Spectral Theory of Cayley Graphs of Finite Groups

A Thesis

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Malavika K S

Roll No: 20201039

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road, Pashan, Pune 411008, INDIA

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Supervisor: Dr. Chandrasheel Bhagwat

Certificate

This is to certify that this dissertation entitled **Spectral Theory of Cayley Graphs of Finite Groups** towards the partial fulfilment of the BS-MS dual degree programme at the **Indian Institute of Science Education and Research, Pune** represents study/work carried out by **Malavika K S** under the supervision of **Dr. Chandrasheel Bhagwat**, Professor, Department of Mathematics, during the academic year 2024-2025.

चं.रा. भागवत

Dr. Chandrasheel Bhagwat

Dr. Leelavati Narlikar

Committee:

Dr. Chandrasheel Bhagwat

Dr. Leelavati Narlikar

This thesis is dedicated to my Ammoommaa

Declaration

I hereby declare that the matter embodied in the report entitled **Spectral Theory of Cayley Graphs of Finite Groups** consists of the results of work carried out by me at the Department of Mathematics, IISER Pune, under the supervision of **Dr. Chandrasheel Bhagwat**, and that the same has not been submitted elsewhere for any other degree.

Malavika K S

April, 2025

Contents

Abstract

This thesis explores the spectral properties of Cayley graphs and their connections to representation theory. Spectral graph theory studies the eigenvalues of adjacency and Laplacian matrices, which reveal structural properties of graphs. When a group acts transitively on a graph, its adjacency and Laplacian spectra are closely related and can often be analyzed through group representations. The Cayley graph of a group, defined with respect to a generating set, provides a natural framework for studying spectral properties using character theory.

Key results on Markov chain theory by J. R. Norris and those of Diaconis, Bayer, and Aldous on card shuffling are examined in this context. Additionally, Lovász's work on the eigenvalues of graphs in terms of character theory is discussed. The thesis concludes with explicit calculations of the spectra of the Cayley graphs of dihedral groups and S_4, using these theoretical insights.

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Chapter 1

Introduction

Graphs serve as fundamental structures in mathematics, encoding relationships between objects across various fields, including combinatorics, computer science, and physics. The eigenvalues of the adjacency and Laplacian matrices of finite graphs are closely related to graph properties such as structure and connectivity. These connections are studied in **spectral graph theory**.

In many cases, graphs arise naturally from algebraic objects such as groups. A particularly important class of graphs, **Cayley graphs**, is constructed from a group G and a generating set S. These graphs provide a direct link between group theory, graph theory, and spectral graph theory. The interplay between group representations and the spectral properties of Cayley graphs has been widely studied, with applications in card shuffling, network design, and theoretical physics.

This thesis examines classical results by **Diaconis, Aldous, and Bayer** on card shuffling, which describe how probability distributions evolve over time on symmetric groups. These ideas have profound implications for understanding the time required for a random process to reach a uniform distribution.

Furthermore, a significant part of this thesis focuses on computing the spectra of Cayley graphs of **dihedral groups** and **the symmetric group** S_4. Using **Lovász's formula**, we derive explicit spectral computations and explore the impact of the choice of generators on eigenvalues. Along the way, we identify limitations in existing formulas and propose refinements to accommodate new cases.

A key motivation for this thesis is the study of the **spectral properties of Cayley graphs of finite groups**, with a particular focus on dihedral groups and the symmetric group S_4. The connection between spectral graph theory and group representation theory allows us to compute eigenvalues explicitly using **character theory**. The adjacency matrix of a Cayley graph is closely related to the **regular representation** of the underlying group, and its eigenvalues can often be expressed in terms of irreducible characters.

1.1 Scope and Structure of the thesis

The main aim of this project was to study the spectral properties of Cayley graphs of finite groups. In this regard, the first part of the project focused on understanding the basics of Cayley graphs, Markov chains, and key results by Diaconis, Bayer, and Aldous on card shuffling.

In the second half of the project, Lovász's results on computing the eigenvalues of a graph G in terms of the characters of a transitive subgroup of its automorphism group were analyzed, and the spectrum of the Cayley graphs of D_{2n} and S_4 was explicitly calculated.

The structure of the thesis is as follows:

Chapter 1 is an Introduction to this thesis and provides an overview of the work presented.

Chapter 2 (Preliminaries) introduces fundamental concepts from probability theory, graph theory, and group representation theory. It covers essential definitions, such as adjacency and Laplacian matrices, shuffling techniques, and group representations, providing a strong foundation for the later chapters.

Chapter 3 (Markov Chains) explores Markov chains in greater depth, discussing key theorems related to mixing times, invariant distributions, and coupling methods. These concepts help in analyzing random walks on groups and understanding their convergence behavior.

Chapters 4 and 5 (Results of Diaconis and Bayer and Results of Aldous and Diaconis) discuss key results from Diaconis, Bayer, and Aldous, particularly in the context of card shuffling. Chapter 4 focuses on the Gilbert-Shannon-Reeds model of riffle shuffling, while Chapter 5 examines the variation distance and strong uniform time, providing rigorous probabilistic estimates for mixing times.

Chapter 6 (Results of Lovász) presents Lovász's spectral results and their applications to finite graphs. This chapter explains how character theory can be used to compute the eigenvalues of adjacency matrices and discusses the limitations of existing methods.

Chapter 7 (Cayley graphs of Dihedral groups and S_4) contains explicit computations of eigenvalues for Cayley graphs of *dihedral groups* and S_4, respectively. Section 7.1 focuses on dihedral groups, detailing the spectral properties of Cayley graphs generated by reflections and rotations. Section 7.2 extends these ideas to the symmetric group S_4, analyzing the spectral properties of its Cayley graph generated by the top-in-at-random shuffle, excluding the identity permutation.

Chapter 8 (Conclusion) concludes with insights into the broader implications of these results and potential future research directions. It discusses possible extensions of this work, including studying spectral properties of other transitive graphs, higher-dimensional representations, and applications to Markov chain mixing times.

1.2 Original Contributions

- Worked out an example where repeated convolutions converge to the uniform distribution (Refer to 5.1).

- Proved that the converse of **Theorem 5.2.7** (Refer to 5.2) is not true .
- Found that Lovász's formula for computing the eigenvalues of the adjacency matrix of a graph holds only for $n_i = 2$. Derived a new formula for the case $n_i = 3$ (Refer to 6).
- Computed all eigenvalues and eigenvectors of the Cayley graphs of dihedral groups generated by $\{r, r^{-1}, s\}$ using Lovász's results (Refer to 7.1).
- Computed all eigenvalues of the Cayley graph of S_4 generated by the top-in-at-random shuffle using Lovász's results (Refer to 7.2).

Chapter 2

Preliminaries

This chapter contains definitions and concepts to prepare the reader for the rest of the thesis.

2.1 Probability and Stochastic Processes

This section introduces fundamental concepts in probability and stochastic processes, which form the foundation for analyzing randomness in structured systems. We cover probability spaces, random variables, stochastic matrices, and key results such as Chebyshev's inequality and the Coupon Collector's problem. In addition, we introduce various shuffling techniques which serve as a basis for the results discussed in later chapters.

