $w \in W$, $w(\mu)$ is also a weight with the same multiplicity as μ

In the analysis so far, we have proved that $(a) \implies (b)$ and $(a) \implies (c)$. The remaining part, is not straight-forward. There are multiple ways to prove this theorem, but we will use the theory of Verma modules that we have developed. The property (c) is motivated from theorem 3.5.1. We can find the example in modules of $sl_2(\mathbb{C})$ in 3.1.

Proof: Let Π be the set of weights of $L(\lambda)$. Assuming (c) as the hypothesis, we can say that Π is closed under the action of the Weyl group W.

Say $\mu \in \Pi$, then we know that $\mu \leq \lambda$. We know that in there is a dominant integral weight in the Weyl orbit of every weight. So assume the $w(\mu)$ is a dominant integral weight of $L(\lambda)$ for some $w \in W$. This means that $\lambda + w(\mu)$ is also dominant integral and $\lambda - w(\mu)$ can be written as an $\mathbb{N}$ sum of simple roots. Therefore, $\langle \lambda + w(\mu), \lambda - w(\mu) \rangle \geq 0 \implies \langle \lambda, \lambda \rangle - \langle w(\mu), w(\mu) \rangle \geq 0$ but we know that $\langle w(\mu), w(\mu) \rangle = \langle \mu, \mu \rangle$. We conclude therefore, if $\mu \in \Pi$ then $\langle \mu, \mu \rangle \leq \langle \lambda, \lambda \rangle$.

Observe that the number of dominant integral weights $\alpha \in \mathfrak{h}^*$ such that, $\langle \alpha, \alpha \rangle \leq \langle \lambda, \lambda \rangle$ is finite. Let that set be denoted as k Therefore, $|\Pi|$ is bounded above by $|W| \times |k|$ and since the upper bound is finite, this means that the number of weights of $L(\lambda)$ are finite. Hence, we have proved that $(c) \implies (a)$.

We will now show $(b) \implies (c)$. We have $L(\lambda)$ with λ dominant integral. This means that there is a natural number $n_\alpha = \lambda(\alpha)$ for a simple root α and if v is the canonical generator of $L(\lambda)$ then $x_\alpha^{n_\alpha+1} \cdot v = 0$. This means that for every simple root α, $L(\lambda)$ contains a finite-dimensional sl_α submodule.

Now we fix a simple root α and by T we denote the sum of all finite-dimensional sl_α submodules of $L(\lambda)$. We have shown above that this sum is non-empty. Take a finite dimensional sl_α submodule S, $\mathfrak{g} \cdot S$ is also an sl_α submodule. This can be easily seen from the commutator relations in $\mathfrak{g}$. So, $\mathfrak{g} \cdot S$ is also a part of the sum T. This means that T is a $\mathfrak{g}$-submodule of $L(\lambda)$. But $L(\lambda)$ is a simple $\mathfrak{g}$-module. This means that $L(\lambda) = T$. From the representation theory of $sl_2(\mathbb{C})$, we know that if an $sl_2(\mathbb{C})$ module is a sum of $sl_2(\mathbb{C})$ modules then it is a (possibly infinite) direct sum of irreducible finite-dimensional $sl_2(\mathbb{C})$ modules. So, $L(\lambda)$ is a direct sum of irreducible finite-dimensional sl_α submodules.

Take a weight μ in the support of $L(\lambda)$ and let $w \in L(\lambda)_\mu$. We can write it as $w = \sum\limits_{i \in I} w_i$ where w_i lies in an irreducible sl_α submodule. If we assume $w \neq 0$ then there is at least one $w_i \neq 0$. Now, $h_\alpha \cdot w = \sum\limits_{i \in I} h_\alpha \cdot w_i \implies h_\alpha \cdot w_i = \mu(h_\alpha) w_i$ i.e. this means that each $w_{i \in} L(\lambda)_\mu$. We know that every finite-dimensional $sl_2(\mathbb{C})$ module has integral eigenvalues and every eigenvalue occurs along with its negative. Therefore, as w_i was in a finite dimensional sl_α submodule, let us assume that $m = \mu(h_\alpha) = 2\frac{\langle \mu, \alpha \rangle}{|\alpha|^2} \in \mathbb{N}$. Also, if $\langle \mu, \alpha \rangle > 0$ then $x_{-\alpha}^m w_i \neq 0$ and consequently $x_{-\alpha}^m w \neq 0$. Similarly if $\langle \mu, \alpha \rangle < 0$ then $x_{-\alpha}^{-m} w_i \neq 0$ and consequently $x_{-\alpha}^{-m} w \neq 0$. This implies the weight space $\mu - m\alpha$ is non-empty and its multiplicity is $\geq$ than the multiplicity of μ.

Observe that $\mu - m\alpha = s_\alpha(\mu)$ and $s_\alpha(s_\alpha(\mu)) = \mu$. Repeating the above arguments for all simple roots we get that the support of $L(\lambda)$ is closed under the action of the Weyl group W and the multiplicity of $\mu =$ the multiplicity of $r(\mu)$ for $r \in W$. This completes the proof of the theorem. ■

Chapter 4

Action of the center

We now aim to study in detail the structure of Verma modules. To study $U(\mathfrak{g})$-modules we have so far made use of the triangular decomposition and the PBW theorem. The study of the center of $U(\mathfrak{g})$ can provide further information. For example, for finite-dimensional representations of $\mathfrak{g}$ the Casimir element, a particular element in the center of $U(\mathfrak{g})$ plays a big part in the proof of Weyl's complete reducibility theorem. By studying the center, we develop a powerful new tool, Harish-Chandra's theorem. Using it we gain new insights into the structure of Verma modules. We denote the center of $U(\mathfrak{g})$ by $Z(\mathfrak{g})$.

4.1 Maximal vectors in Verma modules

We would now like to explore the submodule structure of Verma modules. It is natural to ask the question: Is it possible that a Verma module $M(\lambda)$ has a submodule that is also a Verma module, i.e. is there a submodule of the form $M(\mu)$? In this section we will give a sufficient condition for such a submodule to exist.

We know from (3) of proposition 3.2.2 that if $\mu \in Supp(M(\lambda))$, then $\mu \leq \lambda$. So we will look for maximal vectors of weight $\mu < \lambda$.

Theorem 4.1.1 For a $\lambda \in \mathfrak{h}^*$ and a simple root α, if $\langle \lambda, \alpha^\vee \rangle = n \in \mathbb{Z}^{\geq 0}$, then $M(\mu) \hookrightarrow M(\lambda)$ for $\mu = \lambda - (n+1)\alpha$.

Notice that the role of simple root in the theorem. The multiplicity (the dimension of its weight space) of an arbitrary weight $< \lambda$ can be more than 1. But the multiplicity of a weight $\lambda - k\alpha$ where $k \in \mathbb{Z}$ and α simple, is always 1. So we don't have to worry about (for now) there being multiple maximal vector of the same weight.

Proof: Let v a maximal vector of $M(\lambda)$. Our candidate for the maximal vector is a vector in the weight space $M(\lambda)_\mu$, i.e. the vector $y_\alpha^{n+1} \cdot v$ and we have to show that it is a maximal vector.

Case(1) For a simple root $\beta \neq \alpha$, $x_\beta \cdot y_\alpha^{n+1} \cdot v \in M(\lambda)_{\lambda-(n+1)\alpha+\beta}$ but $\lambda-(n+1)\alpha+\beta \not\leq \lambda$ i.e. it is not a weight of $M(\lambda)$ and so the the vector $x_\beta \cdot y_\alpha^{n+1} \cdot v = 0$.

Case(2) For $\beta = \alpha$, using the commutator relation we can easily show that:

$$x_\alpha \cdot y_\alpha^i \cdot v = (i) y_\alpha^{i-1} \cdot (i - h_\alpha) \cdot v$$

and for $i = n$ we get $(n - \lambda(h_\alpha)) \cdot v = (n - n) \cdot v = 0$. So, $x_\alpha \cdot y_\alpha^{n+1} \cdot v = 0$.

Let W be the submodule generated by $y_\alpha^{n+1} \cdot v$. We showed that for any simple root $\beta : x_\beta \cdot y_\alpha^{n+1} \cdot v = 0$. This makes W a highest weight module with highest weight $\lambda - (n + 1)\alpha$. We know that $U(\eta_-)$ acts injectively on a Verma module. Consider the $U(\mathfrak{g})$-module generated by $y_\alpha^k \cdot v$, call it W. Now because $W \subset M(\lambda)$, $U(\eta_-)$ will also act injectively on W. Hence, $W \cong M(\mu)$ where $\mu = \lambda - (n + 1)\alpha$. So we have showed that $M(\mu) \hookrightarrow M(\lambda)$. ■

Dot action of the Weyl group

Let us look at the previous theorem again. For α simple and $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ implies $M(\mu) \subset M(\lambda)$. Here, $\mu = \lambda - (n+1)\alpha = \lambda - n\alpha - \alpha = s_\alpha(\lambda) - \alpha$ where s_α is the reflection with respect to $\alpha \in \mathfrak{h}^*$. We can again rewrite this using ρ the half sum of positive roots. We know that for any simple root γ, $s_\gamma(\rho) = \rho - \gamma$. So using this formula for $\gamma = \alpha$ we can rewrite μ as $\mu = s_\alpha(\lambda) - \alpha = s_\alpha(\lambda) + s_\alpha(\rho) - \rho = s_\alpha(\lambda + \rho) - \rho$.

This motivates us do define a new action of the Weyl group on $\mathfrak{h}^*$. Let us denote the action by $(\cdot)$ we want this action to be so that $s_\alpha \cdot (\lambda) = s_\alpha(\lambda + \rho) - \rho$.

Definition: We define the dot action of W on $\mathfrak{h}^*$ as $w \cdot \lambda = w(\lambda + \rho) - \rho$.

With this definition at our disposal, we will try to rewrite the previous theorem. Note that if $\langle \lambda + \rho, \alpha^\vee \rangle = 0$ for a simple root α, then $s_\alpha \cdot \lambda = \lambda$ and then, $M(s_\alpha \cdot \lambda) = M(\lambda)$.

Theorem 4.1.1. Let $\lambda \in \mathfrak{h}^*$ and α be a simple root. Then, $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \implies M(s_\alpha \cdot \lambda) \subseteq M(\lambda)$.

The following is an important property of the dot-action.

Proposition 4.1.2. Let $w_1, w_2 \in W$ and $\alpha \in \mathfrak{h}^*$. Then, $(w_1 \circ w_2) \cdot \alpha = w_1 \cdot (w_2 \cdot \alpha)$.

Proof: Using the definition of the dot-action, we get:

$$\begin{aligned} w_1 \cdot (w_2 \cdot \alpha) &= w_1 \cdot (w_2(\alpha + \rho) - \rho) \\ &= w_1(w_2(\alpha + \rho) - \rho + \rho) - \rho \\ &= (w_1 \circ w_2)(\alpha + \rho) - \rho = (w_1 \circ w_2) \cdot \alpha \ \blacksquare \end{aligned}$$

4.2 Harish-Chandra's Theorem

Let v be the canonical generator of $M(\lambda)$. Take $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$,

$$h \cdot z \cdot v = z \cdot h \cdot v = z \cdot (\lambda(h)v) = \lambda(h) z \cdot v$$

i.e., $z \cdot v$ lies in $M(\lambda)_\lambda$, but since it is 1-dimensional, we conclude that $z \cdot v = kv$ where k is some complex number. Since any vector $w \in M(\lambda)$ can be written as $x \cdot v$ where $x \in U(\mathfrak{n}_-)$, we see that $z \cdot w = z \cdot x \cdot v = x \cdot z \cdot v = k\ (x \cdot v) = k\ w$. Therefore, we now know that $Z(\mathfrak{g})$ acts by scalar multiplication on $M(\lambda)$. To capture this, we define central characters.

Definition: A *central character* is an algebra homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$.

Proposition 4.2.1. Highest weight modules have central characters.

Proof: We have already seen that $Z(\mathfrak{g})$ acts on Verma mpdules as scalar multiplication. So, it will act as scalar multiplication on its quotients. We can easily see that this action is associative. This action defines a central character.$\blacksquare$

Consider the central characters of highest weight module, we can ask the following two questions:

(a) Is every algebra homomorphism $Z(\mathfrak{g}) \to \mathbb{C}$ a central character for some highest weight module?

(b) When are central characters of two highest weight modules same?

These questions are answered by Harish-Chandra's theorem.

Theorem 4.2.2. Let χ_λ denote the central character of $M(\lambda)$. Then,

(a) $Z(\mathfrak{g})$ is isomorphic to $S(\mathfrak{h})^W$, the subalgebra of W invariant polynomials in $S(\mathfrak{h})$.

(b) $\chi_\lambda = \chi_\mu \iff \mu = w \cdot \lambda$ where $w \in W$.

(c) Every central character is of the form χ_λ.

A proof of Harish-Chandra's theorem can be found in [12]. The idea is to proove the result for the simpler case of integral weights. Then, using the fact that integral weights are dense in $\mathfrak{h}^*$, extend the result to all weights.

4.3 Formal characters

With the theory we developed in the previous chapter 3.5.2, we are able to determine when $L(\lambda)$ is finite-dimensional. We have no further insights, for example, we don't the dimensions of $L(\lambda)$ and its weight spaces. In this section, we try to find formulas that will help us compute these dimensions. We will develop the notion of a formal character. With it we want to capture the multiplicity of a weight.

Definitions: V be a weight module with finite dimensional weight spaces,

- V be a $\mathfrak{g}$-module and $\lambda \in Supp(V)$. *Multiplicity* of λ is defined as the dimension of the weight space V_λ.
- We define a *formal character* of V to be the map $ch(V) : \mathfrak{h}^* \longrightarrow \mathbb{Z}$ that takes $\lambda \longrightarrow dim(V_\lambda)$.

Using Harish-Chandra's theorem we will build the Weyl character formula. It will help us describe the character of finite-dimensional $L(\lambda)$ using only the Weyl group of $\mathfrak{g}$. Once we know the formal character, it will become easy to calculate the dimensions of $L(\lambda)$ and their weight spaces.

To study formal characters, we will now study the maps from $\mathfrak{h}^* \longrightarrow \mathbb{Z}$.

Definitions:

- By $\mathbb{Z}^{\mathfrak{h}^*}$ we denote the additive group of all functions from $\mathfrak{h}^* \longrightarrow \mathbb{Z}$. The addition is defined point-wise.

- For $f \in \mathbb{Z}^{\mathfrak{h}^*}$ the support of f is defined to be $\{\lambda \in \mathfrak{h}^* \mid f(\lambda) \neq 0\}$.

- By e^λ we mean an element of $\mathbb{Z}^{\mathfrak{h}^*}$ such that its value on λ is 1 and 0 elsewhere.

- By $\mathbb{Z}[\mathfrak{h}^*]$ we denote the the subgroup of $\mathbb{Z}^{\mathfrak{h}^*}$ that consists of functions with finite support. So, any $f \in \mathbb{Z}[\mathfrak{h}^*]$ can be written as $f = \sum\limits_{\lambda \in \mathfrak{h}^*} f(\lambda)e^\lambda$. We will extend this notation to $f \in \mathbb{Z}^{\mathfrak{h}^*}$ for our convenience.

- By $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ we denote the subgroup of $\mathbb{Z}^{\mathfrak{h}^*}$ consisting of functions with support that is contained in a finite union of sets of the form $v - Q_+$ for some $v \in \mathfrak{h}^*$. It is easy to see that $\mathbb{Z}[\mathfrak{h}^*] \subset \mathbb{Z}\langle\mathfrak{h}^*\rangle \subset \mathbb{Z}^{\mathfrak{h}^*}$.

Example : $ch(M(\lambda)) \in \mathbb{Z}\langle\mathfrak{h}^*\rangle$ and $ch(M(\lambda)) = \sum\limits_{\mu \in \mathfrak{h}^*} \beta(\lambda - \mu)e^\mu$.

Ring structure on formal characters

To $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ we can give a ring structure. This will allow us to define multiplication of formal characters. We define the multiplication as convolution product :

$$(\sum_{\lambda \in \mathfrak{h}^*} c_\lambda e^\lambda)(\sum_{\mu \in \mathfrak{h}^*} c'_\mu e^\mu) = \sum_{\nu \in \mathfrak{h}^*} (\sum_{\lambda+\mu=\nu} c_\lambda c'_\mu)e^\nu$$

For this multiplication to be well-defined, the sum $\sum\limits_{\lambda+\mu=\nu} c_\lambda c'_\mu$ should be finite. This is possible because we are working with functions in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$. Let $\lambda = \lambda_0 - q_1$ and $\mu = \mu_0 - q_2$ where $q_1, q_2 \in Q_+$ and we know that there are only finitely many options for λ_0 and μ_0 for us to choose from. So, $\nu = \lambda_0 + \mu_0 - (q_1 + q_2) \implies q_1 + q_2 = \nu - \lambda_0 - \mu_0$. We know that $\beta(q_1 + q_2) < \infty$. As a result we conclude that there are only finitely many ways to express ν as $\lambda + \mu$ and hence the sum in finite.

Defining multiplication this way, makes $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ a commutative ring with identity e^0. We also have multiplication in $\mathbb{Z}[\mathfrak{h}^*]$ defined by the rule : $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$. This coincides with the multiplication in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$. So $\mathbb{Z}[\mathfrak{h}^*]$ is a subring of $\mathbb{Z}\langle\mathfrak{h}^*\rangle$.

Some properties of the formal character

(a) V be a $\mathfrak{g}$-module that has a formal character. V' be a $\mathfrak{g}$-submodule of V and let Φ be the canonical projection. Then, $ch(V) = ch(V') + ch(V/V')$. The result can be easily by looking at individual weight spaces. Extending the basis for the μ weight space for V' we get : $dim(V_\mu) = dim(V'_\mu) + dim(V/V')_\mu$. Thus, the formal character of V is determined completely by the formal characters of its composition series.

(b) We can generalize the first property as follows : Let

$$0 \longrightarrow V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_{n-1}} V_n$$

be an exact sequence of -modules with formal character, then

$$\sum_{j=1}^{n} (-1)^j \quad ch(V_j) = 0$$

To prove this result, we observe the following : For $2 \leq j \leq n-1$ we have for the j^{th} $\mathfrak{g}$-module V_j : $-ch(image(\phi_{j-1})) + ch(V_j) - ch(image(\phi_j)) = 0$ and summing up such expressions for all j with the term $(-1)^j$ multiplied, we get our result.

