ASYMPTOTIC SYMMETRIES AND CONSERVED CHARGES

A Thesis

submitted to

Indian Institute of Science Education and Research Pune

in partial fulfilment of the requirements for the

BS-MS Dual Degree Programme

by

Kanishk Verma

Indian Institute of Science Education and Research Pune

Dr Homi Bhabha Road,
Pashan, Pune 411008, INDIA

April, 2019

Supervisor: **Dr Shailesh Kulkarni**

CERTIFICATE

This is to certify that this dissertation entitled **ASYMPTOTIC SYMMETRIES AND CONSERVED CHARGES** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Kanishk Verma** at Indian Institute of Science Education and Research under the supervision of **Dr Shailesh Kulkarni**, Assistant Professor, Department of Physics, Savitribai Phule Pune University, during the academic year 2018-2019.

20-3-2019

Dr Shailesh Kulkarni

Committee:

Dr Shailesh Kulkarni

Dr Rajeev S. Bhalerao

This thesis is dedicated to my teachers

DECLARATION

I hereby declare that the matter embodied in the report entitled **ASYMPTOTIC SYMMETRIES AND CONSERVED CHARGES** are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of **Dr Shailesh Kulkarni**, Department of Physics, Savitribai Phule Pune University and the same has not been submitted elsewhere for any other degree.

Kanishk Verma

20-3-2019

ACKNOWLEDGEMENTS

First of all, I would like to thank Dr. Shailesh Kulkarni for giving me the opportunity to work on this topic and helping me build sense of doing theoretical physics. I want to express my gratitude towards Dr Rajeev S. Bhalerao for being my thesis advisor and giving me valuable suggestions. I would like to thank my mathematician friend Ajith Nair for helping me through some technical difficulties. Lastly, I would like to thank my family for all their support.

ABSTRACT

Our aim is to learn about asymptotic symmetries and conserved charges so that we may apply its knowledge to know about conserved charges in situations where the notion of conserved charges does not make sense or are difficult to evaluate. We do this specifically by learning about Chern-Simons Theory and the process of finding out the energy of the BTZ black hole in (1+2) dimensions.

CONTENTS

INTRODUCTION

One of the key features to solve a physical system is the need to understand about symmetries of the system. As stated by Noether the symmetries in the system means that there are associated conserved quantities and also, we can think the opposite way that if a physical system has conserved quantities then there will be some sort of symmetries that are associated with them.

To make things concrete when we say a physical quantity is conserved, we mean that the parameters defining that physical quantities do not evolve in time [1]. But the notion of conserved quantities makes sense only when we are in regime of flat spacetime. As we know the theory of general relativity suggests that the physical systems may be in a curved spacetime, thus its difficult to define universally the notion of conserved quantities even though one might be able to peruse some kind of symmetries of the system [2].

One way to know about the conserved quantities of a physical system in curved spacetime is by the concept of asymptotic symmetries which are symmetries corresponding to asymptotic charges. By conserved charges we mean conserved quantities [1]. The notion of asymptotic conserved charges and symmetries makes sense for a physical system only asymptotically which means that we are radially far away from the system. So, we can now talk about conserved charges only asymptotically far away. As soon as we are near the system or break our asymptotic conditions our notion of asymptotic conserved charges and asymptotic symmetries will fail [3].

Having a very general discussion about asymptotic symmetries and conserved charges is not very enlightening. So here we discuss Chern-Simons theory [7] and study its asymptotic symmetries and conserved charges and apply this knowledge to see how the energy of the BTZ black hole in (1+2) dimensions [9],[8] is found, which is nothing but a conserved charge.

CHAPTER 1

PRELIMINARIES

In this chapter we discuss all the important concepts required to understand about conserved quantities of a theory on Minkowski spacetime [6] i.e. what we refer to as flat spacetime. For our discussion we consider all manifolds to be Sudo-Riemannian with Lorentzian signature.

1.1 SYMMETRIES OF MANIFOLD

Consider a manifold labelled as M. The symmetries on the manifold M form a Lie Group which it self is a differentiable manifold. Generators of this Lie Group form a Lie Algebra. Symmetries of manifold M are given by vector fields knows as killing fields. These fields also preserve distances and as known as Isometric fields. For our purposes we will interested in knowing about Isometries of the manifold which form Isometric group.

Let $g_{\mu\nu}$ be the metric and u^{α} be a vector field on M. Now to get all the killing fields on M we use the following equation

$$\mathcal{L}_u g_{\mu\nu} = u^{\lambda}\partial_{\lambda} g_{\mu\nu} + \partial_{\mu} u^{\lambda} g_{\lambda\nu} + \partial_{\nu} u^{\lambda} g_{\mu\lambda} \tag{1.1}$$

where $\mathcal{L}_u$ represents lie derivative of the tensor fields with respect to vector field u and using (1.1) we find out all the vector fields u^α. These fields correspond to symmetries of the manifold M. These field form what is known as killing algebra i.e.

$$[u_i, u_j] = c_{ijk} u_k \tag{1.2}$$

We say that our manifold M is maximally symmetric if it has maximal number of killing fields i.e. for n dimensional manifold we get

$$n + {}^nC_2 = n + \frac{n(n-1)}{2} \tag{1.3}$$

where n killing fields are for translations and $\frac{n(n-1)}{2}$ are for rotations, i.e. we get a total of $n + {}^nC_2$ killing fields.

When we study diffeomorphic covariant theories, which are theories that are invariant under general coordinate transformation then the lie derivate of the canonical fields along the killing fields is equivalent to diffeomorphisms of the theory. Consider a coordinate transformation given by

$$x'^\mu = x^\mu + \varepsilon^\mu(x) \tag{1.4}$$

now the diffeomorphisms are given by lie derivate of the tensor fields $T^{\mu\dots}_{\nu\dots}$ along ε^μ i.e.

$$\delta T^{\mu\dots}_{\nu\dots} = \mathcal{L}_u T^{\mu\dots}_{\nu\dots} \tag{1.5}$$

1.2 SYMMETRIES OF THEORY

A theory on a manifold M, can be describe by scalar, vector and tensor fields present on M. So, a theory is given by its action I which is the integral of scalar Lagrangian. By symmetries of theory we mean symmetries of action I.

For simplicity we consider d-dimensional manifold and a scalar field $\phi(x)$ on it. Now suppose we have a theory

$$I = \int d^d x \, \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.6}$$

Now this field theory has symmetries which are set of all infinitesimal functions $\delta_s\phi(x)$ such that for all $\phi(x)$

$$\delta I(\phi, \delta_s\phi) \equiv I(\phi, \delta_s\phi) - I(\phi) = \int d^d x\, \partial_\mu K^\mu \quad \forall\phi \tag{1.7}$$

These symmetries can be either Noether's symmetries or Gauge symmetry. We would discuss more about Gauge Theories in chapter 2 and chapter 3. For now, we deal with only Non-Gauge theories.

1.3 NOETHER'S THEOREM AND CONSERVED CHARGES

We would now discuss about Noether's theorem which is very important in all of theoretical physics. The basic idea of the Noether's theorem states that symmetry is associated with conservation of some quantity i.e. a quantity which does not evolve in time.

Consider the theory given by (1.6) its equations of motion are given by

$$\varepsilon(\phi(x)) \equiv \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{1.8}$$

These equations are known as Euler Lagrange equations of motion. So, the Noether's Theorem states that if we have a Noether's symmetry in our theory and we compute the variation of action I on shell i.e. where the fields satisfy equation of motion, then we obtain a conserved current.