2.1.1 Fundamental Concepts in Probability and Stochastic Matrices

State-Space and Measures

Let I be a countable set, where each element $\mathbf{i} \in I$ is referred to as a **state**, and the set I itself is known as the **state-space**.

A function $\lambda = (\lambda_{\mathbf{i}})_{\mathbf{i} \in I}$ is said to be a **measure** on I if it satisfies

$$0 \leq \lambda_{\mathbf{i}} < \infty, \quad \forall \mathbf{i} \in I.$$

Moreover, if the total sum of the values is

$$\sum_{\mathbf{i} \in I} \lambda_{\mathbf{i}} = 1,$$

then λ is referred to as a **distribution**.

Probability Spaces and Random Variables

A **probability space** is a mathematical framework for modeling random experiments. It is denoted as:

$$(\Omega, \mathcal{F}, \mathbb{P})$$

where:

- Ω is the **sample space**: the set of all possible outcomes of a random experiment.
- $\mathcal{F}$ is a **sigma-algebra**: a collection of subsets of Ω (called events) that includes Ω itself, the empty set $\emptyset$, and is closed under countable unions, intersections, and complements.
- $\mathbb{P}$ is a **probability measure**: a function that assigns a probability to each event in $\mathcal{F}$, satisfying the following axioms:
  1. $\mathbb{P}(\Omega) = 1$ (the probability of the entire sample space is 1).
  2. $0 \leq \mathbb{P}(A) \leq 1$ for all $A \in \mathcal{F}$.
  3. If $A_1, A_2, A_3, \ldots$ are disjoint events (i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$), then:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

A **random variable** X with values in I is a function

$$X : \Omega \to I.$$

For a state $\mathbf{i} \in \mathbf{I}$, we define

$$\lambda_{\mathbf{i}} = \mathbb{P}(X = \mathbf{i}) = \mathbb{P}(\{\omega : \mathbf{X}(\omega) = \mathbf{i}\}).$$

Thus, λ represents the **distribution of** X, describing the probability of being in state $\mathbf{i}$.

Uniform distribution

A probability distribution is called **uniform** if every outcome in the sample space is equally likely. For a finite set S with $|S| = n$, the uniform distribution assigns probability

$$P(X = x) = \frac{1}{n}, \quad \text{for all } x \in S.$$

The uniform distribution over a finite group G assigns probability $U(g) = \frac{1}{|G|}$ for all $g \in G$.

Chebyshev's Inequality

Let X be a random variable with finite expectation $\mathbb{E}[X]$ and variance $\mathrm{Var}(X)$. Then, for any $k > 0$,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq k\sigma) \leq \frac{1}{k^2},$$

where $\sigma^2 = \mathrm{Var}(X)$ is the variance of X.

In other words, the probability that X deviates from its mean by at least k standard deviations is at most $\frac{1}{k^2}$.

Sufficient Statistic

A **statistic** $T(X)$ is called **sufficient** for a parameter θ if the conditional probability density function (pdf) of the data X, given the statistic $T(X)$, is independent of θ. Formally, $T(X)$ is sufficient if

$$f(X \mid T(X), \theta) = f(X \mid T(X))$$

for all values of X and θ.

Convolution

The **convolution** of two functions f and g is defined as

$$(f*g)(x) = \sum_{y\in\mathbb{Z}} f(y)g(x-y) \quad \text{(discrete case)}, \quad (f*g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy \quad \text{(continuous case)}.$$

The Coupon Collector's Problem

The Coupon Collector's Problem is a classic problem in probability theory. It models a situation where a collector tries to collect all n different types of coupons, with each new coupon obtained being independently and uniformly chosen from the n types.

Define T as the total number of draws needed to collect all n coupons. Let T_i be the number of draws required to obtain a new coupon after already having $i-1$ distinct coupons.

Since each draw has a probability $\frac{n-(i-1)}{n}$ of yielding a new coupon, T_i follows a geometric distribution with expectation:

$$E[T_i] = \frac{n}{n-(i-1)}$$

Thus, the expected total number of draws is:

$$E[T] = \sum_{i=1}^{n} E[T_i] = \sum_{i=1}^{n} \frac{n}{n-(i-1)} = n\sum_{i=1}^{n} \frac{1}{i}$$

Birthday Problem

The **birthday problem** asks: What is the probability that at least two people in a group of k share the same birthday, assuming 365 equally likely birthdays?

Denoting this probability as $P(k)$, we compute the complement (the probability that all birthdays are distinct):

$$P_{\text{distinct}}(k) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-k+1}{365}.$$

Thus, the probability of at least one birthday match is

$$P(k) = 1 - P_{\text{distinct}}(k).$$

Stochastic Matrices

A matrix $P = (p_{ij} : i, j \in I)$ is called **stochastic** if each row

$$(p_{ij} : j \in I)$$

is a probability distribution. That is, for all $i \in I$,

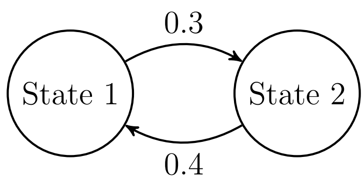
$$0 \leq p_{ij} \leq 1, \quad \sum_{j \in I} p_{ij} = 1.$$

For example, consider the following 2×2 stochastic matrix:

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

Each row sums to 1, which is a defining property of stochastic matrices.

The corresponding transition diagram for the Markov chain (Refer to 3) is:



Mixing time

For any probability measure p on a finite group G, we define the **total variation mixing time** as

$$T(G, p) = T(G, p, 1/(2e)) = \inf \left\{ k : \|p^{(k)} - u\|_{\text{TV}} \leq \frac{1}{2e} \right\}$$

where the more general quantity

$$T(G, p, \varepsilon) = \inf \left\{ k : \|p^{(k)} - u\|_{\text{TV}} \leq \varepsilon \right\}$$

denotes the least number of steps k required for the total variation distance between $p^{(k)}$ and the uniform distribution u to be at most ε. The total variation distance, $\|p^{(k)} - u\|_{\text{TV}}$, quantifies how different the probability distribution $p^{(k)}$ is from u (Refer to 5.1).

2.1.2 Riffle Shuffle: Structure and Probabilistic Properties

In *riffle shuffling*, we start with an ordered deck. The deck is split into two packets of equal size. Next, the two packets are riffled together. The two packets may still be found in the randomized deck as two separate "rising sequences" of face values.

A *rising sequence*, the fundamental invariant of riffle shuffling, is a maximal subset of an ordering of cards, made up of consecutive face values presented in ascending order. Rising sequences do not overlap, and so every ordering of a deck of cards is distinctly the union of its rising sequences.

