Nonlocal Ergodic Control Problem with Near-monotone Cost

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by

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Certificate

This is to certify that this dissertation entitled Nonlocal Ergodic Control Problem with Near-monotone Cost towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune, represents study/work carried out by Bhavana Musunuriat Indian Institute of Science Education and Research under the supervision of Dr. Anup Biswas, Associate Professor, Department of Mathematics during the academic year 2019-2024.

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This thesis is dedicated to my parents, my sister and my mentor.

Declaration

I hereby declare that the matter embodied in the report entitled Nonlocal Ergodic Control Problem with Near-monotone Cost are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Associate Professor Dr. Anup Biswas and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

Bhavana

Bhavana Musunuri

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Abstract

This thesis aims to provide an overview of the Integro-differential equations of fractional type and the associated regularity results for the viscosity solution. In the second part of the thesis, we consider a nonlocal Hamilton-Jacobi equation appearing in the study of ergodic control problems governed by $2s$-stable processes. We show the existence and uniqueness of solution eigen-pair to this equation.

Contents

Introduction

A standard example of a time-independent partial differential operator is the Laplacian operator, often denoted as Δ. This operator also appears as a generator of the Brownian motion (up to a time change). If we replace the Brownian motion by a $2s$-stable process ($s \in (0,1)$), then the corresponding generator gives rise to an integro-differential operator, also known as the s-fractional Laplacian and denoted by $-(-\Delta)^s$. The analysis of this operator is quite different from its local counterpart because of its nonlocal nature. In a series of work [6–8] Caffarelli & Silvestre developed the theory of nonlinear nonlocal operator (these include Harnack inequality, ABP estimates, Hölder regularity, etc). This theory was developed in the framework of viscosity solution (see Chapter 1, for a definition) and then shown that under sufficient regularity assumptions on the nonlocal kernels, the viscosity solutions have $C^{2s+\eta}$ regularity, making the viscosity solution a classical solution. Very recently, the $C^{2s+\eta}$ regularity is established for general Pucci type operators by Serra [15] and $C^{1,\gamma}$ regularity can be found in Biswas & Khan [3]. We review these results in Chapter 1.

Our original contribution to this thesis is given in Chapter 2. We will focus on studying the following nonlocal Hamilton-Jacobi (HJ) equation which is considered in Chapter 2,

$$-(-\Delta)^s w + \min_{u \in \mathbb{U}} \{b(x,u) \cdot \nabla w + c(x,u)\} = \lambda^* \quad \text{in } \mathbb{R}^d.$$

The above HJ equation appears in the study of the ergodic optimal control problem. In the above, $\mathbb{U}$ represents the action (or control) set, that is, an action can only be chosen from $\mathbb{U}$. $b : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d$ is the controlled drift that manipulates the stochastic dynamics by applying the input variable u. $c : \mathbb{R}^d \times \mathbb{U} \to [0,\infty)$ is known as the running cost per unit time. The value λ^* also referred to as the principal eigenvalue, is called the minimal ergodic value. In an ideal situation, that is, when $w \in C^1(\mathbb{R}^d)$, we can extract an optimal control as a minimizing selector of the above HJ equation. This makes the above HJ equation an important tool in the study of optimal control problems. For more details (albeit in the local setting), we refer to the book [2]. Solving the above HJ equation (that is, to find (w, λ^*)) under a general setting seems difficult. Therefore, in practice, we rely on two settings (i) the stable case and (ii) the near-monotone case. Under the stable case, the drift vector is assumed to be stable under every stationary Markov control. This can be assured by imposing the existence of an inf-compact Lyapunov function. In this setting, a complete study of the nonlocal HJ equation can be found in the very recent work of Biswas & Khan [3]. So we focus on the second case, that is, the near-monotone case in this thesis. We say the controlled system is *near-monotone* if

$$\liminf_{|x|\to\infty} \inf_{u\in\mathbb{U}} c(x,u) > \lambda^*.$$

In the first instance, the above definition might appear misleading because λ^* is unknown. In practice, we replace λ^* with the optimal ergodic value. Note that the above inequality always holds if we allow c to blow up at infinity (see Assumption 3(ii) below). Under a reasonable set of hypotheses (Assumption 3), we prove the existence and uniqueness result for the above nonlocal HJ equation. For more details, see Chapter 2 below.

Structure of the thesis

The thesis is split into two parts. Chapter 1 deals with preliminary results of fractional Sobolev space and fractional Laplacian, an overview of fully nonlinear nonlocal equations and the regularity of their solutions. In chapter two, we study a control problem, which is a nonlocal equation. We proved the existence of the solution. The regularity result follows from [3] and [16] and the proof of uniqueness is inspired from [4].

Notation

1. $\mathbb{R}^d$ the d-dimensional real Euclidean space and $\mathbb{R}^1 = \mathbb{R}$.

2. $\mathbb{N}$ denotes the set of natural numbers.

3. For vectors a and b, $a \cdot b$ is the scalar product.

4. The open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we denote the ball by B_r

5. The ball of radius r centered at origin is denoted by B_r.

6. For an integer $k \geq 0$, the space $C^k(A)(C^\infty(A))$ refers to the class of all functions whose partial derivatives up to order k (of any order) exist and are continuous.

7. The set of all compactly supported functions is denoted by $C_c^\infty(A)$,

8. The space $C^{k,r}$ for $r \in (0,1)$ is the collection of functions whose partial derivatives up to order k are Hölder continuous of order r.

9. For any $\gamma > 0$, $C^\gamma(A)$ denotes the space $C^{\lfloor \gamma \rfloor, \gamma - \lfloor \gamma \rfloor}(A)$.

10. Let $u \in C^k(A)$ here A is open and bounded domain and $k \in \mathbb{N}$, we define

$$\|u\|_{C^k(A)} := \sum_{0 \leq \gamma \leq k} \left\|D^k u\right\|_{C^0(A)},$$

$$[D^k u]_{r,A} := \sup_{x \neq y \in A} \frac{|u(x) - u(y)|}{|x - y|^r},$$

$$\|u\|_{C^{k,r}(A)} = u_{C^{k,r}(A)} := \|u\|_{C^k(A)} + \sum_{|\gamma| = k} [D^k u]_{r,A}.$$

Chapter 1

A short survey on the fractional Laplacian

1.1 Fractional Sobolev Space

The fractional Laplacian, denoted by $(-\Delta)^s$, is a fundamental elliptic-linear integro-differential operator of order $2s$. There are many ways to define this operator equivalently, and interested readers can refer to [12] for additional definitions: To understand the properties of the linear space that this operator belongs to, we will refer to [9]. This thesis focuses on the case where s is between 0 and 1, and we will define the fractional Laplacian operator as follows:

$$-(-\Delta)^s u(x) = C(d,s)\int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}dy. \tag{1.1.1}$$

This operator belongs to the Sobolev space of order $2s$. Let Ω be a open, need not be smooth, set in $\mathbb{R}^d$. Fix an $s>0$ and for $p\in[1,\infty)$, we define the fractional-Sobolev space $W^{s,p}$ as:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{(d/p)+s}} \in L^p(\Omega\times\Omega) \right\}.$$

The space $W^{s,p}(\Omega)$ is a Banach space sandwiched between $L^p(\Omega)$ and $W^{1,p}(\Omega)$. This follows from the Proposition 1 below. The norm on this space is given by:

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{(d/p+s)}}\right)^p dxdy + \int_\Omega |u|^p dx\right)^{\frac{1}{p}},$$

and

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dxdy)\right)^{\frac{1}{p}},$$

is called the Gagliardo-seminorm of u. Below are some important propositions and theorems from [9].

Proposition 1. [Proposition 1.1 in [9]] Let Ω be an open set and $u : \Omega \to \mathbb{R}$ be a measurable function. Consider $p \in [1, \infty)$ and $0 < s \leq s' < 1$. Then

$$||u||_{W^{s,p}(\Omega)} \leq C||u||_{W^{s',p}(\Omega)}$$

for some positive constant $C = C(d, s, p) \geq 1$. This implies that the space $W^{s',p}(\Omega)$ is embedded in $W^{s,p}(\Omega)$.

Theorem 1.1.1. For any $s > 0$, the space $C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^d)$.

1.1.1 Extending $W^{s,p}$ functions to $\mathbb{R}^d$

When s is an integer, under certain regularity assumptions on Ω, $W^{s,p}(\Omega)$ can be extended to $W^{s,p}(\mathbb{R}^d)$. There is a similar extension theorem for fractional case. These results are important for proving embedding theorems.

Definition 1. Let $0 < s < 1$ and $1 \leq p < \infty$. The extension domain for the space $W^{(s,p)}$ is an open set $\Omega \subseteq \mathbb{R}^d$, if $\exists$ a constant $C > 0$ and for any $u \in W^{s,p}(\Omega)$ then there exists a function $\tilde{u} \in W^{s,p}(\mathbb{R}^d)$ such that

$$\tilde{u}(x) = u(x)\ \forall x \in \Omega$$

and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^d)} \leq C\,\|u\|_{W^{s,p}(\Omega)}$$

where C depends on d, s, p and Ω,

Lemma 1.1.2. Let $u \in W^{s,p}(\Omega)$ be such that $u \equiv 0$ in $\Omega \setminus K$, where Ω is an open set in $\mathbb{R}^d$ and K a compact subset $K \subseteq \Omega$. Then the extension function $\tilde{u}$ is given by

$$\tilde{u}(x) = \begin{cases} u(x), x \in \Omega, \\ 0, x \in \mathbb{R}^d \setminus \Omega, \end{cases}$$

is in $W^{s,p}(\mathbb{R}^d)$ and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^d)} \leq C \|u\|_{W^{s,p}(\Omega)},$$

where C is a positive constant.

Theorem 1.1.3. Let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set of class $C^{0,1}$. Then $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,p}(\mathbb{R}^d)$. That is, for any $u \in W^{s,p}(\Omega)$, there exists a $\tilde{u} \in W^{s,p}(\mathbb{R}^d)$ such that Ω, $\tilde{u}|_\Omega = u$ and for a constant C (dependent on d, s, p) we have

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^d)} \leq C \|u\|_{W^{s,p}(\Omega).}$$

1.1.2 Fractional Sobolev Inequality

The Gagliardo-Nirenberg Inequality is a significant result in the theory of Sobolev spaces. We have an analogue of this inequality for the fractional case. The proof is taken from [5].

Theorem 1.1.4. Let $s \in (0,1)$ and $p \in (1, \frac{d}{s})$ and $u \in C_c^\infty(\mathbb{R}^d)$, then

$$\|u\|_{L^{\frac{dp}{d-sp}}(\mathbb{R}^d)} \leq C' \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{|u(x) - u(y)|}{|x-y|^{(d/p+s)}} \right)^p dxdy \right)^{\frac{1}{p}},$$

where C' is positive constant dependent on d and s.

Proof. Let r, p^* and p be positive constants and $x \in \mathbb{R}^d$. By triangle inequality we have

$$|u(x)| \leq |u(x) - u(y)| + |u(y)|. \text{ for } y \in \mathbb{R}^d,$$

Now integrate both sides over a ball of radius r, centered at x.

$$\begin{aligned}
|B_r||u(x)| &\le \int_{B_r(x)} |u(x)-u(y)|dy + \int_{B_r(x)} |u(y)|dy \\
&= \int_{B_r(x)} \frac{|u(x)-u(y)|}{|x-y|^{p^*}}|x-y|^{p^*}dy + \int_{B_r(x)} |u(y)|dy \\
&\le r^{p^*}\int_{B_r(x)} \frac{|u(x)-u(y)|}{|x-y|^{p^*}}dy + \int_{B_r(x)} |u(y)|dy.
\end{aligned}$$

Now, we choose $p^* := \frac{d+sp}{p}$. Applying the Hölder inequality on RHS of the above inequality. The Hölder exponents are p, $\frac{p}{p-1}$ and $\frac{dp}{d-sp}$, $\frac{dp}{d(p-1)+sp}$ for the first term and the second term respectively. Then,

$$\begin{aligned}
|B_r||u(x)| \le& \int_{B_r(x)} \left(r^{\frac{d+sp}{p}} \frac{|u(x)-u(y)|}{|x-y|^{\frac{d+sp}{p}}} + |u(y)| \right) dy \\
\le& \, r^{\frac{d+sp}{p}} \left(\int_{B_r(x)} \left(\frac{|u(x)-u(y)|}{|x-y|^{(d/p+s)}} \right)^p dy \right)^{\frac{1}{p}} \left(\int_{B_r(x)} dy \right)^{\frac{p-1}{p}} \\
&+ \left(\int_{B_r(x)} |u(y)|^{\frac{dp}{d-sp}} dy \right)^{\frac{d-sp}{dp}} \left(\int_{B_r(x)} dy \right)^{\frac{d(p-1)+sp}{dp}} \\
\le& \, C' r^{d+2s} \left(\int_{B_r(x)} \left(\frac{|u(x)-u(y)|}{|x-y|^{(d/p+s)}} \right)^p dy \right)^{\frac{1}{p}} + C' r^{\frac{d(p-1)+sp}{p}} \left(\int_{B_r(x)} |u(y)|^{\frac{dp}{d-sp}} dy \right)^{\frac{d-sp}{dp}}
\end{aligned}$$

for some positive constant C'. Now by dividing both sides by r^d we obtain

$$|u(x)| \le C' r^s \left[\left(\int_{B_r(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dy \right)^{\frac{1}{p}} + r^{-\frac{d}{p}} \left(\int_{B_r(x)} |u(y)|^{\frac{dp}{d-sp}} dy \right)^{\frac{d-sp}{dp}} \right].$$

This implies,

$$|u(x)|^{\frac{dp}{d-sp}} \le C' r^{\frac{dsp}{d-sp}} \left(\alpha^{\frac{1}{p}} + r^{\frac{-d}{p}} \beta^{\frac{d-sp}{dp}} \right)^{\frac{dp}{d-sp}},$$

where

$$\begin{aligned}
\alpha &= \int_{B_r(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dy, \\
\beta &= \int_{B_r(x)} |u(y)|^{\frac{dp}{d-sp}} dy.
\end{aligned}$$

Now take $r := \frac{\beta^{\frac{d-sp}{d^2}}}{\alpha^{\frac{1}{d}}}$. Then

$$|u(x)|^{\frac{dp}{d-sp}} \leq C' \alpha \beta^{\frac{sp}{d}}.$$

Integrating over x, we get

$$\begin{aligned}\int_{\mathbb{R}^d} |u(x)|^{\frac{dp}{d-sp}} dx &\leq \int_{\mathbb{R}^d} C' \bigg(\int_{B_r(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dy \bigg) \bigg(\int_{B_r(x)} |u(y)|^{\frac{dp}{d-sp}} dy \bigg)^{\frac{sp}{d}} dx \\ &\leq C' \bigg(\int_{\mathbb{R}^d} \int_{B_r(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dy\, dx \bigg) \bigg(\int_{\mathbb{R}^d} |u(y)|^{\frac{dp}{d-sp}} dy \bigg)^{\frac{sp}{d}} \\ &\leq C' \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dy\, dx \right] \bigg(\int_{\mathbb{R}^d} |u(y)|^{\frac{dp}{d-sp}} dy \bigg)^{\frac{sp}{d}}.\end{aligned}$$

Further simplification will finish the proof.