The on-shell variation is computed as below where $\bar{\phi}$ satisfies equation (1.8)

$$\delta I[\bar{\phi}, \delta\phi] = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi_{,\mu}\right) \tag{1.9}$$

$$\delta I[\bar{\phi}, \delta\phi] = \int d^4x \left(\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right)\right]\delta\phi\right) + \int d^4x \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi\right) \tag{1.10}$$

$$\delta I[\bar{\phi}, \delta\phi] = \int d^4x \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi\right) \tag{1.11}$$

If we have Noethers symmetry with on shell variation, using (1.7) and (1.11) we get conserved current equation

$$\partial_\mu J^\mu = 0 \quad where \quad J^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi - K^\mu \tag{1.12}$$

We can split (1.12) into space+time and we get

$$\partial_t J^0 = -\nabla.\vec{J} \tag{1.13}$$

Integrating both sides of (1.13) and using divergence theorem we obtain

$$\int_V d^3x\, \partial_t J^0 = -\int_V d^3x\, \nabla.\vec{J} = -\int_{\partial V} \vec{J}.d\vec{A} \tag{1.14}$$

And we assume that fields fall of sufficiently rapidly at infinity i.e. when $r \to \infty, \phi \to 0$, where r is the radial coordinate. Generally, this assumption might not be true for Gauge theories. Thus, we get

$$Q = \int_V d^3x\, J^0 \; such\ that \; \frac{dQ}{dt} = 0 \tag{1.15}$$

where Q is our conserved charge i.e. which does not evolve in time.

1.4 ENERGY MOMENTUM TENSOR

The concept of energy momentum tensor is very important. In Minkowski spacetime energy momentum tensor is locally conserved but for a curved spacetime it might not be conserved [2]. So basically, stress energy tensor is the Noether's current associated with symmetry of constant spacetime translations. So, consider the following spacetime translations

$$x'^\mu = x^\mu + \varepsilon^\mu \tag{1.16}$$

where x^μ is the original coordinate system and x'^μ is the transformed coordinate system by constant killing fields ε^μ. Now, corresponding to each translation we would have a conserved current which would be linear in ε^μ and is given by

$$J^\mu = T^\mu{}_\nu \varepsilon^\nu \tag{1.17}$$

So, the coefficient $T^{\mu}{}_{\nu}$ is known as energy momentum tensor. From equation (1.12) we get

$$\partial_\mu T^{\mu}{}_{\nu} = 0 \tag{1.18}$$

which shows that energy momentum tensor is conserved here in flat space time, but it might not be in curved spacetime.

So, in curved spacetime for a given theory i.e. its action I we have an expression of stress energy tensor given by

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta I}{\delta g_{\mu\nu}} \tag{1.19}$$

where $g_{\mu\nu}$ is the metric and g is determinant of $g_{\mu\nu}$ and $\frac{\delta I}{\delta g_{\mu\nu}}$ is the Euler-Lagrange derivative. So, the energy momentum tensor defined by (1.18) is covariantly conserved i.e.

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.20}$$

where ∇_μ is the covariant derivative.

1.5 CONSERVED QUANTITIES OF MINKOWSKI SPACETIME

To put together all the ideas present in this chapter we would compute the conserved charges of a theory in Minkowski spacetime. Here the Minkowski metric is

$$g_{\mu\nu} = diag(-1,1,1,1) \tag{1.21}$$

Now for Minkowski spacetime we compute all its killing fields given by equation

$$\mathcal{L}_\varepsilon g_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \tag{1.22}$$

whose general solution is given by

$$\varepsilon^\mu = a^\mu + b^{[\mu\nu]} x_\nu \tag{1.23}$$

where a^μ correspond to 4 constant translations and $b^{[\mu\nu]} x_\nu$ correspond to 6 rotations due to anti-symmetric matrix. Consider a theory of fields given by action I on Minkowski spacetime, we then compute its energy momentum tensor given by $T^{\mu\nu}$.Now the conserved charge is nothing but integral of J^0 on a Cauchy surface Σ.

Thus, we get

$$P^{\mu} = \int_{\Sigma} d^3x\, T^{\mu 0} \tag{1.24}$$

$$M^{\mu\nu} = \int_{\Sigma} d^3x\, (x^{\mu} T^{\nu 0} - x^{\nu} T^{\mu 0}) \tag{1.25}$$

P^{μ} are 4 conserved quantities corresponding to translations which shows that Minkowski spacetime is homogeneous and $M^{\mu\nu}$ are 6 conserved quantities corresponding to Lorentz transformation which shows that Minkowski spacetime is isotopic and relativistic [6].

CHAPTER 2

DYNAMICS OF CONSTRAINED SYSTEM

The purpose of this chapter is to study about constrained system. Usually if a theory has constraints in it then the action of the theory has Gauge symmetry. We will demonstrate this fact by studying two theories in detail that are electromagnetic field theory and general relativity[1].

2.1 CHARACTERISTICS OF CONSTRAINED SYSTEMS

As we will see in this chapter explicitly a system with constraints have the following characters [4]

- The equations of motion are not independent
- The Hamiltonian of the theory possesses constraints
- The general solution of equations of motion contain arbitrary functions of time
- The action of the theory has gauge symmetry

And we will see that if a theory has first class constraints then the theory has gauge symmetry [4].

2.2 GAUGE THEORIES IN HAMILTONIAN FORM

Here we work with Hamiltonian formulation since it is convenient to use and provides a unified vision to deal with all theories. Thus, the action of all Gauge theories in Hamiltonian form can be written as

$$I[p_i\,, q^i, \lambda^a] = \int dt \left(p_i\dot{q}^i - H_0\left(p_i\,, q^i\right) + \lambda^a \phi_a(p\,, q)\right) \tag{2.1}$$

where

p_i --- Canonical momentum fields

q^i --- Canonical fields

λ^a --- Lagrange multiplier fields

ϕ_a ---Constraints fields

$p_i\,, q^i$ are the dynamical variables of the theory and

$$H_T = H_0\left(p_i\,, q^i\right) - \lambda^a \phi_a(p\,, q) \tag{2.2}$$

is known as the total Hamiltonian of our theory where H_0 is the constraint free part and $\lambda^a \phi_a$ denotes the contribution from the constraints ϕ_a.

Thus, we know that Hamiltonian formulation gives us the following equation of motion of our dynamical variables

$$\dot{q}^i = \frac{\partial H_0}{\partial p_i} - \lambda^a \frac{\partial \phi_a}{\partial p_i} \tag{2.3}$$

$$\dot{p}^i = -\frac{\partial H_0}{\partial q^i} + \lambda^a \frac{\partial \phi_a}{\partial q^i} \tag{2.4}$$

provided with

$$\phi_a(p\,, q) = 0 \tag{2.5}$$

those are constraints in our theory.

So now we need our equations of motion to evolve consistently i.e. at all times we are required to follow equation (2.5). In order to do this Dirac proposed a mathematical method and said that all constraints should be weakly imposed i.e.

$$\phi_a(p\,,q) \approx 0 \tag{2.6}$$

this means we consider that, constraints are not equal to zero and calculate our Poisson's bracket and after the calculation we equate the constraints to be zero.