An *inverse riffle shuffle* can be defined corresponding to each riffle shuffle. Here, we begin with a sorted deck. In inverse riffle shuffling, each card is assigned independently to one of two packets with equal probability. These packets are then placed in order. Then, the two packets are placed in sequence. For example, if we start with the arrangement $A, 6, 2, 3, 7, 8, 4, 5$ and move $A, 2, 3, 4, 5$ one way and $6, 7, 8$ the other way, then place them one after the other, the shuffle we performed is the inverse riffle shuffle corresponding to the riffle shuffle that produced the arrangement $A, 6, 2, 3, 7, 8, 4, 5$.

The Gilbert-Shannon-Reeds shuffle model has other descriptions, and a natural extension to shuffles that start with the deck being cut into a packets, where $a \geq 2$; the different packets are then riffled together.

- **Geometric Description:** The geometric model starts by distributing n points independently and uniformly in the unit interval. The points are indexed in increasing order $x_1 < x_2 < \cdots < x_n$. Given a positive integer a, the map $x \to ax \mod 1$ interchanges the points x_i and thus provides a measure on the symmetric group, to be referred to as an a-shuffle.

- **Maximum Entropy Description:** All possible methods for cutting a deck into a packets and then interleaving the packets are equally likely. There are empty packets permissible.

- **Inverse Description:** All possible ways for reversing a shuffled deck back into a packets are equally probable. There may be empty packets. The following produces an inverse a-shuffle with the appropriate probability: A face-down deck of n cards is grasped. Successive cards are flipped face up and distributed uniformly and independently into one of a piles. Once all cards are dealt out, the piles are stacked up from left to right, and the deck is flipped face down.

- **Sequential Description:** Choose integers $j_1, j_2, \ldots, j_a$ based on the multinomial distribution:
$$P(j_1, \ldots, j_a) = \binom{n}{j_1, j_2, \ldots, j_a} \frac{1}{a^n}.$$
So, $0 \leq j_i \leq n$, $\sum_{i=1}^{a} j_i = n$, and the j_i have the same distribution as the number of balls in box i if n balls are randomly dropped into a boxes.

  Given j_i, cut off the top j_1 cards, the next j_2 cards, and so on, to obtain a or fewer packets. Apply the GSR shuffle to the first two packets. Then, shuffle this mixed packet with packet 3, and continue this way. This is the same as riffling all a heaps simultaneously together, where if there are A_i cards left in heap i, the probability that the next card will fall from heap i is given by
$$\frac{A_i}{A_1 + A_2 + \cdots + A_a}.$$

2.1.3 Top-in-at-Random Shuffle

In Top in at random shuffling, the top card is removed and inserted into the deck at a random position. This procedure is repeated a number of times.

Let T_k be the time it takes for the k^{th} card to be inserted below the original bottom card.

The first time we insert a card below the bottom card can be modeled as a geometric distribution with success probability $\frac{1}{n}$. The expected number of shuffles for the first insertion below the bottom card is the expectation of a geometric random variable with parameter $\frac{1}{n}$.

$$E(T_1) = \frac{1}{\left(\frac{1}{n}\right)} = n$$

Similarly, the k^{th} time we insert a card below the bottom card can be modeled as a geometric distribution with success probability $\frac{k}{n}$. The expected number of shuffles for the first insertion below the bottom card is the expectation of a geometric random variable with parameter $\frac{k}{n}$.

$$E(T_k - T_{k-1}) = \frac{1}{\left(\frac{k}{n}\right)} = \frac{n}{k}$$

To find the expected total number of shuffles for the original bottom card to get inserted at random, ensuring that all permutations are equally likely, we take the sum of expected times for all insertions:

$$E(T) = E(T_1) + E(T_2 - T_1) + \cdots + E(T_n - T_{n-1})$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \approx n \log n$$

By time T_n, all $n!$ permutations are equally likely.

Hence, $n \log n$ shuffles are sufficient to randomize a deck of n cards.

	1	2	3	4	5	6	7	8	9
a	*b*	*c*	*c*	*a*	*b*	*a*	*a*	*d*	*c*
b	*c*	*a*	*a*	*b*	*a*	*d*	*d*	*c*	*d*
c	*a*	*b*	*b*	*d*	*d*	*c*	*c*	*a*	*a*
d	*d*	*d*	*d*	*c*	*c*	*b*	*b*	*b*	*b*
				T_1		T_2		T_3	T

Example of repeated top in at random shuffles of a 4-card deck. Source: Aldous, David; Diaconis, Persi. *Shuffling cards and stopping times.* Amer. Math. Monthly 93 (1986), no. 5, 333–348.

2.2 Introduction to Graph Theory and Spectral Graph Theory

In this section, we introduce fundamental concepts from graph theory and linear algebra that are essential for spectral graph theory. These include adjacency and Laplacian matrices, eigenvalues and eigenvectors, matrix operations like Kronecker products, and the lexicographic product of graphs.

2.2.1 Introduction to Graph Theory

Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph G is defined as an ordered pair:

$$G = (V, E),$$

where V is a set of vertices (or nodes) and E is a set of edges, which are unordered pairs of elements of V. A **null graph** is a graph with a nonempty vertex set but no edges.

The adjacency matrix of a graph G with n vertices is an $n \times n$ matrix $A = (a_{ij})$ where:

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between } v_i \text{ and } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

For an undirected graph, the adjacency matrix is symmetric ($A = A^T$). If the graph is weighted, then a_{ij} represents the weight of the edge (v_i, v_j).

The Laplacian matrix of a graph G is defined as:

$$L = D - A,$$

where D is the degree matrix (a diagonal matrix where d_{ii} is the degree of vertex v_i) and A is the adjacency matrix.

Lexicographic Product of Graphs

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The **lexicographic product** of G and H, denoted by $G[H]$, is a graph with vertex set

$$V(G[H]) = V_G \times V_H.$$

The adjacency relation is defined as follows:

$$(u, x) \sim (v, y) \text{ in } G[H] \text{ if and only if } u \sim v \text{ in } G \text{ or } u = v \text{ and } x \sim y \text{ in } H.$$

2.2.2 Linear Algebra and Matrix Operations for Spectral Graph Theory

Diagonalizable Matrices

A square matrix $A \in \mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$) is called **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ (i.e., $A^T = A$) is always diagonalizable. This follows from the **Spectral Theorem**, which states that every real symmetric matrix can be orthogonally diagonalized, meaning there exists an orthogonal matrix P such that:

$$A = PDP^T,$$

where D is a diagonal matrix containing the eigenvalues of A, and the columns of P are the corresponding orthonormal eigenvectors.

Eigenvalues, Eigenvectors, and Eigensubspaces

Let A be an $n \times n$ matrix with real or complex entries.

A scalar $\lambda \in \mathbb{C}$ is called an **eigenvalue** of A if there exists a nonzero vector $v \in \mathbb{C}^n$ such that

$$Av = \lambda v.$$

Any such nonzero vector v is called an **eigenvector** of A corresponding to the eigenvalue λ.

The eigenvalues of A are the roots of the **characteristic equation**

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix of the same size as A.

The number of times λ appears as a root of the characteristic polynomial of A is called the **algebraic multiplicity** (a_{alg}) of λ.