□

1.1.3 The space $H^s(\mathbb{R}^d)$

The material in this sub-section is from [9]. For the case $p = 2$, the spaces $W^{s,2}(\mathbb{R}^d)$ and $W_0^{s,2}(\mathbb{R}^d)$ are inner product spaces(Hilbert Space). They are usually denoted by $H^s(\mathbb{R}^n)$ and $H_0^s(\mathbb{R}^n)$, respectively. We now define fractional Laplacian:

$$\begin{aligned}(-\Delta)^s u(x) =& C(d,s) \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy \\ =& C(d,s) \lim_{\epsilon \to 0^+} \int_{\mathscr{C} B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy,\end{aligned} \tag{1.1.2}$$

where P.V. denotes 'in the principle value sense' and the constant $C(d,s)$ is precisely given by

$$C(d,s) = \bigg(\int_{\mathbb{R}^d} \frac{1-\cos(\zeta_1)}{|\zeta|^{d+2s}} d\zeta \bigg)^{-1} = \frac{2^{2s} s \Gamma(\frac{d}{2}+s)}{\pi^{\frac{d}{2}} \Gamma(1-s)}. \tag{1.1.3}$$

The readers can refer to [5, Lemma 3.1.3] to find a proof that proves the equivalence of both constants in (1.1.3). To avoid dealing with singularity in (1.1.2), we use (1.1.1) as the definition of fractional Laplacian. The following proposition describes the relation between the space $H^s(\mathbb{R}^d)$ and the fractional Laplacian operator.

Proposition 2. Let $s \in (0,1)$ and let $u \in H^s(\mathbb{R}^d)$, then

$$[u]^2_{H^s(\mathbb{R}^d)} = \frac{1}{C(d,s)} \left\|(-\Delta)^{\frac{s}{2}} u\right\|^2_{L^2(\mathbb{R}^d)}.$$

Proposition 3. Let $u \in C_c^\infty(\mathbb{R}^d)$, then u satisfies the following:

(i) $\lim_{s\to 0^+}(-\Delta)^s u = u$.

(ii) $\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u$.

Proof. Let x be a point in $\mathbb{R}^d$. Let $R_0 > 0$ be such that support of u is in B_{R_0} and $R = R_0 + |x| + 1$. Then by the definition of the fractional Laplacian we have,

$$\begin{aligned}\left|\int_{B_R} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}} dy\right| &\leq \|u\|_{C^2(\mathbb{R}^d)} \int_{B_R} \frac{|y|^2}{|y|^{d+2s}} dy \\ &\leq \omega_{d-1} \|u\|_{C^2(\mathbb{R}^d)} \int_0^R \frac{1}{\rho^{2s-1}} d\rho \\ &= \frac{\omega_{d-1} \|u\|_{C^2(\mathbb{R}^d)} R^{2-2s}}{2(1-s)}.\end{aligned}$$

In the second inequality we applied variable change and ω_{d-1} is $d-1$ dimensional measure of S^{d-1}. To the above inequality we will apply Corollary 4.2 in [9], it states: Let $C(d,s)$ be as in the definition of fractional Laplacian, then following hold,

$$\begin{aligned}&(1)\ \lim_{s\to 1^-} \frac{C(d,s)}{s(1-s)} = \frac{2d}{\omega_{d-1}};\\ &(2)\ \lim_{s\to 0^+} \frac{C(d,s)}{s(1-s)} = \frac{2}{\omega_{d-1}}.\end{aligned}$$

We obtain

$$\lim_{s\to 0^+} -C(d,s) \int_{B_R} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}} dy = 0.$$

Also, for $|y| \geq R$, we have $|x \pm y| \geq |y| - |x| \geq R - |x| > R_0$, this gives $u(x \pm y) = 0$. Now,

$$-\int_{\mathbb{R}^d \setminus B_R} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} dy = 2u(x) \int_R^{+\infty} \omega_{d-1} \frac{1}{\rho^{2s+1}} d\rho$$
$$= \frac{\omega_{d-1} R^{-2s}}{s} u(x).$$

Applying the corollary again proves (i). Now for (ii), when $s \to 1$ and y is outside B_1,

$$\left| \int_{\mathbb{R}^d \setminus B_1} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} dy \right| \leq 4 \|u\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus B_1} \frac{1}{|y|^{d+2s}} dy$$
$$\leq 4\omega_{d-1} \|u\|_{L^\infty(\mathbb{R}^d)} \int_1^\infty \frac{1}{\rho^{2s-1}} d\rho$$
$$= \frac{2\omega_{d-1} \|u\|_{L^\infty(\mathbb{R}^d)}}{s}.$$

Then applying the corollary, we get

$$\lim_{s \to 1^-} -C(d,s) \int_{\mathbb{R}^d \setminus B_R} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} dy = 0.$$

Now, for y in B_1,

$$\left| \int_{B_1} \frac{u(x+y) + u(x-y) - 2u(x) - D^2u(x)y \cdot y}{|y|^{d+2s}} dy \right| \leq 4 \|u\|_{C^3(\mathbb{R}^d)} \int_{B_1} \frac{|y|^3}{|y|^{d+2s}} dy$$
$$\leq \omega_{d-1} \|u\|_{C^3(\mathbb{R}^d)} \int_1^\infty \frac{1}{\rho^{2s-2}} d\rho$$
$$= \frac{2\omega_{d-1} \|u\|_{C^3(\mathbb{R}^d)}}{3 - 2s}.$$

This gives us,

$$\lim_{s \to 1-} -C(d,s) \int_{B_1} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} dy = \lim_{s \to 1-} -C(d,s) \int_{B_1} \frac{D^2u(x)y \cdot y}{|y|^{d+2s}} dy. \tag{1.1.4}$$

Observe that,

$$\int_{B_1} \frac{D^2u(x)y \cdot y}{|y|^{d+2s}} dy = \sum_{i=1}^d \int_{B_1} \frac{\partial_{ii}^2 u(x) y_i^2}{|y|^{d+2s}} dy.$$

Since, for $i \neq j$, $\int_{B_1} \partial^2_{ij} u(x) y_i \cdot y_j dy = 0$, implies

$$\int_{B_1} \frac{\partial^2_{ii} u(x) y_i^2}{|y|^{d+2s}} dy = \frac{\partial^2_{ii} u(x) \omega_{d-1}}{2d(1-s)}. \tag{1.1.5}$$

By the definition of fractional Laplacian, and using equations above, we obtain:

$$\begin{aligned} \lim_{s \to 1^-} (-\Delta)^s u &= \lim_{s \to 1-} -C(d,s) \int_{B_1} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} dy \\ &= \lim_{s \to 1-} -C(d,s) \sum_{i=1}^{d} \int_{B_1} \frac{\partial^2_{ii} u(x) \omega_{d-1}}{2d(1-s)} dy \\ &= \lim_{s \to 1-} \frac{-C(d,s) \omega_{d-1}}{2d(1-s)} \sum_{i=1}^{d} \partial^2_{ii} u(x) = -\Delta u. \end{aligned}$$

□

1.2 Nonlinear Integro-Differential Equations

1.2.1 Introduction

In this section we will look into integro-differential operators with kernels comparable to the fractional Laplacian. The are nonlocal equations because the equation involves the integral operator. The nonlocal equations have diverse applications. Some popular examples are: Optimal control problems with Lévy processes, in financial mathematics-Obstacle problems, Nonlocal electrostatics, Oceanography, Image processing.

We now present results from the paper [6], in which theorems and estimates like, Harnack Inequality, Hölder Regularity, ABP estimates have been extended to the nonlinear elliptic equations.

Lévy processes are generated by linear integro-differential operators, their general form is:

$$Lu(x) = \sum_{ij} a_{ij}\partial_{ij}u + \sum_{i} b_i\partial_i u + \int_{\mathbb{R}^d}(u(x+y) - u(x) - \nabla u(x)\cdot y\chi_{B_1}(y))d\mu.$$

The first term is a second order differential it corresponds to the diffusion, the second term is drift and last term to jump part. In [6], pure jump processes have been studied. Now consider the operators :

$$I'u(x) = \sup_{\alpha} L_\alpha u(x), \tag{1.2.1}$$

$$Iu(x) = \inf_{\beta}\sup_{\alpha} L_{\alpha\beta}u(x). \tag{1.2.2}$$

These are extremal operators which come from stochastic control problems. The operators in (1.2.1) and (1.2.2) are nonlinear operators we want to study. In further sections, regularity properties for a general class of operators will be presented.
Let us consider the operators for which the kernel $K(y)$ is symmetric, that is it satisfies $K(y) = K(-y)$. It is given by:

$$Lu(x) = \int_{\mathbb{R}^d}(u(x+y) + u(x-y) - 2u(x))K(y)dy. \tag{1.2.3}$$

In what follows, we are going to use the following notation for the symmetric difference,

$$\delta(u,x,y) := u(x+y) + u(x-y) - 2u(x).$$

Then

$$Lu(x) = \int_{\mathbb{R}^d} \delta(u, x, y) K(y) dy.$$

The kernel K is a positive function satisfying

$$\int_{\mathbb{R}^d} \frac{|y|^2}{|y|^2 + 1} K(y) dy < \infty. \tag{1.2.4}$$

Here (1.2.4) is the standard Lévy-Khintchine condition.

The nonlocal equations that are used to model stochastic games are of the form:

$$Iu(x) = \inf_\beta \sup_\alpha L_{\alpha\beta} u(x),$$

where is $L_{\alpha\beta}$ is in the class of more general operators as defined in the introduction. Iu is said to be well defined if every kernel satisfies (1.2.4) in uniform sense.

Definition 2. Let ϕ be a function that satisfies the following for some vector v and a positive constant M

$$|\phi(x + y) - \phi(x) - v \cdot y| \leq M|y|^2,$$

for some point x. Then ϕ is said to be $C^{1,1}$ at the point x.

By a solution we mean a **viscosity solution** which is defined as follows.

Definition 3. An upper(lower) semicontinuous function $u : \mathbb{R}^d \to \mathbb{R}$, in $\bar{\Omega}$, is said to be a *subsolution (supersolution)* to $Iu = f$, which is given by

$$Iu \geq f \quad (Iu \leq f),$$

if the conditions given below hold:

- $x \in \Omega$,
- N is a neighbourhood in Ω,
- $\phi \in C^2$ in N,
- $\phi(x) = u(x)$,

- $\phi(y) > u(y)(\phi(y) < u(y)) \forall y \in N \setminus \{x\}$.

Let u be a solution. Here x is the point of contact of the a test function ϕ, that touches u from either from above or from below, in a neighbourhood N of x. Take $\phi = u$ for x outside of N. A solution is such that it is both subsolution and supersolution.

The extremal operators for the class of linear operators $\mathcal{L}$ are defined as follows:

$$M^+_{\mathcal{L}} v(x) = \sup_{L \in \mathcal{L}} Lu(x), \tag{1.2.5}$$

$$M^-_{\mathcal{L}} v(x) = \inf_{L \in \mathcal{L}} Lu(x). \tag{1.2.6}$$

Define $\mathcal{L}_0$ to be the class operators L satisfying (1.2.3) and

$$(2-2s)\frac{\lambda}{|y|^{d+2s}} \leq K(y) \leq (2-2s)\frac{\Lambda}{|y|^{d+2s}}. \tag{1.2.7}$$

We observe that the fractional Laplacian operator is contained in $\mathcal{L}_0$. The maximal and minimal operators on $\mathcal{L}_0$ are given by:

$$\begin{aligned} M^+_{\mathcal{L}_0} v(x) &= (2-2s)\int_{\mathbb{R}^d} \frac{\Lambda\delta(v,x,y)^+ - \lambda\delta(v,x,y)^-}{|y|^{d+2s}} dy, \\ M^-_{\mathcal{L}_0} v(x) &= (2-2s)\int_{\mathbb{R}^d} \frac{\lambda\delta(v,x,y)^+ - \Lambda\delta(v,x,y)^-}{|y|^{d+2s}} dy. \end{aligned} \tag{1.2.8}$$

Using the extremal operators we define elliptic operators.

Definition 4. Consider a class $\mathcal{L}$ of linear integro-differential operators. Assume the kernel satisfies (1.2.4). An *elliptic operator* I with respect to $\mathcal{L}$ has the following properties:

- If $u \in C^{1,1}(x)$ and bounded, then Iu is well defined.
- If $u \in C^2(\Omega)$, then Iu is continuous (Ω is an open set).
- If u, v bounded $C^{1,1}$ functions at x, then

$$M^-_{\mathcal{L}}(u-v)(x) \leq Iu(x) - Iv(x) \leq M^+_{\mathcal{L}}(u-v)(x).$$

The operators satisfying (1.2.7) have an important property. In the definition of viscosity solutions, the operator I need not be evaluated at u. For every x, where u touches the smooth function ϕ, we can find a test function in $C^{1,1}(x)$ to evaluate I at x, say v. Let I be an extremal nonlinear operator as in (1.2.1) or (1.2.2). Then these operators at the points where $u = \phi$, can be evaluated classically at x.

1.2.2 Comparison Principle

The comparison principle is an important property which will be used in proving several results in further chapters. It can be understood as that the difference between supersolution and subsolution satisfies the maximum principle. The solutions we obtain are not classical solutions in C^2. So we cannot compare the solutions point wise. We will impose an ellipticity assumption on the class of operators to define the comparison principle.

Assumption 1. For each $L \in \mathcal{L}$, such that $L\phi > \delta$ in B_R, ($R > R_0$) and there exists a constant $R_0 \geq 1$ so that $R > R_0$ and

$$\phi(x) = \min(1, \frac{x^2}{R^3}).$$

Remark 1.2.1. It can be verified that above assumption holds for the fractional- Laplacian kernel. Clearly $L\phi(x) > 0$ when $\phi = 1$. Then for $|x| < R$ we have,

$$\begin{aligned} L\phi = \int_{\mathbb{R}^d} \left(1 \wedge \frac{|x|^2}{R^3}\right) K(x)dx &\geq \frac{1}{R^3} \int_{|x|<R^{\frac{3}{2}}} |x|^2 K(x)dx \\ &= \frac{1}{R^3} \int_{|z|<1} R^3 |z|^2 K(R^{\frac{3}{2}} z) R^{\frac{3d}{2}} dz \\ &\geq \int_{|z|<1} |z|^2 \lambda \frac{R^{\frac{3d}{2}}}{|z|^{d+2s} R^{\frac{3(d+2s)}{2}}} dz \\ &= \int_{|z|<1} \lambda |z|^{2-d-2s} R^{-3s} dz \\ &= \frac{1}{R^{3s}} \lambda \int_{|z|<1} |z|^{2(1-s)-d} dz = \lambda c(d, s, R), \end{aligned}$$

where in the third step we applied the (1.2.7)- condition for elliptic operator class which contains fractional Laplacian kernel. Taking $\delta = \lambda c(d, s, R)$, we obtain the desired inequality in B_R.