(c) Let V_1 and V_2 be two $\mathfrak{g}$-modules with character in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$. Then the character of $V_1 \otimes V_2$ is $ch(V_1) \cdot ch(V_2)$. We know that $V_1 \otimes V_2$ is a -module with the action : $x \cdot v \otimes w := (x \cdot v) \otimes w - v \otimes (x \cdot w)$ We notice that the tensor-product of weight vectors of V_1 and V_2 is a weight vector of $V_1 \otimes V_2$ and such tensor-product vectors form a basis of $V_1 \otimes V_2$. The desired result is an immediate consequence of this.

(d) We can define an action of the Weyl group on $\mathbb{Z}^{\mathfrak{h}^*}$ as : $w(e^\lambda) = e^{w(\lambda)}$ for all $\lambda \in \mathfrak{h}^*$. In the Theorem of highest weight 3.5.2 we showed that if λ is dominant integral then, $dim\ L(\lambda)_\mu = dim\ L(\lambda)_{w\mu}\ \forall w \in W$, we can restate it now in terms of formal characters as $w(ch\ L(\lambda)) = ch\ L(\lambda)\ \forall w \in W$ and λ dominant integral. A consequence of defining the action this way, is that each $w \in W$ leaves $\mathbb{Z}[\mathfrak{h}^*]$ invariant but does not leave $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ invariant.

Weyl-character formula

Due to highest weight theorem and Weyl's complete reducibility theorem, we can say that: to know the formal characters of finite-dimensional modules, it is enough to determine formal characters of $L(\lambda)$ where λ is dominant integral. To do so, we are going to make use of Harish-Chandra's theorem.

Definitions:

(a) *The Weyl denominator* : $d := e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$ where ρ is the half sum of positive roots.

(b) $\varepsilon(w) := det(w)$ for $w \in W$ the Weyl group of $\mathfrak{g}$.

(c) $K := \sum_{\gamma \in Q^+} \beta(\gamma)\, e^{-\gamma}$.

Theorem 4.3.1. Let V be simple finite-dimensional $\mathfrak{g}$-module i.e. $V \cong L(\lambda)$, for λ dominant integral. Then the character of V is given by the formula:

$$ch(V) = d^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}$$

Remarks:

- The theorem describes the character in terms of the Weyl group of $\mathfrak{g}$.

- From property (c) of formal characters, we know that the characters of $L(\lambda)$ are W-invariant for λ dominant integral and this will reflect in the Weyl-character formula. This property is not true in general. The study of formal characters of $L(\lambda)$ in general proves to be difficult.

Corollaries of Weyl-character formula

Because of 4.3.1 we now have a way to describe character of finite-dimensional $L(\lambda)$. Using that, we now calculate dimension of weight spaces of $L(\lambda)$. This result is known as *Kostant's multiplicity formula*. The proof can be found in [11] and [12].

Theorem 4.3.2. $\lambda \in \mathfrak{h}^*$ be dominant integral. Then the multiplicity of $\mu \in \mathfrak{h}^*$ as a weight in $L(\lambda)$ is given by the formula:

$$dim(L(\lambda)_\mu) = \sum_{w \in W} \varepsilon(w)\beta(\mu - w \cdot \lambda)$$

Once we know all the multiplicities, we can calculate the dimension of $L(\lambda)$. This formula will tell us what are all the possible sizes of irreducible finite-dimensional $\mathfrak{g}$-modules. Using it we can test wether a finite-dimensional module is irreducible or not. The result is called *Weyl's Dimension Formula*. The proof of this result can be found in [11] and [12].

Theorem 4.3.3. The dimension of $L(\lambda)$ for $\lambda \in \mathfrak{h}^*$ dominant integral is given as:

$$dim(L(\lambda)) = \frac{\prod_{\alpha>0} \langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha>0} \langle \rho, \alpha^\vee \rangle}$$

Chapter 5

Submodules of Verma modules

5.1 Composition Series

The complete reducibility theorem due to Weyl says that finite-dimensional modules are decomposable. But, highest weight module need not be. For example, Verma modules are indecomposable. In such a situation, the next best structure we could try to search for is a notion of length of modules. We again rely on Harish-Chandra's theorem to prove such a finite-length type result.

Definition: A finite *filtration* for a module M is a finite chain of submodules $0 \subset M_1 \subset \cdots M_n = M$ such that each successive quotient M_i/M_{i-1} has a known structure.

Definition: A finite *composition series* for a module M, is a finite filtration $0 \subset M_1 \subset \cdots \subset M_n = M$ such that each successive quotient is a simple module. The simple quotients are called *composition factors*. It is also called *Jordan–Hölder* series or a *Jordan flag*.

The Jordan–Hölder theorem states that if a module admits a composition series, then any two composition series have the same length and same composition factors upto permutation and isomorphism. It is well known that a module is both Noetherian and Artinian if and only if it has a finite composition series. We denote by $[M : N]$ the multiplicity of the composition factor N of M.

Theorem 5.1.1. For any $\lambda \in \mathfrak{h}^*$,

(a) The simple subquotients of $M(\lambda)$ are isomorphic to $L(\mu)$ for $\mu \leq \lambda, \mu \in W \cdot \lambda$.

(b) $M(\lambda)$ has a finite Jordan flag ie., composition series.

Proof: Take two submodules of $M(\lambda)$ say N_1 and N_2 such that $N_2 \subset N_1$ and the quotient N_1/N_2 is simple. We know that weights of N_1/N_2 are weights of $M(\lambda)$. This means N_1/N_2 must have a highest weight. Say the highest weight is μ. Take any vector in $v \in (N_1/N_2)_\mu$ weight space. v is a maximal vector. Since N_1/N_2 is simple, it must be generated by v. Hence, N_1/N_2 is a simple highest weight module. Therefore, N_1/N_2 is isomorphic to $L(\mu)$ where $\mu \leq \lambda$. Since, the central character of N_1/N_2 is the same as that of $M(\lambda)$, by 4.2.2, we can say that $\mu \in W \cdot \lambda$. This proves (a).

From 2.4.2 we know that $U(\mathfrak{g})$ is a Noetherian ring and since $M(\lambda)$ is singly generated by $U(\mathfrak{g})$ it is a Noetherian module. Therefore, every submodule N of a Verma moudule has a submodule N' such N/N' is simple. Take a Verma module $M(\lambda)$ and let $M(\lambda) = N_0 \supset N_1 \supset N_2 \supset \cdots$ be a chain of submodules such that each successive quotient N_i/N_{i+1} is simple. Then by part (a), each such quotient is isomorphic to $L(\mu)$ where $\mu \leq \lambda$ and $\mu \in W \cdot \lambda$. If there are infinitely many such quotients then one of the weights of $M(\lambda)$ will become infinite dimensional. We know this is not possible because weight spaces of Verma modules are finite-dimensional (see 3.2.2). Hence, a Verma module must admit a Jordan flag. ■

Corollary 5.1.1.1. Verma modules are Artinian.

5.2 Socle of $M(\lambda)$

Definition: *Socle* of a module M over a ring R is the sum of the minimal non-zero submodules of M.

When a module is semi-simple, it is equal to its socle. In the case of indecomposable modules, the socle can be thought of as the maximal semi-simple part. The following result determines the socle of $M(\lambda)$.

Theorem 5.2.1. Consider the Verma module $M(\lambda)$. Then,

(a) There exists a unique smallest non-zero submodule V in $M(\lambda)$.

(b) V is a simple Verma module.

Proof: $U(\mathfrak{n}_-) \cong M(\lambda)$ as $U(\mathfrak{n}_-)$ modules. Therefore, submodules of $M(\lambda)$ correspond to left ideals in $U(\mathfrak{n}_-)$. Two simple submodules of $M(\lambda)$. Take the corresponding left ideals in $U(\mathfrak{n}_-)$. Two non-trivial left ideals in $U(\mathfrak{n}_-)$ have a non-trivial intersection (see chapter 3 of [9]). Their intersection is a left ideal. But this is a contradiction because simple submodules must have non-trivial intersection.

Because $M(\lambda)$ has a composition series where each subquotient is a highest weight module, we get that $V \cong L(\mu)$ for some μ. We know that $U(\mathfrak{n}_-)$ acts injectively on $M(\lambda)$ hence also on $L(\mu)$. Therefore, the projection map $Q : M(\mu) \to L(\mu)$ is isomorphism. ■

Since there is only one simple submodule of $M(\lambda)$, by definition it is the unique socle. Recall that the action of $U(\mathfrak{n}_-)$ on $M(\lambda)$ is free. As a result the action of $U(\mathfrak{n}_-)$ on the socle is also free. Hence, the socle of a Verma module is a simple Verma module. The above result puts a significant restriction on what homomorphisms can exist between Verma modules. We get the following result as a consequence.

Theorem 5.2.2. Let $\lambda, \mu \in \mathfrak{h}^*$. Then,

(a) The vector space $Hom_{\mathfrak{g}}(M(\mu), M(\lambda))$ is null or one-dimensional.

(b) Every non-zero element of $Hom_{\mathfrak{g}}(M(\mu), M(\lambda))$ is injective.

Proof: Let v_μ & v_λ be maximal vectors in $M(\mu)$ and $M(\lambda)$ respectively.

Injectivity: Let $f \neq 0 \in \mathrm{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda))$. Then $f(v_\mu) \neq 0$ because otherwise f would be trivial as v_μ generates $M(\mu)$. Let $f(v_\mu) = u \cdot v_\lambda$ for some $u \in U(\mathfrak{n}_-)$. Thus, we can think of f as a map from $U(\mathfrak{n}_-) \to U(\mathfrak{n}_-)$ that maps $u' \mapsto u' \cdot u$. Because $U(\mathfrak{n}_-)$ is an integral domain f has to be injective. This proves part (a).

Let $M(\alpha)$ be the simple submodule of $M(\mu)$. Take $f, g \in \mathrm{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda))$ both non-zero. So by the above part we can say that they are injective. Let $f(M(\alpha)) = L_1$ and $g(M(\alpha)) = L_2$. Since f and g are injective, both L_1 & L_2 are isomorphic to $M(\alpha)$. Since $M(\lambda)$ has a unique simple submodule, $L_1 = L_2$. So $f|_{M(\alpha)}$ and $g|_{M(\alpha)}$ are endomorphism of $M(\alpha)$. Then they must be scalars. So there exists a complex number z such that $f|_{M(\alpha)} = z \cdot g|_{M(\alpha)}$. This means $f - z \cdot g \in \mathrm{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda))$ has a non trivial kernel. So by part (a) $f - z \cdot g = 0$. Therefore, f and g are linearly dependent. Hence, $\dim(\mathrm{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda))) \leq 1$. ■

5.3 Existence of Embeddings

From the previous section we know that if a homomorphism exists between Verma modules then it has to be an embedding. We would like to now enquire when such embeddings exist? Let us recall theorem 4.1.1. It says that if $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$, then $M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$ for a simple root α. When such an embeddings exist, we know that $L(s_\alpha \cdot \lambda)$ will be a composition factor of $M(\lambda)$. So we will now try to generalize theorem 4.1.1 to obtain a sufficient condition for exsistence of embeddings of Verma modules. The below theorem is such a result for the special case of dominant integral weight. The result is a direct consequnce of theorem 4.1.1, although it requires some carefull book-keeping.

Theorem 5.3.1. If $\lambda + \rho \in \Lambda^+$ i.e., dominant intergal, then $M(w \cdot \lambda) \subset M(\lambda) \forall w \in W$.

Proof: Let λ be a dominant integral weight. First we start with a reduced form expression of the weyl group element w. It can be written as $w = s_{\alpha_n} \circ \cdots \circ s_{\alpha_1}$, where each α_i is a simple root (refer [9]). Now set $\lambda_0 := \lambda$ and define $\lambda_i := s_{\alpha_i} \cdot \lambda_{i-1}$. We claim that $\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_n$ and $\lambda_i(h_{\alpha_{i+1}}) \in \mathbb{Z}_{\geq 0}$. If this is true, then using 4.1.1, we can say that $M(\lambda_0) \supset M(\lambda_1) \supset ... \supset M(\lambda_n)$. It is easy to see that the dot-action has the property $(w_1 \circ w_2) \cdot \mu = w_1 \cdot (w_2 \cdot \mu)$ for any weight $\mu \in \mathfrak{h}^*$ and $w_1, w_2 \in W$. Therefore, if our claim is true then $M(w \cdot \lambda) \subset M(\lambda)$ is also true.

Now we will prove our claim. We know that:

$$\begin{aligned} \lambda_i = s_{\alpha_i} \cdot \lambda_{i-1} &= s_{\alpha_i}(\lambda_{i-1} + \rho) - \rho = s_{\alpha_i}(\lambda_{i-1}) + s_{\alpha_i}(\rho) - \rho \\ &= \lambda_{i-1} - (\lambda_{i-1}(h_{\alpha_i}))\alpha_i + \rho - (\rho(h_{\alpha_i}))\alpha_i - \rho \\ &= \lambda_{i-1} - (\lambda_{i-1} + \rho)(h_{\alpha_i})\alpha_i \end{aligned}$$

Now if $(\lambda_{i-1} + \rho)(h_{\alpha_i}) \in \mathbb{Z}_{\geq 0}$ then, $\lambda_i \leq \lambda_{i-1}$ and $M(\lambda_i) \subset M(\lambda_{i-1})$. Let $t \in W$ be $t = s_{\alpha_{i-1}} \circ \cdots \circ s_{\alpha_1}$ an element of the Weyl group. It is easy to see that its inverse is $t^{-1} = s_{\alpha_1} \circ \cdots \circ s_{\alpha_{i-1}}$. Now,

$$\begin{aligned} (\lambda_{i-1} + \rho)(h_{\alpha_i}) = \langle \lambda_{i-1} + \rho, \alpha_i^\vee \rangle &= \langle (t \cdot \lambda_0) + \rho, \alpha_i^\vee \rangle \\ &= \langle t(\lambda_0 + \rho) - \rho + \rho, \alpha_i^\vee \rangle = \langle t(\lambda_0 + \rho), \alpha_i^\vee \rangle \\ &= \langle \lambda_0, t^{-1}(\alpha_i)^\vee \rangle \end{aligned}$$

We know that if $\alpha > 0$ and $w \in W$ such that $l(w \circ s_\alpha) > l(w)$ then, $w(\alpha) > 0$ (see chapter 0.3 from [12]). This means that $t^{-1}(\alpha_i)$ is a positive root. Because λ_0 is dominant integral, we can conclude that $(\lambda_{i-1}+\rho)(h_{\alpha_i}) \in \mathbb{Z}_{\geq 0}$. This completes the proof. ■

We get the following corollary by applying theorem 5.2.1 to the above result.

Corollary 5.3.1.1. Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight. Then the unique smallest submodule of $M(\lambda)$ is $M(w_0 \cdot \lambda)$ where w_0 is the longest element of the Weyl group.

When we try to generalize the above theorem for arbitrary $\lambda \in \mathfrak{h}^*$, we run into a problem that $s_\alpha \cdot \lambda$ is not always $< \lambda$. So, instead we choose a positive root α such that $s_\alpha \cdot \lambda \leq \lambda$. We now try to see if we can get an embedding theorem in this hypothesis. But first we state a lemma that will be useful to prove that result.

Proposition 5.3.2. Let $\lambda, \mu \in \mathfrak{h}^*$ and $\langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}$ for some simple root α. Assume that $M(s_\alpha \cdot \mu) \subset M(\mu) \subset M(\lambda)$. Then, $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$.

Proof: From the hypothesis we can say that $M(s_\alpha \cdot \lambda) \subset M(\lambda)$. Let v_λ and v_μ be the generators of $M(\lambda)$ and $M(\mu)$ respectively. Let $n = \langle \lambda+\rho, \alpha^\vee \rangle$.

Now, if $n \leq 0$ then, $-n = \langle \lambda+\rho, -\alpha^\vee \rangle = \langle \lambda+\rho, s_\alpha(\alpha)^\vee \rangle = \langle s_\alpha(\lambda+\rho), \alpha^\vee \rangle$. Therefore, $-n = \langle s_\alpha \cdot \lambda + \rho, \alpha^\vee \rangle$. And using 4.1.1 this means $M(\lambda) \subset M(s_\alpha \cdot \lambda)$. So the claim of the proposition is true.

Now assume that $n > 0$. Then we know from 4.1.1 that $M(s_\alpha \cdot \lambda) \subset M(\lambda)$. We denote by v_λ a maximal vector in $M(\lambda)$. The, $y_\alpha^n \cdot v_\lambda$ will be a maximal vector of $M(s_\alpha \cdot \lambda)$. Because of the hypothesis $M(\mu) \subset M(\lambda)$, we can say that $\exists u \in U(\mathfrak{n}_-)$ be such that $u \cdot v_\lambda = v_\mu$ where v_μ is a maximal vector of $M(\mu)$.

Notice that $\mathfrak{n}_-$ is a nilpotent Lie algebra. So, using the commutator relation, we can say that for $u \in U(\mathfrak{n}_-)$ and some positive integer n, there exists another positive integer t such that $y_\alpha^t \cdot u \in U(\mathfrak{n}_-)y_\alpha^n$ (the integer t depends on u and y_α). This gives us $y_\alpha^t \cdot u \cdot v_\lambda = U(\mathfrak{n}_-)\, y_\alpha^n \cdot v_\lambda \in M(s_\alpha \cdot \lambda)$.

Let $s = \langle \mu+\rho, \alpha^\vee \rangle$. So without loss of generality we can assume that $t > s$. Now, $y_\alpha^t \cdot u \cdot v_\lambda = y_\alpha^t \cdot v_\mu \in M(s_\alpha \cdot \lambda)$. So, $x_\alpha \cdot y_\alpha^t \cdot v_\mu = [x_\alpha, y_\alpha^t] \cdot v_\mu = c\, y_\alpha^{t-1} \cdot v_\mu$ for some scalar c. But we can keep on substracting from t such that we get s and $y_\alpha^s \cdot v_\mu \in M(s_\alpha \cdot \lambda)$. We know that $y_\alpha^s \cdot v_\mu$ generates $M(s_\alpha \cdot \mu)$ $\therefore$ $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$. ■

Theorem 5.3.3. (Verma) Let $\lambda \in \mathfrak{h}^*$. Given $\alpha > 0$ suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$ then $M(\mu) \subset M(\lambda)$.

Proof: We give a detailed version of the proof given in [8] and [12]. The proof is given in two steps. First we will try to prove this for the case when λ is an integral weight and then we extend it to arbitrary weights.