For a given eigenvalue λ, the set of all eigenvectors corresponding to λ, together with the zero vector, forms a subspace of $\mathbb{C}^n$, called the **eigensubspace** (or eigenspace) associated with λ. Mathematically, the eigenspace is given by:

$$E_\lambda = \{v \in \mathbb{C}^n \mid Av = \lambda v\} = \ker(A - \lambda I).$$

This eigensubspace is a vector space whose dimension is called the **geometric multiplicity** (a_{geo}) of λ.

- Always:
$$1 \leq a_{\text{geo}}(\lambda) \leq a_{\text{alg}}(\lambda).$$
- A is **diagonalizable if and only if**
$$a_{\text{geo}}(\lambda) = a_{\text{alg}}(\lambda) \quad \text{for all eigenvalues } \lambda.$$
- If $a_{\text{geo}}(\lambda) < a_{\text{alg}}(\lambda)$ for some λ, then A is **not diagonalizable**, and λ has associated **generalized eigenvectors**.

Kronecker Product of Matrices

Let X be an $a \times b$ matrix and Y be a $c \times d$ matrix. The **Kronecker product** of X and Y, denoted by $X \otimes Y$, is an $(ac) \times (bd)$ block matrix defined by:

$$X \otimes Y = \begin{bmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1b}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2b}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{a1}Y & x_{a2}Y & \cdots & x_{ab}Y \end{bmatrix}.$$

Each block $x_{ij}Y$ is obtained by multiplying the scalar x_{ij} with the entire matrix Y.

If X and Y are square matrices with eigenvalues $\lambda_i(X)$ and $\mu_j(Y)$, respectively, then the eigenvalues of $X \otimes Y$ are given by:

$$\lambda_i(X)\mu_j(Y), \quad \forall i, j.$$

2.3 Introduction to Group Theory and Representation Theory

This section provides an overview of fundamental algebraic structures, including groups, rings, fields, and vector spaces, followed by an introduction to group homomorphisms, representations, and character theory. Special emphasis is given to key concepts in the representation theory of finite groups, such as irreducibility, conjugacy classes, and character orthogonality.

2.3.1 Introduction to Group Theory

Introduction to Abstract Algebra

A set G equipped with a binary operation $*$ is called a *group* if the following properties hold:

- **Closure:** For all $a, b \in G$, the product $a * b$ is also in G.

$$\forall a, b \in G, \quad a * b \in G.$$

- **Associative Law:** For all $a, b, c \in G$, the operation satisfies associativity:

$$(a * b) * c = a * (b * c).$$

- **Identity Element:** There exists an element $e \in G$ such that for all $a \in G$,

$$e * a = a * e = a.$$

- **Inverse Law:** For each $a \in G$, there exists an element $a^{-1} \in G$ such that

$$a * a^{-1} = a^{-1} * a = e.$$

If, in addition, the operation $*$ is *commutative*, i.e.,

$$a * b = b * a, \quad \forall a, b \in G,$$

then G is called an **abelian group**.

Let H be a subset of G. Then, H is called a **subgroup** of G if H is itself a group under the same operation $*$ as G.

For any element $g \in G$, the **left translate** and **right translate** of H by g are defined as:

$$gH = \{gh \mid h \in H\}$$

$$Hg = \{hg \mid h \in H\}$$

A **ring** $(R, +, \cdot)$ is a set R equipped with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that:

1. $(R, +)$ is an abelian group.

2. $(R, \cdot)$ is a semigroup (i.e., multiplication is associative).

3. Multiplication is distributive over addition:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a, \quad \forall a, b, c \in R.$$

A **field** is a set $\mathbb{F}$ equipped with two binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the following properties:

1. $(\mathbb{F}, +, \cdot)$ is a commutative ring with unity.

2. The set $\mathbb{F} \setminus \{0\}$ forms an abelian group under multiplication.

3. Distributive property holds:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad \forall a, b, c \in \mathbb{F}.$$

A **vector space** over a field $\mathbb{F}$ is a set V equipped with two operations:

1. Vector addition: $+ : V \times V \to V$.

2. Scalar multiplication: $\cdot : \mathbb{F} \times V \to V$.

These operations satisfy the following axioms for all $u, v, w \in V$ and all scalars $a, b \in \mathbb{F}$:

- $(V, +)$ is an abelian group.

- Distributivity of scalar multiplication:

$$a(u + v) = au + av, \quad (a + b)u = au + bu.$$

- Associativity of scalar multiplication:

$$a(bu) = (ab)u.$$

- Identity element of scalar multiplication:

$$1u = u, \quad \text{where 1 is the multiplicative identity in } \mathbb{F}.$$

Let $(G, *)$ and $(H, \cdot)$ be two groups. A function $h : G \to H$ is called a **group homomorphism** if for all $u, v \in G$, the following properties hold:

1. **Preservation of operation:**

$$h(u * v) = h(u) \cdot h(v).$$

2. **Identity preservation:**

$$h(e_G) = e_H.$$

3. **Inverse preservation:**
$$h(u^{-1}) = \big(h(u)\big)^{-1}.$$

If h is also bijective, then it is called a **group isomorphism**.

If the isomorphism is from a group G to itself then it's called an **automorphism** of G.

General Linear Group $GL(V)$: Let V be a vector space over a field $\mathbb{F}$. The **general linear group** of V, denoted $GL(V)$, is the group of all bijective linear transformations from V to itself, with function composition as the group operation. That is,

$$GL(V) = \{T : V \to V \mid T \text{ is linear and invertible}\}.$$

Conjugacy Classes in a Group

Let G be a group, and let $g, h \in G$. The **conjugate** of an element g by h is given by:

$$hgh^{-1}.$$

The **conjugacy class** of an element g in G is the set of all elements in G that are conjugate to g, i.e.,

$$\mathrm{Cl}(g) = \{hgh^{-1} \mid h \in G\}.$$

Commutator Subgroup of a Group

Let G be a group. The **commutator** of two elements $g, h \in G$ is defined as:

$$[g, h] = ghg^{-1}h^{-1}.$$

The **commutator subgroup** of G, denoted by $[G, G]$ is the subgroup generated by all such commutators:

$$[G, G] = \langle [g, h] \mid g, h \in G \rangle.$$

Quotient of a Group

Let G be a group and let N be a **normal subgroup** of G (i.e., $N \triangleleft G$, meaning $gNg^{-1} = N$ for all $g \in G$). The **quotient group** of G by N, denoted by G/N, is the set of left cosets of N in G:

$$G/N = \{gN \mid g \in G\}.$$

The group operation on G/N is defined as:

$$(gN) \cdot (hN) = (gh)N, \quad \text{for all } g, h \in G.$$

The identity element in G/N is the coset $N = eN$, and the inverse of gN is $g^{-1}N$.

Abelianization of a Group

Let G be a group. The **abelianization** of G, denoted by G^{ab} is the quotient group:

$$G^{\text{ab}} = G/[G, G].$$

2.3.2 Introduction to Representation Theory

Groups play an important role in various parts of mathematics via their actions on sets with different structures e.g. geometric objects like manifolds and algebraic varieties, other groups, combinatorial structures like graphs, root systems etc. Any algebraic object can be studied through how it interacts with other mathematics objects of similar kind. In Representation theory of groups, we study the groups through the way they transform vectors in vector spaces via invertible linear maps.