Assumption 1 plays a key role in establishing the comparison principle.

Theorem 1.2.1 (Comparison Principle)**.** Let $\mathcal{L}$ be a class of linear operators satisfying Assumption 1. Let I be an elliptic operator as defined in Definition 4. Let Ω be a bounded open set and u, v be functions such that

(i) u, v are bounded in $\mathbb{R}^d$,

(ii) u is upper-semicontinuous $\forall\ x \in \bar{\Omega}$,

(iii) v is lower-semicontinuous $\forall\ x \in \bar{\Omega}$,

(iv) $Iu \geq f$ and $Iv \leq f$ in Ω for some continuous function f and

(v) $u \leq v$ in $\mathbb{R}^d \setminus \Omega$.

Then $u \leq v$ in $\mathbb{R}^d$.

The above theorem is proved with the help of the following two results.

Theorem 1.2.2. Let I be an elliptic operator. Let u and v be two bounded functions satisfying the following conditions:

1. u be upper-semicontinuous in $\bar{\Omega}$ and v be lower-semicontinuous in $\bar{\Omega}$.

2. $Iu \geq f$ and $Iv \leq g$ in the viscosity sense in Ω, for some continuous f and g. Then $M^+_{\mathcal{L}}(u - v) \geq f - g$ in Ω, in the viscosity sense.

Lemma 1.2.3. Grant the setting of Theorem 1.2.1. Let u be an upper-semicontinuous $\forall$ $x \in \bar{\Omega}$ and bounded in $\mathbb{R}^d$. Assume that $M^+_{\mathcal{L}}(u) \geq 0$, that is u is a sub-solution in the viscosity sense in Ω. Then we have $\sup_{\Omega} u \leq \sup_{\mathbb{R}^d \setminus \Omega} u$.

Below are some definitions and properties semicontinuous functions, which are used in the proofs.

Definition 5. A sequence of lower-semicontinuous functions $\{u_k\}$ is said to be $\Gamma-$ convergent to u in Ω, if the following hold

- If $x_k \to x$,in Ω, then $\liminf_{k\to\infty} u_k(x_k) \geq u(x)$.

- If $\forall x \in \Omega$, then $\exists$ a sequence $x_k \to x$ satisfying

$$\limsup_{k\to\infty} u_k(x_k) = u(x).$$

Definition 6. Let u be an upper-semicontinuous function. We define the sup-convolution of approximation as:

$$u^\epsilon(x) = \sup_y u(x+y) - \frac{|y|^2}{\epsilon}.$$

Similarly, for a lower-semicontinuous function v, the inf-convolution is given as:

$$v_\epsilon(x) = \inf_y v(x+y) + \frac{|y|^2}{\epsilon}.$$

Proof of Theorem 1.2.2. First we will prove the theorem, when u, v are continuous in $\mathbb{R}^d$. From Definition 6, fix an x and y be corresponding point where supremum is attained, note that

$$u^\epsilon(x) = \sup_y u(x+y) - \frac{|y|^2}{\epsilon}$$

and

$$u^\epsilon(x) \geq u(x+y+z-z) - \frac{|y-z|^2}{\epsilon} = u^\epsilon(x) - \frac{|z|^2}{\epsilon} + 2\frac{zy}{\epsilon}.$$

Hence, $u^\epsilon(x)$ is semi-convex. We just need to show that $M^+_{\mathcal{L}}(u^\epsilon(x) - v_\epsilon(x)) \geq f - g - 2d_\epsilon$ where $\epsilon > 0$ and $x \in \Omega$. This follows from stability property of the solutions under Γ limit (Lemma 4.5 in [6]).

Let ϕ be the test function touching $(u^\epsilon - v_\epsilon)$ from above at x. Since $u^\epsilon, -v_\epsilon$ are semi-convex there exists a paraboloid touching $(u^\epsilon - v_\epsilon)$ from below $\forall x$. If ϕ is C^2 then by the definition of $C^{1,1}$ function we have both $u^\epsilon, -v_\epsilon$ as $C^{1,1}$ functions at the point x. Using [6, Lemma 5.7], $Iu^\epsilon(x)$ and Iv_ϵ are defined in classical at the point x. Then

$$M^+_{\mathcal{L}}\phi(x) = M^+_{\mathcal{L}}(u^\epsilon(x) - v_\epsilon(x)) \geq Iu^\epsilon(x) - Iv_\epsilon(x) \geq f - g - 2d_\epsilon.$$

That is $M^+_{\mathcal{L}}(u^\epsilon(x) - v_\epsilon(x)) \geq f - g - 2d_\epsilon$ in Ω in the viscosity-sense. Taking $\epsilon \to 0$ completes the first step of the proof. Now let u be upper semicontinuous in $\bar{\Omega}$ and v be lower semicontinuous in $\bar{\Omega}$.

Claim: There exists sequences u_k, v_k (upper semi-continuous and lower semi-continuous resp.) and $u_k = u$ and $v_k = v$ in $\bar{\Omega}$ $\forall$ d, such that $u_k \to u$ and $v_k \to v$ in $\mathbb{R}^d \setminus \bar{\Omega}$ and

$$Iu_k \geq f_k, \;\; Iv_k \leq g_k$$

where f_k and g_k uniformly converges to the functions f and g respectively. The existence of the sequences u_k, v_k, satisfying this claim can be obtained by mollification of u and v outside the domain Ω. Now, $u_k - u = 0$ for $x \in \Omega$, this implies that for $x \in \Omega^c$,

$$M^-_{\mathcal{L}}(u_k - u)(x) \geq -2 \int |u_k(x+y) - u(x+y)| K(y) dy =: h_k(x).$$

Here, $h_k \to 0$ as $k \to \infty$ in Ω. Let ϕ be $C^{1,1}(x)$ where ϕ touches u_k at the point x from above. Then, the function $(\phi + u - u_k)$ is a sub-solution. That is,

$$I(\phi + u - u_k)(x) \geq f(x).$$

This implies,

$$I\phi(x) \geq I(\phi + u - u_k)(x) + M^-_{\mathcal{L}}(u_k - u)(x) \geq f(x) + h_k(x).$$

Take $f_k = f + h_k$, this is required sequence f_k, that proves the theorem for u_k. Similarly, can also find g_k for v_k. Use the first step of this proof on u_k, v_k completes the proof.

$\square$

Proof of Lemma 1.2.3 . For $R > R_0$, such that $\Omega \subset B_R$, and $\epsilon > 0$, define

$$\phi_M(x) = M + \epsilon(1 - \min(1, \frac{|x|^2}{R^3})).$$

The function ϕ_M is bounded in $\mathbb{R}^d$. By Assumption 1, $M^+_{\mathcal{L}}\phi_M(x) \geq -\delta\epsilon$, forall x in B_R. Lets define $M_0 := \{\min_{\mathbb{R}^d} M : \phi_M \geq u\}$. We claim that

$$M_0 \leq \sup_{\mathbb{R}^d \setminus \Omega} u.$$

Suppose not, then for some point $x_0 \in \Omega$, $M_0 > \sup_{\mathbb{R}^d \setminus \Omega} u$. Then, ϕ_{M_0} touche u from above at x_0. This leads to contradiction as, $M^+_{\mathcal{L}}\phi_{M_0} < 0$, but by definition $M^+_L \geq 0$. Therefore,

$\forall x \in \mathbb{R}^d$, we have

$$M_0 + \epsilon \leq \sup_{\mathbb{R}^d \setminus \Omega} u + \epsilon.$$

As $\epsilon \to 0$, proves the claim. By definition,

$$u(x) \leq \phi_{M_0}.$$

in $\mathbb{R}^d$. Hence, $\sup_\Omega u \leq \sup_{\mathbb{R}^d \setminus \Omega} u$. □

Proof of Theorem 1.2.1. The proof follows by combining Theorem 1.2.2 and Lemma 1.2.3. □

1.3 Regularity Results

1.3.1 Nonlocal Alexandrov-Bakelman-Pucci Estimate

Harnack inequality and Hölder regularity for nonlinear second-order equations are some of the key results in the paper [6]. For the proof of Harnack inequality, we need nonlocal version of the ABP estimate. Recall (1.2.7) and (1.2.8). Let M^+ and M^- denote $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$. For a function u we define its concave envelope as follows...

Definition 7 (Concave Envelope)**.** Let $u : \mathbb{R}^d \to \mathbb{R}$ be such that $u \leq 0$ for $x \in B_1^c$. The concave envelope Γ of u in B_3 is given by:

$$\Gamma(x) := \begin{cases} \min\{p(x) : \text{for all planes } p \geq u^+\} & \text{in } B_3, \\ 0 & \text{in } \mathbb{R}^d \setminus B_3. \end{cases} \tag{1.3.1}$$

Theorem 1.3.1 (Theorem 8.7 in [6])**.** Let u be a non-positive function outside unit ball and Γ denote the concave envelope of u in B_3. There exists a family of open cubes $Q_j(j = 1, ..., m$) with diameter d_j and j is finite satisfying the following hold:

1. For $i \neq j$, $Q_j \cap Q_i = \varnothing$.
2. $u = \Gamma \subset \cup_{j=1}^m \bar{Q}_j$.

3. $u = \Gamma \cap \bar{Q}_j \neq \varnothing$ for any Q_j.

4. $d_j \leq \rho_0 2^{\frac{-1}{(2-2s)}}$ where $\rho = \frac{1}{8\sqrt{d}}$.

5. $|\nabla\Gamma(\bar{Q}_j)| \leq C(\max_{Q_j} f)^d |Q_j|$.

6. $|y \in 8\sqrt{d}\, Q_j : u(y) > \Gamma(y) - C(\max_{\bar{Q}_j} f) d_j^2| \geq \mu |Q_j|$.

Here $C > 0$ and $\mu > 0$ are constants depending on d, Λ, λ.

The nonlocal ABP estimate is part (5) of the above theorem, that is

$$|\nabla\Gamma(\bar{Q}_j)| \leq C(\max_{Q_j} f)^d |Q_j|.$$

One of the most important applications of the above theorem is the following point estimate which will play the central role in our Hölder and Harnack estimate.

The following special function will be useful for our calculation.

Lemma 1.3.2. Fix s_0, any value in $(0, 2)$, then $\exists$ a function ϕ such that the following holds:

- ϕ is a continuous function in $\mathbb{R}^d$,
- $\phi(x) = 0$ for $x \in B^c_{2\sqrt{d}}$,
- $\phi(x) > 2$ for $x \in Q_3$, and
- $M^-\phi - \psi(x)$ in $\mathbb{R}^d$ for some positive function ψ, and supp(ϕ) is in $\bar{B}_{1/4}$.

for every $s < s_0$.

1.3.2 Point estimates

The ABP estimate and the point estimate given in this section connects an estimate of u which is point wise to an estimate in measure.

Lemma 1.3.3. Let $s > s_0 > 0$. Suppose

1. $\inf_{Q_3} u \leq 1$,

2. u is non-negative in $\mathbb{R}^d$,

3. $M^- u \leq \epsilon_0$ in $Q_{4\sqrt{d}}$,

where $\epsilon_0 > 0,\ 0 < \mu < 1$, and $M > 1$ are constants (dependent on s_0, λ, Λ, d) then we have that $|\{u \leq M\} \cap Q_1| > \mu$.

Here $Q_r(x)$ denotes the open cube given by $\{y : |y_j - x_j| \leq \frac{r}{2}$ for all $j\}$.

Proof. Lets define $v := \phi - u$ (ϕ is the special function in Lemma 1.3.2). We will apply the ABP estimate from Theorem 1.3.1 on v. Observe that $M^+ v \geq M^- \phi - M^- u \geq -\psi - \epsilon_0$.

Let Q_j be the family of cubes as in Theorem 1.3.1. We have

$$\begin{aligned}
\max v \leq C|\nabla\Gamma(B_{2\sqrt{d}})|^{\frac{1}{d}} &\leq C\bigg(\sum_j C|\nabla\Gamma(\bar{Q}_j)|\bigg)^{\frac{1}{d}} \\
&\leq \bigg(C\sum_j (\max_{Q_j}(\psi+\epsilon_0)^+)^d |Q_j|\bigg)^{\frac{1}{d}} \\
&\leq C\epsilon_0 + C\bigg(\sum_j (\max_{Q_j}\psi^+)^d |Q_j|\bigg)^{\frac{1}{d}}.
\end{aligned}$$

In above inequalities Γ is the concave envelope of v in $B_{6\sqrt{d}}$. By hypotheses $\inf_{Q_3} u \leq 1$ and by definition $\psi \geq 2$ this gives a lower bound of v, that is $\max v \geq 1$, this implies

$$1 \ \leq C\epsilon_0 + C\bigg(\sum_j (\max_{Q_j}\psi^+)^d |Q_j|\bigg)^{1/d}.$$

Taking ϵ_0 small enough, we obtain

$$\frac{1}{2} \leq CC\bigg(\sum_j (\max_{Q_j}\psi^+)^d |Q_j|\bigg)^{1/d}.$$

$\because \psi$ is a bounded function and supp $\psi \subset \bar{B}_{1/4}$ we get

$$\frac{1}{2} \leq C\bigg(\sum_{Q_j \cap B_{1/4} \neq \varnothing} |Q_j|\bigg)^{1/d}.$$

The inequality above is a lower bound for the sum of the volumes of the cubes Q_j which intersect $B_{1/4}$,

$$\sum_{Q_j \cap B_{1/4} \neq \varnothing} |Q_j| \geq c. \tag{1.3.2}$$

The diameters d_j of the cubes Q_j for all j, are bounded by $\rho_0 2^{-1/(2-s)}$, and

$$\rho_0 2^{-1/(2-s)} < \rho_0 = 1/(8\sqrt{d}).$$

Hence, whenever $Q_j \cap B_{1/4}$, the cube $4\sqrt{d}Q_j$ will be contained in $B_{1/2}$. Here, $kQ_r(x) := Q_{kr}(x)$. Let $M_0 := \max_{B_{1/2}} \phi$. By, Theorem 1.3.1, we get

$$|\{x \in 4\sqrt{d}Q_j : v(x) \geq \Gamma(x) - Cd_j^2\}| \geq c|Q_j| \tag{1.3.3}$$

and $Cd_j^2 < C\rho_0^2$. Consider the cubes $\sqrt{d}Q_j$ for all j, Q_j that intesrets $B_{1/4}$. This gives an open cover the closure of the cubes contaoned in $B_{1/4}$. Take a subcover with finite overlapping, which covers the closure of cubes. From (1.3.2) and (1.3.3),

$$|\{x \in B_{1/2} : v(x) \geq \Gamma(x) - C\rho_0^2\}| \geq c.$$

This implies,

$$|\{x \in B_{1/2} : u(x) \leq M_0 + C\rho_0^2\}| \geq c.$$

Since, $B_{1/2} \subset Q_1$ and taking $M = M_0 + C\rho_0^2$, we have

$$|\{x \in Q_1 : u(x) \leq M\}| \geq c,$$

completes the proof. $\square$

The following lemma is a consequence of Lemma 1.3.3.