Integral Case: Assume $\lambda \in \Lambda$ i.e., an integral weight and $\mu = s_\alpha \cdot \lambda$ for some positive root α such that $\mu < \lambda$. Therefore, μ is also an integral weight. We know that for any weight γ there exists one and only one dominant weight in its Weyl group orbit (see 13.2 from [9]). So for integral weights, there exists a unique dominant integral element in their Weyl group orbit. Assume that $\lambda' + \rho$ and $\mu' + \rho$ are the unique dominant integral weights in the Weyl group orbits of $\lambda + \rho$ and $\mu + \rho$ respectively. Now,

$$\begin{aligned} w(\mu + \rho) &= \mu' + \rho \\ \implies \mu + \rho &= w^{-1}(\mu' + \rho) \\ \implies \mu &= w^{-1}(\mu' + \rho) - \rho = w^{-1} \cdot \mu' \end{aligned}$$

Let $w^{-1} = s_{\alpha_n} \circ \cdots \circ s_{\alpha_1}$ be the reduced form expression of w. Now define $\mu_t := s_{\alpha_t} \cdot \mu_{t-1}$ and $\lambda_t := s_{\alpha_t} \cdot \lambda_{t-1}$ where $\mu_0 = \mu'$ and $\lambda_0 = \lambda'$, and $\mu_n = \mu$ and $\lambda_n = \lambda$. We now do the following calculation:

$$\begin{aligned} \mu_n = s_\alpha \cdot \lambda_n \Rightarrow s_{\alpha_n} \cdot \mu_{n-1} &= s_\alpha \cdot s_{\alpha_n} \cdot \lambda_{n-1} \\ \Rightarrow \mu_{n-1} &= s_{\alpha_n}^{-1} \cdot s_\alpha \cdot s_{\alpha_n} \cdot \lambda_{n-1} \\ &= (s_{\alpha_n}^{-1} \circ s_\alpha \circ s_{\alpha_n}) \cdot \lambda_{n-1} \end{aligned}$$

Similarly, we can write

$$\mu_k = (w_k^{-1} \circ s_\alpha \circ w_k) \cdot \lambda_k \text{ where } w_k = s_{\alpha_n} \circ \cdots \circ s_{\alpha_{k+1}}$$

But, $w_k^{-1} \circ s_\alpha \circ w_k$ is the reflection $s_{w_k^{-1}(\alpha)}$. So, let $s_{\beta_k} := w_k^{-1} \circ s_\alpha \circ w_k$ where $\beta_k = w_k^{-1}(\alpha)$. Hence, we can say that $\mu_k = s_{\beta_k} \cdot \lambda_k$ for some root β_k. Particularly, $\mu_k - \lambda_k$ is an integral multiple of the root β_k. This means that μ_k and λ_k are in the same dot-action orbit of the Weyl group. Now orbserve that $\mu' > \lambda'$ because μ' is dominant integral and $\mu = \mu_n < \lambda_n = \lambda$ from the hypothesis. This means there exists an integer k such that $\mu_k > \lambda_k$ and $\mu_{k+1} < \lambda_{k+1}$. This means that:

$$\begin{aligned}\mu_{k+1} - \lambda_{k+1} &= s_{\alpha_{k+1}} \cdot \mu_k - s_{\alpha_{k+1}} \cdot \lambda_k \\ &= s_{\alpha_{k+1}}(\mu_k - \lambda_k)\end{aligned}$$

Notice that $\mu_{k+1} - \lambda_{k+1}$ is a multiple of root β_{k+1} while $\mu_k - \lambda_k$ is a multiple of root β_k. Also, $\mu_{k+1} - \lambda_{k+1} < 0$ while $\mu_k - \lambda_k > 0$. But the only positive root that $s_{\alpha_{k+1}}$ maps to negative is α_{k+1}. So, $\beta_k = \alpha_{k+1} = \beta_{k+1}$. So we now have $\mu_{k+1} < \lambda_{k+1}$ and $\mu_{k+1} = s_{\alpha_{k+1}} \cdot \lambda_{k+1}$. Therefore, by 4.1.1, we have $M(\mu_{k+1}) \subset M(\lambda_{k+1})$. We now use the previous proposition. Since $\mu_{k+2} = s_{\alpha_{k+2}} \cdot \mu_{k+1}$, this implies:

$$M(\mu_{k+2}) \subset M(s_{\alpha_{k+2}} \cdot \lambda_{k+1}) = M(\lambda_{k+2})$$

Using this iteratively we get $M(\mu) \subset M(\lambda)$. This completes the proof for the integral case.

General Case: When proving $\chi_\lambda = \chi_{w\cdot\lambda}$ we first proved it for $\lambda \in \Lambda$ the integral case, then we used tools from algebraic geometry to say Λ is dense in $\mathfrak{h}^*$ and conclude the result for arbitrary λ.

We will follow the same method here. Consider $\lambda \in \Lambda$ and α a positive root. Then, if $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ i.e., $s_\alpha \cdot \lambda \leq \lambda$ then 4.1.1 says that $M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$. We fix a positive root α and $n \in \mathbb{Z}^+$ and define the following:

$$H_{\alpha,n} := \{\nu \in \mathfrak{h}^* \mid \langle \nu + \rho, \alpha^\vee \rangle = n \in \mathbb{Z}^+\}$$

$$X_{\alpha,n} := \{\mu \in \mathfrak{h}^* \mid M(\mu - n\alpha) \subset M(\mu)\}$$

If $\mu \in X_{\alpha,n}$ then that means that μ and $\mu - n\alpha$ are linked. Then it implies that $X_{\alpha,n} \subset H_{\alpha,n}$. H is an affine hyperplane in $\mathfrak{h}^*$. If $\mathfrak{h}^* \cong \mathbb{C}^l$ then $H \cong \mathbb{C}^{l-1}$. It is easy to see that the intersection $H_{\alpha,n} \cap \Lambda$ is isomorphic to $\mathbb{Z}^{l-1}$. Hence we can say that $H_{\alpha,n} \cap \Lambda$ is dense in $H_{\alpha,n}$. If $X_{\alpha,n}$ was closed, then that would mean:

$$\overline{H_{\alpha,n} \cap \Lambda} \subset \overline{X_{\alpha,n}} \Rightarrow H_{\alpha,n} \subset X_{\alpha,n}$$

completing the proof.

We know that $U(\mathfrak{n}_-)$ is a weight module under the adjoint action of $\mathfrak{h}$. So we define $S := U(\mathfrak{n}_-)_{\lambda-\mu}$. If $u \in S$ then it is easy to see that $u \cdot v \in M(\lambda)_{\lambda-\mu}$ for some $v \in M(\lambda)$. This generates a bijection between S and $M(\lambda)_{\lambda-\mu}$ given by $v \mapsto u \cdot v$. Hence, S is finite-dimensional because weight spaces of Verma modules are finite dimensional.

We now look at the following:

$$\begin{aligned} M(\lambda-\mu) \subset M(\lambda) &\iff \text{there exists a maximal vector in} M(\lambda)_{\lambda-\mu} \\ &\iff \exists u \in S,\ u \neq 0 \text{ such that } u \cdot v \text{ is a maximal vector} \\ &\iff \exists u \in S,\ u \neq 0 \text{ such that } [x_\alpha, u] \cdot v = 0, \alpha \text{ simple} \end{aligned}$$

By PBW theorem, we now have $[x_\alpha, u] \in U(\mathfrak{n}_-) \oplus U(\mathfrak{n}_-)h_\alpha$. Let $[x_\alpha, u] = u_\alpha + u'_\alpha h_\alpha$. Then $[x_\alpha, u] \cdot v = (u_\alpha + u'_\alpha \lambda(h_\alpha)) \cdot v$. So, we define the map

$$\begin{aligned} f_\alpha^\lambda : U(\mathfrak{n}_-) &\to U(\mathfrak{n}_-) \\ u &\mapsto (u_\alpha + u'_\alpha \lambda(h_\alpha)) \end{aligned}$$

This is a linear map because ad_{x_α} is linear. Now,

$$[x_\alpha, u] \cdot v = 0 \implies (u_\alpha + u'_\alpha \lambda(h_\alpha)) \cdot v = 0 \Rightarrow f_\alpha^\lambda(u) = 0$$

Let $\{\alpha_1, \ldots, \alpha_l\}$ be simple roots. Then define:

$$\begin{aligned} g^\lambda : U(\mathfrak{n}_-) &\to U(\mathfrak{n}_-)^l \\ u &\mapsto (f_{\alpha_1}^\lambda(u), \ldots, f_{\alpha_l}^\lambda(u)) \end{aligned}$$

This is a linear map since each $f_{\alpha_i}^\lambda$ is linear. Therefore, we can write

$$M(\lambda-\mu) \subset M(\lambda) \iff \exists u \in S \setminus \{0\} \text{ such that } f_{\alpha_i}^\lambda(u) = 0\ \forall i$$

This implies that rank $g^\lambda|_S < \dim S$, which means there are minors of the matrix of g^λ with determinant 0. A determinant is a polynomial. So we get polynomials in terms of λ. Therefore, for all $\lambda \in X_{\alpha,n}$ we have a family of polynomials that vanish on λ. Hence, $X_{\alpha,n}$ is Zariski closed. ■

5.4 The BGG theorem

Verma's theorem proved in the previous section gives us sufficient conditions for when $L(\mu)$ ocuurs as a composition factor of $M(\lambda)$ i.e., when $[M(\lambda) : L(\mu)] \neq 0$. In their paper [2], I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, gave the necessary and sufficient conditions for $[M(\lambda) : L(\mu)] \neq 0$. Before we dicuss their theorem, we look at a useful result.

We get an intresting class of modules when we tensor Verma modules with finite dimensional modules. The following result will be useful in the last section. Based on this property of $M(\lambda) \otimes F$ we define a new type of flag of submodules called the Verma flag or a standard filtration.

Lemma 5.4.1. Let F be a finite-dimensional $\mathfrak{g}$-module and let $(f_1, \ldots, f_k)$ be a basis of F consisting of weight vectors such that, for all i, we have $f_i \in F_{\mu_i}$. We assume that the indexing is such that $\mu_j > \mu_i$ when $j < i$ (it is easy to see that such a numbering exists). Then, the $U(\mathfrak{g})$-module $M(\lambda) \otimes F$ has a flag:

$$M(\lambda) \otimes F = M_k \supset M_{k-1} \supset \cdots \supset M_0 = 0$$

such that, for all $i \in \{1, 2, \ldots, k\}$,

$$M_i/M_{i-1} \cong M(\lambda + \mu_i).$$

Proof: We present the proof given in [8] with added detail. Let v be a maximal vector in $M(\lambda)$ and define $a_i := v \otimes f_i$. So a_i's are weight vectors in $M(\lambda) \otimes F$ with weight $\lambda + \mu_i$. Define

$$M_i := U(\mathfrak{g})a_1 + \cdots + U(\mathfrak{g})a_i.$$

Then $v \otimes F \subset M_k$. It is easy to see for all $i \in \{1, ..., k\}$ that $M_i \subset M(\lambda) \otimes F$.

Now observe that M_k is closed under the action of $U(\mathfrak{g})$. Therefore $M(\lambda) \otimes F \subset M_k$ because v generates $M(\lambda)$. Hence, $M_k = M(\lambda \otimes F)$. Take a positive root α and let $e \in \mathfrak{g}_\alpha$. Observe the following:

$$e \cdot a_i = e \cdot (v \otimes f_i) = e \cdot v \otimes f_i + v \otimes e \cdot f_i = v \otimes (e \cdot f_i) \in v \otimes F_{\mu_i + \alpha}.$$

Since $\mu_{i+k} < \alpha$, using the hypothesis we can say that $F_{\mu_i+\alpha}$ lies in the $\mathbb{C}$ span of $\{f_1, \ldots, f_{i-1}\}$. Therefore we can say that $e \cdot a_i$ lies in the span of $\{a_1, \ldots, a_{i-1}\}$ and hence we can rewrite

$$M_i = U(\mathfrak{n}_-)a_1 + \cdots + U(\mathfrak{n}_-)a_i.$$

So a_i is a maximal vector of weight $\lambda + \mu_i$ in M_i/M_{i-1}. If we now show that each M_i is a free $U(\mathfrak{n}_-)$ module with $\{a_1, \ldots, a_i\}$ as basis, then it will mean that M_i/M_{i-1} is also a free $U(\mathfrak{n}_-)$ module generated by a maximal vector a_i. Hence, by the universal property of Verma modules (3.3.2), we get:

$$M_i/M_{i-1} \cong M(\lambda + \mu_i).$$

which will complete the proof. We will now show that M_i is a free $U(\mathfrak{n}_-)$ module with basis $\{a_i\}$. Take $u_1, \ldots, u_i$ not all zero. Then we will have to show that the sum $u_1a_1 + \cdots + u_ia_i \neq 0$. Let p be the largest filtration among the non-zero u_j's. Then we have:

$$u_1a_1 + \cdots + u_ia_i = \sum_{a=1}^{i} u_a v \otimes f_a + \sum_{b=1}^{i} v \otimes u_b f_b.$$

Now $U(\mathfrak{n}_-)f_b$ lies in the $\mathbb{C}$ span of $\{f_1, \ldots, f_k\}$, so the second sum is of the form

$$\sum_{b=1}^{k} z_b v_b \otimes f_b$$

for some complex numbers z_b. Split the u_b's in the first term in two parts as follows:

$$u_1a_1 + \cdots + u_ia_i = \sum_{a=1}^{i}(w_a \cdot v) \otimes f_a + \sum_{b=1}^{i} \tilde{w}_b \otimes f_b + \sum_{c=1}^{k} z_c(v \otimes f_c).$$

Here, w_a's have filtration p and $\tilde{w}_b$ have filtration less than p. Let $j \in \{1, \ldots, i\}$ be such that $w_j \neq 0$. So the term:

$$(w_j + \tilde{w}_j + z_j) \cdot v \otimes f_j \neq 0.$$

So the sum

$$u_1a_1 + \cdots + u_ia_i \neq 0.$$

Hence proved that M_i is a free $U(\mathfrak{n}_-)$ basis over $\{a_1, \ldots, a_i\}$. This completes the proof. ■ We know that Verma modules admit a finite composition series (5.1.1). This gives us the following corollary:

Corollary 5.4.1.1. $M(\lambda) \otimes F$ admits a finite composition series.

Definition: A $U(\mathfrak{g})$-module M is said to have a Verma flag/standard filtration if there is a finite chain of submodules

$$M = M_0 \supset M_1 \supset \cdots M_n = 0$$

such that each subquotient is isomorphic to a Verma modules $M(\lambda)$. We denote the multiplicity of $M(\lambda)$ in the Verma flag of M by $(M : M(\lambda))$.

Theorem 5.4.2. (BGG Theorem) The following are equivalent:

(a) $M(\lambda) \subset M(\lambda')$.

(b) $[M(\lambda') : L(\lambda)] \neq 0$.

(c) $\exists\, w_1, \ldots, w_n \in W$ the Weyl group of $\mathfrak{g}$ such that

$$\lambda' \geq w_1 \cdot \lambda' \geq \cdots \geq w_n \cdots w_1 \cdot \lambda' = \lambda.$$

Using any submodule of a finite length module we can make a composition series and it is easy to see that composition factors of the submodule are also composition factors of the parent module. This means that $(a) \implies (b)$. Verma's theorem proved in the previous section shows that $(c) \implies (a)$. The proof of $(b) \implies (c)$ is highly technical. It makes use of the previous lemma (5.4.1) and the concept of admissible triplets. It is detailed in [8].

BGG theorem does not discuss what exactly the multiplicities of simple modules $L(\lambda)$ in the composition series of Verma modules are. It only tells us when the multiplicity is non-zero. The problem is of great importance because it allows us to calculate the formal characters of $L(\lambda)$ resursively. Refer to [12] for the problem of calculating exact multiplicities which is not in the scope of this thesis.

Chapter 6

The BGG Category $\mathcal{O}$

We aim to study in detail the structure and relationships between highest weight modules. To do so we would like to work with an appropriate module category. The category $U(\mathfrak{g})$-Mod is too large a category to do this. We would like to work with a smaller category of $U(\mathfrak{g})$ modules where we are to be able to use tools like homological algebra. In their paper [2], Bernstein, I. Gelfand, and S. Gelfand studied a restricted class of modules that encompassed the representation theory of highest weight modules. This was subsequently named as Category $\mathcal{O}$.

Throughout this chapter, $\mathfrak{g}$ is a finite-dimensional complex semi-simple Lie algebra.

6.1 Axioms

In the first section we define the axioms of the BGG Category $\mathcal{O}$. We will show that highest weight modules are building blocks of this category and determine all the simples modules in this category.

Definition: The BGG Category $\mathcal{O}$ is the full subcategory of $\mathfrak{g}$-modules consisting of modules which satisfy the following conditions:

(a) M is finitely generated.

(b) M is $\mathfrak{h}$-semisimple.

(c) M is locally $U(\mathfrak{n}_+)$ finite.

Condition (b) says that any module $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where M_λ is the λ weight space of M. The condition (c) means that for any $v \in M$ the subspace $U(\mathfrak{n}_+) \cdot v$ is finite-dimensional.

Remarks:

- Finitely generated condition in the algebraic equivalent of compactness.
- The second axiom ensures the category contains weight modules. This means we have a well-defined notion of formal characters for these modules.
- The third conditions ensures that all modules have highest weight vectors. If we replace $U(\mathfrak{n}_+)$ by $U(\mathfrak{n}_-)$, then we would be working with lowest weight vectors instead.
- Some examples of modules in this category are finite-dimensional $\mathfrak{g}$-modules, highest weight modules and Verma modules. All of which are of importance to us.

We will now look at some immediate consequences of the axioms.

Theorem 6.1.1. Let $M \in \mathcal{O}$. Then:

(a) Category $\mathcal{O}$ is Noetherian.

(b) Any module has a finite generating set consisting of weight vectors.

(c) M is finitely generated as $U(\mathfrak{n}_-)$ module.

(d) The weights of M are contained in a finite union of sets of the form $\lambda - \Gamma$, where Γ is defined as the $\mathbb{Z}^+$-linear span of simple roots.

(e) All weight spaces of M are finite-dimensional.

(f) M is $Z(\mathfrak{g})$-finite.

(g) $\mathcal{O}$ is an abelian category.

(h) For a finite dimensional $U(\mathfrak{g})$-module L, $M \mapsto M \otimes L$ defines an exact endofunctor of $\mathcal{O}$.

Proof: We know that $U(\mathfrak{g})$ is a noetherian ring (see 2.4.2), and that finitely generated modules over $U(\mathfrak{g})$ are noetherian. Due to the second axiom, we can write any generator as a finite sum of weight vectors. This proves (a) and (b).