Introduction to Representation theory of finite groups

A **representation** of a group Γ on a vector space V (over a field F) is a group homomorphism:

$$\pi : \Gamma \to GL(V)$$

Γ is a **transitive** group of automorphisms of the (finite) graph G if for every pair of vertices $v_i, v_j \in V_G$, there exists an element $\gamma \in \Gamma$ such that $\gamma v_i = v_j$ and Γ is **regular** if $|\Gamma| = |G|$.

Character of a Representation (χ_π) is the trace of π. i.e. $\chi_\pi(h) = tr(\pi(h))$, $\forall\, h \in H$.

W is an H**-invariant**(H-stable) subspace of V if W is a subspace of V such that

$$\pi(h)w \in W \quad \forall \quad w \in W$$

e.g. V and (0) are H-stable.

W is **irreducible** if there does not exist any H-stable $W' \subseteq W$ apart from (0) and W.

A representation $\pi : \Gamma \to GL(V)$ is called **irreducible** if there does not exist a nontrivial proper Γ-invariant subspace $W \subset V$. That is, the only Γ-invariant subspaces of V are

$$\{0\} \quad \text{and} \quad V.$$

If a representation is not irreducible, it is called **reducible**, meaning it admits a nontrivial invariant subspace.

A representation $\pi : \Gamma \to GL(V)$ is said to have an **irreducible decomposition** if the vector space V can be written as a direct sum of irreducible Γ-invariant subspaces. That is,

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

where each W_i is an irreducible subrepresentation, meaning there is no nontrivial Γ-invariant subspace within any W_i.

Column orthogonality of characters

Proposition 2.3.2.1: Let C_i and C_j be conjugacy classes in G. Then

$$\sum_{\pi \in \mathrm{Irr} G} \chi_\pi(C_i)\overline{\chi_\pi(C_j)} = \frac{|G|}{|C_i|}\delta_{ij} \quad \text{for all } i, j.$$

2.4 Introduction to Cayley Graphs

In this section, we discuss the spectral theory of a special family of graphs called Cayley graphs. The Cayley graph of a group G is defined with respect to a fixed generating set in G and the spectral theory is closely related to the representations of the group.

Definition 2.4.1: (Refer to [1]) Let H be a group and let S be a subset of H. The *Cayley graph* of H generated by S, denoted $\mathrm{Cay}(H, S)$, is a directed graph $G = (V, E)$ where the vertex set is given by

$$V = H$$

and the edge set is defined as

$$E = \{(x, xs) \mid x \in H, s \in S\}.$$

If S is closed under inversion, meaning $S = S^{-1}$, then $\mathrm{Cay}(H, S)$ is an undirected graph.

Let H be a finite abelian group. **A character** of H is a homomorphism $\psi : H \to \mathbb{C}^\times$.

Theorem 2.4.2: (Refer to [1]) If Ψ is a character of H, then Ψ is an eigenvector of the adjacency matrix of $G = \mathrm{Cay}(H, S)$, with eigenvalue $\sum_{s \in S} \Psi(s)$.

If H is a finite abelian group, $H \cong \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$ for some integers n_i. In this case, for each $\mathbf{a} = (a_1, \ldots, a_k) \in \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$, we have a character $\Psi_{\mathbf{a}} : H \to \mathbb{C}^\times$ given by

$$\Psi_{\mathbf{a}}(h_1, \ldots, h_k) = \prod_{i=1}^k \omega_{n_i}^{a_i h_i},$$

where $\omega_t = e^{2\pi i/t}$.

If χ is a character of H, then there exists a tuple $\mathbf{a} = (a_1, \ldots, a_k) \in \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$ such that $\chi = \Psi_{\mathbf{a}}$. Furthermore, for distinct $\mathbf{a}, \mathbf{b} \in \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$, $\Psi_{\mathbf{a}}$ and $\Psi_{\mathbf{b}}$ are orthogonal. Since there are $|H|$ such characters, this provides a complete description of the eigenvalues of H.

Chapter 3

Markov Chains

This chapter introduces the fundamental concepts, definitions, and key theorems of Markov chains. Markov chains are the most fundamental mathematical models for random phenomena that develop over time. A Markov chain is a stochastic process in which the future state depends only on the present state, not on the past states. The simple structure of Markov chains makes it possible to say a great deal about their behavior. At the same time, the class of Markov chains is sufficiently diverse to be applicable in several contexts. A random walk is a special case of a Markov chain where the transition probabilities are determined by the structure of a graph or lattice.

3.1 Basic Definitions and Theorems

Definition 3.1.1: (Refer to [5]) A **Markov kernel** on a finite state space $\mathcal{X}$ is a function

$$K : \mathcal{X} \times \mathcal{X} \to [0, 1]$$

such that, for all $x \in \mathcal{X}$, the transition probabilities sum to 1:

$$\sum_{y \in \mathcal{X}} K(x, y) = 1.$$

Definition 3.1.2: (Refer to [4]) Let $(\mathcal{X}, K)$ be a Markov chain with a finite state space $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. The **transition matrix** P of the Markov chain is the $n \times n$ matrix defined by:

$$P = (P_{ij})_{1 \le i,j \le n}, \quad P_{ij} = K(x_i, x_j).$$

This implies that P_{ij} represents the probability of transitioning from state x_i to state x_j in one step: i.e.

$$P_{ij} = \mathbb{P}(X_{t+1} = x_j \mid X_t = x_i).$$

Definition 3.1.3: (Refer to [4]) Let $(X_n)_{n \ge 0}$ be a *Markov chain* with *initial distribution* λ and *transition matrix* P (i.e. $(X_n)_{n \ge 0}$ is Markov(λ, P)) . Then,

(i) $\mathbb{P}(X_0 = i_0) = \lambda_{i_0}$;

(ii) $\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n) = p_{i_n i_{n+1}}$.

Theorem 3.1.4: (Refer to [4]) A discrete-time random process $(X_n)_{0 \le n \le N}$ is Markov(λ, P) if and only if for all $i_0, \dots, i_N \in I$,

$$P(X_0 = i_0, X_1 = i_1, \dots, X_N = i_N) = \lambda_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{N-1} i_N}$$

The theorem is proved by first assuming that $(X_n)_{0 \le n \le N}$ follows the Markov property and showing that the given probability formula holds using the definition of conditional probabilities. Then, the reverse direction is established by induction, demonstrating that if the formula holds, the process satisfies the Markov property.

Theorem 3.1.5 (Markov property): (Refer to [4]) Let $(X_n)_{n \ge 0}$ follow a Markov process with parameters (λ, P). Then, given that $X_m = i$, the sequence $(X_{m+n})_{n \ge 0}$ follows a Markov process with initial distribution δ_i and transition matrix P, and it remains independent of the random variables $X_0, \dots, X_m$.