Now we do our analysis in general and look at the time evolution of the constraints i.e. compute its Poisson's bracket with the total Hamiltonian and thus we get

$$\frac{d}{dt}\phi_a(p\,,q) = [\phi_a\,,H_T] = [\phi_a\,,H_0] - [\phi_a\,,\phi_b]\lambda^b \tag{2.7}$$

Since the constraints will be imposed weakly, we get

$$[\phi_a\,,H_0] - [\phi_a\,,\phi_b]\lambda^b \approx 0 \tag{2.8}$$

which can also be written as

$$[\phi_a\,,H_0] - C_{ab}\lambda^b \approx 0 \tag{2.9}$$

Since we are doing our analysis in general, we can now have the following cases:

CASE I: Non-Gauge Theories

If the matrix C_{ab} is invertible than using (2.8) we can find $\lambda^a(t)$ i.e.

$$\lambda^a(t) = C_{ab}[\phi_a\,,H_0] \tag{2.10}$$

This is known as the theory with second class constraints. Thus, we are able to solve our theory.

CASE II: Gauge Theories

If the matrix C_{ab} is zero or $C_{ab} \approx 0$ on the surface $\phi_a = 0$, then $\lambda^a(t)$ remains undetermined. This shows that the dynamical variables of the theory contain arbitrary functions of time which gives us a signature of gauge theories. If $C_{ab} = 0$ then $[\phi_a\,,H_0] = 0$.

Constraints are said to be first class if they satisfy

$$[\phi_a , H_0] = K_{ab}\phi_b \tag{2.11}$$

$$[\phi_a , \phi_b] = K_{abc}\phi_c \tag{2.12}$$

CASE III: Mixed Case

We can also have a theory which has both first-class and second-class constraints thus representing a mixed case.

We now state a very important result [1]:

If the constraints and Hamiltonian of our theory satisfy (2.11) and (2.12) then our action (2.1) will be invariant under the following transformations:

$$\delta q^i = [q^i, \phi_a]\epsilon^a(t) \tag{2.13}$$

$$\delta p^i = [p_i , \phi_a]\epsilon^a(t) \tag{2.14}$$

$$\delta\lambda^c = \dot{\epsilon}^c + \epsilon^a(t)K_a{}^c - \lambda^a\epsilon^b(t)K_{ab}{}^c \tag{2.15}$$

These transformations are nothing but gauge symmetry of the action.

2.3 ELECTROMAGNETIC FIELD THEORY

So, the electromagnetic field action is given by [5]

$$I_{EM} = -\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu} \tag{2.16}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.17}$$

After doing a (1+3) split action can be written as

$$I_{EM} = \int d^4x \left(\frac{1}{2}\dot{A}_i\dot{A}_i - \dot{A}_i\partial^i A_0 + \frac{1}{2}\partial_i A_0 \partial^i A^0 - \frac{1}{4}F_{ij}F^{ij}\right) \tag{2.18}$$

Here A_i are canonical fields and so we define our canonical momentum field as

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \tag{2.19}$$

Thus, we get

$$\pi^0 = 0 \quad , \quad \pi^i = \dot{A}^i - \partial^i A^0 \tag{2.20}$$

Here $\pi^0 = 0$ is known as the primary constraint of the theory since it follows from the definition of canonical momenta.

Thus, our Hamiltonian will be

$$H(p, A) = \pi_i \dot{A}^i - \mathcal{L} \tag{2.21}$$

After calculating we get

$$H(p, A) = \frac{1}{2}\pi_i\pi^i + \frac{1}{4}F_{ij}F^{ij} - A^0\partial^i\pi_i \tag{2.22}$$

where we have dropped the boundary term.

Here

$$H_0 = \frac{1}{2}\pi_i\pi^i + \frac{1}{4}F_{ij}F^{ij} \tag{2.23}$$

The Poisson's brackets of our theory are

$$\left[\pi_i(x), \pi_j(y)\right] = 0 \tag{2.24}$$

$$\left[A_i(x), A_j(y)\right] = 0 \tag{2.25}$$

$$\left[A_i(x), \pi_j(y)\right] = \delta_{ij}\delta^3(x - y) \tag{2.26}$$

For consistency we need to check time evolution of π^0 so we use the equation (2.7) and we get

$$[\pi^0 \,, H_T] \approx 0 \tag{2.27}$$

$$[\pi^0(x), H_0(y)] + \lambda^b[\pi^0(x), \pi^0(y)] \approx 0 \tag{2.28}$$

Doing this calculation, we get

$$\partial_i \pi^i \approx 0 \tag{2.29}$$

This is known as secondary constraint of our theory. We have to again do a consistency check on $\partial_i \pi^i$ i.e.

$$\left[\partial_i \pi^i , H_T\right] \approx 0 \tag{2.30}$$

Thus, we get $0 = 0$ so this means that our theory has no more secondary constraints.

We can see by using equation (2.11) and (2.12) both of these are first class constraints as shown by algebra below

$$[\pi^0 , H_0] = 0 \quad \left[\partial_i \pi^i , H_0\right] = 0 \tag{2.31}$$

$$\left[\pi^0 , \partial_i \pi^i\right] = 0 \quad \left[\partial_i \pi^i , \partial_j \pi^j\right] = 0 \tag{2.32}$$

Since there are two constraints, the action of theory is

$$I_{EM}[\pi_i , A_i , A_0 , \gamma] = \int d^4x \left[\pi_i \dot{A}^i - \left(\frac{1}{2}\pi_i \pi^i + \frac{1}{4}F_{ij}F^{ij}\right) + A_0 \partial_i \pi^i + \gamma \pi^0\right] \tag{2.33}$$

Here

π_i-----------Canonical momentum fields

A_i-----------Canonical fields

A_0 , γ-------Lagrange multiplier fields

$\partial_i \pi^i, \pi^0$----Constraint fields

Using physical arguments one can say that the term $\gamma \pi^0$ will give us no additional information so, we can drop this term from our action. This process is gauge fixing in a theory and the process of gauge fixing has to be done according to the situation, i.e. here we fixed our gauge $\pi^0 = 0$ [5].

So finally, we have

$$I_{EM}[\pi_i , A_i , A_0] = \int d^4x \left[\pi_i \dot{A}^i - \left(\frac{1}{2}\pi_i \pi^i + \frac{1}{4}F_{ij}F^{ij}\right) + A_0 \partial_i \pi^i\right] \tag{2.34}$$

which is in the general form equation (2.1)

So, equations of motion are

$$\dot{\pi}_i = -\frac{\partial H}{\partial A^i} \quad , \quad \dot{A}^i = \frac{\partial H}{\partial \pi^i} \quad , \quad \phi = \partial_i \pi^i = 0 \tag{2.35}$$

Now we use equation (2.7) i.e.

$$\frac{d\phi}{dt} = [\phi, H_T] = 0$$

$$[\phi(x), H_T(x')] = \left[\partial_i \pi^i(x), \frac{1}{2}\left(\pi_i \pi^i(x') + F_{ij}F^{ij}(x') - \frac{1}{2}A_0 \partial_i \pi^i(x')\right)\right] \tag{2.36}$$

which gives us

$$\frac{d\phi}{dt} = 2\partial_k \partial_i' [\pi_k(x'), A_j(x')] F^{ij}(x')$$

$$\frac{d\phi}{dt} = -2\big(\partial_k \partial_i' \delta^3(x - x')\big) F^{ij}(x') = 0 \tag{2.37}$$