Lemma 1.3.4. Let u, μ and M be as in above Lemma 1.3.3. Then

$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k$$

for $k = 1, 2....,$. Then we get,

$$|\{u > t\} \cap Q_1| \leq kt^{-\epsilon},$$

for positive constants d, t.

Using a covering argument on the above lemma we get the following theorem:

Theorem 1.3.5. Let $u \geq 0$ in $\mathbb{R}^d$, $u(0) \leq 1$, and $M^- u \leq \epsilon_0$ in $B_{1/2}$, that is u is a supersolution. Suppose $s \geq s_0$ for some $s_0 > 0$. Then

$$|\{u > t\} \cap B_1| \leq Ct^{-\epsilon} \text{ for every } t > 0,$$

where the C is a constant dependent on λ, Λ, d and s_0.

The next theorem states that the above theorem can be scaled to B_r for any $r > 0$.

Theorem 1.3.6. Let $u \geq 0$ in $\mathbb{R}^d$, $u(0) \leq 1$, be a supersolution in B_{2r} that is $M^- u \leq C_0$ in B_{2r} . Suppose $s \geq s_0$ for some $s_0 > 0$. Then

$$|\{u > t\} \cap B_r| \leq Cr^d t^{-\epsilon}(u(0) + C_0 r^2 s)^\epsilon \;\text{ for every } t > 0$$

where C is a constant that depends on λ, Λ, d and s_0.

1.3.3 Harnack Inequality for Nonlocal Equations

The following proof of the Harnack Inequality for nonlocal equations is taken from [6], by Caffarelli and Silvestre.

Theorem 1.3.7. Let $u \geq 0$ in $\mathbb{R}^d$, $M^- u \leq C_0$ and $M^+ u \geq C_0$ in B_2. For some $s_0 > 0$ assume $s \geq s_0$. Then $u(x) \leq C(u(0) + C_0)$ for all $x \in B_{\frac{1}{2}}$.

Proof. We divide u by $u(0) + C_0$, and observe that considering $u(0) \leq 1$ and $C_0 = 1$ is enough. Let $\epsilon > 0$ be from the Theorem 1.3.6. Let $\gamma = \frac{n}{\epsilon}$ and consider the smallest value of t such that

$$u(x) \leq h_t(x) := t(1 - |x|)^{-\gamma} \text{ for every } x \in \; B_1.$$

Since, t is the minimum then there exists some $x_0 \in B_1$ such that $u(x_0) = h_t(x_0)$. Let k denote the distance between x_0 and ∂B_1 given by $d = (1 - |x_0|)$.

We are estimating the portion of the ball $B_r(x_0)$ that is covered by two sets: $\{u < u(x_0)/2\}$ and $\{u > u(x_0)/2\}$, where $r = \frac{k}{2}$. Our objective is to prove that t cannot be too large. This proof will establish the result of the theorem, which states that if the upper bound $t < C$, then $u(x) < C(1 - |x|)^{-\gamma}$.

Consider the set $A := \{u > u(x_0)/2\}$. By Theorem 1.3.5, we have

$$|A \cap B_1| \leq C|\frac{2}{u(x_0)}|^{\epsilon} \leq Ct^{-\epsilon}k^d,$$

since $|B_r| = Ck^d$, for large values of t , A covers a small portion of $B_r(x_0)$,

$$|\{u > u(x_0)/2\} \cap B_r(x_0)| \leq Ct^{-\epsilon}|B_r|. \tag{1.3.4}$$

We can arrive at a contradiction, if we prove that $|\{u < u(x_0)/2\} \cap B_r(x_0)| \leq (1 - \delta)B_r$ for $\delta > 0$. We estimate $|\{u < u(x_0)/2\} \cap B_{\theta r}(x_0)|$ for small $\theta > 0$. For all $x \in B_{\theta r}(x_0)$ we get

$$u(x) \leq h_t(x) \leq t\left(\frac{k - k\theta}{2}\right)^{-\gamma} \leq u(x_0)\left(\frac{1 - \theta}{2}\right)^{-\gamma},$$

with $((1 - \theta)/2)^{-\gamma}$ close to 1.

Let us consider

$$v(x) = \left(1 - \frac{\theta}{2}\right)^{-\gamma} u(x_0) - u(x).$$

Then, in $B_{\theta r}$ $v \geq 0$ and $M^{-}v \leq 1$ since $M^{+}u \geq -1$. Theorem 1.3.6 cannot be applied on v, since v is positive only on $B_{\theta r}$. To apply the theorem we instead consider $w = v^{+}$. Now we find the upper bound for $M^{-}w = M^{-}v^{+}$. We know that

$$M^{-}v(x) = (2 - 2s)\int_{\mathbb{R}^d} \frac{\lambda\delta(v, x, y)^{+} - \Lambda\delta(v, x, y)^{-}}{|y|^{d+2s}} dy \leq 1.$$

Therefore, for $x \in B_{\theta r/2}(x_0)$,

$$
\begin{aligned}
M^- w(x) &= (2-2s)\int_{\mathbb{R}^d} \frac{\lambda\delta(w,x,y)^+ - \Lambda\delta(w,x,y)^-}{|y|^{d+2s}}dy \\
&\leq 1 + 2(2-2s)\int_{\mathbb{R}^d\cap\{v(x+y)<0\}} -\Lambda\frac{v(x+y)}{|y|^{d+2s}}dy \\
&\leq 1 + 2(2-2s)\int_{\mathbb{R}^d\setminus B_{\theta r/2}(x-x_0)} \Lambda\frac{(u(x+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+}{|y|^{d+2s}}dy.
\end{aligned} \tag{1.3.5}
$$

Now if $u(x) \geq g_\tau := \tau(1-|4x|^2)$ for largest $\tau > 0$. Then there exists a point x_1 in $B_{1/4}$ such that $u(x_1) = \tau(1-|4x_1|^2)$. The value of $\tau \leq 1$, since $u(0) \leq 1$. Thus we the upper bound

$$
(2-2s)\int_{\mathbb{R}^d} \frac{\delta(u,x,y)^-}{|y|^{d+2s}}dy \leq (2-2s)\int_{\mathbb{R}^d} \frac{\delta(g_\tau,x_1,y)^-}{|y|^{d+2s}}dy \leq C \tag{1.3.6}
$$

and C is a constant which does not depend on $2s$. Since $M^- u(x_1) \leq 1$, then

$$
(2-2s)\int_{\mathbb{R}^d} \frac{\delta(u,x,y)^+}{|y|^{d+2s}}dy \leq C.
$$

Since $u(x_1) \leq 1$ and $u(x_1 - y) \geq 0$,

$$
(2-2s)\int_{\mathbb{R}^d} \frac{(u(x_1+y)-2)^+}{|y|^{d+2s}}dy \leq C.
$$

Using the above inequality we estimate (1.3.5).Suppose that $u(x_0) > 2$, if not t will not be large enough.

$$
\begin{aligned}
&(2-2s)\int_{\mathbb{R}^d\setminus B_{\theta r}(x-x_0)} \Lambda\frac{(u(x+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+}{|y|^{d+2s}}dy \\
&\leq (2-2s)\int_{\mathbb{R}^d\setminus B_{\theta r}(x-x_0)} \Lambda\frac{(u(x_1-x_1+x+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+}{|y-x_1+x|^{d+2s}}\frac{|y-x_1+x|^{d+2s}}{|y|^{d+2s}}dy \\
&\leq C(\theta r)^{-d-2s}.
\end{aligned}
$$

We obtain, $M^- w \leq C(\theta r)^{-d-2s}$ in $B_{\theta r/2}(x_0)$. We now apply the Theorem1.3.6 to w in $B_{\theta r/2}(x_0)$.

Recall that

$$w(x_0) = \left(1 - \frac{\theta}{2}\right)^{-\gamma} u(x_0).$$

We have

$$\begin{aligned}
|\{u > u(x_0)/2\} \cap B_{\frac{\theta r}{4}}(x_0)| &= \left|\{w > u(x_0)\left(\left(1 - \frac{\theta}{2}\right)^{-\gamma} - \frac{1}{2}\right)\} \cap B_{\theta r/4}(x_0)\right| \\
&\leq C(\theta r)^d \left(\left(\left(1 - \frac{\theta}{2}\right)^{-\gamma} - 1\right) u(x_0) + C(\theta r)^{-d-2s}(r\theta)^{2s}\right)^{\epsilon} \\
&\left(u(x_0)\left(\left(1 - \frac{\theta}{2}\right)^{-\gamma} - \frac{1}{2}\right)\right)^{-\epsilon} \\
&\leq C(\theta r)^d \left(\left(\left(1 - \frac{\theta}{2}\right)^{-\gamma} - 1\right)^{\epsilon} + \theta^{-d\epsilon} t^{-\epsilon}\right).
\end{aligned}$$

Fix a $\theta > 0$ such that the first term is small:

$$C(\theta r)^d \left(\left(1 - \frac{\theta}{2}\right)^{-\gamma} - 1\right)^{\epsilon} \leq \frac{1}{4}|B_{\theta r/4}(x_0)|.$$

Here the choice of θ is independent of t. When θ is fixed, observe that for large values of t, we also have

$$C(\theta r)^d \theta^{-d\epsilon} t^{-\epsilon} \leq \frac{1}{4}|B_{\theta r/2}|$$

and hence,

$$\left|\{u < u(x_0)/2\} \cap B_{\frac{\theta r}{4}}(x_0)\right| \leq \frac{1}{4}|B_{\theta r/4}(x_0)|,$$

which implies for t large

$$\left|\{u > u(x_0)/2\} \cap B_{\frac{\theta r}{4}}(x_0)\right| \geq c|B_r|,$$

This contradicts 1.3.4. Therefore, t cannot be large, this finishes the proof. □

1.3.4 Hölder Estimates

We refer to [6] for the following theorem.

Theorem 1.3.8. Let $s < s_0$ for some positive $s_0 > 0$. Let u be a bounded function in $\mathbb{R}^d$.

If,

$$M^{-}u \leq C_0 \text{ in } B_1,$$

$$M^{+}u \geq C_0 \text{ in } B_1$$

then there exists an $\alpha > 0$ such that $u \in C^{\alpha}(B_{1/2})$ and

$$u_{C^{\alpha}(B_{1/2})} \leq C(\sup_{\mathbb{R}^d} u + C_0)$$

for $C_0 > 0$ is a constant and α is dependent on $(\lambda, \Lambda, d, s_0)$.

The theorem follows from following lemma by scaling argument.

Lemma 1.3.9. Let $s < s_0$ for some positive $s_0 > 0$. Let u be a function satisfying

$$M^{-}u \leq \epsilon_0 \text{ in } B_1,;$$

$$M^{+}u \geq -\epsilon_0 \text{ in } B_1,$$

and also $u(x) \in [-1/2, 1/2]$ for all $x \in \mathbb{R}^d$. Then there exists a positive number α which depends on $(\lambda, \Lambda, d, s_0)$, such that $u \in C^{\alpha}$ at origin. That is,

$$|u(x) - u(0)| \leq C|x|^{\alpha}$$

where C is a constant.

Proof. We will prove that there exists sequences m_k and M_k such that $m_k \leq u \leq M_k$ in $B_{4^{-k}}$, also show that if

$$M_k - m_k \leq 4^{-k\alpha}.$$

then the Theorem 1.3.8 holds for $C = 4^{\alpha}$. Let $k > 0$, fix $m_0 = -\frac{1}{2}$ and $M_0 = \frac{1}{2}$. Since, $m_0 \leq u \leq M_0$ in $\mathbb{R}^d$. We want to find subsequences the of M_k and m_k using induction.

Suppose that, for $k \in \mathbb{N}$, we have sequences up to m_k and M_k. We will prove that the sequences can be continued, by finding m_{k+1} and M_{k+1}. In the ball $B_{4^{-k-1}}$, either $u \geq (M_k + m_k)/2$ in atleast half points (of the measure) or $u \leq (M_k + m_k)/2$ in atleast half points. Suppose,

$$\left| \{u \geq \frac{M_k + m_k}{2}\} \cap B_{4^{-k-1}} \right| \geq \frac{|B_{4^{-k-1}}|}{2}.$$

Take,

$$v(x) := \frac{u(4^{-k}x - m_k)}{(M_k - m_k)/2}.$$

Then $v(x)$ is positive when $x \in B_1$ and

$$|\{v \geq 1\} \cap B_{1/4}| \geq |B_{1/4}|/2.$$

Also, $M^- u \leq \epsilon_0$ in B_1. Then in B_{4^k},

$$M^- v \leq \frac{4^{-(2s)k}\epsilon_0}{(M_k - m_k)/2} = 2\epsilon_0 4^{k(2s-\alpha)} \leq 2\epsilon_0 \text{ for } \alpha < 2s.$$

By induction, for $j \geq 1$, we obtain

$$v \geq \frac{m_{k-j} - m_k}{(M_k - m_k)/2} \geq \frac{m_{k-j} - M_{k-j} + M_k - m_k}{(M_k - m_k)/2} \geq -2 \cdot 4^{\alpha j} + 2 \geq 2(1 - 4^{\alpha j}) \text{ in } B_{2^j}.$$

Then we have, $v(x) \geq 2(1 - |4^{\alpha j}x|)$ in B_1^c. Now define $w(x) := \max(v, 0)$ then $M^- w \leq M^- v + 2\epsilon_0$ in $B_{3/4}$ for small α. For some $x \in B_{1/4}$, we apply Theorem 1.3.6 in $B_1(x)$ which gives,

$$C(w(x) + 2\epsilon_0)^{\epsilon} \geq |\{w > 1\} \cap B_{1/2}(x)| \geq 1/2|B_{1/4}|.$$

As ϵ_0 be arbitrarily small, we get $w \geq \theta$ in $B_{1/4}$ for some $\theta > 0$. If $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta((M_{k+1} - m_k)/2)$, we obtain $m_{k+1} \leq u \leq M_{k+1}$ in $B_{2^{k+1}}$. So, taking α, θ small enough, such that $(1 - \theta) - 4^{-\alpha}$, we get $M_{k+1} - m_{k+1} = 4^{-\alpha(k+1)}$.