(We denote *unit mass* at i by $\delta_i = (\delta_{ij} : j \in I)$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases})$$

3.2 Stopping time

Definition 3.2.1: (Refer to [4]) A random variable $T : \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$ is called a *stopping time* if the event $\{T = n\}$ is determined solely by the values of $X_0, X_1, \dots, X_n$ for each $n = 0, 1, 2, \dots$. Intuitively, this means that by observing the process, you can determine exactly when T occurs. If you are required to stop at T, you will know the precise moment to do so.

For example, The *first passage time*

$$T_j = \inf\{n \ge 1 : X_n = j\}$$

is a stopping time since

$$\{T_j = n\} = \{X_1 \ne j, \dots, X_{n-1} \ne j, X_n = j\}.$$

Theorem 3.2.2 (Strong Markov property): (Refer to [4]) Let $(X_n)_{n \ge 0}$ be a Markov process with parameters (λ, P), and let T be a stopping time for $(X_n)_{n \ge 0}$. Then, given that T is finite and $X_T = i$, the process $(X_{T+n})_{n \ge 0}$ follows a Markov process with initial distribution δ_i and transition matrix P, and it remains independent of $X_0, X_1, \dots, X_T$.

The proof follows by conditioning on the stopping time T, applying the standard Markov property at T, summing over all possible values of T, and normalizing to obtain the desired conditional probability result.

3.3 Recurrence and Transience

A fundamental question in the study of Markov chains is whether a given state will be visited infinitely often or only a finite number of times. This leads to the classification of states as recurrent or transient.

Definition 3.3.1: (Refer to [4]) Consider a Markov chain $(X_n)_{n\geq 0}$ with transition matrix P. A state i is called **recurrent** if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n \mid X_0 = i) = 1.$$

A state i is called **transient** if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n \mid X_0 = i) = 0.$$

3.4 Invariant distributions

The long-term behavior of Markov chains is often associated with the concept of an invariant distribution or measure. A measure λ is a row vector $(\lambda_i : i \in I)$ consisting of non-negative components.

Definition 3.4.1: (Refer to [4]) A measure λ is called *invariant* if it satisfies

$$\lambda P = \lambda.$$

The terms *equilibrium* and *stationary* are often used interchangeably to describe this property.

Theorem 3.4.2: Let $(X_n)_{n\geq 0}$ be Markov (λ, P) and suppose that λ is invariant for P. Then $(X_{m+n})_{n\geq 0}$ is also Markov (λ, P).

This can be proved using Theorem 3.1.4 and the fact that, conditional on $X_{m+n} = i$, the random variable X_{m+n+1} is independent of $X_m, X_{m+1}, \ldots, X_{m+n}$ and has distribution $(P_{ij} : j \in I)$.

3.5 Coupling time and Strong Stationary Time

Coupling time and Strong Stationary Time are two probabilistic approaches that yield quantitative estimates related to the convergence to stationarity of finite Markov chains. Both consist in the construction and investigation of certain ”stopping times” and possess theoretical and practical interest.

Coupling Time

Let K be a Markov transition kernel on a finite state space $\mathcal{X}$ with an invariant distribution π.

A **coupling** consists of a sequence of random variable pairs (X_n^1, X_n^2) taking values in $\mathcal{X}$, where each marginal process (X_n^i) for $i = 1, 2$ follows a Markov chain governed by the transition kernel K.

The two chains can have distinct starting distributions, with one often chosen as the stationary distribution π. Although the joint process (X_n^1, X_n^2) may not always be Markovian, in many practical setups, it typically is.

For a given coupling (X_n^1, X_n^2), define

$$T = \inf \left\{ n : X_k^1 = X_k^2 \quad , \forall k \geq n \right\}.$$

This quantity T is referred to as the *coupling time*.

Theorem 3.5.1 (Refer to [5]) Let μ_i^n represent the distribution of X_i^n for $i = 1, 2$. Then, the total variation distance satisfies

$$d_{\mathrm{TV}}(\mu_1^n, \mu_2^n) \leq \mathbb{P}(T > n).$$

Applying this result to random walks on finite groups, we derive the following.

Theorem 3.5.2 (Refer to [5]) Consider a probability measure p on a finite group G. Let (X_n^1, X_n^2) be a coupling of the random walk induced by p, where (X_n^1) begins at the identity and (X_n^2) follows the stationary distribution. Then, the total variation distance satisfies

$$d_{\mathrm{TV}}(p^{(n)}, u) \leq \mathbb{P}(T > n).$$

One of the theoretical advantages of the coupling method is that it allows for the construction of a coupling where the inequalities in the previous theorems become equalities. This makes coupling particularly well-suited for analyzing convergence in total variation.

In practice, **Theorem 3.5.2** simplifies the task of estimating the total variation distance between a random walk and the uniform probability measure on G. It reduces the problem to constructing a coupling for which the probability $\mathbb{P}(T > n)$ can be effectively estimated.

For example, consider the random-to-top shuffling scheme, where a card is chosen at random and placed on top. This shuffle can be viewed as a random walk on the symmetric group S_n driven by a measure on the set of cyclic permutations of varying lengths.

We imagine two decks: one in a given order and another that is perfectly shuffled. The coupling proceeds by picking a card at random from the first deck, moving it to the top, and then performing the same operation on the corresponding card in the second deck by matching face values. This procedure helps track which cards in the first deck are "matched" with their counterparts in the perfectly shuffled deck.

Once a card is matched, it remains matched for the rest of the process. A "match" occurs when a card in the first deck is in the same position as its counterpart in the second deck. Importantly, a card that has been matched stays matched permanently with its counterpart. However, matches involving unchecked cards may fluctuate until they become permanently checked.

The coupling time T is the time at which all cards in the first deck have been matched with their counterparts in the shuffled deck. Using the coupon collector's problem (Refer to 2.1.1) , we can estimate the probability that there exists any unchecked card after k steps as

$$\mathbb{P}(T > k) \leq ne^{-k/n}.$$

Thus, after approximately $n \log n$ steps, all cards are likely to have been matched, providing an upper bound for the mixing time of the random-to-top shuffle.

Strong Stationary Time

Consider a Markov kernel K on a finite state space $\mathcal{X}$ with an invariant distribution π. The separation function is defined as:

$$\operatorname{sep}_K(x, n) = \max_{y \in \mathcal{X}} \left(1 - \frac{K^n(x, y)}{\pi(y)}\right),$$

and its maximum over all x gives:

$$\operatorname{sep}_K(n) = \max_{x \in \mathcal{X}} \operatorname{sep}_K(x, n).$$

This function, denoted as $\operatorname{sep}(n)$, measures the maximal separation between K^n and π.

Since the total variation distance is given by:

$$d_{\mathrm{TV}}(K^n(x, \cdot), \pi) = \sum_{y: K^n(x,y) \leq \pi(y)} (\pi(y) - K^n(x, y)),$$

it follows that:

$$d_{\mathrm{TV}}(K^n(x, \cdot), \pi) \leq \operatorname{sep}_K(x, n).$$

Thus, the separation function provides an upper bound on the total variation distance.