Now we use the equations (2.13), (2.14) and (2.15) and find the gauge symmetry of the theory. But before that instead of using $\phi(x) = \partial_i \pi^i(x)$ we use $\Phi[\wedge(x)]$ i.e. we attach a weight function also known as test function as follows

$$\Phi[\wedge(x)] = \int d^3x \wedge(x) \partial_i \pi^i(x) \tag{2.38}$$

Calculating the poisons bracket we find

$$\delta A_i(x) = \big[A_i(x,t)\, , \Phi[\wedge(x')]\big] \tag{2.39}$$

$$\delta A_i(x) = \int d^3x' \wedge(x',t) \partial_i' \big[A_i(x,t), \pi^j(x',t)\big]$$

$$= \int d^3x' \wedge(x',t) \delta_{ji} \partial_j' \delta^3(x - x')$$

$$= -\partial_i \wedge(x,t) \tag{2.40}$$

and

$$\delta \pi_i(x,t) = \big[\pi_i\, , \Phi[\wedge(x')]\big] = \int d^3x' \wedge(x',t) \big[\pi_i\, , \partial_j \pi^j\big] = 0 \tag{2.41}$$

Now to find δA_0 we calculate the variation of our action i.e.

$$\delta I = \int d^4x \left[\pi_i \delta \dot{A}^i + \delta A_0 \phi\right]$$

$$= \int d^4x \left[-\pi_i \partial^i \partial_0 \wedge + \delta A_0 \partial_i \pi^i\right]$$

$$= \int d^4x \left[\partial_i\left(-\pi^i \partial_0 \wedge\right) + \partial_i \pi^i \partial_0 \wedge + \delta A_0 \partial_i \pi^i\right]$$

$$= \int d^4x \left[\partial_\mu K^\mu + \partial_i \pi^i (\delta A_0 + \partial_0 \wedge)\right] \quad (2.42)$$

For the variation to be the boundary term we need $\delta A_0 = -\partial_0 \wedge$. Thus, we have found out δA_0.and we have proved that $\delta A_i(x)$ together with δA_0 form the gauge transformation of electromagnetic field i.e.

$$A_\mu \rightarrow A_\mu - \partial_\mu \wedge (x) \quad (2.43)$$

which was generated by the constraint $\partial_i \pi^i = 0$ [5].

2.4 GENERAL RELATIVITY

We have a manifold $\mathcal{M}$ whose metric is given by $g_{\mu\nu}$ and the Einstein Hilbert action is given by [2]

$$I_{EH} = \frac{1}{16\pi} \int R\sqrt{-g} d^4x \quad (2.44)$$

To write this action in Hamiltonian form we do an ADM decomposition of the spacetime [3]. Spacetime is divided into 3 dimensional surfaces Σ_t^3 where g_{ij} is the intrinsic metric on Σ_t^3. The lapse function is defined as $N(\vec{x}, t)$ such that $N(\vec{x}, t)dt$ takes us from Σ_t^3 to Σ_{t+dt}^3 and the shift function is defined as $N^i(\vec{x}, t)$ such that $N^i(\vec{x}, t)dt$ measures the shift produced at constant time between $\vec{x} + d\vec{x}$ and the point that will eventually hit $(\vec{x} + d\vec{x}, t + dt)$ by projecting Ndt Fig(2.1)

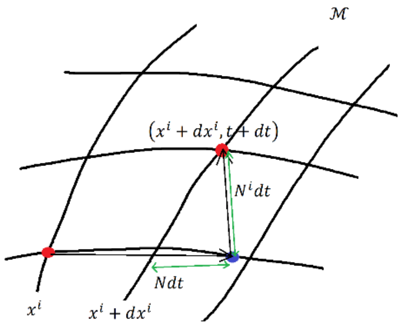

Fig (2.1) Lapse and Shift Functions

Action can now be written as

$$I_{EM}[N, N^i, g_{ij}] = \int d^4x\, N\sqrt{-^{(3)}g}\left(^{(3)}R - K^2 + K^{ij}K_{ij}\right) \tag{2.45}$$

where we have neglected the boundary term. Here K_{ij} is the extrinsic curvature and K is the Ricci scalar on Σ_t^3 given by

$$K_{ij} = \frac{1}{2N}\left[-\dot{g}_{ij} + N_{i;j} + N_{j;i}\right] \tag{2.46}$$

$$K = g^{ij}K_{ij} \tag{2.47}$$

Now we can define canonical momentum fields as

$$\Pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}} \tag{2.48}$$

which we calculate using δI and we get

$$\Pi^{ij} = \sqrt{|g|}\left[Kg^{ij} - K^{ij}\right] \tag{2.49}$$

Taking the trace of (2.49) we get

$$K = \frac{\Pi}{\sqrt{|g|}} \qquad where\ \Pi = \Pi^{ij}g_{ij} \tag{2.50}$$

where K_{ij} is given as

$$K^{ij} = \frac{1}{\sqrt{|g|}}\left(-\Pi^{ij} + \frac{\Pi}{2}g^{ij}\right) \tag{2.51}$$

Using (2.46) and (2.51) we calculate $\dot{g}^{ij}$ and we get the Hamiltonian as

$$H = \int d^3x \left(\Pi^{ij}\dot{g}_{ij} - \mathcal{L}\right) \tag{2.52}$$

which after calculating is

$$= \int d^3x \left[N\left[\frac{\Pi^{ij}\Pi_{ij}}{\sqrt{|g|}} - \frac{1}{2}\frac{\Pi^2}{\sqrt{|g|}} - \sqrt{|g|}R\right] - 2\Pi^{ij}{}_{;j}N_i\right]$$

Thus, our Hamiltonian is

$$H = \int d^3x \left(N\mathcal{H} + N_i\mathcal{H}^i\right) \tag{2.53}$$

Where

$$\mathcal{H} = \frac{1}{\sqrt{|g|}}\left(\Pi^{ij}\Pi_{ij} - \frac{\Pi^2}{2}\right) - \sqrt{|g|}R \qquad \mathcal{H}^i = -2\Pi^{ij}{}_{;j} \tag{2.54}$$

So, the Einstein Hilbert action can be written in form (2.1) as

$$I_{ADM}\left[g_{ij}\,, \Pi^{ij}, N, N_i\right] = \int d^4x \left[\Pi^{ij}\dot{g}_{ij} - N\mathcal{H} - N_i\mathcal{H}^i\right] \tag{2.55}$$

Where

g_{ij}--------Cannonical Fields

Π^{ij}--------Canonical Momentum fields

N, N_i------Lagrange Multiplier Fields

$\mathcal{H}, \mathcal{H}^i$----Contraint Fields

and the Poisson's bracket for our theory is

$$\left[g_{ij}(x), \pi^{kl}(x')\right] = \frac{1}{2}\left(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k\right)\delta^3(x - x') \tag{2.56}$$

since both the tensors are symmetric.