Now, for the second case, that is

$$\left| \{u \leq \frac{M_k + m_k}{2}\} \cap B_{4^{-k}} \right| \geq \frac{|B_{4^{-k}}|}{2}.$$

Now v' be a function defined as below:

$$v'(x) := \frac{M_k - u(4^{-k}x)}{(M_k - m_k)/2}.$$

Following similar steps, we can obtain $M^+ u \geq -\epsilon_0$. $\square$

1.3.5 $C^{1,\gamma}$ Regularity of solutions

The regularity result in [3] of viscosity solutions is more generalised than that of in [6]. We will consider the following equation:

$$\mathcal{A}u(x) := \inf_{t\in T} \sup_{j\in J} \left[I_{t,j}[u](x) + b_{tj}(x) \cdot \nabla u(x) + g_{tj}(x) \right] = 0 \tag{1.3.7}$$

where T, J are indexing sets and

$$I_{t,j}[u](x) + b_{tj}(x) = \int_{\mathbb{R}^d} \delta(u, x, y) \frac{k_{t,j}(x, y)}{|y|^{d+2s}} dy.$$

The kernel k_{tj} is symmetric in y and satisfies

$$(2-2s)\lambda \le k_{t,j}(x, y) \le (2-2s)\Lambda$$

$\forall x, y \in \mathbb{R}^d$ and $0 < \lambda < \Lambda$. The kernel k_{tj} is symmetric in y and satisfies ellipticity condition as in Definition 4. The coefficients of (1.3.7) are under the following assumptions:

Assumption 2.
- $b_{tj},\ g_{tj}$ are continuous and

$$\sup_{t,j} \|b_{tj}\|_{L^\infty} < \infty, \text{ and} \sup_{t,j} \|g_{tj}\|_{L^\infty} < \infty.$$

- $k_{t,j}(x, y)$ is continuous, uniformly in x, t, j and y.

We prove the following regularity result in this section.

Theorem 1.3.10. Let $s \in (1/2, 1)$. Assumptions 2 hold in B_1 and

$$\sup_{t,j} \|b_{t,j}\|_{L^\infty}(B_1) \le C_0.$$

Then a $\gamma \in (0, 2s-1)$ exists so that for $u \in C(\mathbb{R}^d)$ a bounded viscosity solution to (1.3.7) in B_1 we get

$$\|u\|_{C^{1,\gamma}}(B_{\frac{1}{2}}) \le C\bigg(\|u\|_{L^\infty(\mathbb{R}^d)} + \sup_{t,j} \|g_{t,j}\|_{L^\infty(B_1)} \bigg),$$

where C depends on d, s, λ, Λ.

Lemma 1.3.11. Let$s > 1/2$ and I be an integro-differential operator which is elliptic in the above sense. Given $x_0 \in \mathbb{R}^d$, $r > 0, c > 0$, and $p(x) = a \cdot x + b$, define $\tilde{I}$ by

$$\tilde{I}\bigg(\frac{v(x_0 + r\cdot) - p(x_0 + r\cdot)}{c}\bigg)(x) = \frac{r^{2s}}{c} I(cv)(x_0 + rx).$$

Then $\tilde{I}$ is elliptic with the same ellipticity constants.

Proof of Theorem 1.3.10. We argue by contradiction. Suppose there exists $I_k, u_k, b^k_{t,j}$ and $\{g^k_{t,j}\}_k$ satisfying

$$\sup_{t,j} |b^k_{t,j}| \leq C_0 \text{ and } \|u_k\|_{L^\infty(\mathbb{R}^d)} + \sup_{t,j} \left\|g^k_{t,j}\right\|_{L^\infty(B_1)} = 1,$$

$$\text{but } \|u_k\|_{C^\beta(B_{1/2})} \to k \to \infty + \infty, \text{ where } \beta = 1 + \gamma.$$

By Lemma 5.3 of [3], there exists $\mu^*(k, r, z)$ and $\nu^*(k, r, z)$ such that

$$\sup_k \sup_{r>0} \sup_{B_{1/2}} r^{-\beta} \|u_k - p(k, r, z)\|_{L^\infty(B_r(z))} = +\infty, \tag{1.3.8}$$

where

$$(\mu^*(k, r, z), \nu^*(k, r, z)) = \arg\, \min_{(\mu,\nu)\in\mathbb{R}^d\times\mathbb{R}} \int_{B_r(z)} (u_k(x) - \mu \cdot (x - z) + \nu)^2 dx$$

and

$$l(k, r, z) = \mu^*(k, r, z) \cdot (x - z) + \nu^*(k, r, z).$$

Now for any $r > 0$, define Θ as:

$$\Theta(r) := \sup_k \sup_{r'\geq r} \sup_{z\in B_{1/2}} (r')^{-\beta} \|u_k - p_{k,r',z}\|_{L^\infty(B'_r(z))}.$$

We know that u is bounded then, $\|u_k\|_{L^\infty} (\mathbb{R}^d)$ is finite. Observe that for all $r > 0$ $\Theta(r) < \infty$. Then $\Theta(r)$ is well defined. From (1.3.8), for any $M > 0$, we can find a $\tilde{r} > 0, \tilde{k} \in \mathbb{N}$ and $\tilde{z} \in B_{1/2}$ so that

$$(\tilde{r})^{-\beta} \left\|u_{\tilde{k}} - p_{\tilde{k},r,\tilde{z}}\right\|_{L^\infty(B_{\tilde{r}}(\tilde{z}))} > M.$$

Since $\Theta(r) \geq \Theta(\tilde{r}) > M$ for any $0 < r < \tilde{r}$, we have $\Theta(r)$ increasing to ∞ when r is decreasing to 0. Then there exists $r_m \geq 1/m, k_m \in \mathbb{N}$ and $z_m \in B_{1/2}$ such that

$$\frac{1}{2}\Theta(r_m) \leq \frac{1}{2}\Theta(1/m) < (r_m)^{-\beta} \left\|u_{k_m} - p_{k_m,r_m,z_m}\right\|_{L^\infty(B_{r_m}(z_m))}.$$

Then we see that, r_m converges to 0. Now, we define a function v_m as :

$$v_m(x) := \left(\frac{u_{km} - p_{k_m,r_m,z_m}}{r_m^\beta \Theta(r_m)}\right)(z_m + r_m x). \tag{1.3.9}$$

By minimality condition, we get

$$\int_{B_1} v_m dx = 0; \int_{B_1} v_m x_i dx = 0; \text{ and } \|v_m\|_{L^\infty(B_1)} \geq 1/2. \tag{1.3.10}$$

Claim:

$$\|v_m\|_{L^\infty(B_R)} \leq CR^\beta, \ \forall R \geq 1. \tag{1.3.11}$$

To get the growth bound, observe that for $z \in B_{1/2}, k \in \mathbb{N}$ and $r' \geq r > 0$,

$$\|u_k - p_{k,r',z}\|_{L^\infty(B_{r'}(z))} \leq (r')^\beta \Theta(r')$$

and

$$\|u_k - p_{k,2r',z}\|_{L^\infty(B_{2r'}(z))} \leq (2r')^\beta \Theta(2r').$$

This implies,

$$\begin{aligned}\|p_{k,2r',z} - p_{k,r',z}\|_{L^\infty(B_{r'}(z))} &\leq \|u_k - p_{k,r',z}\|_{L^\infty(B_{r'}(z))} + \|u_k - p_{k,2r',z}\|_{L^\infty(B_{2r'}(z))} \\ &\leq (r')^\beta \Theta(r') + (2r')^\beta \Theta(2r').\end{aligned}$$

Hence by monotonicity property of Θ, for any $r' \geq r > 0$

$$\|p_{k,2r',z} - p_{k,r',z}\|_{L^\infty(B_{r'}(z))} \leq (2^\beta + 1)(r')^\beta \Theta(2r).$$

If we take $R = 2^N$, $N \geq 1$ and $r'_j = 2^j r$ in the above inequalities, then we get

$$\begin{aligned}
\frac{|\mu^*(k, Rr, z) - \mu^*(k, r, z)|}{r^{\beta-1}\Theta(r)} &\leq \sum_{j=1}^{N} \frac{|\mu^*(k, r'_j, z) - \mu^*(k, r'_{j-1}, z)|}{r^{\beta-1}\Theta(r)} \\
&= \sum_{j=1}^{N} 2^{(\beta-1)(j-1)} \frac{|\mu^*(k, r'_j, z) - \mu^*(k, r'_{j-1}, z)|}{2^{(\beta-1)(j-1)} r^{\beta-1}\Theta(r)} \\
&\leq C \sum_{j=1}^{N} 2^{(\beta-1)(j-1)} \leq C \frac{(2^{(\beta-1)N} - 1}{2^{(\beta-1)} - 1} \leq \frac{C}{2^{(\beta-1)} - 1} R^{\beta-1}.
\end{aligned} \tag{1.3.12}$$

We can similarly prove that

$$\frac{|\nu^*(k, Rr, z) - \nu^*(k, r, z)|}{r^{\beta}\Theta(r)} \leq CR^{\beta}. \tag{1.3.13}$$

From the definition of v, we have

$$\|v_m\|_{L^\infty(B_r)} \leq \frac{\|u_{k_m} - p_{k_m, r_m, z_m}\|_{L^\infty(B_{r_m}(z_m))}}{(r_m)^{\beta}\Theta(r_m)} \leq R^{\beta}\frac{\Theta(Rr_m)}{\Theta(r_m)} + \frac{\|p_{k_m, Rr_m, z_m} - p_{k_m, r_m, z_m}\|_{L^\infty(B_{r_m}(z_m))}}{(r_m)^{\beta}\Theta(r_m)}.$$

Also, for any $x \in B_{Rr_m}(z_m)$

$$\begin{aligned}
|p_{k_m, Rr_m, z_m} - p_{k_m, r_m, z_m}| &< |\mu^*(k_m, Rr_m, z_m) - \mu^*(k_m, r_m, z_m)||x - z_m| + |\nu^*(k_m, Rr_m, z_m) - \nu^*(k_m, r_m, z_m)| \\
&\leq Rr_m|\mu^*(k_m, Rr_m, z_m) - \mu^*(k_m, r_m, z_m)| + |\nu^*(k_m, Rr_m, z_m) - \nu^*(k_m, r_m, z_m)|.
\end{aligned}$$

Using, (1.3.12) and (1.3.13), we obtain

$$\begin{aligned}
\frac{\|p_{k_m, Rr_m, z_m} - p_{k_m, r_m, z_m}\|_{L^\infty(B_{r_m}(z_m))}}{(r_m)^{\beta}\Theta(r_m)} &\leq R\frac{|\mu^*(k_m, Rr_m, z_m) - \mu^*(k_m, r_m, z_m)|}{r_m^{\beta-1}\Theta(r_m)} \\
&+ \frac{|\nu^*(k_m, Rr_m, z_m) - \nu^*(k_m, r_m, z_m)|}{r_m^{\beta-1}\Theta(r_m)} \leq C(\beta)R^{\beta} + c(\beta)R^{\beta}.
\end{aligned}$$

This gives (1.3.11). As Θ is monotone in r and from above inequality, we conclude that

$$\|v\|_{L^\infty(B_R)} \leq (1 + 2C(\beta))R^{\beta}. \tag{1.3.14}$$

Now, see that by Lemma 1.3.11, v_m satisfies the following equation in the vsicosity sense

$$M^- v_m - r_m^{2s-1} C_0 |\nabla v_m| - r_m^{2s-\beta} \frac{|b_{tj}^k \cdot \mu_m^*| + |g_{tj}^k|}{\Theta(r_m)} \leq 0 \text{ in } B_R, \tag{1.3.15}$$

$$M^+ v_m + r_m^{2s-1} C_0 |\nabla v_m| - r_m^{2s-\beta} \frac{|b_{tj}^k \cdot \mu_m^*| + |g_{tj}^k|}{\Theta(r_m)} \geq 0 \text{ in } B_R, \tag{1.3.16}$$

where $\mu_m^* := \mu_m^*(k_m, r_m, z_m)$.

Claim:

$$|\mu_m^*| \leq C(1 + \Theta(r_m)).$$

To prove this choose $l_m \in \mathbb{N}$ so that $2^{-l_m} \leq r_m \leq 2^{-(l_m-1)}$, then we observe that,

$$\begin{aligned}
|\mu^*(k_m, r_m, z_m) - \mu^*(k_m, 2^{(l_m-1)} r_m, z_m)| &\leq \sum_{j=1}^{l_m} |\mu^*(k_m, 2^j r_m, z_m) - \mu^*(k_m, 2^{(j-1)} r_m, z_m)| \\
&\leq \sum_{j=1}^{l_m} (2^{(j-1)} r_m)^{\beta-1} \frac{|\mu^*(k_m, 2^j r_m, z_m) - \mu^*(k_m, 2^{(j-1)} r_m, z_m)|}{(2^{(j-1)} r_m)^{\beta-1}} \\
&\leq \sum_{j=1}^{l_m} (2^{(j-1)} r_m)^{\beta-1} \Theta(2^{(j-1)} r_m) \leq \Theta(r_m)(r_m)^{\beta-1} \frac{2^{(\beta-1)l_m} - 1}{2^{\beta-1} - 1} \\
&\leq \Theta(r_m) \frac{2^{(\beta-1)} - 1}{2^{\beta-1}}.
\end{aligned}$$

Again, by the monotonicity property of Θ,

$$|(u_{k_m} - p_{k_m, r_m, z_m})(x)| \leq \Theta(1/2)(2^{l_m-1} r_m)^\beta,$$

for all $x \in B_{2^{l_m-1} r_m}(z_m)$, then we have

$$|(u_{k_m} - \nu^*_{k_m,(2^{l_m-1}) r_m, z_m})(x)|| \leq \Theta(1/2)(2^{l_m-1} r_m)^\beta.$$

This further implies, for small r_m,

$$|\mu^*_{k_m,(2^{l_m-1}) r_m, z_m}| \leq C\Theta(1/2),$$

then

$$|\mu^*_{k_m,r_m,z_m}| \leq |\mu^*_{k_m,r_m,z_m} - \mu^*_{k_m,(2^{l_m-1})r_m,z_m}| + |\mu^*_{k_m,(2^{l_m-1})r_m,z_m}| \leq C\Theta(r_m) + 1.$$

This proves our claim. Therefore, as b_{tj}, g_{tj} are bounded we have

$$r_m^{2s-\beta} \frac{|b^k_{tj} \cdot \mu^*_m| + |g^k_{tj}|}{\Theta(r_m)} \to 0 \quad \text{as} \quad m \to \infty.$$

Applying the regularity result from [14, Theorem 7.2] on v ,then from (1.3.14)-(1.3.16), we have that the functions $\{v_m : m \geq 1\}$ are locally Hölder continuous uniformly in m. By Arzela-Ascoli, standard diagonalization method we find a convergent subsequence v_m. It follows from (1.3.14),

$$\|v\|_{L^\infty(B_R)} \leq C(1 + R^\beta).$$

Now, We claim that existence of a translation invariant elliptic operator I_0 such that

$$I_0(v) = 0 \text{ in } \mathbb{R}^d. \tag{1.3.17}$$

Once we prove (1.3.17), it follows from [16, Theorem 3.1], that v(x)=ax+b. Pass the limit in first and second equation of (1.3.10). We get $a = 0, b = 0$. This contradicts the third inequality in (1.3.10). Hence, we have a contradiction to (1.3.17).