Let (X_k) be a Markov chain with kernel K. A **strong stationary time** is a randomized stopping time T for (X_k) such that

$$\forall k, \quad \forall y \in \mathcal{X}, \quad \mathbb{P}(X_k = y \mid T = k) = \pi(y).$$

Theorem 3.5.3: (Refer to [5]) Let T be a strong stationary time for the Markov chain starting at $x \in \mathcal{X}$. Then for all n,

$$\mathrm{sep}_K(x, n) \leq \mathbb{P}_x(T > n).$$

Moreover, there exists a strong stationary time for which this inequality holds as an equality.

For a random walk on a finite group G with probability measure p, the separation function simplifies to:

$$\mathrm{sep}(k) = \mathrm{sep}_p(k) = \max_{x \in G} \left(1 - |G| p^{(k)}(x)\right).$$

This result is particularly useful when comparing separation and total variation distances in the context of finite group random walks.

Theorem 3.5.4: (Refer to [5]) Let p be a probability measure on a finite group G. Then the total variation distance satisfies:

$$d_{\mathrm{TV}}(p^{(k)}, u) \leq \mathrm{sep}(k).$$

If T is a strong stationary time for the random walk starting at the identity e, then:

$$d_{\mathrm{TV}}(p^{(k)}, u) \leq \mathrm{sep}(k) \leq \mathbb{P}_e(T > k).$$

For example, Consider the strong stationary time for top-to-random shuffle. Let q_1 represent the top-to-random measure on S_n. Consider the first instance, denoted as T_1, when a card is placed beneath the bottom card. This follows a geometric waiting time distribution with an expected value of n. Next, let T_2 be the first time a second card is inserted below the original bottom card. Clearly, the difference $T_2 - T_1$ is also a geometric waiting time with an expected value of $n/2$ and is independent of T_1.

Furthermore, the ordering of these two cards relative to each other is equally probable to be high-low or low-high. Extending this idea, we determine that the first time T when the bottom card is moved to the top and then inserted randomly is a strong stationary time. This stopping time can be expressed as

$$T = T_n = T_1 + (T_2 - T_1) + \cdots + (T_n - T_{n-1}),$$

where each difference $T_i - T_{i-1}$ follows an independent geometric waiting time distribution with an expected value of n/i. Consequently, we can estimate $\mathbb{P}_e(T > k)$, which is at most

$$\mathbb{P}_e(T > k) \leq ne^{-k/n}.$$

Thus, from **Theorem 3.5.4**, we obtain the following bound:

$$d_{\mathrm{TV}}(q_1^{(k)}, u) \leq \mathrm{sep}(k) \leq \mathbb{P}_e(T > k) \leq ne^{-k/n}.$$

This result aligns with the bound obtained earlier through the coupling argument. In this particular case, the coupling method used before and the strong stationary time approach give the same estimate for the mixing time.

Chapter 4

Results of Diaconis and Bayer

This chapter discusses various results on the number of rising sequences, the basic invariant of riffle shuffling.

Theorem 4.1.1: (Refer to [2]) All four descriptions of Gilbert-Shannon-Reeds model (Refer to 2.1.2) for shuffling generate the same permutation distribution. Moreover, in each model an a-shuffle followed by a b-shuffle is equivalent to an ab-shuffle.

The lemma is a consequence of the fact that every description of the shuffle yields a multinomial distribution of the packet sizes and the packet interleavings are equally probable. The product rule is a consequence of the inverse description: Lexicographic combination of an inverse a-shuffle pile assignments and an inverse b-shuffle pile assignments yields uniform and independent pile assignments for an inverse ab-shuffle.

Theorem 4.1.2: (Refer to [2])The probability that an a-shuffle will result in the permutation π is

$$\frac{\binom{a+n-r}{n}}{a^n},$$

where r is the number of rising sequences in π.

This can be proven using the maximum entropy description.

Corollary 4.1.3: (Refer to [2]) If a deck of cards is given a sequence of m shuffles of types $a_1, a_2, \ldots, a_m$, then the chance that the deck is in arrangement π is given by

$$\frac{\binom{n+a-r}{n}}{a^n},$$

where $a = a_1 + a_2 + \cdots + a_m$, and r is the number of rising sequences in π.

Theorem 4.1.1 combined with **Theorem 4.1.2** gives the proof of **Corollary 4.1.3** .

The following theorem is an immediate corollary to this.

Theorem 4.1.4: (Refer to [2]) If cards are shuffled m times, then the chance that the deck is in arrangement π is

$$\frac{\binom{2^m+n-r}{n}}{2^{mn}},$$

where r is the number of rising sequences in π.

Corollary 4.1.5: (Refer to [2]) Consider a Markov chain on the symmetric group that starts at the identity and evolves through successive independent a-shuffles, each sampled according to the Gilbert-Shannon-Reeds measure. Then, the number of rising sequences, $R(\pi)$, forms a Markov chain.

This can be proven by coupling **Theorem 4.1.2** with a completeness condition on the induced family of distributions for the process of rising sequences.

Proposition 4.1.6: (Refer to[2]) Let

$$Q^{(m)}(r) = \frac{\binom{2^m+n-r}{n}}{2^{mn}},$$

be the probability of a permutation with r rising sequences after m shuffles from the GSR distribution. Let $r = \frac{n}{2} + h$, $-\frac{n}{2} + 1 \leq h \leq \frac{n}{2}$. Let $m = \log(n^{3/2}c)$ with $0 < c < \infty$ fixed. Then

$$Q^{(m)}(r) = \frac{1}{n!} \exp\left\{ \frac{1}{c\sqrt{n}} \left(-h + \frac{1}{2} + O_c\left(\frac{h}{n}\right) \right) - \frac{1}{24c^2} - \frac{1}{2}\left(\frac{h}{cn}\right)^2 + O_c\left(\frac{1}{n}\right) \right\}.$$

Proposition 4.1.7: (Refer to [2]) With the notation as in **Proposition 4.1.6**, let h^* be an integer such that $Q^{(m)}\left(\frac{n}{2} + h\right) > \frac{1}{n!}$ for $h < h^*$. Then, for any fixed c, as $n \to \infty$,

$$h^* = \frac{-\sqrt{n}}{24c} + \frac{1}{12c^2} + B + O_c\left(\frac{1}{\sqrt{n}}\right),$$

where $-1 < B < 1$.

We concluded that the number of rising sequences is a sufficient statistic (Refer to 2.1.1) for both $Q^{(m)}$ and U after reading some results of Diaconis and Bayer. The following is a proof of this.

Lemma 4.1.8: The number of rising sequences is a sufficient statistic for both $Q^{(m)}$ and U.

Proof:

Let π be a permutation with r rising sequences after m shuffles from the GSR distribution.

$$P_{Q^m}(\pi \,|\, \text{number of rising sequences} = r) = \frac{P_{Q^m}(\pi \cap \text{number of rising sequences} = r)}{P_{Q^m}(\text{number of rising sequences} = r)}$$

$$= \frac{\binom{2^m+n-r}{n}/2^{mn}}{R_{n,h}\left(\binom{2^m+n-r}{n}\right)/2^{mn}} = \frac{1}{R_{n,h}},$$

where $R_{n,h}$ is the number of permutations with $\frac{n}{2} + h = r$ rising sequences. This is independent of Q^m.