And the equations of motion are

$$\frac{\delta I}{\delta N} = \mathcal{H} = 0 \qquad , \qquad \dot{g}_{ij} = \frac{\delta H}{\delta \pi^{ij}} \tag{2.57}$$

$$\frac{\delta I}{\delta N_i} = \mathcal{H}^i = 0 \qquad , \qquad \dot{\pi}_{ij} = -\frac{\delta H}{\delta g_{ij}} \tag{2.58}$$

So, there are 4 constraints $\mathcal{H} = 0$, $\mathcal{H}^i = 0$ and 4 Lagrange multipliers N, N_i. These constraints are first class constraints and they satisfy the Dirac Algebra

$$[\mathcal{H}(x), \mathcal{H}(y)] = g^{ij}\mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - g^{ij}\mathcal{H}_j(y)\frac{\partial}{\partial y^i}\delta(x,y) \tag{2.59}$$

$$[\mathcal{H}_j(x), \mathcal{H}(y)] = \mathcal{H}(y)\frac{\partial}{\partial x^i}\delta(x,y) \tag{2.60}$$

$$[\mathcal{H}_j(x), \mathcal{H}_j(y)] = \mathcal{H}_j(y)\frac{\partial}{\partial y^i}\delta(x,y) - \mathcal{H}_i(y)\frac{\partial}{\partial y^j}\delta(x,y) \tag{2.61}$$

As done in section of electromagnetic field theory we write down the generator as

$$\Phi_{\mathcal{H}^i}[\xi^i] = \int d^3x\, \xi_i(x)\, \mathcal{H}^i \tag{2.62}$$

$$= -2\int d^3x\, \xi_i \Pi^{ij}_{\ ;j}$$

$$= 2\int d^3x\, \xi_{i;j}\Pi^{ij} \tag{2.63}$$

for the constraint $\mathcal{H}^i$ where ξ^i is the test function. Now using Poisson's bracket, we find

$$\delta g_{ij} = \left[g_{ij}, \Phi_{\mathcal{H}^i}[\xi^i]\right] \tag{2.64}$$

$$= 2\int d^3x'\, \xi_{k;l}(x')\left[g_{ij}(x), \pi^{kl}(x')\right]$$

$$\delta g_{ij} = \varepsilon_{i;j} + \varepsilon_{j;i} = \mathcal{L}_\varepsilon g_{ij} \tag{2.65}$$

which is simply the lie derivate of g_{ij} on Σ_t^3 that are diffeomorphisms which are gauge transformations in general relativity [2].

CHAPTER 3

ASYMPTOTIC SYMMETRIES AND CONSERVED CHARGES

In this chapter we deal with gauge theories i.e. where fields have effect at large distances [1]. Thus, the analysis of theory includes contribution from boundary terms which we had neglected in chapter 1 and chapter 2.

3.1 BOUNDARY CONDITIONS AND BOUNDARY TERMS

Let us now explain what we mean by boundary conditions and boundary terms. For simplicity consider a theory of some real scalar field ϕ on some non-compact manifold $\mathcal{M}$ which is given by

$$I = \int_{\mathcal{M}} d^4x\, \mathcal{L}\big(\phi, \partial_\mu \phi\big) \tag{3.1}$$

Now the variation of the action is given by

$$\delta I = \int_{\mathcal{M}} d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi_{,\mu} \right) \tag{3.2}$$

$$\delta I = \int_{\mathcal{M}} d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) \right) \delta\phi + \int_{\mathcal{M}} d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi \right) \tag{3.3}$$

Here the boundary term is

$$B = \int_{\mathcal{M}} d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi \right) \tag{3.4}$$

which using divergence theorem can be written as

$$B = \int_{\mathcal{M}} d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi \right) = \int_{\partial\mathcal{M}} d\Sigma_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi \right) \tag{3.5}$$

So, to visualize the situation better let us consider a 3-dimensional case, with 1-spatial dimension suppressed (Fig 3.1).

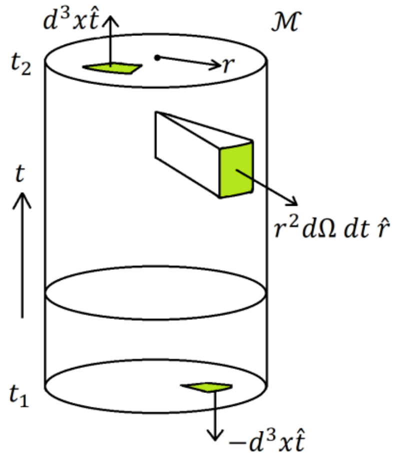

Fig (3.1): The Manifold

From the figure (3.1) we can see that the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ have three pieces, the two covers at constant time t_1 and t_2 and where $d\Sigma_\mu = d^3x$ pointing upwards and downwards in time respectively and the cylinder at $r \to \infty$ where $d\Sigma_\mu = r^2 d\Omega dt\hat{r}$ where $d\Omega$ is the solid angle. The boundary term then becomes

$$B = \int \frac{\partial \mathcal{L}}{\partial \phi_{,0}} \delta\phi \, d^3x|_{t_1}^{t_2} + \int \frac{\partial \mathcal{L}}{\partial \phi_{,r}} \delta\phi \, r^2 d\Omega dt|_{r\to\infty} \tag{3.6}$$

Since the initial and final conditions are fixed the first term vanishes i.e. $\delta\phi(t_1) = \delta\phi(t_2) = 0$. And we are left with

$$B = \int \frac{\partial \mathcal{L}}{\partial \phi_{,r}} \delta\phi \, r^2 d\Omega dt|_{r\to\infty} \tag{3.7}$$

Usually we deal with Non-Gauge theories where field have compact support i.e. when $r \to \infty$ then $\phi \to 0$, but in case of Gauge theories this is not true i.e. when $r \to \infty$ then $\phi \neq 0$. Even if physical fields do tend to zero the presence of Lagrange multipliers with no dynamical equations restricting them make it difficult to solve our theory.

3.2 CHERN SIMONS THEORY

The action of Chern-Simons theory in (1+2) dimensions in component notation is given as [7]

$$I[A] = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} \, Tr\left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda\right) \tag{3.8}$$

Here

$\epsilon^{\mu\nu\lambda}$--- Totally Antisymmetric tensor

k------- Constant

Tr----- Represents trace of the matrix

A_μ----- Are known as the connections

The connection takes values in some given algebra usually associated with some Gauge group i.e.

$$A_\mu = A^a{}_\mu(x) J_a \tag{3.9}$$

here J_a are generators of the Gauge group and $A^a{}_\mu(x)$ are scalars.

We split the Chern-Simons action in (1+2) where $A_\mu = (A_0, A_i)$ and we get

$$I_{CS}[A_0\,, A_i] = \frac{k}{8\pi}\int d^3x\, \epsilon^{ij}\eta_{ab}\left[-A^a{}_i\dot{A}^b{}_j + A^a{}_0 F^b{}_{ij}\right] \tag{3.10}$$

This action has the general structure of (2.1), where A_i are two dynamical fields. Here $A^a{}_0$ are the Lagrange multipliers, $F^b{}_{ij}$ are the constraints and η_{ab} is the flat metric. The poisons bracket is given by

$$\left[A^a{}_i(x), A^b{}_j(x')\right] = \frac{4\pi}{k}\epsilon_{ij}\eta^{ab}\delta(x-x') \tag{3.11}$$

and the poisons bracket of ant two functions $L_1(A)$,$L_2(A)$ of the fields $A^a{}_i$ is given by

$$[L_1(A), L_2(A)] = \frac{4\pi}{k}\int d^2x\, \frac{\delta L_1(A)}{\delta A^a{}_i}\epsilon_{ij}\eta^{ab}\frac{\delta L_2(A)}{\delta A^b{}_j} \tag{3.12}$$

and the equation of motion are

$$\dot{A}^a{}_i = D_i A^a{}_0 \tag{3.13}$$

$$F^a{}_{ij} = 0 \tag{3.14}$$

where $D_i A^a{}_0 = \partial_i A^a{}_0 + \epsilon^a{}_{bc}A^b{}_i A^c{}_0$ is the covariant derivative.