Using property (b) select a finite generating set of M consisting of weight vectors. Let V be the $U(\mathfrak{n}_+)$ submodule generated by those generators. We know it is finite-dimensional due to the third axiom. From [4] we know that modules generated by weight vectors are $\mathfrak{h}$-semisimple. Hence, V is a $U(\mathfrak{b})$-module. Using the PBW theorem 1.4, it is easy to see that M is generated by a finite basis of V as a $U(\mathfrak{n}_-)$-module (c).

We know that V is finite-dimensional and $\mathfrak{h}$-semisimple. So, it has a finite basis consisting of weight vectors. Let λ_i be the those weights. We know from (c) that M is generated by these weight vectors as a $U(\mathfrak{n}_-)$-module. So, all the weights of M will be of the form $\lambda - \Gamma$. This proves (d).

Using the PBW theorem (1.4), we can show that $U(\mathfrak{n}_-)$ is a weight module under the adjoint action of $\mathfrak{h}$ and each of its weight spaces are finite-dimensional. We also showed that as a $U(\mathfrak{n}_-)$-module, M has a finite generating set consisting of weight vectors. Using these two facts we can see that all weight spaces of M are finite-dimensional (e).

$z \in Z(\mathfrak{g})$ commutes with the action of $U(\mathfrak{h})$. So, it maps a vector from M_λ to itself. We know that weight spaces of M are finite-dimensional. This proves (f).

Since M is noetherian, every submodule of M is finitely generated. It is easy to that any submodule of M is and locally $U(\mathfrak{n}_+)$ finite. From [6], we know that submodules of $\mathfrak{h}$-semisimple modules are $\mathfrak{h}$-semisimple modules. Thus, category $\mathcal{O}$ is closed under taking submodules. It is easy to see that it is closed under taking quotients and finite direct sums. We also know that $U(\mathfrak{g})$-Mod is an abelian category. This proves that $\mathcal{O}$ is an abelian category (g).

Let $\{v_1, ..., v_n\}$ be the generators of M and let $\{w_1, ..., w_m\}$ be basis of L. Then the set $\{v_i \otimes w_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ generates $M \otimes L$. Hence, $M \otimes L$ is finitely generated. It is also $\mathfrak{h}$-semisimple and locally $U(\mathfrak{n}_+)$ finite. This proves (h). ■

Corollary 6.1.1.1. For $M, N \in \mathcal{O}$, $dim(Hom_{\mathcal{O}}(M, N)) < \infty$.

Proof: Start by taking a finite generating set of M consisting of weight vectors. We know that $\mathfrak{g}$-module homomorphisms map weight vectors to weight vectors of same weight. So a generator of weight λ will be mapped to a vector in N_λ. Since weight spaces of modules in $\mathcal{O}$ are finite dimensional. Hence, $dim(Hom_{\mathcal{O}}(M, N)) < \infty$.■

We know that the BGG Category $\mathcal{O}$ is not a semi-simple module category. For example, all Verma modules are indecomposable and most of them are not simple. Hence, we need to search for a finite length type result. The following theorem is such a result. It shows why we can understand highest weight modules as building blocks of Category $\mathcal{O}$.

Theorem 6.1.2. A module $M \in \mathcal{O}$ has a finite filtration where each successive quotient is a highest weight module.

Proof: Start with a finite generating set of M consisting of weight vectors. This is possible due to 6.1.1. Let V be the finite dimensional $U(\mathfrak{b})$-submodule generated by those generators. We will prove the theorem using induction on dimension of V. If it is one dimensional, then M is a highest weight module. Now assume it is true for $dim(V) = k - 1$. If $dim(V) = k$ then select a highest weight vector $v \in V$. N be the submodule generated by v. Consider the image of V in M/N. It's dimension will be less than k. So by induction hypothesis, M/N has a finite filtration. Consequently, M has a finite filtration. This completes the induction. ■

Corollary 6.1.2.1. Every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

Proof: By the provious theorem we know that every module in $\mathcal{O}$ has a filtration where the quotients are highest weight modules. A simple module will have such a filtration of length 1. So it has to be a simple highest weight module $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. ■

The filtration we obtained here could be refined further. From 5.1.1 we can see that Verma modules have a composition series. We now prove that modules in $\mathcal{O}$ also admit composition series.

Theorem 6.1.3. Category $\mathcal{O}$ is Artinian.

Proof: From 5.1.1.1 we know that Verma moduels are artinian. Because highest weight modules are quotients of Verma modules (see 3.3.2), they will also satisfy the descending chain condition. Now, 6.1.2 says that each $M \in \mathcal{O}$ has a finite filtration where each subquotient is a highest weight module. Therefore, each $M \in \mathcal{O}$ satisfies descending chain condition. Hence, Category $\mathcal{O}$ is Artinian ■

We now know that Category $\mathcal{O}$ is both Noetherian as well as Artinian. Recall that a module is of finite length i.e., it admits a composition series if and only if it is both Noetherian and Artinian. A standard result called the Krull-Remak-Schmidt-Azumaya theorem states that if a module has finite length, then it can be decomposed into a finite direct sum of indecomposable modules with the direct summands being unique up to isomorphism and rearrangement. All these results are explored in great detail in [19].

Corollary 6.1.3.1. Modules in $\mathcal{O}$ admit a finite composition series.

Corollary 6.1.3.2. Category $\mathcal{O}$ is Krull-Schmidt.

6.2 Blocks of $\mathcal{O}$

In section 4.2 we saw how $Z(\mathfrak{g})$ acted on highest weight modules. We know that any $M \in \mathcal{O}$ is build up by a finite number of highest weight modules (see 6.1.2), so we can expect that the action of $Z(\mathfrak{g})$ on M will be governed by a finite number of central characters. In this section we will make this relation precise.

Definition: For $z \in Z(\mathfrak{g})$ and central character χ we define:

$$M^\chi := \{v \in M | (z - \chi(z))^n \cdot v = 0 \text{ for some } n > 0 \text{ depending on } v, z\}$$

M^χ is therefore the generalized eigenspace for the action of the commutative algebra $Z(\mathfrak{g})$ with eigenvalue given by the central character χ. Rewrite $z \in Z(\mathfrak{g})$ as $\chi(z) + z - \chi(z)$ then we can say that z acts as scalar plus a nilpotent operator on M^χ.

Because elements of $Z(\mathfrak{g})$ commute with elements of $U(\mathfrak{g})$, we can see that M^χ will be a submodule of M. It is also clear that for different cental characters the subspaces M^χ are linearly independant.

We know that $Z(\mathfrak{g})$ commutes with the action of $U(\mathfrak{h})$. So, it maps a vector from M_λ to itself. That means $Z(\mathfrak{g})$ is a family of commuting endomorphisms of weight spaces of M. Since we are working over complex numbers, we can write $M(\mu)$ as a direct sum of generalized eigenspaces, i.e. $M_\mu = \bigoplus\limits_{\chi} (M_\mu \cap M^\chi)$. Therefore, $M = \bigoplus\limits_{\chi} M^\chi$. From 6.1.1, we know that M has a finite generating set consisting of weight vectors. This will ensure that this direct sum decomposition is finite. Because of 4.2.2 we know that every central character is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$. Hence, $M = \bigoplus\limits_{\chi_\lambda} M^{\chi_\lambda}$

Definition: Denote by $\mathcal{O}_\chi$ the full subcategory of $\mathcal{O}$ consisting of modules M such that $M = M^\chi$.

For example, a highest weight module with highest weight λ lies in $\mathcal{O}_{\chi_\lambda}$. From the above analysis, we can write the following theorem.

Theorem 6.2.1. Category $\mathcal{O}$ is the direct sum of full subcategories $\mathcal{O}_{\chi_\lambda}$.

This direct sum decomposition of Category $\mathcal{O}$ is due to the action of the center $Z(\mathfrak{g})$ and catupres the spectral data of central characters. So the study of Category $\mathcal{O}$ now boils down to study of the subcategories $\mathcal{O}_{\chi_\lambda}$ which are more managable. Let us explore some of the advantages of this decomposition.

Proposition 6.2.2. The following are true for subcategories $\mathcal{O}_\chi$:

(a) Each simple module in $\mathcal{O}$ lies in exactly one subcategory $\mathcal{O}_{\chi_\lambda}$.

(b) The simple modules in $\mathcal{O}_{\chi_\lambda}$ are $\{L(w \cdot \lambda) | \; w \in W\}$. Hence, there are finitely many simples in $\mathcal{O}_{\chi_\lambda}$ up to isomorphism.

(c) $\mathcal{O}_\chi$'s are Serre subcategories i.e., they are closed under taking subobjects, quotients and extensions.

(d) The subcategories $\mathcal{O}_\chi$'s are orthogonal i.e., if $M \in \mathcal{O}_\chi$ and $N \in \mathcal{O}_{\chi\prime}$ such that $\chi \neq \chi\prime$. Then $Hom_{\mathcal{O}}(M, N) = 0$ and $Ext^1_{\mathcal{O}}(M, N) = 0$.

Proof: From 6.1.2.1 we know that $\{L(\lambda)|\lambda \in \mathfrak{h}^*\}$ are the simple modules in Category $\mathcal{O}$ up to isomorphism. Since these are highest weight modules they have a central character (see 4.2.1) i.e., $Z(\mathfrak{g})$ acts on them as scalar multiplication. Hence, each simple module will lie in exactly one subcategory $\mathcal{O}_{\chi_\lambda}$. This proves (a).

The central character of $L(\lambda)$ will be χ_λ. Harish-Chandra's theorem (4.2.2) says that $\chi_\lambda = \chi_\mu \iff \mu = w \cdot \lambda$ where $w \in W$. Now, $\{L(w \cdot \lambda)| \; w \in W\}$ are the only simples with central character χ_λ. This proves (b).

If $M = M^\chi$ then by definition $N = N^\chi$ for any submodule N of M. Hence, $\mathcal{O}_\chi$ is closed under taking subobjects. Since $U(\mathfrak{g})$-module homomorphisms preserve the action of $Z(\mathfrak{g})$ it can be said that $\mathcal{O}_\chi$ is also closed under taking quotients. Assume that

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{O}$ such that $A, C \in \mathcal{O}_\chi$. We can see that the set of composition factors of B is equal to the union set of composition factors of A and C. Hence, $C \in \mathcal{O}_\chi$. Thus, we have proved (b).

Assume $M \in \mathcal{O}_\chi$ and $N \in \mathcal{O}_{\chi\prime}$ such that $\chi \neq \chi\prime$. We know that both modules have finite composition length (see 6.1.3.1) so we will use induction on their lengths. If M, N are simple then by Schur's lemma we know that if $f \in Hom(M, N)$, then f is either 0 or an isomorphism. This means that M and N lie in both $\mathcal{O}_\chi$ and $\mathcal{O}_{\chi\prime}$. This contradicts (a) which says a simple module in $\mathcal{O}$ lies in exactly one $\mathcal{O}_{\chi_\lambda}$. We now consider this to be true for N of length k. Now Assume N has composition length k+1. We can write a short exact sequence

$$0 \to N_1 \to N \to N/N_1 \to 0$$

such that N/N_1 is simple and apply $Hom(M, \_)$ to this. We get:

$$0 \to Hom(M, N_1) \to Hom(M, N) \to Hom(M, N/N_1)$$

Here, $Hom(M, N_1)$ is 0 by induction step since N_1 has length k and $(M, N/N_1)$ is also 0 since both M and N/N_1 are simple. Therefore $Hom(M, N) = 0$. Applying similar induction to the length of M, we can conclude that $Hom(M, N) = 0$ for all $M \in \mathcal{O}_\chi, N \in \mathcal{O}_{\chi\prime}$ and $\chi \neq \chi\prime$.

Take the short exact sequenece $0 \to M \to E \to N \to 0$ where $M \in \mathcal{O}_\chi$ and $N \in \mathcal{O}_{\chi\prime}$ such that $E \in \mathcal{O}$. From the previous theorem, we can give a direct sum decomposition $E = \bigoplus_\psi E^\psi$. So, $M \subset E^\chi$. We also know that $E/M \cong N = N^{\chi\prime}$ this means that $M \cong E^\chi$ and $N \cong E^{\chi\prime}$. Therefore, $E = M \oplus N$ and hence the short exact sequence splits. Since we started off with an arbitrary short exact sequence, we can say that $Ext^1_{\mathcal{O}}(M, N) = 0$ for all $M \in \mathcal{O}_\chi$ and $N \in \mathcal{O}_{\chi\prime}$ where $\chi \neq \chi\prime$. This proves (c). ■

Recall from 6.1.3.2 that $\mathcal{O}$ is a Krull-Schmidt category. This means that every module can be decomposed into a finite direct sum of indecomposable submodules. The idea now is to extend this notion to the level of category. We want a certain collection of subcategories called *blocks*, to separate indecomposable modules that are homologically unrelated i.e., they have no extensions of any degree between them. If we want these blocks to do so as effectively as possible, then these blocks must be indecomposable themselves. Each indecomposable module will belong to a unique block. This notion is especially useful in categories like $\mathcal{O}$ which fail to be semisimple.

Definition: Consider two simple modules M and N. We say $M \sim N$ if there exists a non-trivial extension of them i.e., $Ext^1(M,N) \neq 0$ or $Ext^1(N,M) \neq 0$. Let $\mathcal{R}$ the smallest equivalence relation generated by this binary relation. A *block of* $\mathcal{O}$ is a full subcategory of $\mathcal{O}$ containing modules whose composition factors lie in the same equivalence class generated by $\mathcal{R}$.

We can think of a block as a collection of modules generated by virtices in the connected components of the $Ext^1_{\mathcal{O}}$-graph of simple modules in $\mathcal{O}$.

Theorem 6.2.3. Let $\mathcal{B}_i$ be blocks of Category $\mathcal{O}$ as defined above. Then:

(a) Each simple object of $\mathcal{O}$ lies in a single block.

(b) Each block $\mathcal{B}_i$ of $\mathcal{O}$ is a Serre subcategory. Furthermore, these blocks are orthogonal to each other.

(c) Category $\mathcal{O}$ decomposes as a direct sum of blocks up to permutation. Each block is indecomposable as an abelain category.

Proof: The relation $\mathcal{R}$ defined above makes equivalence classes of simple modules in $\mathcal{O}$. Hence, it lies in only one block. This proves (a).

Assume that $0 \to M \to E \to N \to 0$ is a short exact sequence in $\mathcal{O}$. We know that the set of composition factors of E will be a union of composition factors of M and N. This immediately implies that blocks are closed under taking submodules and quotients and extensions. Hence, blocks are Serre subcategories.

Take two simples in different blocks. From Schur's lemma we know that the homomorphisms spacce between these two simple modules in either 0 or consists of isomorphisms. We know that blocks form an equivalent class of simple modules, so two isomorphic simples cannot lie in two different blocks. By using induction on the length of modules in two blocks just like we did in the proof of 6.2.2, we can show that $Hom(M,N) = 0$ when M and N lie in two different blocks. Similary if A, B are two simples in two different blocks, then $Ext^1_{\mathcal{O}}(A,B) = 0$, otherwise they will be in the same block. We then use induction on the length of both modules A and B to show that $Ext^1_{\mathcal{O}}(A,B) = 0$ when A and B are any modules in two different blocks. This completes the proof of part (b).

Take $M \in \mathcal{O}$. Let M_i be the submodule of M whose composition factors lie in the block $\mathcal{B}_i$. We can say that $M = \sum_i M_i$ since each composition factor of M lies in some block. Now we know that $M_i \cap M_j = 0$ because otherwise we have a submodule whose composition factors lie in two different blocks which is not possible. Hence the sum is a direct sum. Therefore, Category $\mathcal{O}$ is a direct sum of its blocks.

We know that Category $\mathcal{O}$ is abelian (see 6.1.1) and a block is a Serre subcategory of $\mathcal{O}$. Then, a block is also an abelian subcategory. Now assume that a block is decomposable i.e., there exists full subcategories A_1 and A_2 of a block $\mathcal{B}$ such that $\mathcal{B} = A_1 \oplus A_2$. Then, each simple module in $\mathcal{B}$ will lie in either A_1 or A_2. We know that any two simples in a block have a non-trivial extension. So let E be a non-trivial extension of two simple modules, one in A_1 and the other in A_2. Now E decomposes into a direct sum $E_1 \oplus E_2$ where $E_i \in A_i$. But this means that E is a trivial extension which is a contradiction. Therefore, a block is indecomposable. This completes the proof of (c) ■

Our previous direct sum decomposition of Category $\mathcal{O}$ seems like a step in the direction of a block decomposition. We have shown that the subcategories $\mathcal{O}_\chi$'s are Serre subcategories and are orthogonal to each other. This hints that they could potentially act as blocks of $\mathcal{O}$. The following result discusses the case when this is true.

Theorem 6.2.4. Let $\lambda \in \Lambda$ be integral. Then, $\mathcal{O}_{\chi_\lambda}$ is a block of Category $\mathcal{O}$.

Proof: The simple objects in $\mathcal{O}_{\chi_\lambda}$ (up to isomorphism) are $\{L(w \cdot \lambda)|\ w \in W\}$. In 6.2.2(c) we have shown that these are not $Ext^1_{\mathcal{O}}$-connected to other simple modules. If we are able to show that there is a non-trivial extension of $L(\lambda)$ and $L(w \cdot \lambda)$ for any element w of the Weyl group, then we are done.

Take $w = s_\alpha$ for some positive root α. Without loss of generality we can assume $\mu := s_\alpha \cdot \lambda$ such that $\mu < \lambda$. From 3.4.3 we know that a Verma module has a unique maximal submodule. Let $N_\mu, N(\lambda)$ be the maximal submodules of $M(\mu), M(\lambda)$ respectively. From 5.3.3 we know of the existence of the embedding $M(\mu) \hookrightarrow M(\lambda)$. Combining these two facts we have $M(\mu) \hookrightarrow N(\lambda) \hookrightarrow M(\lambda)$. Let K be the image of $N(\mu)$ in $N(\lambda)$. Therefore, the quotient $M(\lambda)/N$ has a submodule isomorphic to $M(\mu)/N(\mu) \cong L(\mu)$. Since $M(\lambda)/N$ is a highest weight module with highest weight λ, it will have $L(\lambda)$ as its composition factor. We know that highest weight modules are not decomposable, therefore $M(\lambda)/N$ is a non-trivial extension of $L(\mu)$ and $L(\lambda)$.