Thus, we just need to show (1.3.10). Let $z_m \to 0$ and $v_m \to v$ be subsequences converging uniformly on compact sets. Now, by the definition $\tilde{I}^m$, in any bounded domain Ω we have

$$\tilde{I}^m v_m - r_m^{2s-1} K_0 |\nabla v_m| - r_m^{2s-\beta} \frac{|b^k_{tj} \cdot \mu^*_m| + |g^k_{tj}|}{\Theta(r_m)} \leq 0 \text{ in } \Omega, \tag{1.3.18}$$

and,

$$\tilde{I}^m v_m + r_m^{2s-1} K_0 |\nabla v_m| - r_m^{2s-\beta} \frac{|b^k_{tj} \cdot \mu^*_m| + |g^k_{tj}|}{\Theta(r_m)} \geq 0 \text{ in } \Omega. \tag{1.3.19}$$

By [3, Lemma 5.4], $\tilde{I}^{m_k}$ weakly converges to a translation invariant elliptic operator I_0. Therefore, from above equations we see that the claim holds.

□

1.3.6 C^{2s+} Regularity of Solutions

The following result is from [15]. The operators considered in [15] satisfy the Assumptions 3, in chapter 2. We will borrow this result to complete the proof of existence of a classical solution of Theorem 2.1.1. We will consider the following nonlocal dirichlet problem:

$$\begin{cases} I(u,x) = 0 \text{ in } B_1, \\ u = g \text{ in } \mathbb{R}^d \setminus B_1. \end{cases} \tag{1.3.20}$$

I is an elliptic operator with respect to $\mathcal{L}_0$, that is it satisfies the Definition 4. Let I be of the form,

$$I(u,x) := \inf_{a\in\mathcal{A}} \left(\int_{\mathbb{R}^d} \delta(u,x,y)K(x,y)dy + c_a(x) \right)$$

where $\mathcal{A}$ is some indexing set. Assume that for every $a \in \mathcal{A}$, then the kernel K satisfies (1.2.7) and for all $x, x' \in \mathbb{R}^d$ we have:

$$\int_{B_{2r}\setminus B_r} |K_a(x,y) - K_a(x',y)|dy \leq A_0|x-x'|^\alpha \frac{2-2s}{r^{2s}},$$

and

$$\|c_a\|_{C^\alpha(B_1)} \leq C_0,$$

where A_0, C_0 are given constants.

Theorem 1.3.12. Let I be an operator satisfying the nonlocal Dirichlet problem (1.3.20), and the above mentioned assumptions hold, and g is a bounded function in $C(\mathbb{R}^d)$. Assume that either

$$(a)\ g \in C^\alpha(\mathbb{R}^d \setminus B_1)$$

or

$$(b)\ I \text{ satisfies,}$$

$$[K_a(x,y)]_{C^\alpha(\mathbb{R}^d\setminus B_\rho)} \leq \Lambda(2-2s)\rho^{-d-2s-\alpha}$$

for all $\rho > 0$.

Then there exists a classical solution $u \in C^{2s+\alpha}(B_1) \cap C(\mathbb{R}^d)$ of the Dirichlet problem (1.3.20). As a consequence, the solution u is the unique viscosity solution same(1.3.20).

Furthermore, if the solution u satisfies (a), then

$$\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C(C_0 + \|g\|_{C^\alpha(\mathbb{R}^d \setminus B_1)})$$

and in case of (b),

$$\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C(C_0 + \|g\|_{L^\infty(\mathbb{R}^d)}),$$

where C_0 is the constant in above the Hölder assumption on the kernel and C is dependent on $d, s, \alpha, \lambda, \Lambda$ and A_0.

Chapter 2

Nonlocal Ergodic Control Problem with Near-Monotone Cost

2.1 Introduction

In this chapter our main objective is to show the existence-uniqueness results for the fractional HJ (Hamilton-Jacobi) equation coming from an ergodic control problem in $\mathbb{R}^d$. More precisely, we consider the equation

$$-(-\Delta)^s w(x) + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla w + c(x,u)\} = \lambda^*, \tag{2.1.1}$$

where $s \in (\frac{1}{2}, 1)$, $\lambda^* \in \mathbb{R}, \mathbb{U}$ is a compact set in $\mathbb{R}^d$. The functions $c(x,u)$ and $b(x,u)$ referred to as the running cost and drift term, respectively. As before, a solution should be understood in the viscosity sense. Under suitable hypotheses , we show that any viscosity solution is in fact, a classical solution. For the ease of notation let us denote,

$$Lw = -(-\Delta)^s w(x) + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla w + c(x,u)\},$$

where $-(-\Delta)^s w(x)$ is the fractional Laplacian, is given by

$$-(-\Delta)^s w(x) = C(n,s)\int_{\mathbb{R}^n}(w(x+y)+w(x-y)-2w(x))\frac{1}{|y|^{d+2s}}dy.$$

Also, let $\omega_s = \frac{1}{1+|y|^{d+2s}}$. The *eigenvalue* λ^* in (2.1.1) is defined as

$$\lambda^* = \inf\{\lambda \, : \, \exists \, \text{non-negative } u \in C(\mathbb{R}^d) \cap L^1(\omega_s) \text{ satisfying } Lu \leq \lambda \text{ in } \mathbb{R}^d\}. \qquad (2.1.2)$$

Assumption 3. We will study (2.1.1) under the following assumptions

(i) There exists $V \in C^2(\mathbb{R}^d)$, a non-negative and inf-compact function, such that

$$LV < -\epsilon \quad \text{in } K^c$$

for some $\epsilon > 0$ and compact set K.

(ii) The running cost $c(x,u)$ satisfies a near-monotone type condition in the sense

$$\liminf_{|x|\to\infty} \min_{u\in\mathbb{U}} c(x,u) = \infty.$$

(iii) $b(x,u)$ is a bounded function.

(iv) For every compact set $\mathcal{K} \in \mathbb{R}^d$, there exists $\beta \in (0,1)$ such that

$$\sup_{x,x'\in\mathcal{K}} \sup_{u\in\mathbb{U}} \left[\frac{|b(x,u) - b(x',u)|}{|x-x'|^\beta} + \frac{|c(x,u) - c(x',u)|}{|x-x'|^\beta}\right] \leq C_{\mathcal{K}}.$$

Example 2.1.1. Fix $\gamma \in (1, 2s)$. Consider $V(x) \in C^2(\mathbb{R}^d)$ such that $V(x) = |x|^\gamma$ and $|x| \geq 1$.

Now for $x \in B_r$, and $r > 2$,

$$\begin{aligned} -(-\Delta)^s V &= C(d,s) \int_{B_{r+1}} \frac{|x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma}{|y|^{d+2s}} dy \\ &\leq \kappa \left\|D^2 V(x)\right\|_{L^\infty(B_r)} (r+1)^{2-2s}, \end{aligned}$$

for some constant κ. When $y \in B^c_{r+1}$, we see that $|x \pm y| \geq 1$, then we obtain

$$\begin{aligned} \left|\int_{B^c_{r+1}} \frac{|x+y|^\gamma + |x-y|^\gamma - 2V(x)}{|y|^{d+2s}} dy\right| &\leq \int_{B^c_{r+1}} \frac{2|x|^\gamma + 2|y|^\gamma - 2V(x)}{|y|^{d+2s}} dy \\ &\leq \kappa_1 (r+1)^{\gamma-2s} (r^\gamma + \|V\|_{L^\infty(B_r)}) r^{-2s} \end{aligned}$$

for some constant κ_1. Next, assume that $x \in B_r^c$. Then,

$$\int_{B_{r/2}} \frac{|x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma}{|y|^{d+2s}} dy \leq \left\| D^2 V(x) \right\|_{L^\infty(B_r)} C'.$$

Now, we will compute the integral on $B_{r/2}^c$.

$$\begin{aligned}
&\int_{B_{r/2}^c} \frac{V(x+y) + V(x-y) - 2V(x)}{|y|^{d+2s}} dy \\
&= \int_{r/2<|y|<|x|/2} \frac{V(x+y) + V(x-y) - 2V(x)}{|y|^{d+2s}} dy + \int_{|y|>|x/2|} \frac{V(x+y) + V(x-y) - 2V(x)}{|y|^{d+2s}} dy.
\end{aligned}$$

Below, we take the estimates of integral from [Example1.1 in [3]]. In the first integral, we see that $|x \pm y| > r/2 > 1$. Therefore,

$$\int_{r/2<|y|<|x|/2|} \frac{|x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma}{|y|^{d+2s}} dy \leq C|x|^{\gamma-2s},$$

and rewrite the second integral as

$$\left(\int_{|y|>|x/2|} \frac{V(x+y) + V(x-y) - 2V(x)}{|y|^{d+2s}} - \frac{|x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma}{|y|^{d+2s}} dy \right)$$

$$+ \int_{|y|>|x/2|} \frac{|x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma}{|y|^{d+2s}} dy \leq C|x|^{\gamma-2s}.$$

We are only interested in the case when $x \in B_r^c$. Choose $\gamma \in (1, 2s)$. Consider a bounded b so that $b(x,u) = \frac{x}{|x|}u$ for $|x| \geq 1$ and $\mathbb{U} = \{-1, 1\}$ (bang-bang type control set). Then, $b(x,u) \cdot \nabla V$ for $V = |x|^\gamma$ is,

$$b(x,u) \cdot \nabla V = u\gamma|x|^{\gamma-1}.$$

Fixing $u = -1$, we obtain

$$b(x,u) \cdot \nabla V = -\gamma|x|^{\gamma-1},$$

for $|x| \geq 1$. Also, choose $c : \mathbb{R}^d \times \mathbb{U} \to [0, \infty)$ so that

$$\limsup_{|x|\to\infty} \frac{c(x,-1)}{|x|^{\gamma-1}} = 0.$$

Hence

$$\begin{aligned} LV(x) &\le C(1+|x|^{\gamma-2s}) + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla V(x) + c(x,u)\} \\ &\le C(1+|x|^{\gamma-2s}) + b(x,-1)\cdot\nabla V(x) + c(x,-1) \\ &\le -\epsilon, \end{aligned}$$

for some $\epsilon > 0$ and $|x| \ge r_\circ$. We set $K = \overline{B_{r_\circ}}$. We observe the Assumption 3 holds with the above V, K and ϵ.

Without any loss of generality, we assume that $c(x,u)$ is non-negative. In what follows, $C^{2s+}_{\mathrm{loc}}(\mathbb{R}^d)$ denotes the collection of continuous function f on $\mathbb{R}^d$ satisfying the following: for every compact set $\mathcal{K}$ there exists $\eta \in (0,1)$ such that $f \in C^{2s+\eta}(\mathcal{K})$. The following is the theorem of existence and uniqueness of the solution which is the main result of this chapter.

Theorem 2.1.1. Suppose that Assumption 3 holds. Then there exists a pair $(w,\lambda^*) \in C^{2s+}_{\mathrm{loc}}(\mathbb{R}^d) \times \mathbb{R}$ satisfying

$$Lw = \lambda^* \text{ in } \mathbb{R}^d \tag{2.1.3}$$

for some non-negative w satisfying $w(0) = 0$. Furthermore, the solution w is unique up to scalar addition. Here λ^* is the *ergodic constant* given by (2.1.2).

We give the proof of the above Theorem 2.1.1 in two steps:

1. Showing the existence of a solution w_α to the α-discounted problem for $\alpha \in (0,1)$
2. Prove that the normalised solution $\bar{w}_\alpha(x) := w_\alpha(x) - w_\alpha(0)$ converges to a solution w to (2.1.3).

Brief overview of the literature and comparison

One of the main motivations to study (2.1.1) comes from the stochastic ergodic control problems, where the random noise in the controlled dynamics corresponds to some $2s$-stable process. More detailed discussion can be found in [3]. The principal eigenvalue λ^* corresponds to the optimal ergodic cost with the running cost function c. In practice, there are two types of situations that are normally considered to study ergodic control problems. These are (i) near-monotone condition, and (ii) stability condition (cf. [2]). Case (ii) is recently settled in [3] whereas linear-quadratic case with fractional Laplacian is considered in [4]. In this thesis, we consider case (i) which remained unaddressed in the literature.

2.2 α-discounted Problem

We solve the associated α-discounted problem in this section. To this goal, we let

$$Iv(x) = -(-\Delta)^s v(x) + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla v(x) + c(x,u)\} - \alpha v(x) \text{ in } \mathbb{R}^d, \tag{2.2.1}$$

where $\alpha \in (0,1)$. We borrow the Comparison principle from [1], in which a very similar model has been studied (Lemma 2.1 of [3] is the comparison principle).

Lemma 2.2.1. Suppose that Assumption 3 hold. Then there exists a viscosity solution W that satisfies

$$I\,W = 0 \quad \text{in } B_r \quad \text{and} \quad W = 0 \quad \text{in } B_r^c. \tag{2.2.2}$$

Also, W is in $C^{2s+}_{loc}(B_r)$. Furthermore, $0 \le W \le \frac{k_0}{\alpha} + V$ for some κ_0 independent of α.

Proof. The existence of a viscosity solution to (2.2.2), in bounded domain, that is in B_r for $r < \infty$ follows from [13, Corollary 5.7]. □

The constant k_0 above can be taken as $\max_{x\in K}|LV(x)|$ where K is the compact set in Assumption 3(i). Now we will extend the domains B_r to $\mathbb{R}^d$. Let W_n be a viscosity solution to (2.2.2) in B_n. Set $\overline{V} = \frac{k_0}{\alpha} + V$, where k_0 is the positive constant mentioned above. Then from Assumption 3(i) it follows that

$$\begin{aligned} I\overline{V}(x) &= -(-\Delta)^s\overline{V}(x) + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla\overline{V} + c(x,u)\} - \alpha\overline{V}(x) \\ &\le -\epsilon\, 1_{\{x\in K^c\}} - k_0 - \alpha\overline{V} \\ &\le -\epsilon 1_{\{x\in K^c\}} - \alpha V. \end{aligned}$$

In particular, we get

$$I\overline{V} \le 0 \quad \text{in } \mathbb{R}^d.$$

Then $\overline{V}$ is a super-solution to (2.2.2). It is easily seen that that 0 is subsolution to (2.2.2). By comparison principle we then have

$$0 \leq W_n \leq \overline{V} \quad \text{in } \mathbb{R}^d. \tag{2.2.3}$$

Also, from [3, Theorem 2.2], $W_n \in C^{1,\gamma}(B_{\frac{n}{2}})$ for some $\gamma \in (0,1)$.