3.2.1 BOUNDARY TERMS

As we had shown in chapter 2 in case of electromagnetic field theory and general relativity, if we look at the action (3.10) we could say that the generator of gauge transformation with parameter ξ^a is the integral of constraint with a test function ξ is

$$G_0[\xi] = \frac{k}{8\pi}\int d^2x\, \epsilon_{ij}\xi_a F^a{}_{ij} \tag{3.15}$$

which would have been correct if we had no boundaries on the manifold but here the case is different, we do have manifold boundaries and the correct Gauge transformation with parameter ξ^a acting on $A^a{}_i$ as we know is

$$\delta A^a{}_i = D_i \xi^a \tag{3.16}$$

If the equation (3.15) is correct then the following should hold from (3.11) i.e.

$$\left[A^a{}_i, G_0[\xi]\right] = \epsilon^{ij}\eta^{ab}\,\frac{\delta G_0[\xi]}{\delta A^b{}_j} = D_i\xi^a \tag{3.17}$$

Now to check whether the equality hold true or not we calculate the variation

$$\delta G_0[\xi] = \frac{k}{8\pi}\int d^2x\,\epsilon_{ij}\xi_a\delta F^a{}_{ij} \tag{3.18}$$

which after calculating we get

$$\delta G_0[\xi] = \delta B[\xi] - \frac{k}{4\pi}\int d^2x\,\epsilon_{ij}D_i\xi^a\delta A^a{}_j \tag{3.19}$$

where the first term which is a boundary term is

$$B[\xi] = \frac{k}{4\pi}\oint_{r\to\infty} d\varphi\xi_a A^a{}_\varphi \tag{3.20}$$

In order to make sense of equation (3.17) we do the following by taking $\delta B[\xi]$ on the LHS of equation (3.19) and we get

$$\delta(G_0[\xi] - B[\xi]) = -\frac{k}{4\pi}\int d^2x\,\epsilon_{ij}D_i\xi^a\delta A^a{}_j \tag{3.21}$$

This suggests that instead of considering $G_0[\xi]$ we consider $G[\xi] \equiv G_0[\xi] - B[\xi]$ so we get

$$G[\xi] = \frac{k}{8\pi}\int d^2x\,\epsilon_{ij}\xi_a F^a{}_{ij} - \frac{k}{4\pi}\oint_{r\to\infty} d\varphi\xi_a A^a{}_\varphi \tag{3.22}$$

whose variation is well-defined as

$$\frac{\delta G[\xi]}{\delta A^a{}_j} = -\frac{k}{4\pi}\epsilon_{ij}D_i\xi^a \tag{3.23}$$

So, we conclude by saying that in case of boundaries on manifold the generator of gauge transformation is $G[\xi]$ instead of $G_0[\xi]$ and $G[\xi]$ is known as the improved generator. Now if we consider the equation (3.17) we see that using $G[\xi]$ makes sense i.e.

$$\left[A^a{}_i, G[\xi]\right] = \epsilon^{ij}\eta^{ab}\,\frac{\delta G[\xi]}{\delta A^b{}_j} = D_i\xi^a \tag{3.24}$$

and indeed, the equality holds true.

We should be aware that the value of $B[\xi]$ depends on gauge parameter ξ^a so it can in principle take any value. Thus, not all gauge transformations are on equal footing so we can make the following classification

- PROPER GAUGE TRANSFORMATION: Choices of ξ^asuch that $B[\xi] = 0$. These form the class of Pure Gauge transformations and do not change the physical state of the system and their generator is purely a constraint.
- IMPROPER GAUGE TRANSFORMATION: Choices of ξ^asuch that $B[\xi] \neq 0$. These form the class of impure gauge transformations and they do change the physical state of the system and the transformation corresponding to them is physical which can be realized physically.

3.2.2 ASYMPTOTIC SYMMETRIES

Suppose we have a field with specified boundary conditions. We say that asymptotic symmetry group of this field is the set of all symmetry transformation that preserve the asymptotic boundary conditions and these symmetries possess non-zero conserved charges. For our case to say whether $B[\xi]$ is zero or non-zero depends on asymptotic conditions. These conditions depend on the physical situations. We will explain these ideas with the help of examples below.

Example 1: Consider the set of all connections $A_0\ A_r\ A_\varphi$ such that we have the following boundary conditions

$$A_0 = 0\,, \qquad A_r = 0 \qquad at\ r \to \infty \tag{3.25}$$

We find all the gauge transformation that leave this boundary condition invariant i.e.

$$\delta A_\mu = D_\mu \xi \quad \Rightarrow \quad D_t \xi = D_r \xi = 0 \tag{3.26}$$

and the solution is given by the parameter ξ^a which depends only on φ i.e.

$$\xi^a = \xi^a(\varphi) \tag{3.27}$$

At boundary the only non-zero field is $A_\varphi(\varphi)$ and our theory is invariant under the transformations whose parameter depend only on φ i.e.

$$\delta A_\varphi = D_\varphi \xi \tag{3.28}$$

which is set of asymptotic symmetries for these boundary conditions.

Example 2: We have boundary condition known as chiral condition which is use d to study black holes and CFTs are given as

$$A_0 = A_\varphi\,, \qquad A_r = 0 \quad at\; r \to \infty \tag{3.29}$$

once again, we find all the gauge transformation that leave this boundary condition invariant i.e.

$$\delta A_\mu = D_\mu \xi \tag{3.30}$$

and the solution is given by the parameter

$$\xi^a = \xi^a(t+\varphi) \tag{3.31}$$

At the boundary the only non-zero field is the chiral field i.e. $A_\varphi(t+\varphi)$ and our theory is invariant under the chiral transformations whose parameter depend only on $\xi^a(t+\varphi)$

3.2.3 ASYMPTOTIC ALGEBRA

Now we calculate the asymptotic algebra of the generators of the gauge transformations since now we have well defined variations for $G[\xi]$ so computing the algebra we get

$$\big[G[\xi], G[\rho]\big] = \frac{k}{4\pi}\int d^2x\epsilon_{ij}\eta^{ab}\,\frac{\delta G[\xi]}{\delta A^a{}_i(x)}\frac{\delta G[\rho]}{\delta A^b{}_j(x)} \tag{3.32}$$

$$= \frac{k}{4\pi}\int d^2x\eta^{ab}\epsilon_{ij}\epsilon^{in}\epsilon^{jm}\,(D_n\xi_a)(D_m\rho_b)$$

$$= \frac{k}{4\pi}\int d^2x\epsilon^{nm}\partial_n(\xi_a D_m\rho^a) - \frac{k}{4\pi}\int d^2x\epsilon^{nm}\xi_a D_n D_m\,\rho^a$$

$$= \frac{k}{4\pi}\oint_{r\to\infty} d\varphi\xi_a D_\varphi\rho^a - \frac{k}{4\pi}\int d^2x\frac{1}{2}\xi_a\epsilon^a{}_{bc}F^b{}_{ij}\,\rho^c\epsilon^{ij}$$

$$= \frac{k}{4\pi}\oint_{r\to\infty} d\varphi\xi_a\partial_\varphi\rho^a - \frac{k}{4\pi}\oint_{r\to\infty} d\varphi\epsilon_{abc}\xi^a\rho^c A^b{}_\varphi + \frac{k}{8\pi}\int d^2x\;\epsilon_{abc}\xi^a\rho^c\epsilon^{ij}F^b{}_{ij} \tag{3.33}$$

where we have used strokes theorem, $\epsilon^{nm} D_n D_m \rho^a = \frac{1}{2}\epsilon^{nm} F_{nm}\rho^a$ and $\xi_a \rho^a$ is scalar under the gauge group. The last two terms group to form G again thus finally we get

$$\big[G[\xi], G[\rho]\big] = G\big[[\xi,\rho]\big] + \frac{k}{4\pi}\oint_{r\to\infty} d\varphi \xi_a \partial_\varphi \rho^a \tag{3.34}$$

Where we have used $([\xi,\rho])_b = \epsilon_{abc}\xi^a\rho^c$ for ξ, ρ in gauge algebra. The last term is called the central extension of the algebra.