Write a reduced form of $w \in W$. Then, we can iterate this process to show that there is a non-trivial extension of $L(w \cdot \lambda)$ and $L(\lambda)$. This completes the proof. ■

Remark: Not all $\mathcal{O}_{\chi_\lambda}$ are blocks because when λ is not integral, $\mathcal{O}_{\chi_\lambda}$ might fail to be indecomposable. We can see this with an example. Take $\mathfrak{g} = sl_2(\mathbb{C})$. We know from 3.4.1 that when λ is not integral, $M(\lambda)$ is simple. So is $M(-\lambda-2)$. We know that the weight $-\lambda-2$ is linked to λ by the non-trivial element of the Weyl group of $sl_2(\mathbb{C})$. Notice that the support of $M(\lambda)$ is disjoint from the support of $M(-\lambda-2)$ in $\mathfrak{h}^*$. This means that any non-trivial extension of the two simple modules $M(\lambda)$ and $M(-\lambda-2)$ in Category $\mathcal{O}$ will always split. So, $\mathcal{O}_{\chi_\lambda}$ cannot be a block. This happens because the λ and $-\lambda-2$ lie in two different cosets of $\mathfrak{h}^*/\Lambda_r$ (where Λ_r is the root lattice).

In the next section we discuss the further decomposition of $\mathcal{O}_{\chi_\lambda}$ into blocks whenever possible. This will give a description of blocks of Category $\mathcal{O}$. Note that since each $\mathcal{O}_\chi$ has a finite number of simples, this means that each block will also have finite number of simple objects up to isomorphism.

6.3 Dominant and Antidominant weights

As the remark above suggests, to get a better desccription of blocks, we need to look at linked weights that lie in the same cosets of $\mathfrak{h}^*/\Lambda_r$ where Λ_r is the $\mathbb{Z}$-span of roots Φ. The following definitions introduced by Jantzen are a step in this direction.

Definition: For $\lambda \in \mathfrak{h}^*$ we define,

$$W_{[\lambda]} := \{w \in W \mid w(\lambda) \equiv \lambda \pmod{\Lambda_r}\}$$
$$\Phi_{[\lambda]} := \{\alpha \in \Phi \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$$

Hence, $w \in W_{[\lambda]}$ if and only if $w(\lambda) - \lambda \in \Lambda_r$.

Remarks:

(a) It is easy to see that for an integral weight α, $\Phi_{[\alpha]} = \Phi$ and $W_{[\alpha]} = W$.

(b) Using the above fact, simple calculations show that if $\lambda \equiv \mu \pmod{\Lambda}$, then:

$$\Phi_{[\lambda]} = \Phi_{[\mu]} \text{ and } W_{[\lambda]} = W_{[\mu]}.$$

(c) Recall that $\rho \in \Lambda$ i.e., ρ is an integral weight. So we get:

$$\begin{aligned} W_{[\lambda]} &= \{w \in W \mid w \cdot \lambda - \lambda \in \Lambda_r\} \\ \Phi_{[\lambda]} &= \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}. \end{aligned}$$

(d) Let λ and μ be in different $\mathfrak{h}^*/\Lambda_r$ cosets such that $\mu = w \cdot \lambda$ for some $w \in W$. Then,

$$\begin{aligned} \Phi_{[\mu]} &= w(\Phi_{[\lambda]}) \\ W_{[\mu]} &= wW_{[\lambda]}w^{-1} \end{aligned}$$

(e) If $[M(\lambda) : L(\mu)] > 0$, then we know that $\mu = w \cdot \lambda$ for $w \in W$. Because $\mu \leq \lambda$, we get $\lambda \equiv \mu \pmod{\Lambda_r}$. Therefore, $w \in W_{[\lambda]}$.

Proposition 6.3.1. $\Phi_{[\lambda]}$ is a root system in its $\mathbb{R}$-span denoted by $E(\lambda) \subset \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$.

Proof: If $\alpha \in \Phi_{[\lambda]}$ then by definition we have $-\alpha \in \Phi_{[\lambda]}$. If $\alpha, \beta \in \Phi_{[\lambda]}$ then $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ because α and β are from the root system Φ. It only remains to show that $\Phi_{[\lambda]}$ is closed under reflections $\{s_\alpha \mid \alpha \in \Phi_{[\lambda]}\}$. Let $\beta \in \Phi_{[\lambda]}$. We have to show that $s_\alpha(\beta) \in \Phi_{[\lambda]}$ i.e., in other words, $\langle \lambda, s_\alpha(\beta)^\vee \rangle \in \mathbb{Z}$.

$$\begin{aligned} \langle \lambda, s_\alpha(\beta)^\vee \rangle &= \langle \lambda, (\beta - \langle \beta, \alpha^\vee \rangle \alpha)^\vee \rangle \\ &= 2\frac{\langle \lambda, \beta - \langle \beta, \alpha^\vee \rangle \alpha \rangle}{\langle s_\alpha(\beta), s_\alpha(\beta) \rangle} = \underbrace{2\frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle}}_{= \langle \lambda, \beta^\vee \rangle \in \mathbb{Z}} - 2\langle \beta, \alpha^\vee \rangle \frac{\langle \lambda, \alpha \rangle}{\langle \beta, \beta \rangle}. \end{aligned}$$

Now,

$$2\langle \beta, \alpha^\vee \rangle \frac{\langle \lambda, \alpha \rangle}{\langle \beta, \beta \rangle} = 4\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \frac{\langle \lambda, \alpha \rangle}{\langle \beta, \beta \rangle} = 4\frac{\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \langle \alpha, \beta^\vee \rangle \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}.$$

Therefore, $s_\alpha(\beta) \in \Phi_{[\lambda]}$. Hence proved that $\Phi_{[\lambda]}$ is a root system. ■

A natural question arises about the nature of the Weyl group of the root system $\Phi_{[\lambda]}$. It is easy to see that $W_{[\lambda]}$ is a subgroup of the Weyl group W of Φ. Infact, it is the Weyl group of the root system $\Phi_{[\lambda]}$ and it is generated by reflections $\{s_\alpha \mid \alpha \in \Phi_{[\lambda]}\}$. The proof of this fact is subtle and involves knowledge of the Weyl group and reflection groups. An outline of the proof is given in [7] as an exercise. A simplified version of Jantzen's proof for field $\mathbb{C}$ can be found in [12].

We know that if α is a root in a root system Φ then the only multiple of α in Φ is $-\alpha$. So, the set $\Phi_{[\lambda]} \cap \Phi^+$ can be considered to be a set of positive roots in $\Phi_{[\lambda]}$. We denote by $\Delta_{[\lambda]}$ the simple roots of $\Phi_{[\lambda]}$ corresponding to the positive roots $\Phi_{[\lambda]} \cap \Phi^+$.

We know that the socle of a Verma module $M(\lambda)$ is another Verma module $M(\mu)$ where $\mu \leq \lambda$ and $\mu = w \cdot \lambda$ for some $w \in W_{[\lambda]}$. It is easy to see that if λ is the lowest in its $W_{[\lambda]}$ orbit, then $M(\lambda)$ has to be simple. Our intuition is that this sufficient condition for simplicity of Verma modules could also be necessary. Therefore, to capture this idea, we define antidominant weights.

Definition: A weight $\lambda \in \mathfrak{h}^*$ is called an *antidominant* weight if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi^+$. It will be called dominant if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Phi^+$.

Notice: In the case of integral weights, the above definition of dominant weights is slightly different from the notion of dominant integral weights (Λ^+) that we have been using. In the above notion, the set $\Lambda^+ - \rho$ will be the dominant weights in Λ while $-\rho$ will be both dominant as well as antidominant.

The following proposition establishes the idea that an anti-dominant weight λ is the lowest weights in its $W_{[\lambda]}$ orbit.

Proposition 6.3.2. Let $\lambda \in \mathfrak{h}^*$. Let $\Phi_{[\lambda]}$, $W_{[\lambda]}$ and $\Delta_{[\lambda]}$ be as defined above. Then, λ is antidominant if and only if one of the following three equivalent conditions is true:

(a) $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta_{[\lambda]}$.

(b) $\lambda \leq s_\alpha \cdot \lambda$ for all $\alpha \in \Delta_{[\lambda]}$.

(c) $\lambda \leq w \cdot \lambda$ for all $w \in W_{[\lambda]}$.

Proof: By the definition of dot-action, it is easy to see that (a) and (b) are equivalent. In their reduced form, length 1 elements of Weyl group are just reflections with respect to simple roots. So, (c) implies (b). Inducing on length of the reduced form it is easy to show that (b) implies (c). So, the three statements are equivalent.

Recall that $\Phi_{[\lambda]} = \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$. If λ is antidominant, then $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$. Therefore, λ is antidominant if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$ for all for all $\alpha \in \Phi^+$. Hence proved that λ is antidominant if and only if one of the following three equivalent conditions is true. ■

Corollary 6.3.2.1. There exists a unique antidominant weight in the orbit $W_{[\lambda]} \cdot \lambda$.

We can now characterize $W_{[\lambda]}$ orbits by the unique antidominant weight lying in them. Similarly, we can also show that they will have a unique maximal (dominant) element. The next theorem shows that antidominant weights are in bijection with simple Verma modules.

Theorem 6.3.3. The Verma module $M(\lambda)$ is simple if and only if λ is antidominant.

Proof: First assume λ is anti-dominant. If $M(\lambda)$ is not simple, then there exists $\mu \in \mathfrak{h}^*$ such that $M(\mu) \subset M(\lambda)$ We know that $\mu \leq \lambda$ and $\mu \in W_{[\lambda]} \cdot \lambda$. But $w \cdot \lambda \geq \lambda$ for all $w \in W_{[\lambda]}$ since λ is anti-dominant. Therefore $M(\lambda)$ is simple.

Now assume $M(\lambda)$ is simple. If λ is not anti-dominant, then there exists $\alpha \in \Phi^+$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$. So, let $\mu := \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha$. Therefore $\mu \leq \lambda$ and $\mu \in W \cdot \lambda$. This satisfies the hypothesis of Verma's theorem (5.3.3) and hence there exists the embedding $M(\mu) \subset M(\lambda)$. This contradicts our assumption that $M(\lambda)$ is simple. Therefore, λ is anti-dominant. ■

Finally using the language we built in this section, we can now refine the subcategories $\mathcal{O}_{\chi_\lambda}$ when they fail to be indecomposable. In this way we obtain a description of blocks of Category $\mathcal{O}$ in terms of antidominant (or alternatively, dominant) weights.

Proposition 6.3.4. Blocks of $\mathcal{O}$ are full subcategories where the objects are modules whose composition factors all have highest weights linked by $W_{[\lambda]}$ for some antidominant weight λ. Hence, blocks are in bijection with antidominant (or alternatively, dominant) weights.

Proof: Let $\mathcal{B}$ be a block of Category $\mathcal{O}$. If $L(\mu)$ is a simple in this block than take the Verma module $M(\mu) \in \mathcal{B}$. We know from 5.2.1, that $M(\mu)$ has a socle that is a simple Verma module say $M(\lambda)$. By the previous theorem, we can say that λ is anti-dominant. We also know that all composition factors of $M(\mu)$ are linked to μ by $W_{[\mu]}$. Now, because

$$\lambda \leq \mu \implies \lambda \equiv \mu \pmod{\Lambda} \implies W_{[\lambda]} = W_{[\mu]}.$$

This means that all composition factors of $M(\mu)$ including $L(\mu)$ are linked to λ by $W_{[\lambda]}$. So we have shown that highest weights of simples in a block are linked to an antidominant weight.

Now assume that λ is anti-dominant. Then we know from the previous proposition that $\lambda \leq w \cdot \lambda$ for all $w \in W_{[\lambda]}$. Now consider the Verma modules $M(w \cdot \lambda)$ where $w \in W_{[\lambda]}$. Because $w \in W_{[\lambda]}$, we can see $w \cdot \lambda$ and λ lie in the same coset of $\mathfrak{h}^*/\Lambda_r$. So, $W_{[w\cdot\lambda]} = W_{[\lambda]}$. So, all composition factors of $M(w \cdot \lambda)$ are linked to λ by $W_{[\lambda]}$. Therefore, the unique simple submodule of $M(w \cdot \lambda)$ is $M(\lambda)$.

Let K_w be the maximal submodule of $M(w \cdot \lambda)$. Then, we have seen in the proof of 6.2.4, that $M(w \cdot \lambda)/K_w$ is a non-trivial extension of $L(w \cdot \lambda)$ and $L(\lambda)$. Hence, the simple modules $L(w \cdot \lambda)$ and $L(\lambda)$ lie in the same block. This shows that simples with highest weight linked to an antidominant weight lie in a block. ■

6.4 Grothendieck Group of $\mathcal{O}$

Definition: The Grothendieck group of $\mathcal{O}$ (denoted as $K(\mathcal{O})$) is the free abelian group $\mathcal{F}$ on symbols $[M]$, where $M \in \mathcal{O}$, quotient by the submodule R generated by the relation $[B] = [A] + [C]$ where $0 \to A \to B \to C \to 0$ is a short exact sequence.

Theorem 6.4.1. The elements $\{[L(\lambda)] \mid \lambda \in \mathfrak{h}^*\}$ form a $\mathbb{Z}$-basis of $K(\mathcal{O})$.

Proof: Let $0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset M_{n+1} = M$ be a composition series of M. Then,

$$0 \to M_n \to M \to M/M_n \to 0$$

is exact. This gives:

$$[M] = [M/M_n] + [M_n].$$

Doing this iteratively over the composition flag, we get:

$$[M] = \sum_{j=1}^{n+1} [M_j/M_{j-1}] = \sum_{\lambda \in \mathfrak{h}^*} \underbrace{([M : L(\lambda)])}_{\in \mathbb{Z}} [L(\lambda)].$$

This shows that the every element in $K(\mathcal{O})$ can be written as a $\mathbb{Z}$-linear combination of $\{[L(\lambda)] \mid \lambda \in \mathfrak{h}^*\}$. Now we show their linear independance.

Define the map Φ_λ that takes $M \in \mathcal{O}$ to $[M : L(\lambda)] \in \mathbb{Z}$. Recall that $[M : L(\lambda)]$ is the multiplicity of $L(\lambda)$ in composition series of M. This gives a homomorphism from $\mathcal{F} \to \mathbb{Z}$. Because formal characters are additive under short exact sequences:

$$\begin{aligned} &0 \to A \to B \to C \to 0 \\ \implies &[B : L(\lambda)] = [A : L(\lambda)] + [C : L(\lambda)] \end{aligned}$$

Hence, $B - A - C$ lies in the kernel of Φ_λ. But such elements of $\mathcal{F}$ are generators of the relation that defines $K(\mathcal{O})$ i.e., $R \subseteq ker(\Phi_\lambda)$. This means that we can extend Φ_λ to a group homomorphism:

$$\begin{aligned} \Psi_\lambda : K(\mathcal{O}) &\to \mathbb{Z} \\ [M] &\mapsto [M : L(\lambda)] \end{aligned}$$

Now consider the following finite sum:

$$\sum_\mu n_\mu [L(\mu)] = 0$$

where $n_\lambda \in \mathbb{Z}$ not all zero. Then applying Ψ_λ we get:

$$\Psi_\lambda(\sum_\mu n_\mu [L(\mu)]) = \sum_\mu n_\mu \Psi_\lambda(L(\mu)) = \sum_\mu n_\mu \delta_{\mu,\lambda} = 0$$

Hence, $n_\lambda = 0$ and similary we can show all coefficients are zero. Therefore, we have proved that $\{[L(\lambda)] \mid \lambda \in \mathfrak{h}^*\}$ is a $\mathbb{Z}$ basis of $K(\mathcal{O})$. ■

The theorem is also true for finite length abelian categories (see Theorem 7.87 in[18]).

Corollary 6.4.1.1. For $M, N \in \mathcal{O}$, $[M] = [N]$ if and only if $[M : L(\lambda)] = [N : L(\lambda)]$ for all simples $L(\lambda)$.

Because exact functors take exact sequences to exact sequences, we can say:

Proposition 6.4.2. Any exact functor induces an endomorphism of $K(\mathcal{O})$.

Since modules in Category $\mathcal{O}$ are $U(\mathfrak{g})$-modules with finite-dimensional weight spaces, we can also define the notion of formal characters on modules in $\mathcal{O}$. Recall from 4.3 that formal characters are also additive on short exact sequences. So we can use the same technique as in the proof of 6.4.1 to give a homomorphism:

$$\begin{aligned} \Psi_{ch} : K(\mathcal{O}) &\to \mathbb{Z}\langle \mathfrak{h}^* \rangle \\ [M] &\mapsto \operatorname{ch}(M) \end{aligned}$$

Theorem 6.4.3. The following are equivalent for $M, N \in \mathcal{O}$:

(a) $[M] = [N]$.

(b) formal characters of M and N are same i.e., $\operatorname{ch}(M) = \operatorname{ch}(N)$.

(c) $[M : L(\lambda)] = [N : L(\lambda)]$ for all simples $L(\lambda)$.

Proof: The corollary of 6.4.1 says that (a) and (c) are equivalent statements. From 4.3, we know that formal characters are additive over short exact sequences. So, a formal character of $M \in \mathcal{O}$ can be written as:

$$\operatorname{ch}(M) = \sum_{\lambda} [M : L(\lambda)] \operatorname{ch}(L(\lambda))$$

Non-isomorphic simple modules in $\mathcal{O}$ will have distinct highest weights, so their formal characters are linearly independant in $\mathbb{Z}\langle \mathfrak{h}^* \rangle$. These results show that (b) and (c) are equivalent. ■

Chapter 7

Methods in Category $\mathcal{O}$

The BGG Category $\mathcal{O}$ is a module ctegory with 3 axioms. This means that the standard tools used for categories like $R-Mod$ need not work for $\mathcal{O}$. In this chapter, we build tools like duality functor for Category $\mathcal{O}$. We see if $\mathcal{O}$ is closed under extensions of modules. We will see that $\mathcal{O}$ has enough projective and injective modules and prove the BGG reciprocity theorem given by Bernstein-Gelfand-Gelfand in their original paper [3].

7.1 Hom and Ext

We will see how the Hom and Ext functors will act in Category $\mathcal{O}$. First and foremost we ask the question, if an extension of two modules in Category $\mathcal{O}$ lies in $\mathcal{O}$? The answer is no, and the following is an example. Hence, from here onwards when we use $Ext_{\mathcal{O}}(A, B)$ we will mean extensions of A by B in Category $\mathcal{O}$.

Example: Take $\mathfrak{g} = sl_2(\mathbb{C})$ and start with a 2 dimensional $U(\mathfrak{b})$-module N where the action is given as:

$$e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Let $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$. Then, we get the following short exact sequence:

$$0 \to M(\lambda) \to M \to M(\lambda) \to 0$$

So, M is an extension of the Verma module $M(\lambda)$ by itself but because the action of h is not semi-simple, it is not a module in Category $\mathcal{O}$.