Fix $m < n$. Define $\psi_n = W_n\chi$, where χ is a smooth cut-off function such that $\chi = 1$ when $x \in B_{\frac{3m}{4}}$ and $\chi = 0$ when $x \in B_m^c$. Let $-(-\Delta)^s W_n = h$. Then we have,

$$-(-\Delta)^s \psi_n = h + (-\Delta)^s(1-\chi)W_n \quad \text{in} \quad B_{\frac{m}{2}}.$$

Define

$$\eta(x) = -(-\Delta)^s(1-\chi)W_n(x) = \int_{B^c_{\frac{m}{4}}} \frac{(1-\chi)(W_n(x+y)+W_n(x-y))}{|y|^{d+2s}}dy$$

for $x \in B_{\frac{m}{2}}$ and $y \in B^c_{\frac{m}{4}}$. Let

$$\eta_1(x) = \int_{B^c_{\frac{m}{4}}} \frac{(1-\chi(x+y))w(x+y)}{|y|^{d+2s}}dy.$$

To simplify the notation denote $\xi = (1-\chi)w$. Then for $|x_1 - x_2| < 1/8$, we have

$$\begin{aligned}
&\eta_1(x_1) - \eta_1(x_2) \\
&= \int_{B^c_{\frac{m}{4}}(x_1)} \frac{\xi(z)}{|x_1-z|^{d+2s}}dz - \int_{B^c_{\frac{m}{4}}(x_2)} \frac{\xi(z)}{|x_2-z|^{d+2s}}dz \\
&= \int_{B^c_{\frac{m}{4}}(x_1)} \frac{\xi(z)}{|x_1-z|^{d+2s}}dz - \int_{B^c_{\frac{m}{4}}(x_1)} \frac{\xi(z)}{|x_2-z|^{d+2s}}dz + \int_{B^c_{\frac{m}{4}}(x_1)} \frac{\xi(z)}{|x_2-z|^{d+2s}}dz - \int_{B^c_{\frac{m}{4}}(x_2)} \frac{\xi(z)}{|x_2-z|^{d+2s}}dz \\
&\leq \kappa_1 \int_{B^c_{\frac{m}{4}}(x_1)} |\xi(z)| \frac{(x_1-x_2)}{|x_1-z|^{d+2s+1}}dz + \kappa_1 \max_{B_{\frac{m}{4}+1}} |\xi| \, |B_{\frac{m}{4}(x_1)} \Delta B_{\frac{m}{4}(x_2)}|,
\end{aligned}$$

for some constant κ_1. Note that

$$|B_{\frac{m}{4}(x_1)} \Delta B_{\frac{m}{4}(x_2)}| \leq \kappa_2 |x_1 - x_2|,$$

for some $\kappa_2 > 0$.

Thus η_1 is a locally Lipschitz function with Lipschitz constant independent of n.

Similarly, we can prove that $\eta_2 = \int_{B^c_{\frac{m}{4}}(x)} \frac{(1-\chi)w(x-y)}{|y|^{d+2s}} dy$ is also a locally Lipschitz function. In particular, this gives Then $(-\Delta)^s(1-\chi)W_n \leq C_0$ in $B_{\frac{n}{2}}$. Letting

$$h(x) = \min_{u \in \mathrm{U}} \{b(x,u) \cdot \nabla W_n + c(x,u)\} - \alpha(x) W_n(x),$$

we see that $h(x) \in C^\beta(\bar{B}_1)$ from [3].

Also, by Lemma 2.2.1 we have

$$\alpha W_n \leq (k_0 + V) \quad \text{in } \mathbb{R}^d.$$

Now applying [14, Theorem 7.2] on (2.2.2), we obtain

$$\|W_n\|_{C^\varsigma(\mathcal{K})} \leq C(\mathcal{K}),$$

for every compact set $\mathcal{K}$. If $\mathcal{K}$ is a compact set in B_n, we have the family $\{W_n\}$ equicontinuous on $\mathcal{K}$. Now we apply Arzelá-Ascoli theorem on W_n to extract a convergent subsequence $W_{n_k} \to W \in C(\mathbb{R}^n)$ as $n_k \to \infty$, locally uniformly on compacts.

The above Lipschitz regularity of η can be used to show that $W \in C^{2s+}_{\text{loc}}(\mathbb{R}^d)$, by [15, Theorem 1.3]. Combining the above discussion we obtain the following result.

Theorem 2.2.2. Suppose that Assumption 3 hold. Then there exists a viscosity solution W that satisfies

$$I\,W = 0 \quad \text{in } \mathbb{R}^d. \tag{2.2.4}$$

Also, W is in $C^{2s+}_{loc}(\mathbb{R}^d)$ and $0 \leq W \leq \frac{k_0}{\alpha} + V$ for some κ_0 independent of α.

2.3 Ergodic HJ equation

Lemma 2.3.1. Let w_α be a solution obtained in Theorem 2.2.2 for $\alpha \in (0,1)$ and $w_\alpha \geq 0$ in $\mathbb{R}^d$, then w_α is inf-compact, that is, for $\kappa > 0$, we have $\{w_\alpha \leq \kappa\}$ compact.

Proof. Define, for $\theta > 0$, $\eta(x)$ as

$$\eta(x) = \begin{cases} \theta[(1-|x-x_0|^2)] & \text{in } \ B_2(x_0), \\ \\ -3\theta & \text{in } B_2^c(x_0). \end{cases} \tag{2.3.1}$$

Then, for some constant κ, we have $|-(-\Delta)^s\eta(x)| \leq \kappa\theta$ in $B_1(x_0)$. Therefore, in $B_1(x_0)$ we have

$$\begin{aligned} L\eta - \alpha\eta &\geq -\kappa\theta + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla\eta + c(x,u)\} - \alpha\theta \\ &\geq -(\kappa+1)\theta - \|b\|_{L^\infty}\theta 2 + \min_{u\in\mathbb{U}} c(x,u) \\ &\geq -\theta(\kappa+1+2\|b\|_{L^\infty}) + \min_{B_1(x_0)}[\min_{u\in\mathbb{U}} c(x,u)]. \end{aligned}$$

If we choose

$$\theta = \frac{1}{2(\kappa+1+2\|b\|_{L^\infty})}\min_{B_1(x_0)}[\min_{u\in\mathbb{U}} c(x,u)]$$

then, in $B_1(x_0)$, we have

$$L\eta - \alpha\eta \geq 0 \quad \text{in}\, B_1(x_0). \tag{2.3.2}$$

Hence, η is a subsolution to (2.2.4), and also

$$w_\alpha \geq \eta \quad \text{in } B_1^c(x_0).$$

Applying comparison principle we obtain

$$w_\alpha \geq \eta \quad \text{in}\, \mathbb{R}^d.$$

In particular, $w_\alpha(x_0) \geq \eta(x_0) = \theta$. Since $\theta \to \infty$ as $|x_0| \to \infty$, we have the result. $\square$

Now we want to show the convergence of normalized solution $\bar{w}_\alpha(x) := w_\alpha(x) - w_\alpha(0)$ to a solution of (2.1.1). Towards this we first prove

Lemma 2.3.2. Under the stated hypotheses above we have a compact set $\mathcal{K}_1$, independent of α, such that

$$-\sup_{\mathcal{K}_1}|\bar{w}_\alpha| \leq \bar{w}_\alpha(x) \leq \sup_{\mathcal{K}_1}|\bar{w}_\alpha| + V(x) \quad \text{in } \mathbb{R}^d, \tag{2.3.3}$$

where V is given by Assumptions 3.

Proof. From Lemma 2.3.1, we see that $w_\alpha(x) \to \infty$, as $|x| \to \infty$. Let x_α be a point where

$w_\alpha(x)$ attains its minimum. Let x_α be the point where, w_α minimum is attained. Then, w_α being a classical solution (see Theorem 2.2.2) we obtain

$$Lw_\alpha(x_\alpha) - \alpha w_\alpha(x_\alpha) = 0, \tag{2.3.4}$$

$$\Rightarrow -(-\Delta)^s w_\alpha(x_\alpha) + \min_u \{b(x,u)\cdot\nabla w_\alpha(x_\alpha) + c(x,u)\} = \alpha w_\alpha(x_\alpha).$$

Since $-(-\Delta)^s w_\alpha(x_\alpha) \geq 0$, We observe that

$$\min_{u\in\mathbb{U}} c(x_\alpha,u) \leq \alpha w_\alpha(x_\alpha) \leq \alpha w_\alpha(0) \leq k_0 + \alpha V(0) \leq k_0 + V(0), \tag{2.3.5}$$

where in above we use the bound from Theorem 2.2.2. From Assumption 3(ii), we can find a compact set $B \in \mathbb{R}^d$, independent of α, that contains x_α. In particular, this implies

$$\begin{aligned} \bar{w}_\alpha(x) &= w_\alpha(x) - w_\alpha(0) \\ &\geq w_\alpha(x_\alpha) - w_\alpha(0) \\ &= \bar{w}_\alpha(x_\alpha) \\ &\geq -\sup_B |\bar{w}_\alpha(x)|. \end{aligned} \tag{2.3.6}$$

This proves one-side of the inequality.

To prove the other side, we define u_α as: $u_\alpha := \bar{V} - w_\alpha$. Here, w_α is a classical solution and $\bar{V} = \frac{k_0}{\alpha} + V$. Next we show that u_α is inf-compact. Consider the following operator in K^c, where K is a compact set given by Assumption 3,

$$\mathcal{G}(\cdot) = -(-\Delta)^s(\cdot) + \min_u \{b(x,u)\cdot\nabla(\cdot)\}$$

This implies,

$$\begin{aligned} \mathcal{G}u_\alpha =& -(-\Delta)^s(\bar{V} - w_\alpha) + \min_u \{b(x,u)\cdot\nabla(\bar{V} - w_\alpha)\} \\ &\leq -(-\Delta)^s\bar{V} + \min_u \{b(x,u)\cdot\nabla\bar{V} + c(x,u)\} + (-\Delta)^s w_\alpha + \min_u \{b(x,u)\cdot\nabla w_\alpha + c(x)\}. \end{aligned} \tag{2.3.7}$$

By Assumptions 3 we see that on K^c we have

$$(-\Delta)^s\bar{V}+\min_u \{b(x,u)\cdot\nabla\bar{V}+c(x,u)\}+(-\Delta)^s w_\alpha+\min_u \{b(x,u)\cdot\nabla w_\alpha+c(x)\}\leq -\epsilon-\alpha w_\alpha.$$

Let $h_\alpha=\epsilon+\alpha w_\alpha$ which is a non-negative function, going to infinity as $|x|\to\infty$. From (2.3.7), we get

$$-(-\Delta)^s u_\alpha+\min_u (b(x,u)\cdot\nabla u_\alpha)+h_\alpha\leq 0 \quad \text{in } K^c$$

In particular, u_α is a supersolution to above equation.

Repeating the argument of Lemma 2.3.1 we see that u_α is an inf-compact function. Let z_α be the point where it attains its minimum. Then,

$$u_\alpha(z_\alpha)\leq u_\alpha(x)$$

$$\Rightarrow w_\alpha(x)\leq V(x)-V(z_\alpha)+w_\alpha(z_\alpha)$$

Take $\phi(x)=V(x)-V(x_\alpha)+w_\alpha(z_\alpha)$ as a test function. From (2.2.4) we get that

$$L\phi(z_\alpha)-\alpha w_\alpha(z_\alpha)\geq 0 \quad \Rightarrow LV(z_\alpha)-\alpha w_\alpha(z_\alpha)\geq 0 \quad \Rightarrow LV(z_\alpha)\geq 0.$$

If $z_\alpha\in K^c$, then from Assumption 3(i), we have $LV(z_\alpha)\leq -\epsilon$, contradicting the above inequality. Hence, $z_\alpha\in K$. Therefore

$$\bar{w}_\alpha(x)=w_\alpha(x)-w_\alpha(0)\leq V(x)+w_\alpha(x)-w_\alpha(0)\leq V(x)+\sup_{z\in K}|\bar{w}_\alpha(z)|.$$

Taking $\mathcal{K}_1=B\cup K$, we have the proof. $\square$

Lemma 2.3.3. It holds that

$$\sup_{\alpha\in(0,1)}\sup_{\mathcal{K}_1}|\bar{w}_\alpha|<\infty.$$

Proof. Suppose, on the contrary, there exists a subsequence α_n, such that

$$\sup_{\alpha\in(0,1)}\sup_{\mathcal{K}_1}|\bar{w}_{\alpha_n}|\to\infty,$$

as $\alpha_n\to 0$. Let

$$\rho_n=\sup_{\mathcal{K}_1}|\bar{w}_{\alpha_n}|.$$

From Lemma 2.3.2, we have that

$$|\bar{w}_\alpha(x)| \leq \rho_n + V(x) \tag{2.3.8}$$

Let $v_n = \frac{\bar{w}_\alpha(x)}{\rho_n}$. Now, from (2.2.4), we then obtain

$$-(-\Delta)^s v_n + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla v_n + \frac{c(x,u)}{\rho_n}\} - \alpha v_n - \frac{\alpha_n}{\rho_n} w_{\alpha_n}(0) = 0. \tag{2.3.9}$$

From (2.3.8) we also have

$$-1 \leq v_n \leq 1 + \frac{V}{\rho_n}.$$

Hence, v_n is locally uniformly Hölder continuous by [14] and there exists a $v \in C(\mathbb{R}^d)$ such that,

$$v_n \to v,$$

uniformly converging on compact sets. Also by (2.3.8), $|v| \leq 1$, and by definition

$$\max_{\mathcal{K}_1} |v| = 1. \tag{2.3.10}$$

Also, by the stability property of viscosity solutions, we have

$$-(-\Delta)^s v + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla v\} = 0 \quad \text{in } \mathbb{R}^d.$$

This implies v is a constant function. But we know that $v(0) = 0$, a contradiction to (2.3.10). Therefore,

$$\sup_{\alpha\in(0,1)} \sup_{\mathcal{K}_1} |\bar{w}_\alpha| < \infty.$$

□

Now, we will proceed to the proof of the existence of a solution to the ergodic HJ equation.