3.2.4 HAMILTONIAN AND ENERGY

For Chern-Simons action given by (3.10) we compute its variation i.e.

$$\delta I_{CS} = \frac{k}{8\pi}\delta \int d^3x\, \epsilon^{ij}\, Tr\big(A_i \dot{A}_j - A_0 F_{ij}\big) \tag{3.35}$$

$$= \frac{k}{8\pi}\int d^3x\, \epsilon^{ij}\, Tr\left(-\delta A_0 F_{ij} + \delta A_i \dot{A}_j - \dot{A}_i \delta\, A_j - A_0 \delta F_{ij} + \frac{d}{dt}\big(A_i \delta A_j\big)\right)$$

$$= -\frac{k}{8\pi}\int d^3x\, \epsilon^{ij}\, Tr\big(\delta A_0 F_{ij} + 2\dot{A}_i \delta\, A_j + A_0 \delta F_{ij}\big) \tag{3.36}$$

where the time derivative is zero since the initial and final conditions are held fixed and we used the cyclic property of the trace. And we use

$$\epsilon^{ij}\, Tr\big(A_0 \delta F_{ij}\big) = 2\epsilon^{ij}\, Tr\big[\partial_i\big(A_0 \delta A_j\big) - D_i A_0 \delta A_j\big] \tag{3.37}$$

and thus, we get

$$\delta I_{CS} = -\frac{k}{8\pi}\int d^3x\, \epsilon^{ij}\, Tr\left(\delta A_0 F_{ij} + 2\big(\dot{A}_i - D_i A_0\big)\delta A_j + 2\partial_i\big(A_0 \delta A_j\big)\right) \tag{3.38}$$

$$= \frac{k}{8\pi}\int (e.o.m) - \frac{k}{4\pi}\int dt \int d^2x\, \partial_i Tr\left(\epsilon^{ij}\big(A_0 \delta A_j\big)\right)$$

$$\delta I_{CS} = \frac{k}{8\pi}\int (e.o.m) - \frac{k}{4\pi}\int dt \int_{r\to\infty} d\varphi\, Tr\big(A_0 \delta A_\varphi\big) \tag{3.39}$$

where in the last step we used Strokes theorem.

Now we define a new action by passing the boundary term on the left side of equation (3.39), now the equations of motion define an extremum for this new action i.e.

$$\delta I_{CS} + \frac{k}{4\pi}\int dt \int_{r\to\infty} d\varphi\, Tr\left(A_0 \delta A_\varphi\right) = \frac{k}{8\pi}\int (e.o.m) \tag{3.40}$$

But the process done above is equivalent to defining a new Hamiltonian i.e.

$$H = H_0 + E \tag{3.41}$$

$$H = \frac{k}{8\pi}\int \epsilon^{ij} A_0 F_{ij} + E \tag{3.42}$$

where

$$\delta E = \frac{k}{4\pi}\int dt \int_{r\to\infty} d\varphi\, Tr\left(A_0 \delta A_\varphi\right) \tag{3.43}$$

This boundary term is known as Energy, since H_0 is a constraint, E gives us energy for any solution to equations of motion.

Now finding E depends on boundary conditions

Case I: Let's consider $A_0 = 0$ and $A_r = 0$ when $r \to \infty$ so this means that our energy $E = 0$

Case II: Lets consider Chiral boundary conditions i.e. $A_0 = A_\varphi$ and $A_r = 0$ when $r \to \infty$ so now using (3.43) we get

$$\delta E = \frac{k}{4\pi}\int dt \int_{r\to\infty} d\varphi\, Tr\left(A_\varphi \delta A_\varphi\right) = \frac{k}{4\pi}\delta \int dt \int_{r\to\infty} d\varphi\, Tr\left(\frac{1}{2} {A_\varphi}^2\right) \tag{3.44}$$

Thus, our energy will be

$$E\left[A_\varphi\right] = \frac{k}{8\pi}\int dt \int_{r\to\infty} d\varphi\, Tr\left({A_\varphi}^2\right) \tag{3.45}$$

3.2.5 HAMILTONIAN GENERATES TIME TRANSLATIONS

The Hamiltonian H we defined in the previous section is the correct Hamiltonian is the generator of time translations which are symmetry transformations corresponding to conserved energy. The equations of motion are now given by

$$\dot{A}^a{}_i = [A^a{}_i\,, H] = \epsilon^{ij}\eta^{ab}\frac{\delta H}{\delta A^b{}_j} = D_i A^a{}_0 \tag{3.46}$$

3.2.6 CONSERVED CHARGES

As we have seen that improper gauge transformations are not generated by constraints, they change the physical state of the system and do give rise to non-trivial conserved charge. Thus, its generator should not evolve in time so we need to see if $\frac{d}{dt}G[\xi]$ is zero or not. So, the time derivative of $G[\xi]$ is

$$\frac{d}{dt}G[\xi] = [G[\xi], H] + \int d^2x\,\frac{\delta G[\xi]}{\delta\xi}\dot{\xi} \tag{3.47}$$

since $G[\xi]$ depends on A_i and the parameter ξ which may also depend on time. Now from the previous section we know that Hamiltonian is gauge generator with the parameter A_0 i.e.

$$H = G[A_0] \tag{3.48}$$

Here we are dealing with chiral boundary conditions (3.29) so we get

$$\frac{d}{dt}G[\xi] = \big[G[\xi], G[A_0]\big] + \int d^2x\,\frac{\delta G[\xi]}{\delta\xi}\dot{\xi} \tag{3.49}$$

$$= \int d^2x\,\epsilon^{ij}[\xi, A_0]F_{ij} + \int_{r\to\infty} d\varphi\,\xi\,\partial_\varphi A_0 + \int d\varphi\,A_\varphi\dot{\xi}$$

$$= 0 + \int d\varphi\,\xi\big(\partial_\varphi - \partial_0\big)A_\varphi$$

$$\frac{d}{dt}G[\xi] = 0 \tag{3.50}$$

Thus equation (3.50) shows us that $G[\xi]$ i.e. the generator of symmetry is conserved, in our case the conserved charge is given by

$$Q[\xi] = -\frac{k}{4\pi}\int_{r\to\infty} d\varphi\, \xi_a A^a{}_\varphi \tag{3.51}$$

which is independent of time.