Proposition 7.1.1. Let $\lambda, \mu \in \mathfrak{h}^*$.

(a) Let M be a highest weight module of weight μ. If $\lambda \not< \mu$ (i.e. $\lambda \geq \mu$ or λ is incomparable with μ), then

$$Ext^1_{\mathcal{O}}(M(\lambda), M) = 0.$$

(b) If $\lambda > \mu$ and $N(\lambda)$ is the maximal submodule of $M(\lambda)$, then

$$Hom(N(\lambda), L(\mu)) \cong Ext^1_{\mathcal{O}}(L(\lambda), L(\mu)).$$

(c) $Ext^1_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$.

Proof: For part (a) assume a non-split short exact sequence $0 \to M \xrightarrow{f} E \xrightarrow{g} M(\lambda) \to 0$. Let v be the canonical highest weight generator of $M(\lambda)$. Since g is surjective, choose $\tilde{v} \in E$ such that $g(\tilde{v}) = v$. We know that v has weight λ, therefore $\tilde{v}$ is a weight vector of weight λ. This is because g is a $U(\mathfrak{g})$-module homomorphism. Since $U(\eta_+) \cdot v = 0$ this means $U(\eta_+) \cdot \tilde{v} \in M$. The weight of such a vector contradicts $\lambda \not< \mu$. Hence, $U(\eta_+) \cdot \tilde{v} = 0$ i.e., $\tilde{v}$ is a maximal vector of highest weight λ. Hence $\tilde{v}$ generates a submodule of E isomorphic to $M(\lambda)$. This provides a splitting of the short exact sequence. Therefore, $Ext^1_{\mathcal{O}}(M(\lambda), M) = 0$. This proves part (a).

Take $\lambda > \mu$ and $N(\lambda)$ be the maximal submodule of $M(\lambda)$. Start with the short exact sequence $0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0$. Since Hom is contravariant in the first argument, we get:

$$\begin{aligned} \cdots \to Hom(M(\lambda), L(\mu)) &\to Hom(N(\lambda), L(\mu)) \\ &\to Ext^1_{\mathcal{O}}(L(\lambda), L(\mu)) \to Ext^1_{\mathcal{O}}(M(\lambda), L(\mu)) \to \cdots \end{aligned}$$

Now, $Hom(M(\lambda), L(\mu)) = 0$ since $L(\mu)$ is the unique simple subquotient of M and $Ext^1_{\mathcal{O}}(M(\lambda), L(\mu)) = 0$ from (a). Therefore, $Hom(N(\lambda), L(\mu)) \cong Ext^1_{\mathcal{O}}(L(\lambda), L(\mu))$. This proves (b).

To prove (c), we will use the same technique as above. Consider the short exact sequence $0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0$. And since Hom is contravariant in the first argument, we get:

$$\begin{aligned} \cdots \to Hom(M(\lambda), L(\lambda)) \to Hom(N(\lambda), L(\lambda)) \\ \to Ext^1_{\mathcal{O}}(L(\lambda), L(\lambda)) \to Ext^1_{\mathcal{O}}(M(\lambda), L(\lambda)) \to \cdots \end{aligned}$$

From 3.4.3, recall that $N(\lambda)$ does not have a λ weight space, so $Hom(N(\lambda), L(\lambda)) = 0$. Using $\lambda = \mu$ in part (a), we can say that $Ext^1_{\mathcal{O}}(M(\lambda), L(\lambda)) = 0$. This proves (c). ■

Remark: As an immediate corollary of part (a) of the above theorem, we get:

(a) $Ext^1_{\mathcal{O}}(M(\lambda), M(\lambda)) = 0$.

(b) $Ext^1_{\mathcal{O}}(M(\lambda), L(\lambda)) = 0$.

(c) If $\mu \leq \lambda$, then $Ext^1_{\mathcal{O}}(M(\lambda), L(\mu)) = 0$.

7.2 Duality in Category $\mathcal{O}$

In module categories the notion of duality plays an important role. A duality functor is exact and contravariant. So it allows one to study injective and projective modules in tandom. It also helps simplify the study of homomorphisms into study of tensor products. In this section we will try to build a duality functor for Cateogry $\mathcal{O}$. Here are some problems that we need to address when we try to define dual modules in $\mathcal{O}$.

(a) If a vector space V has $dim(V) = \infty$ then the dimension of its dual vector space blows up. This is a problem because most modules in Category $\mathcal{O}$ do not have finite dimension.

(b) If M is a module over a non-commutative ring R, then its dual module M^* is naturally a right R-module. Objects in Category $\mathcal{O}$ are left $U(\mathfrak{g})$-modules.

(c) We want the dual of a module in $\mathcal{O}$ to satisfy the axioms of Category $\mathcal{O}$.

Take a module $M \in \mathcal{O}$. First we will try to make M^* into a left $U_{\mathfrak{g}}$-module. To do so we can use anti-automorphisms of of $U_{\mathfrak{g}}$.

Definition: An anti-automorphism ϕ of an algebra $(A, \cdot)$ is an automorphism $\phi : A \to A$ such that $\phi(x \cdot y) = \phi(y) \cdot \phi(x)$.

Take a vector $f \in M^*$ and $v \in M$ and define an action of $x \in U_{\mathfrak{g}}$ on M^* as:

$$x \cdot f(v) := f(\phi(x) \cdot v)$$

Take $x, y \in U_{\mathfrak{g}}$, $f \in M^*$ and $v \in M$. Then,

$$\begin{aligned}(x \cdot y) \cdot f(v) &= x \cdot (y \cdot f)(v) = y \cdot f(\phi(x) \cdot v) \\ &= f(\phi(y) \cdot \phi(x) \cdot v) = f(\phi(x \cdot y) \cdot v)\end{aligned}$$

The above calculation shows how M^* can now be viewed as a left $U_{\mathfrak{g}}$-module. We can now define weight spaces on dual modules. However, working with any anti-automorphism is not ideal. We want an anti-automorphism that will help us keep dual modules locally $U(\mathfrak{n}_+)$ finite. To illustrate this, consider the map $\psi : M \to M$ that takes $x \mapsto -x$. Consider $v \in M_\lambda$ and v^* be its dual vector in M^*. Let $w \in M_\mu$ and $e \in \mathfrak{g}_\alpha$ for a simple root α. Then,

$$\begin{aligned}e \cdot v^*(w) &= v^*(\psi(e) \cdot w) = v^*(-e \cdot w) \\ &= -v^*(e \cdot w) = -\delta_{\lambda, \mu+\alpha}\end{aligned}$$

So, $v^* \in M^*_{\lambda-\alpha}$. So under the ψ action, e acts by raising weights of module M but lowers the weights of its dual module . Since, there is no lower bound on weights of M, we can say that under the left action induced by ψ, M^* is not locally $U(\mathfrak{n}_+)$ finite. To tackle this problem, we can use a special anti-automorphism generated by twisting the action of $\mathfrak{g}$.

Definition: Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an anti-involution map that fixes pointwise $\mathfrak{h}$ and sends $e_\alpha \mapsto f_\alpha$ for all simple roots α. This extends canonically to an anti-automorphism of $U_{\mathfrak{g}}$ (see [9] for details).

Now, in the same calculation as above, when we use τ instead of ψ, we get:

$$\begin{aligned}e \cdot v^*(w) &= v^*(\tau(e) \cdot w) = v^*(f \cdot w) \\ &= \delta_{\lambda, \mu-\alpha}\end{aligned}$$

So, $v^* \in M^*_{\lambda+\alpha}$. This means that raising and lowering operators behave the same on modules and their duals. This solves the problem we highlighted with the ψ action.

We now inspect the weight spaces of dual modules. We know that modules in $\mathcal{O}$ need not be finite-dimensional, but their weight spaces are. Take $M \in \mathcal{O}$. We know that:

$$(M_\lambda)^* \cong \{f \in M^* \mid f(M_\mu) = 0 \ \forall \ \mu \neq \lambda\}$$

Take $f \in M^*_\lambda, v \in M_\mu$ and $h \in \mathfrak{h}$. Then we observe that:

$$\begin{aligned} h \cdot f(v) &= f(\tau(h) \cdot v) = f(h \cdot v) = f(\mu(h)v) \\ &= \lambda(h)f(v) \quad (\text{because } f \in M^*_\lambda) \\ &\implies (\lambda(h) - \mu(h))f(v) = 0 \end{aligned}$$

Which means that f vanishes on M_μ when $\mu \neq \lambda$. Hence, $M^*_\lambda \subset (M_\lambda)^*$. Similarly, an element $g \in (M_\lambda)^*$ is identified by its action on M_λ, which implies that, $(M_\lambda)^* \subset M^*_\lambda$. Hence, under the τ action of $U_\mathfrak{g}$, $M^*_\lambda = (M_\lambda)^*$. Because M is locally $U(\mathfrak{n}_+)$ finite, under this τ action, M^* also becomes locally $U(\mathfrak{n}_+)$ finite. We know that M_λ is finite dimensional, so M^*_λ is also finite-dimensional. This motivates us to consider the following:

Definition: Let $\mathcal{C}$ be the full subcategory of $U_\mathfrak{g}$-Mod where the objects are $\mathfrak{h}$-semisimple modules with finite-dimensional weight spaces. We define the dual of $M \in \mathcal{C}$ as:

$$M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^*} M^*_\lambda.$$

Notice that by definition, $M^\vee$ is $\mathfrak{h}$-semisimple and under the τ action of $U_\mathfrak{g}$, $M^\vee$ is a left $U_\mathfrak{g}$-module. We have also shown that under the τ action of $U_\mathfrak{g}$, $M^*_\lambda = (M_\lambda)^*$. So, weight spaces of $M^\vee$ are finite-dimensional. Therefore, $M^\vee \in \mathcal{C}$.

Theorem 7.2.1. For the module category $\mathcal{C}$ defined above,

$$\begin{aligned} \Phi : \mathcal{C} &\to \mathcal{C} \\ M &\mapsto M^\vee \end{aligned}$$

is an exact and contravariant functor.

Proof: We have seen that $M^\vee$ is $\mathfrak{h}$-semisimple and has finite dimensional weight spaces. Let $f : M \to N$ be a homomorphism of $U_\mathfrak{g}$-modules in $\mathcal{C}$. Then, we define the dual morphism as:

$$\begin{aligned} f^\vee : N^\vee &\to M^\vee \\ g &\mapsto g \circ f \end{aligned}$$

For Φ to be a functor, we have to show that $f^\vee$ is a $U_\mathfrak{g}$-module homomorphism and it satisfies the functor axioms. Take $x \in U_\mathfrak{g}$, $g \in N^\vee$ and $v \in M$. Then,

$$\begin{aligned} x \cdot f^\vee(g)(v) &= x \cdot (g \circ f)(v) = g \circ f(\tau(x) \cdot v) \\ &= g(\tau(x) \cdot f(v)) \quad \text{(since f is } U_\mathfrak{g}\text{linear)} \\ &= x \cdot g(f(v)) = f^\vee(x \cdot g)(v) \end{aligned}$$

This shows that $f^\vee$ is a $U_\mathfrak{g}$-module homomorphism. It is easy to see that $(Id_N)^\vee = Id_{N^\vee}$. Consider $: N \to S$ and $s \in S^\vee$. Then,

$$\begin{aligned} (h \circ f)^\vee(s) &= s \circ (h \circ f) = (s \circ h) \circ f \\ &= f^\vee(s \circ h) = f^\vee \circ (h^\vee(s)) \\ &= f^\vee \circ h^\vee(s) \end{aligned}$$

So we have shown that the dual morphism also respects composition. Hence, Φ is a well-defined functor on $\mathcal{C}$.

Now we will show that this functor Φ is exact and contravariant. Recall that $U_\mathfrak{g}$-module homomorphisms preserve weight spaces and in $\mathcal{C}$, modules have finite-dimensional weight spaces. So, a morphism $f : M \to N$ in $\mathcal{C}$ can be described by a collection of maps $f_\lambda : M_\lambda \to N_\lambda$. Similarly its dual can be described by $f_\lambda^* : N_\lambda^* \to M_\lambda^*$. Now consider a short exact sequence in $\mathcal{C}$:

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

whic gives rise to the sequences:

$$0 \to L_\lambda \xrightarrow{f_\lambda} M_\lambda \xrightarrow{g_\lambda} N_\lambda \to 0$$

Using the standard duality functor we get:

$$0 \to (N_\lambda)^* \xrightarrow{g_\lambda^*} (M_\lambda)^* \xrightarrow{f_\lambda^*} (L_\lambda)^* \to 0$$

Now taking direct sum over all weights we get:

$$0 \to \bigoplus_\lambda (N_\lambda)^* \xrightarrow{\oplus (g_\lambda)^*} \bigoplus_\lambda (M_\lambda)^* \xrightarrow{\oplus (f_\lambda)^*} \bigoplus_\lambda (L_\lambda)^* \to 0$$

Which gives us:

$$0 \to N^\vee \xrightarrow{g^\vee} M^\vee \xrightarrow{f^\vee} L^\vee \to 0$$

This shows that Φ is exact and contravariant. ■

We get the following corollary as a direct consequence of the theorem:

Corollary 7.2.1.1. Let $M, N \in \mathcal{C}$. Then,

(a) M is simple if and only if $M^\vee$ is simple.

(b) $(M \oplus N)^\vee = M^\vee \oplus N^\vee$.

(c) $M \to M^{\vee\vee}$ defines a natural isomorphism $\Phi^2 \cong \mathrm{Id}_{\mathcal{C}}$. Since Φ is contravariant, it is an anti-equivalence on $\mathcal{C}$.

Theorem 7.2.2. For $M \in \mathcal{O}$, the module $M^\vee \in \mathcal{O}$. Hence,

$$\begin{aligned} \mathcal{D} : \mathcal{O} &\to \mathcal{O} \\ M &\mapsto M^\vee \end{aligned}$$

is an exact and contravariant functor on Category $\mathcal{O}$.

Proof: We know that $M^\vee$ is $\mathfrak{h}$-semisimple. We have shown that under the τ action of $U_\mathfrak{g}$, $M^\vee$ is locally $U(\mathfrak{n}_+)$ finite. It remains to show that $M^\vee$ is finitely generated. We will prove this by induction on length of M. Since modules in $\mathcal{O}$ are also objects in $\mathcal{C}$, we can make use of the above two results.

If M has length one, i.e., it is a simple $U_\mathfrak{g}$-module, then so is $M^\vee$ (due to the above cororllary). So, $M^\vee$ has finite length. Let us say this is true for modules of length k. We will prove this for modules of length $k+1$. Let $N \in \mathcal{O}$ have length $k+1$. If N'

is the simple submodule of N then we can write the following short exact sequence:

$$0 \to N' \to N \to N/N' \to 0$$

and since taking the dual in $\mathcal{C}$ is exact and contravariant, we get:

$$0 \to (N/N')^{\vee} \to N^{\vee} \to N'^{\vee} \to 0$$

From the induction hypothesis, $(N/N')^{\vee}$ has length k and from the previous corollary $N'^{\vee}$ is simple and has length 1. So, $N^{\vee}$ has length $k+1$. So, dual object sof modules in $\mathcal{O}$ has finite length. Thus, they have to be finitely generated. From the theorem, we can say that $\mathcal{D}$ is an exact and contravariant functor on $\mathcal{O}$. ■

It is easy to see that properties listed in 7.2.1.1 carry over to Category $\mathcal{O}$ as well. So we get the following corollary:

Corollary 7.2.2.1. The canonical map $M \to M^{\vee\vee}$ induces a natural isomorphism $\mathcal{D}^2 \cong \mathrm{Id}_{\mathcal{O}}$. In particular, $\mathcal{D}$ is an anti-equivalence of Category $\mathcal{O}$.

We look at two more properties of this duality functor.

Proposition 7.2.3. $\mathcal{D}$ be the duality functor on $\mathcal{O}$ as defined above. Then, for $M \in \mathcal{O}$, the following are true:

(a) M and $M^{\vee}$ have the same formal character, so they have the same composition factor multiplicities. In particular, $L(\lambda) \cong L(\lambda)^{\vee}$.

(b) For any central character χ, $(M^{\vee})^{\chi} \cong (M^{\chi})^{\vee}$. Hence, if $M \in \mathcal{O}_{\chi}$ the $M^{\vee} \in \mathcal{O}_{\chi}$.

Proof: We know that $M_{\lambda}^{*} = (M_{\lambda})^{*}$. Since both M and $M^{\vee}$ are $\mathfrak{h}$-semisimple modules, this mean that they have the same formal characters. We know from 7.2.1.1 that $L(\lambda)$ is simple if and only if $L(\lambda)^{\vee}$ is simple. If $L(\lambda)$ and $L(\lambda)^{\vee}$ have the same formal character then that means they have the same highest weight. By the universal property of simple highest weight modules (3.4.4), we can say that $L(\lambda) \cong L(\lambda)^{\vee}$.

Start with a composition series for M. Say:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

Then we get a short exact sequence:

$$0 \to M_1 \to M \to M/M_1 \to 0$$

and applying the duality functor gives:

$$0 \to (M/M_1)^\vee \to M^\vee \to M_1^\vee \to 0$$

So, $ch(M^\vee) = ch(M_1) + ch((M/M_1)^\vee)$ since M_1 is simple and has the same formal character as $M_1^\vee$. Because these modules have finite length, we can do this iteratively and see that composition factors of M and $M^\vee$ are the same. This proves part (a).

Recall that:

$$M^\chi = \{v \in M \mid (z - \chi(z))^n \cdot v = 0 \quad \text{for some } n \text{ depending on } z \;\&\; v\}$$

$$(M^\vee)^\chi = \{f \in M^\vee \mid (z - \chi(z))^n \cdot f = 0 \quad \text{for some } n \text{ depending on } z \;\&\; f\}$$

Say $f \in (M^\vee)^\chi$ and take $v \in M^{\chi'}$. From the definition we have:

$$\begin{aligned} ((z-\chi(z))^n \cdot f)(v) &= f((z-\chi(z))^n \cdot v) \quad \because \tau \text{ preserves } Z(\mathfrak{g}) \text{ pointwise} \\ &= f((\chi'(z) - \chi(z))^n \cdot v) \\ &= (\chi'(z) - \chi(z))^n f(v) = 0 \end{aligned}$$

This means that $f = 0$ for all $v \in M^{\chi'}$ where $\chi' \neq \chi$. Therefore, f is determined completely by its action on M^χ. So, $f \in (M^\chi)^\vee$ and hence, $(M^\vee)^\chi \subset (M^\chi)^\vee$.