Theorem 2.3.4. Suppose Assumptions 3 holds. Then there exists a solution $(w, \hat{\lambda}) \in C(\mathbb{R}^d) \times \mathbb{R}$ satisfying

$$Lw = \hat{\lambda} \quad \text{in } \mathbb{R}^d.$$

Also, w is inf-compact and $w \in C^{2s+}(\mathbb{R}^d)$.

Proof. From (2.2.4) we get that

$$L\bar{w}_\alpha(x) - \alpha\bar{w}_\alpha(x) - \alpha w(0) = 0 \quad \text{in } \mathbb{R}^d. \tag{2.3.11}$$

From Lemma 2.3.2 and [14], we have w_α an locally equicontinuous family, for $\alpha \in (0,1)$. Hence there exists a sequence, $\alpha_n \to 0$, such that

$$\lim_{n\to\infty} \alpha_n w_{\alpha_n}(0) = \hat{\lambda}$$

and

$$\lim_{n\to\infty} w_{\alpha_n} = w,$$

uniformly on compact sets. From (2.3.11), applying stability property of viscosity solutions and taking $\alpha \to 0$, we obtain

$$Lw - \hat{\lambda} = 0 \text{ in } \mathbb{R}^d. \tag{2.3.12}$$

By Lemma 2.3.2, we see that w is bounded from below and therefore, by Lemma 2.3.1, w is inf-compact. Proof of Theorem 2.2.2 gives $w \in C^{2s+}_{loc}(\mathbb{R}^d)$. □

Next we show that the limit $\hat{\lambda}$ is the above result is actually the principal eigenvalue in the sense of (2.1.2).

Lemma 2.3.5. It holds that $\hat{\lambda} = \lambda^*$ where λ^* is given by (2.1.2).

Proof. It is easily seen that $\lambda^* \leq \hat{\lambda}$. For a contradiction assume that $\lambda^* < \hat{\lambda}$. By definition, there exists a λ^* such that, $\tilde{\lambda} \in (\lambda^*, \hat{\lambda})$ and non-negative $u \in C(\mathbb{R}^d) \cap L^1(\omega_s)$ and

$$Lu - \tilde{\lambda} \leq 0 \quad \text{in } \mathbb{R}^d.$$

Let $(w, \hat{\lambda})$ be the solution obtained in Theorem 2.3.4, where $w = \lim_{k\to\infty} w_{\alpha_k}$.

Fix $M > 0$. Since $\alpha_k w_{\alpha_k}(0) \to \hat{\lambda}$, there exists $k_0 \in \mathbb{N}$, such that

$$\tilde{\lambda} < \alpha_k w_{\alpha_k}(0) - \alpha_k M \text{ for all } k \geq k_0.$$

Choose a $k \geq k_0$ and let $\alpha_k = \tilde{\alpha}$ and $\bar{w}_{\alpha_k} = \tilde{w}$. By Lemma 2.2.1, we can find a sequence of

W_l such that, the following holds:

$$\begin{aligned} LW_l - \tilde{\alpha} W_l &= 0 \text{ in } B_l, \\ W_l &= 0 \text{ in } B_l^c, \end{aligned} \tag{2.3.13}$$

such that as $l \to \infty$ W_l uniformly converges to w_{α_k} in compact sets. Since $W_l \geq 0$ lets define $\zeta_l(x) = W_l(x) - W_l(0) + M$. Then, by (2.3.13),

$$\begin{aligned} L\zeta_l(x) - \tilde{\alpha}\zeta_l &= \tilde{\alpha}\, W_l(0) - \tilde{\alpha} M \text{ in } B_l, \\ \zeta_l &= -W_l(0) + M \leq M \text{ in } B_l^c. \end{aligned} \tag{2.3.14}$$

If l is large , we get $\tilde{\lambda} < \tilde{\alpha} W_l(0) - \tilde{\alpha} M$. From Lemma 2.3.1 it follows that

$$u(x) \to \infty \text{ as } |x| \to \infty.$$

So for sufficiently large l, u is a strict super-solution to (2.3.14). Therefore, by the comparison principle, we conclude that $u \geq \zeta_l$ in $\mathbb{R}^d$. Now if we let $l \to \infty$, we get

$$w_{\alpha_k}(x) - w_{\alpha_k}(0) + M \leq u$$

for all $k \geq k_0$. Letting $k \to \infty$, we thus get $w + M \leq u$ in $\mathbb{R}^d$. However, w is fixed, and M can be chosen arbitrarily. We arrive at a contradiction as such a u cannot exist.

Hence $\hat{\lambda} = \lambda^*$.

□

Now we focus on the uniqueness part of Theorem 2.1.1. The following lemma is the key for that goal.

Lemma 2.3.6. Let w be the solution obtained in Theorem 2.3.4, and suppose w satisfies the hypotheses of Theorem 2.3.4. Then there exists $R > 0$ such that, if $v \in C^1(\mathbb{R}^d) \cap L^1(\omega_s)$, and $v > -\lambda^*$, solving the inequality :

$$Lv - \lambda^* \leq 0 \quad \text{in } \mathbb{R}^d. \tag{2.3.15}$$

Then we get

$$w \leq v \quad \text{ in } \mathbb{R}^d.$$

Here, by solution we mean the viscosity solution and w satisfies $w \leq v$ in B_R.

Proof. We have $v \in C(\mathbb{R}^d) \cap L^1(\omega_s)$ such that $w \leq v$ in B_R. Choose R so that $\min_u c(x,u) > \lambda^*$ in B_R^c. We want to prove that $w \leq v$ in $\mathbb{R}^d$.
Suppose not, then $\exists$ some $x_0 \in \bar{B}_R^c$, such that $v(x_0) < w(x_0)$. We can find a $\mu \in (\frac{1}{2}, 1)$ close to 1, such that

$$\sup_{\bar{B}_R} (\mu w - v) < \mu w(x_0) - v(x_0), \quad \mu w(x_0) - v(x_0) > 0. \tag{2.3.16}$$

By Theorem 2.3.4, and standard diagonalisation method, we can find a sequence (W_{n_k}, α_{n_k}) such that

$$\begin{cases} LW_{n_k} - \alpha_{n_k} W_{n_k} = 0 & \text{in } B_{n_k}, \\ W_{n_k} = 0 & \text{in } B_{n_k}^c, \end{cases}$$

and

$$\lim_{n_k \to \infty} \alpha_{n_k} W_{n_k}(0) = \lambda^*, \quad \lim_{n_k \to \infty} \bar{W}_{n_k} = w \tag{2.3.17}$$

uniformly over compact sets. Let B_{n_k} contains x_0, for large values of n_k. Then by (2.3.16) and (2.3.17),

$$\max_{\bar{B}_R} (\mu \bar{W}_{n_k} - v) < \mu \bar{W}_{n_k}(x_0) - v(x_0), \quad \mu \bar{W}_{n_k}(x_0) - v(x_0) > 0 \text{ and } v > 0 \text{ in } B_{n_k}^c \tag{2.3.18}$$

where the last inequality follows from Lemma 2.3.1. Define, $\hat{v}_{n_k}(x) = v + \mu W_{n_k}(0)$. By (2.3.15), $L\hat{v} - \lambda^* \leq 0$ in $\mathbb{R}^d$. Since, $W_{n_k} = 0$ in $B_{n_k}^c$, we obtain $\mu W_{n_k} < \hat{v}_{n_k}$ in $B_{n_k}^c$. By (2.3.18),

$$\sup_{B_{n_k}} (\mu W_{n_k} - \hat{v}_{n_k}) = \sup_{\mathbb{R}^d} (\mu W_{n_k} - \hat{v}_{n_k}) > 0.$$

Let $\kappa_{n_k} = \sup_{\mathbb{R}^d} (\mu W_{n_k} - \hat{v}_{n_k})$, and it is attained at a point, say $x_{n_k} \in B_{n_k} \setminus B_R$. By Assumption 3 and hypotheses $v, w \in C^1(\mathbb{R}^d)$ and $b(x,u)$ is a bounded function. Then we have

$$\begin{aligned} -(-\Delta)^s(\hat{v}_{n_k} - \mu W_{n_k}) &= -(-\Delta)^s(\hat{v}) + (-\Delta)^s(\mu W_{n_k}) \\ &\leq -\min_u \{b \cdot \nabla v + c\} + \lambda^* + \min_u \{b\mu \cdot \nabla W_{n_k} + \mu c\} - \alpha_{n_k}(\mu W_{n_k}) \\ &\leq k|\nabla(\hat{v} - \mu W_{n_k})| + \sup_u (\mu - 1)c(x,u) - \alpha_{n_k}(\mu W_{n_k}). \end{aligned}$$

In the above inequality, we observe that at x_{n_k}, $\nabla \hat{v} = \nabla \mu W_{n_k}$. So, we test the above inequality at x_{n_k} to get

$$0 \leq k \cdot 0 + \sup_u (\mu - 1)c(x_{n_k}, u) - \alpha_{n_k}\mu(W_{n_k}(x_{n_k})) + \lambda^*.$$

There are two possibilities. Suppose $|x_{n_k}| \to \infty$ as $n_k \to \infty$. Then we have,

$$0 \leq \sup_u (\mu - 1)c(x_{n_k}, u) + \lambda^* = (\mu - 1)\min_u c(x_{n_k}, u) + \lambda^* < 0,$$

by Assumption 3(ii) and the fact $\mu < 1$. This is a contradiction. Suppose we have x_{n_k} is a bounded sequence. We can assume that $x_{n_k} \to \tilde{x}$, as $n_k \to \infty$ along some sub-sequence. Then,

$$0 \leq (\mu - 1)\min_u c(x_{n_k}, u) - \alpha_{n_K}\mu(\bar{W}_{n_k}(x_{n_k})) - \alpha\mu W_{n_k}(0) + \lambda^*$$

$$\Rightarrow 0 \leq (\mu - 1)[\min c(\tilde{x}, u) - \lambda^*] < 0.$$

This is a contradiction to the choice of R. Therefore, $w \leq v$ in $\mathbb{R}^d$. $\square$

Theorem 2.3.7. Suppose that Assumption 3 holds. Let $v \in C^1(\mathbb{R}^d)$ be such that $v \geq 0$ in $\mathbb{R}^d$ and satisfies

$$Lv - \lambda^* \leq 0 \text{ in } \mathbb{R}^d$$

a super-solution to (2.1.3) and w be the solution obtained in Theorem 2.3.4. Then we have that $v = w + C$, where C is a scalar.

Proof. Let B_R be the ball as in Lemma 2.3.6. Let $\min_{\bar{B}_R} v - w = C$. Let $v_1 = v - C$, and by definition of L. we get

$$Lv_1 - \lambda^* \leq 0 \text{ in } \mathbb{R}^d.$$

Here v_1 touches w from above in $\bar{B}_R$. By Theorem 2.3.4, we have $v_1 \geq w$ in $\mathbb{R}^d$. Take $\varphi = v_1 - w$, we see that

$$-(-\Delta)^s\varphi - \sup_u\{b(x, u) \cdot \nabla\varphi\} \leq 0 \text{ in } \mathbb{R}^d. \tag{2.3.19}$$

Since,

$$\inf_{\mathbb{R}^d} \varphi = \min_{\bar{B}_R} \varphi = 0.$$

Therefore

$$\varphi(\bar{x}) = 0 = \nabla\varphi(\bar{x}),$$

for some $\bar{x} \in \bar{B}_R$. So if we test at the point $\bar{x}$, from (2.3.19) we obtain $-(-\Delta)^s\varphi(\bar{x}) \geq 0$. Using the definition of fractional-Laplacian, we obtain that φ is constant. Hence proved. □

We finish the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. The existence part follows from Theorem 2.3.4 and Lemma 2.3.5 whereas the uniqueness follows from Theorem 2.3.7. □

2.4 Conclusion and Future directions

2.4.1 Conclusion

We started by looking at the regularity theory for the nonlinear integro-differential equations from [6]. For fully nonlinear integro-differential equations of the type

$$Lu = \sup_{\alpha} \inf_{\beta} \int_{\mathbb{R}^d} (u(x+y) + u(x-y) - 2u(x)) K_{\alpha\beta}(y) dy,$$

[6] provides the nonlocal ABP estimate, Harnack Inequality, Hölder estimates and $C^{1+\alpha}$ estimates. The equations considered here are associated with pure jump processes.

Next, we studied [3], in which a more general class of equations with drift and nonlocal diffusion terms have been considered. These types of equations are motivated by stochastic ergodic control problems. [3] provides a complete study of the equation under the Lyapunov stability condition. In [3], $C^{1,\gamma}$ regularity and C^{2s+} regularity results, and the existence-uniqueness of the solution, are given.
In this thesis, we considered an equation similar to [3], with a fractional Laplacian kernel. We gave proof of the existence and uniqueness of the solution for the equation under the near-monotonicity assumption of the cost and we do not impose any stability criterion.

2.4.2 Future Directions

$$-(-\Delta)^s w(x) + \min_{u \in \mathbb{U}} \{b(x,u) \cdot \nabla w(x) + c(x,u)\} = \lambda^*. \tag{2.4.1}$$

Let $\mathcal{L}_u$ be denoted as,

$$\mathcal{L}_u[w](x) = -(-\Delta)^s w(x) + \min_{u \in \mathbb{U}} \{b(x,u) \cdot \nabla w(x)\}.$$

To complete the picture we should be able to prove that the *eigenvalue* λ^* in the above equation is actually the optimal ergodic value. To elaborate, let us consider the following stochastic control problem. Suppose the control set $\mathbb{U}$ is a metric space and $\mathcal{U}$ be the set of all stationary Markov controls, that is collection of all Borel measurable functions $v : \mathbb{R}^d \to \mathbb{U}$.

Define $\mathcal{L}^v$ as

$$\mathcal{L}^v[f](x) = \mathcal{L}_{v(x)}[f](x).$$

Assume that the martingale problem corresponding to the operator $\mathcal{L}^v$ is well posed. This means for every $v \in \mathcal{U}$ there exists a family of probability measures $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ on the space of Càdlàg functions on $[0,\infty)$ taking values in $\mathbb{R}^d$, such that $(\mathbb{P}_x^v, \mathcal{X}_v)$ solves the martingale problem, where $\mathcal{X}_v$ denotes the canonical coordinate process. As defined in the previous chapter, $\mathbb{R}^d \times \mathbb{U} \ni (x,u) \to c(x,u)$ denote the running cost, we want to minimize the *ergodic cost criterion*

$$\mathcal{J}[v] := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_x^v\bigg[\int_0^T c(\mathcal{X}_v, u)\bigg],$$

over $\mathcal{U}$. We denote the optimal value by λ_{opt}. We need to show that $\lambda^* = \lambda_{\mathrm{opt}}$, where λ^* is from (2.4.1).

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