3.3 BTZ BLACK HOLE

The Einstein-Hilbert action in (1+2) dimensions is given by [2]

$$I = \frac{1}{16\pi}\int_M d^3x\,\sqrt{-g}(R - 2\lambda) \tag{3.52}$$

where $\lambda = -\frac{1}{l^2}$ is the cosmological constant. Extremizing this action with respect to the metric $g_{\mu\nu}$ gives us Einstein's vacuum field equations that are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{l^2}g_{\mu\nu} \tag{3.53}$$

where $R_{\mu\nu}$ is Ricci tensor and R is Ricci scalar.

Now the equations (3.53) in (1+2) dimensions can be obtained from the action given by I_{CSG} (Chern-Simons Gravity) which is reformulated in terms of two copies of Chern-Simons theory [7],[8],[9] i.e.

$$I_{CSG} = I[A^+] - I[A^-] \tag{3.54}$$

Since our covering space is anti-de Sitter due to negative cosmological constant λ, here our gauge group is $SO(2,2)$ whose algebra is given as [7]

$$[J_a\,, J_b] = \epsilon_{abc}J^c \tag{3.55}$$

$$[J_a\,, P_b] = \epsilon_{abc}P^c \tag{3.56}$$

$$[P_a\,, P_b] = \lambda\epsilon_{abc}J^c \tag{3.57}$$

This algebra can be simplified by introducing

$$J_a^{\pm} = \frac{1}{2}\left(J_a \pm \frac{P_a}{\sqrt{\lambda}}\right) \tag{3.58}$$

since $SO(2,2) \equiv SL(2,R) \times SL(2,R)$, the algebra then becomes

$$[J_a^+ , J_b^+] = \epsilon_{abc} J_c^+ \tag{3.59}$$

$$[J_a^- , J_b^-] = \epsilon_{abc} J_c^- \tag{3.60}$$

$$[J_a^+ , J_b^-] = 0 \tag{3.61}$$

So, our connections are given by

$$A^{a\pm}_j = \omega^a{}_j \pm \sqrt{\lambda} e^a{}_j \tag{3.68}$$

where $e^a{}_j$ and $\omega^a{}_j$ are the triads and spin connections given by $e^a = e^a{}_i dx^i$ and $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{ibc} dx^i$ respectively. The covariant derivative is

$$D_j = \partial_j + J_a^+ A^{a+}_j + J_a^- A^{a-}_j \tag{3.69}$$

One of the solutions of (3.53) is given by a black hole known as BTZ black hole and its metric is given by [9]

$$ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2\left(d\phi + N^{\phi} dt\right)^2 \tag{3.70}$$

where

$$f^2 = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) \qquad N^{\phi} = -\frac{J^2}{2r^2} \tag{3.71}$$

Here f is the lapse function and N^{ϕ} is the shift function with ($|J| \leq Ml$). This metric is stationary and axially symmetric with killing fields ∂_t and ∂_{ϕ}. ADM mass and angular momentum are given by M and J respectively. This metric is asymptotically AdS.

The BTZ black hole with mass M and angular momentum J in the Chern-Simons formulation can be described by constant connections $A_{\varphi}^{\pm}$ which follow [8]

$$Tr\left(A_{\varphi}^{\pm}\right) = \pm\frac{2}{k}(M \pm J) \tag{3.72}$$

We are interested in finding out the energy of BTZ black hole using the ideas we studied in this chapter. We know the energy corresponding to asymptotic symmetry provided with asymptotic boundary conditions in our case which was chiral condition was given by expression (3.29). Here our boundary conditions are given as

$$A_0^+ = A_\varphi \qquad A_0^- = -A_\varphi^- \qquad at\ r \to \infty \tag{3.73}$$

and these condition lead to what are known as holomorphic and anti-holomorphic currents. Now using (3.45) for (3.54) we get

$$E[A_\varphi , A_\varphi^-] = \frac{k}{8\pi}\int dt \int_{r\to\infty} d\varphi\, Tr\left(\left(A_\varphi^+\right)^2 - \left(A_\varphi^-\right)^2\right) \tag{3.74}$$

Therefore, in Euclidean time where $0 \leq t \leq 1$ the energy of the black hole is

$$E_{BTZ} = \frac{k}{8\pi}\int_{r\to\infty} d\varphi \left(\frac{2\pi}{k}(M+J) + \frac{2\pi}{k}(M-J)\right) = M \tag{3.75}$$

which is the mass of the (1+2) dimensional BTZ black hole.

CHAPTER 4

CONCLUSIONS

In this chapter we will summarize all three chapters and try to conclude our work.

So, in the first chapter we saw what we meant by symmetries and conserved charges of a theory specifically by learning about them for Minkowski spacetime. In chapter 2 we studied dynamics of constrained system. But our discussion was limited to the theories excluding the boundary terms. We learnt that if a theory has constraints in it and if these constraints are first class then this implies that our theory has gauge symmetry, and this gauge symmetry does not have a corresponding conserved charge. We discussed these ideas via study of electromagnetic field theory and general relativity.

Chapter 3 had all the core ideas about asymptotic symmetries and conserved charges. Now here we dealt with gauge theories specifically Chern Simons theory with boundary terms and saw how the boundary terms interfere with the generators of gauge symmetry. So, we had two possibilities, first was proper gauge transformations and second was improper gauge transformations. Proper gauge transformations are the ones where the boundary term was zero and they corresponded to symmetry which did not change the physical state of the system. Whereas improper gauge transformations were the ones where we had non zero boundary terms and they corresponded to symmetry which did change the physical state of the system and we had the corresponding conserved charges.

The idea of improper gauge transformation provided with specific boundary conditions resulted in what are known as asymptotic symmetries. We saw how important these boundary conditions were, as the generator of asymptotic symmetries depends on them. We were required to find out those gauge transformations which left the boundary condition invariant. One important boundary condition was chiral boundary condition which is used in study of CFTs and black hole physics. We then explicitly calculated the asymptotic algebra.

One of the corresponding conserved charges for asymptotic symmetries is the energy. The Hamiltonian is the generator of energy which corresponds to asymptotic time translations. We saw that under certain boundary conditions we might even have a trivial energy. Thus, boundary conditions are really important for understanding the system. Lastly, we applied all the knowledge to see the calculation of the energy of the BTZ black hole in (1+3) dimensions, which was nothing but the asymptotically conserved charge corresponding to the chiral boundary conditions.

BIBLIOGRAPHY

[1] Banados, Max, and Ignacio Reyes. "A short review on Noether’s theorems, gauge symmetries and boundary terms." *International Journal of Modern Physics D* 25.10 (2016): 1630021.

[2] Blau, Matthias. "Lecture notes on general relativity." (2011).

[3] Poisson, Eric. "A relativist's toolkit: the mathematics of black-hole mechanics." (2004).

[4] Dirac, Paul Adrien Maurice. "Lectures on quantum mechanics." (2001).

[5] Das, Ashok. "Lectures on quantum field theory." (2008).

[6] Compère, Geoffrey. "Advanced Lectures on General Relativity." (2018).

[7] Witten, Edward. "2+ 1 dimensional gravity as an exactly soluble system." *Nuclear Physics B* 311.1 (1988): 46-78.

[8] Banados, Maximo. "Three-dimensional quantum geometry and black holes." *AIP Conference Proceedings*. Vol. 484. No. 1. AIP, 1999.

[9] Carlip, Steven. "The (2+ 1)-dimensional black hole." *Classical and Quantum Gravity* 12.12 (1995): 2853.