Now let $g \in (M^\chi)^\vee$. We know that $M^\vee$ decomposes into a direct sum of weight spaces. So, any $f \in M^\vee$ is non zero on finitely many weight spaces. And since each weight space is finite-dimensional, we can say that support of f is finite-dimensional. Therefore, support of $g \in (M^\chi)^\vee$ is also finite-dimensional and lies in M^χ. That means we can choose a natural number N big enough such that for

$$z \in Z(\mathfrak{g}) \;,\; (z - \chi(z))^N \cdot \text{supp}(g) = 0.$$

Because τ preserves $Z(\mathfrak{g})$ pointwise, we can say that for

$$z \in Z(\mathfrak{g}) \;,\; (z - \chi(z))^N \cdot g = 0.$$

We know that $g \in M^\vee$, so we get $g \in (M^\vee)^\chi$. This means $(M^\chi)^\vee \subset (M^\vee)^\chi$, which completes the proof of (b). ■

Theorem 7.2.4. Let $\mu, \lambda \in \mathfrak{h}^*$. Then,

(a) $\dim \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = \delta_{\mu,\lambda}$.

(b) $\mathrm{Ext}^1_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = 0$

Proof: For $M(\lambda)$ we have the short exact sequence:

$$0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0$$

where $N(\lambda)$ is its unique maximal submodule (see (3.4.3)). Applying the dualtiy functor $\mathcal{D}$ we get:

$$0 \to L(\lambda)^\vee \to M(\lambda)^\vee \to N(\lambda)^\vee \to 0.$$

The above proposition says that $L(\lambda) \cong L(\lambda)^\vee$, so we get that $M(\lambda)^\vee$ has a simple submodule isomorphic to $L(\lambda)$. Notice that it is the unique simple submodule of $M(\lambda)^\vee$. Because if V is another such simple submodule then its dual is a simple quotient of $M(\lambda)$ and we know that Verma modules have unique simple quotient because they have a unique maximal submodule.

Now assume that $f \in \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee)$ is non-zero, then $M(\lambda)^\vee$ has a highest weight submodule with highest weight μ. Since, $M(\lambda)^\vee$ has a unique simple submodule $L(\lambda)$, this means that $\mu \geq \lambda$. The above proposition says that $M(\lambda)$ and $M(\lambda)^\vee$ have the same composition factors. So if, $L(\gamma)$ is a composition factor of $M(\lambda)^\vee$ then, $\gamma \leq \lambda$. If we quotient of $f(M(\mu))$ with $f(N(\mu))$, we get that $L(\mu)$ is a composition factor multiplicity of $M(\lambda)^\vee$. Hence, $\mu \leq \lambda$. Therefore, $\dim \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = 1$ if and only if $\mu = \lambda$. This proves (a).

Take the short exact sequence:

$$0 \to M(\lambda)^\vee \to M \xrightarrow{\pi} M(\mu) \to 0$$

Let v be a maximal vector of $M(\mu)$ and w be its preimage in M. Now, w is a vector of weight μ. If w is also maximal vector then by universal property of Verma modules we get a homomorphism from v to w which gives us the splitting. If not, then take

$x \in \mathfrak{g}_\alpha$ for some positive root α. Then,

$$\pi(x \cdot w) = x \cdot \pi(w) = x \cdot v = 0.$$

So, $x \cdot w \in \ker(\pi)$ which we know is isomorphic $M(\lambda)^\vee$. From the previous result we know that , formal character of $M(\lambda)$ and $M(\lambda)^\vee$ are the same. So all weights of $M(\lambda)^\vee$ are $\leq \lambda$. The vector $x \cdot w$ has weight $\mu + \alpha$ so $\mu + \alpha \leq \lambda$. This means that $\mu < \lambda$. Now, applying duality functor on the above short exact sequence we get:

$$0 \to M(\mu)^\vee \to M^\vee \to M(\lambda) \to 0.$$

So, λ has to be a highest weight of $M^\vee$. Using the universal property of Verma modules again, we can show that this short exact sequence splits. If λ is not the highest weight of $M^\vee$, then using the same calculation as above, we get that $\lambda < \mu$ which contradicts the fact that $\mu < \lambda$. Thus, the short exact sequence splits. Similarly its dual short exact sequence:

$$0 \to M(\lambda)^\vee \to M \to M(\mu) \to 0$$

also splits. This completes the proof of (b). ■

7.3 Projectives in $\mathcal{O}$

To make effective use of homological techniques in Category $\mathcal{O}$ we need to study projective modules. For example, to show that projective resolutions of modules in $\mathcal{O}$ exists, we need to show that it contains enough projectives. Once we do that, then we can talk about indecomposable projective modules. We will end the section with a characterization of indecomposable modules in $\mathcal{O}$.

We have shown the existence of a duality functor $\mathcal{D}$ on $\mathcal{O}$. It will send projective modules to injective modules. Hence, having enough projectives in $\mathcal{O}$ will automatically imply the existence of enough injectives. First we will show the existence of projective modules in Category $\mathcal{O}$ with explicit examples and then we will show that we can make more projective modules using those.

Theorem 7.3.1. There are some projective objects in Category $\mathcal{O}$.

(a) Let $\lambda \in \mathfrak{h}^*$ be dominant weight. Then, $M(\lambda)$ is projective.

(b) If L is finite dimensional and P is projective, then $P \otimes L$ is also projective.

Proof: Assume $M, N \in \mathcal{O}$ such that

$$0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} M(\lambda) \to 0$$

is a short exact sequence. Now take v to be the canonical generator of $M(\lambda)$ and $\bar{v}$ be its pre-image in N. Because $U_{\mathfrak{g}}$-module homomorphisms take weight vectors to weight vectors, we can say that $\bar{v}$ is a weight vector of weight λ. Note that $\bar{v} \notin M$. We know that $M(\lambda) \in \mathcal{O}^{\chi}$ which implies that $\bar{v} \in N^{\chi}$. Now, $U(\mathfrak{n}_+) \cdot \bar{v}$ is finite dimensional and it must contains a maximal vector. The weight of this maximal vector will then be linked to λ. But our assumption was that λ is dominant. So, there can be no weight $> \lambda$ that is linked to λ. This forces $U(\mathfrak{n}_+) \cdot \bar{v} = 0$. Therefore, $\bar{v}$ is a highest weight vector. By universal property of Verma modules (3.3.2), there exists $\gamma : M(\lambda) \to N$ a $U(\mathfrak{g})$ homomorphism that maps $v \mapsto \bar{v}$ which implies that $\beta \circ \gamma = \mathrm{Id}_{M(\lambda)}$. So, the short exact sequence splits and $M(\lambda)$ is proved to be projective. This completes the proof of part (a).

Take M, L and $P \in \mathcal{O}$, such that L is finite-dimensional and P is projective. Then, using the adjoint associativity of the Hom functor we get:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}}(P \otimes L, M) &\cong \mathrm{Hom}_{\mathcal{O}}(P, \mathrm{Hom}_{\mathcal{O}}(L, M)) \\
&\cong \mathrm{Hom}_{\mathcal{O}}(P, L^* \otimes M)
\end{aligned}$$

We know that $\mathrm{Hom}_{\mathcal{O}}(P, L^* \otimes -)$, $L^* \otimes -$ and $\mathrm{Hom}_{\mathcal{O}}(P, -)$ are exact functors. Therefore, we can say that $\mathrm{Hom}(P \otimes L, -)$ is exact and $P \otimes L$ is projective. This proves part (b). ■

Remarks: Note the following about the theorem:

(a) We have showed that $\lambda \in \mathfrak{h}^*$ being dominant is sufficient for $M(\lambda)$ to be projective. We will later show that this is also necessary.

(b) Using part (b) and the duality functor, we can show that $Q \otimes L$ is injective when Q is injective and L is finite-dimensional.

Definition: A module category is said to have enough projectives when every module in the category is a quotient of a projective module.

Theorem 7.3.2. Category $\mathcal{O}$ has enough projectives.

Proof: We will prove this by induction on the length of modules. For the first step we have to show that the simple modules $\{L(\lambda) \mid \lambda \in \mathfrak{h}^*\}$ are quotients of projective modules. For $\lambda \in \mathfrak{h}^*$ we know that the weight $\mu := \lambda + n\rho$ will be dominant for large enough n. Therefore from the previous theorem $M(\mu)$ is projective. Since $n\rho$ is a dominant integral weight, by highest weight theorem (3.5.2), we know that $L(n\rho)$ is finite-dimensional. So, by the previous theorem, $M(\mu) \otimes L(n\rho)$ is also projective.

Notice that the lowest weight of $L(n\rho)$ is $-n\rho$. Because $-n\rho = w_0(n\rho)$ where w_0 is the longest element of W which flips all positive roots to negative. From 5.4.1 we know that $M(\mu) \otimes L(n\rho)$ has subquotients isomorphic to $M(\mu + \nu)$ where ν is a weight of $L(n\rho)$. Since $-n\rho$ is the lowest weight of $L(n\rho)$, the module $M(\mu) \otimes L(n\rho)$ has $M(\mu - n\rho) = M(\lambda)$ as a quotient. Which means $L(\lambda)$ is also a quotient of $M(\mu) \otimes L(n\rho)$.

Now assume this is true for modules of length k. Consider a module M of length $k+1$. Now since M has as socle that is a simple Verma module (see 5.2.1), we get the following short exact sequence:

$$0 \to L(\lambda) \to M \to N \to 0$$

Because of the induction hypothesis, there exists projective modules P and P' such that $L(\lambda)$ is a quotient of P and N is a quotient of P' respectively. Because P' is projective, we get a $U_{\mathfrak{g}}$-homomorphism π such that the following diagram commutes:

$$\begin{tikzcd} & P' \arrow[dl, dashed, "\exists\pi"'] \arrow[d, "\beta"] \\ M \arrow[r, two heads, "\alpha"'] & N \end{tikzcd}$$

Since the diagram commutes, we can say that:

$$\alpha \circ \pi = \beta \implies \alpha(\mathrm{Img}(\pi)) = N$$

$$\begin{aligned}
&\text{case 1)} \quad \ker(\alpha) \cap \mathrm{Img}(\pi) = 0 \implies \mathrm{Img}(\pi) \cong N \implies M = L(\lambda) \oplus N \\
&\text{case 2)} \quad \ker(\alpha) \cap \mathrm{Img}(\pi) \neq 0 \implies L(\lambda) \subset \mathrm{Img}(\pi) \text{ since } \ker(\alpha) = L(\lambda) \text{ which is simple} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \implies \mathrm{Img}(\pi) = M
\end{aligned}$$

In the first case we can see that M will be a quotient of $P \oplus P'$ whereas in the second case it is the quotient of P'. Hence, we have shown that M is a quptient of a projective module. This completes the proof. ■

We have shown that every module in Category $\mathcal{O}$ is a quotient of a projective module. The projective module however is not unique. If P is a projective module with a module M as a quotient, then M is also a quotient of $P \oplus P'$ for any other projective module P'. In such a situation it makes sense to talk about minimal projective covers of modules. Note that in the literature minimal projective covers are also called projective covers.

Definition: A minimal projective cover of a module M is a projective module P such that M is a quotient of P but not a quotient of any proper submodule of P. We say the epimorphism $P \to M$ is essential.

Theorem 7.3.3. In a Krull-Schmdit abelian category with enough projectives, every object has a minimal projective cover.

The proof of this theorem can be found in the expository article [15]. Since Category $\mathcal{O}$ is abelain and Krull-Schmdit and has enough projective objects, we can say that every object in $\mathcal{O}$ has a minimal projective cover.

Proposition 7.3.4. If M is left module over a ring R and it has a projective cover, then the projective cover is unique upto isomorphism.

Proof: We follow the proof given in [16]. Suppose M is a module that has two minimal projective covers, P_1 and P_2. Using the fact that P_1 is projective, we get the map $\pi : P_1 \to P_2$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P_2 \arrow[d, two heads, "\beta"] \\
P_1 \arrow[ur, "\pi"] \arrow[r, two heads, "\alpha"'] & M
\end{tikzcd}$$

Then, $\beta \circ \pi = \alpha$ and $\mathrm{Img}(\pi) \subset P_2$. We claim that π is surjective:

$$\beta(\mathrm{Img}(\pi)) = \beta \circ \pi(P_1) = \alpha(P_1) = M$$

Since P_2 is a minimal projective cover, $\mathrm{Im}(\pi) = P_2$.
Now, we have the following short exact sequence:

$$0 \to \ker(\pi) \to P_1 \to P_1/\ker(\pi) \to 0$$

But $P_1/\ker(\pi) \cong \mathrm{Im}(\pi) = P_2$. Therefore:

$$0 \to \ker(\pi) \to P_1 \to P_2 \to 0$$

The short exact sequence splits because P_2 is projective. We get $P_1 \cong P_2 \oplus \ker(\pi)$. But that means M is a quotient of a proper submodule of P_1, which is a contradiction since P_1 is a minimal projective cover. Hence, $P_1 \cong P_2$. ■

Definition: We denote by $P(\lambda)$ the minimal projective cover of $L(\lambda)$.

From 6.1.3.2 we know that Category $\mathcal{O}$ is Krull Schmidt. That means we can decompose a projective module as a direct sum of indecomposable modules in $\mathcal{O}$. But, we also know that direct summands of projective modules are projective. So we know that there are indecomposable projective modules in Category $\mathcal{O}$. The following theorem gives a characterization of indecomposable projective modules in $\mathcal{O}$.

Theorem 7.3.5. Every indecomposable projective module in Category $\mathcal{O}$ is isomorphic to some minimal projective cover $P(\lambda)$. This gives a one-to-one mapping between the collection of indecomposable modules and simple modules in Category $\mathcal{O}$.

Proof: First we will show that $P(\lambda)$ is indecomposable. Let $\pi_\lambda : P(\lambda) \to L(\lambda)$ be the surjection of the projective cover onto the simple module. Then, $ker(\pi_\lambda)$ will be a maximal submodule in $P(\lambda)$. If N is another maximal submodule different from $ker(\pi_\lambda)$, then π_λ maps N onto $L(\lambda)$. Since, $L(\lambda)$ is simple, the image of N under π_λ is either zero or $L(\lambda)$. In the first case, N coincides with $ker(\pi_\lambda)$ which is contrary to our assumption. The second case contradicts the fact that $P(\lambda)$ is minimal projective cover. Hence, $N = ker(\pi_\lambda)$.

So, $P(\lambda)$ has a unique maximal submodule i.e., $P(\lambda)$ is a local module. We will now show that it is indecomposable. Let $P(\lambda) = A \oplus B$ where A and B are non-trivial submodules. Because $A, B \in \mathcal{O}$ they have finite length (see 6.1.3.1). This means either they are simple or they have a maximal submodule.

Case 1) If A and B are simple then, $P(\lambda)$ has two distinct maximal submodules A and B which is not possible.

Case 2) If A and B both have maximal submodules m_A and m_B then $m_A \oplus B$ and $A \oplus m_B$ are two distinct maximal submodules of $P(\lambda)$.

Case 3) If only A has a maximal submodule m_A and B is simple. Then, A and $m_A \oplus B$ are distinct maximal submodules of $P(\lambda)$ which is a contradiction.

This proves that $P(\lambda)$ is indecomposable.

Now consider an indecomposable projective module $P \in \mathcal{O}$. We know that it will have a composition series (see 6.1.3.1), so it will have a quotient isomorphic to a simple module in $\mathcal{O}$, say $L(\mu)$. Let $g : P \to L(\mu)$ be the quotient map. Then consider $P(\mu)$ the minimal projective cover of $L(\mu)$. Because P is projective, this will give us a map $f : P \to P(\mu)$ such that $\pi_\mu \circ f = g$. Then, image of P under π_μ maps surjectively onto $L(\mu)$. So, f has to be surjective because $P(\mu)$ is minimal projective cover of $L(\mu)$.

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, "g"] & \\ P(\mu) \arrow[rr, "\pi_\mu"'] & & L(\mu) \end{tikzcd}$$

This gives us the short exact sequence:

$$0 \to ker(f) \to P \to P(\mu) \to 0$$

and since $P(\mu)$ is projective, it splits. But this contradicts the hypothesis that P is indecomposable. Hence, $ker(f)$ has to be 0. Therefore, $P \cong P(\mu)$ and this completes the proof. ■

We will now look at the some properties of the modules $P(\lambda)$ which will be useful in the next section.

Proposition 7.3.6. The following are true for modules $P(\lambda)$, $\lambda \in \mathfrak{h}^*$:

(a) $\dim \mathrm{Hom}_{\mathcal{O}}(P, L(\lambda))$ = number of times $P(\lambda)$ appears as direct summand of P.

(b) $\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$.

Note: The proposition requires the assumption that we are working with algebraically closed field. This is true because we are working with $\mathbb{C}$.

Proof: We know that Category $\mathcal{O}$ is Krull-Schmidt (see 6.1.3.2), so we can write any projective module as a finite direct sum of indecomposable projective module. So, using the previous theorem we get:

$$P = \bigoplus_{i=1}^{k} P(\lambda_i)^{\oplus n_i}.$$

Because of additivity of Hom we have:

$$\mathrm{Hom}_{\mathcal{O}}(P, L(\lambda)) = \bigoplus_{i=1}^{k} \mathrm{Hom}_{\mathcal{O}}(P(\lambda_i), L(\lambda))^{\oplus n_i}.$$

We know that

$$\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), L(\mu)) = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

This is because $\ker(\pi_\lambda)$ is the unique maximal submodule of $P(\lambda)$ and $L(\lambda)$ is the unique simple quotient of $P(\lambda)$. Therefore, we can say that $\dim \mathrm{Hom}_{\mathcal{O}}(P, L(\lambda))$ equals the number of times $P(\lambda)$ occurs as a direct summand of P. This proves part (a).
For $M \in \mathcal{O}$ consider the following short exact sequence:

$$0 \to M_1 \to M \to M_2 \to 0$$

Since $P(\lambda)$ is projective, $\mathrm{Hom}_{\mathcal{O}}(P(\lambda), -)$ is exact and covariant, we get:

$$0 \to \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M_1) \to \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) \to \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M_2) \to 0.$$

Therefore, as vector spaces:

$$\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) = \dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M_1) + \dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M_2).$$

Now applying this iteratively on length of M, we obtain:

$$\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) = \sum_{i=1}^{n} \dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), L(\mu_i)).$$

But we know that:

$$\dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), L(\mu)) = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ 1 & \text{if } \lambda = \mu. \end{cases}$$

Hence, $\dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$. This proves (b). ■

7.4 Standard Filtration

In this section we want to illustrate some properties of modules in $\mathcal{O}$ that admit a standard filtration (Verma flag). The results produces in this section will be used in the next section. We have already showed that for finite dimensional module F, $M(\lambda) \otimes F$ admits a standard filtration. We also know from 6.1.1 that Category $\mathcal{O}$ is closed under taking tensor product with finite dimensional $U(\mathfrak{g})$-modules. So, there are modules in $\mathcal{O}$ that have a standard filtration.

If a module $M \in \mathcal{O}$ admits a standard filtration i.e., Verma flag (see 5.4), then using additivity of formal characters over short exact sequences (see 4.3), we get:

$$\begin{aligned} 0 = M_0 \subset M_1 \subset \cdots \subset M_n = M &\text{ and } M_i/M_{i-1} \cong M(\lambda_i) \\ \Rightarrow \operatorname{ch}(M) &= \operatorname{ch}(M/M_{n-1}) + \operatorname{ch}(M_{n-1}) \\ &= \operatorname{ch}(M/M_{n-1}) + \operatorname{ch}(M_{n-1}/M_{n-2}) + \cdots + \operatorname{ch}(M_1) \\ &= \sum_{i=1}^{n} \operatorname{ch}(M_i/M_{i-1}) \end{aligned}$$

and we can say that $\operatorname{ch}(M) = \sum_i (M : M(\lambda_i)) \operatorname{ch}(M(\lambda_i))$ where $(M : M(\lambda))$ is the Verma flag multiplicity of $M(\lambda)$ in M. Since non-isomorphic Verma modules have distinct highest weights, their formal characters are linearly independant in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$. Hence, the following lemma follows:

Lemma 7.4.1. Assume that $M, N \in \mathcal{O}$ admit a standard filtration. The the following are equivalent:

(a) M and N have the same formal character.

(b) $(M : M(\lambda)) = (N : M(\lambda))$ for all $\lambda \in \mathfrak{h}^*$.

Corollary 7.4.1.1. Let $M \in \mathcal{O}$ be a module with standard filtration. The length of Verma flag and multiplicity of Verma modules in its Verma flag are independant of the choice of the standard filtration of M.

So for example, a standard filtration length of a module is 1 if and only if it is a Verma module. The corollary follows directly from the above theorem. A different style of proof of this corollary can be found in BGG's original paper [3].

Theorem 7.4.2. Let $M \in \mathcal{O}$ have standard filtration.

(a) If $\lambda \in \mathfrak{h}^*$ is a maximal weight in the support of M, then M has a submodule isomorphic to $M(\lambda)$ and $M/M(\lambda)$ has a standard filtration.

(b) M is a free $U(\mathfrak{n}_-)$ module.

(c) M has standard filtration if and only if its direct summands have standard filtration.

Proof: Let $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a Verma flag of M. Because of the hypothesis, we have a maximal vector v of weight λ. Let i be the smallest index such that $v \in M_i$. Therefore $v \notin M_{i-1}$. By universal property of Verma modules (3.3.2) there exists $f : M(\lambda) \to M_i$ a non-zero homomorphism mapping $1 \otimes 1 \mapsto v$. Define

$$g := \pi \circ f : M(\lambda) \xrightarrow{f} M_i \xrightarrow{\pi} M_i/M_{i-1}.$$

π is non-zero since $\pi(v) \neq 0$. Hence g is also non-zero. The hypothesis says that M_i/M_{i-1} is isomorphic to a Verma module say $M(\mu)$. Because g is a map between Verma modules, using theorem 5.2.2 we can say that g is injective. Then, f has to be an injective map because $ker(f) \subseteq ker(g) = 0$. So we have an embedding $f : M(\lambda) \hookrightarrow M$.

Now, we have to show that $M/M(\lambda)$ has a standard filtration. Since g is an injective map between Verma modules, we have $\lambda \leq \mu$. Notice that λ and μ are both weights in the support of M. The hypothesis says that λ was a highest weight in support of M, so $\lambda = \mu$ i.e., g is an isomorphism. Notice that the short exact sequence:

$$0 \to M_{i-1} \to M_i \xrightarrow{\pi} M(\lambda) \to 0$$

splits. This is because we have shown that $f : M(\lambda) \to M_i$ is injective and $\pi \circ f = g$ is an endomorphism of Verma module $M(\lambda)$ so it is a scalar multiple of identity map on $M(\lambda)$. Thus, we get a splitting: $M_i = M(\lambda) \oplus M_{i-1}$.

We have $M(\lambda) \subseteq M_i \subseteq M$ and $M_i/M(\lambda) \cong M_{i-1}$. Using the third isomorphism theorem for modules we get a short exact sequence:

$$0 \to M_{i-1} \to M/M(\lambda) \to M/M_i \to 0$$

Both M_i and M/M_i have standard flag. Therefore, $M/M(\lambda)$ has standard filtration. This proves (a).

We will prove (b) using induction on composition length of M. Assume it has length 1, i.e. $M = M(\lambda)$. So, $U(\mathfrak{n}_-)$ acts freely on M. Now assume this is true for length k. We will show for M with length $k+1$. Take a composition series for M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$$

Here, $M_1 \cong M(\mu)$ for some μ. Hence, $U(\mathfrak{n}_-)$ acts freely on M_1. Now M/M_1 has standard filtration of length k and using induction hypothesis, $U(\mathfrak{n}_-)$ acts freely on M/M_1. Therefore, $U(\mathfrak{n}_-)$ acts freely on M. This proves (b).

Given $M' \supset M'_1 \supset \cdots \supset M'_k = 0$ and $M'' \supset M''_1 \supset \cdots \supset M''_k = 0$ be Verma flags of M' and M'' respectively. Then, the direct sum $M := M' \oplus M''$ also has a standard filtration $M' \oplus M'' \supset M'_1 \oplus M'' \supset \cdots \supset 0 \oplus M'' \supset 0 \oplus M''_k \supset \cdots \supset 0$.

Now, say M has a standard filtration, we want to show M' and M'' have standard filtration. We will do this by induction on the standard filtration length of M. When M has length 1 i.e. $M = M(\lambda)$, either M' or M'' has to be 0 as Verma modules are indecomposable. Assume this is true for length k, we will show it is true for modules with standard filtration length $k+1$. Let $M \in \mathcal{O}$ be a module with standard filtration: $0 = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$ $M = M' \oplus M''$ in $\mathcal{O}$. Let μ be a maximal weight in the support of M. Without loss of generality, assume that M'_μ is non-zero. Then, there exists a maximal vector w of weight μ in M'. Using part (a) we can say that there exists $M(\mu) \subset M$ and since $w \in M'$ generates $M(\mu)$, we get $M(\mu) \subset M'$. Therefore, $M/M(\lambda) \cong M'/M(\lambda) \oplus M''$ has a standard filtration because of part (a).And by induction hypothesis $M'/M(\lambda)$ and M'' have standard filtration. This completes the proof of (c). ■

Remarks: The hypothesis that λ is maximal weight of M in part (a) is necessary. If M is a module with standard filtration such that $M(\lambda) \subset M$ for some arbitrary $\lambda \in \mathfrak{h}^*$, then $M/M(\lambda)$ need not have standard filtration. The simplest example is for $\mathfrak{g} = sl_2(\mathbb{C})$, $M(-n-2) \hookrightarrow M(n)$ for a natural number n. Their quotient is the simple highest weight module $L(n)$ which does not admit a Verma flag.

Proposition 7.4.3. If $M \in \mathcal{O}$ admits a standard filtration. Assume $\phi : M \twoheadrightarrow M(\lambda)$, a $U(\mathfrak{g})$-module homomorphism. Then $\ker \phi$ admits a standard filtration.

Proof: If ϕ is trivial then its kernel is entire module M which has standard filtration. We will assume that the map is non-zero. We will prove this by induction on length of standard filtration of M. If standard filtration length is 1 then M is a Verma module and we know homomorphims between Verma modules are either null or injective (5.2.2). Since we assumed that ϕ is non-zero, we can say that it is injective and its kernel is trivial i.e., has standard filtration length 0. We now assume the fact to be true for modules with standard filtration length k. Let M now have standard filtration length $k+1$ and $\phi : M \twoheadrightarrow M(\lambda)$ be a $U(\mathfrak{g})$-module homomorphism.

Take a Verma flag $0 = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M$ of M. Let j be the largest index such that $M_j \subset \ker(\phi) = 0$. Using the third isomorphism theorem for modules over rings, we get that:

$$M/ker\phi \cong (M/M_j)/(\ker(\phi)/M_j).$$

Now, $M/\ker(\phi)$ is a Verma module by hypothesis. This gives a surjective map from M/M_j to $M/\ker(\phi)$. Here, M/M_j has standard filtration of length less that $k+1$. So, by induction hypothesis, $\ker(\phi)/M_j$ has a standard filtration. So we have the short exact sequence:

$$0 \to M_j \to \ker(\phi) \to (\ker(\phi)/M_j) \to 0$$

where M_j and $\ker(\phi)/M_j$ have a Verma flag. So, $\ker(\phi)$ also has a Verma flag using the lattice isomorphism theorem for modules. This completes the proof. ■

Theorem 7.4.4. Let $M \in \mathcal{O}$ has standard filtration, then for all $\lambda \in \mathfrak{h}^*$ we have:

$$(M : M(\lambda)) = \dim \mathrm{Hom}_{\mathcal{O}}(M, M(\lambda)^{\vee})$$

Proof: We follow the proof given in [12]. We use induction on standard filtration length of M. When length $= 1$, we know that M is a Verma module say $M(\mu)$, then $(M : M(\lambda)) = \delta_{\mu,\lambda}$. From (7.2.4) we know that $\dim \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = \delta_{\mu,\lambda}$. This proves the base case. Assume the result is true for modules with standard filtration length k. Take M to have standard filtration length $k+1$.

M has a submodule N such that we get the following short exact sequence:

$$0 \to N \to M \to M(\mu) \to 0.$$

Applying $\mathrm{Hom}_{\mathcal{O}}(-, M(\lambda)^\vee)$ we get:

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) &\to \mathrm{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee) \\ &\to \mathrm{Hom}_{\mathcal{O}}(N, M(\lambda)^\vee) \to \mathrm{Ext}^1_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) \to \cdots \end{aligned}$$

Using (7.2.4) we can say that the Ext term vanishes, $\dim \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = \delta_{\mu,\lambda}$.

Using induction hypothesis we have $\dim \mathrm{Hom}_{\mathcal{O}}(N, M(\lambda)^\vee) = (N : M(\lambda))$. So, $\dim \mathrm{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee) = (N : M(\lambda)) + \delta_{\mu,\lambda}$. Since formal characters are additive over short exact sequences, we can say that $\mathrm{ch}(M) = \mathrm{ch}(N) + \mathrm{ch}(M(\mu))$. This says that:

$$(M : M(\lambda)) = (N : M(\lambda)) + (M(\mu) : M(\lambda)) = (N : M(\lambda)) + \delta_{\mu,\lambda}.$$

This completes the proof. ■

7.5 BGG Reciprocity

We want to show that projective modules in Category $\mathcal{O}$ have a standard filtration. We then use the properties of standard filtration to prove the BGG reciprocity theorem given in [3]. Irving's paper [14] provides generalizations of many results in this section.

Lemma 7.5.1. *$P(\lambda)$ is the minimal projective cover of $M(\lambda)$.*

Proof: Recall from theorem (7.3.4) that minimal projective covers are unique upto isomorphism. We know that $P(\lambda)$ is the minimal projective cover of $L(\lambda)$ and $M(\lambda)$ maps surjectively onto $L(\lambda)$ under the quotient map α. Because $P(\lambda)$ is projective we get the map $\pi : P(\lambda) \to M(\lambda)$ such that the following diagram commutes:

$$\begin{tikzcd} P(\lambda) \arrow[d, "\pi"] \arrow[dr, "f"] & \\ M(\lambda) \arrow[r, "\alpha"] & L(\lambda) \end{tikzcd}$$

Assume $\mathrm{im}(\pi) \subsetneq M(\lambda)$. $M(\lambda)$ has a unique maximal submodule $\ker(\alpha)$ (see (3.4.3)). Therefore, $\mathrm{im}(\pi) \subset \ker(\alpha)$. This implies that

$$\alpha \circ \pi(P(\lambda)) = f(P(\lambda))0.$$

Hence, $\mathrm{im}(\pi) = M(\lambda)$. If P is a minimal projective cover of $M(\lambda)$ then it also has $L(\lambda)$ as a projective cover. Then we have two cases:

case 1) If P is indecomposable then $P \cong P(\lambda)$ using (7.3.5)

case 2) If P is not indecomposable then (7.3.6) says that it has $P(\lambda)$ as a direct summand. We just showed that it maps surjectively on $M(\lambda)$ which contradicts the minimality of P.

Hence proved that $P(\lambda)$ is also the minimal projective cover of $M(\lambda)$. ■

Theorem 7.5.2. Each projective module in $\mathcal{O}$ has a standard filtration (Verma flag). In particular, $(P(\lambda) : M(\mu)) \neq 0$ only if $\mu \geq \lambda$ and $(P(\lambda) : M(\lambda)) = 1$.

Proof: We know that Category $\mathcal{O}$ is Krull-Schmidt (see 6.1.3.2), and each projective module will decompose as a finite direct sum of indecomposable projective modules. From 7.4.2, we know that a module has standard filtration if and only if its direct summands have standard filtration. Thereom 7.3.5 says that indecomposable projectives in $\mathcal{O}$ are of minimal projective covers of simple modules. So if we prove the theorem for projective modules $P(\lambda)$ for all $\lambda \in \mathfrak{h}^*$, we are done.

Let $\mu := \lambda + n\rho$ be dominant. Then using the highest weight theorem (3.5.2) we know that, $\dim L(n\rho) < \infty$ because $n\rho$ is dominant integral. The theorem (7.3.1) says that $P := M(\mu) \otimes L(n\rho)$ is projective. From (5.4.1) we know that P has standard filtration.

We know that $M(\lambda)$ and $L(\lambda)$ are quotients of P. So, $\dim \mathrm{Hom}_{\mathcal{O}}(P, L(\lambda)) \neq 0$. Using (7.3.6) we know that $P(\lambda)$ occurs as a direct summand of P. We know that

P has a standard filtration and theorem 7.4.2, implies that its direct summands have standard filtration. So, $P(\lambda)$ has a standard filtration. ■.

Theorem 7.5.3. (**BGG Reciprocity**) For $\lambda, \mu \in \mathfrak{h}^*$, we have:

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$$

Remarks: From 7.2.3 we know that $M(\mu)$ and $M(\mu)^\vee$ have the same composition factor multiplicities. So we can substitue $[M(\mu) : L(\lambda)]$ with $[M(\mu)^\vee : L(\lambda)]$ in this theorem.

Notice that the theorem allows us to compute Verma flag multiplicities of indecomposable projective modules using composition factor multiplicities of a Verma modules.

Proof: Substituting $M = P(\lambda)$ in 7.4.4 (b) we get:

$$(P(\lambda) : M(\mu)) = \dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee)$$

From 7.3.6 we get:

$$\dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee) = [M(\mu)^\vee : L(\lambda)]$$

which completes the proof. ■

Appendix A

A.1 Some Theory of Algebras

1. An algebra over a field k is a vector space over k with a bilinear map $(V, \cdot)$:

$$\cdot : V \times V \to V.$$

   - If $\cdot$ is associative, then it is an associative algebra.
   - If $\cdot$ is commutative, then it is a commutative/symmetric algebra.
   - If $\cdot$ is anti-commutative and satisfies the Jacobi identity, it is a Lie algebra.

2. A map $f : A \to B$ is called a k-algebra homomorphism between algebras A and B over a field k if:

   - f is k-linear.
   - f preserves the product: $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in A$.

3. A vector subspace $W \subset A$ of a k-algebra A is called a subalgebra if it is closed under the product:

$$w_1 \cdot w_2 \in W \quad \forall w_1, w_2 \in W.$$

4. A vector subspace $I \subset A$ of a k-algebra A is called a left ideal if:

$$a \cdot x \in I \quad \forall a \in A, x \in I.$$

A.2 Basics of Semi-Simple Lie algebras

Semi-Simple Lie Algebras

For a Lie algebra L, the derived series of L is the series of ideals:

$$L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq L^{(3)} ...$$

such that $L^{(1)} = L'$ and $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$. If for some $n \in \mathbb{N}$ if $L^{(n)} = 0$ then we say L is *solvable*. The largest solvable ideal of L is called the *radical* of L, denoted $\mathrm{Rad}(L)$.

A Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ (or $\mathbb{R}$) is called *semi-simple* if it has a trivial radical.

Cartan Subalgebras

A *Cartan subalgebra* ($\mathfrak{h}$) is a maximal abelian subalgebra consisting of semisimple (diagonalizable) elements. $\mathfrak{h}$ is nilpotent and self-normalizing. The dimension of $\mathfrak{h}$ is called the *rank* of $\mathfrak{g}$.

Dual of the Cartan Subalgebra

The dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ can be naturally identified with a Euclidean space:

- The *Killing form* of a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form:

  $$B(X, Y) = \mathrm{Tr}\big(\mathrm{ad}(X)\,\mathrm{ad}(Y)\big), \quad X, Y \in \mathfrak{g},$$

  where $\mathrm{ad}(X)$ denotes the adjoint endomorphism $\mathrm{ad}(X)(Z) = [X, Z]$ for $Z \in \mathfrak{g}$. The Killing form is invariant under the adjoint action.

- Cartan's criterion says that $\mathrm{Rad}(\mathfrak{g}) = 0$ is equivalent to the Killing form of $\mathfrak{g}$ being non-degenerate. Hence, semi-simple Lie algebras decompose into a direct sum of simple ideals orthogonal with respect to the Killing form.

- The restriction of the Killing form to $\mathfrak{h}$ is non-degenerate, which induces an inner product on $\mathfrak{h}^*$.

- With this inner product, $\mathfrak{h}^*$ is isomorphic to a Euclidean space.

Cartan Decomposition

For a semi-simple Lie algebra $\mathfrak{g}$, the Cartan decomposition refers to the root space decomposition relative to a Cartan subalgebra $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where:

- $\Phi \subset \mathfrak{h}^*$ is the set of *roots*.
- Each root space $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$.
- The decomposition is orthogonal with respect to the Killing form.

Simple Roots

Within the root system Φ, one can choose a basis of *simple roots* $\Delta \subset \Phi$ with the following properties:

- Every root $\alpha \in \Phi$ can be expressed as an integer linear combination of elements of Δ, with all coefficients either non-negative or non-positive.
- The set Δ is minimal with this property, and its cardinality equals the rank of $\mathfrak{g}$.
- The choice of Δ determines a partition of Φ into positive and negative roots.

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