Similarly, we can also prove that the number of rising sequences is a sufficient statistic for U.

Chapter 5

Results of Aldous and Diaconis

This chapter discusses various results on variation distance and strong uniform time associated with shuffling of cards.

5.1 Variation distance

Repeated shuffling can be viewed as a random walk on the permutation group S_n. More generally, consider an arbitrary finite group G. Suppose elements of G are chosen randomly according to a specific selection scheme, where $Q(g)$ represents the probability of selecting g. The set of values $\{Q(g) : g \in G\}$ constitutes a probability distribution, satisfying $Q(g) \geq 0$ and $\sum Q(g) = 1$. When elements are selected independently following this distribution, we obtain a sequence of random elements $g_1, g_2, g_3, \ldots$ in G. Now, define their products.

$$\begin{aligned} X_0 &= \text{identity}, \\ X_1 &= g_1, \\ &. \\ &. \\ X_k &= g_k X_{k-1} = g_k g_{k-1} \cdots g_1. \end{aligned}$$

The sequence of random variables $X_0, X_1, X_2, \ldots$ represents a random walk on G with step distribution Q. Let X_k denote the location of a randomly moving particle at time k. The probability distribution of X_2, given by $P(X_2 = g)$ for $g \in G$, is obtained through the convolution

$$P(X_2 = g) = (Q * Q)(g) = \sum_{h \in G} Q(h)Q(gh^{-1}).$$

For $Q(h)Q(gh^{-1})$, this represents the probability that the element h was chosen initially, followed by the selection of gh^{-1}. Since this holds for any h, their product results in g. Likewise, the probability $P(X_k = g)$ is given by $Q^{k*}(g)$, where Q^{k*} is the repeated convolution:

$$Q^{k*} = (Q * Q * \cdots * Q) = \sum_{h \in G} Q(h) Q^{(k-1)*}(gh^{-1}).$$

A fundamental result is that **repeated convolutions converge to the uniform distribution U:**

$$Q^{k*}(g) \to U(g) = \frac{1}{|G|} \quad \textbf{as} \quad k \to \infty,$$

unless Q is concentrated on a coset of some subgroup.

We worked out an example where we considered a random walk on S_3 with Q concentrated on $\{I, (12), (321)\}$ and $Q(I) = Q((12)) = Q((321)) = \frac{1}{3}$. Here, Q is not concentrated on any coset of a subgroup of S_3, and we found that Q^{n*} converges to the uniform distribution as $n \to \infty$.

To prove this, let $p(n)$ be the statement:

$$Q^{n*}(g) = \begin{cases} \frac{\frac{3^{n-1}-1}{2}+1}{3^n} = \frac{1}{6} + \frac{1}{2(3^n)}, & \text{if } Q(g) = \frac{1}{3}, \\ \frac{\frac{3^{n-1}-1}{2}}{3^n} = \frac{1}{6} - \frac{1}{2(3^n)}, & \text{if } Q(g) = 0. \end{cases}$$

We can observe that $p(1)$ is true. Assume that $p(k)$ is true. Then, for $g \in G$ such that $Q(g) = \frac{1}{3}$,

$$Q^{k*}(g) = \frac{\frac{3^{k-1}-1}{2} + 1}{3^k}.$$

Then,

$$Q^{k+1*}(g) = \frac{2}{3} \cdot \frac{\frac{3^{k-1}-1}{2} + 1}{3^k} + \frac{1}{3} \cdot \frac{\frac{3^{k-1}-1}{2}}{3^k} = \frac{\frac{3^k-1}{2} + 1}{3^{k+1}}.$$

For $g \in G$ such that $Q(g) = 0$,

$$Q^{k*}(g) = \frac{\frac{3^{k-1}-1}{2}}{3^k}.$$

Then,

$$Q^{k+1*}(g) = \frac{1}{3} \cdot \frac{\frac{3^{k-1}-1}{2} + 1}{3^k} + \frac{2}{3} \cdot \frac{\frac{3^{k-1}-1}{2}}{3^k} = \frac{\frac{3^k-1}{2}}{3^{k+1}}.$$

Hence $p(k+1)$ is true.

Thus, by induction, $p(n)$ holds for all $n \in \mathbb{N}$.

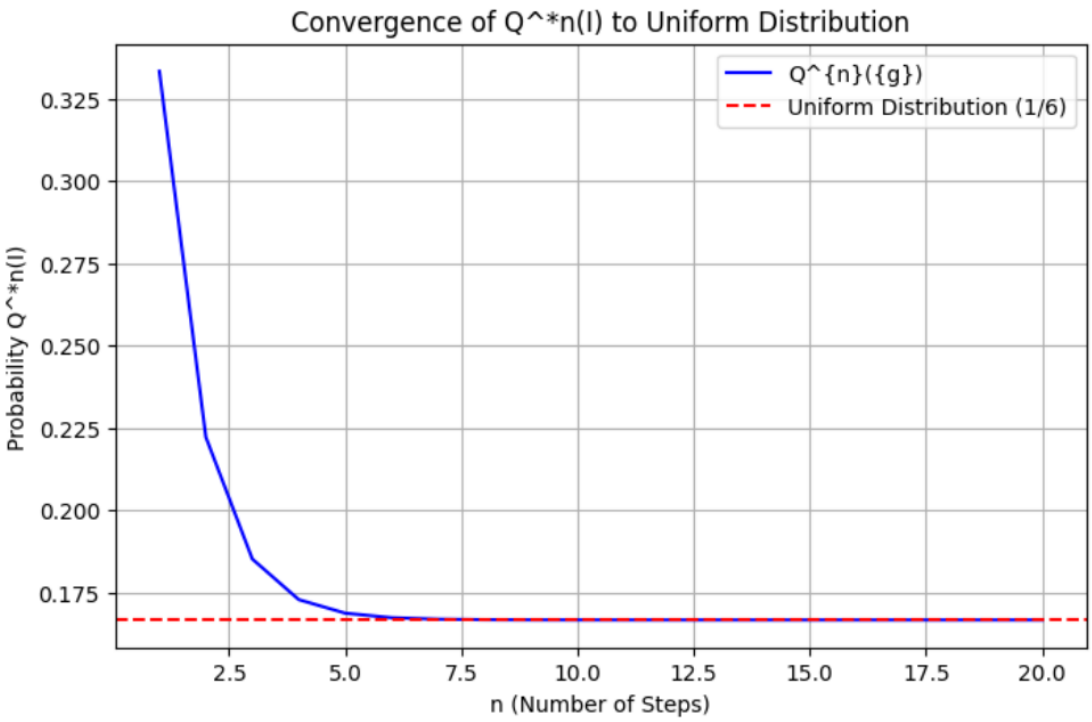
$$\lim_{n \to \infty} \frac{1}{3^n} = 0.$$

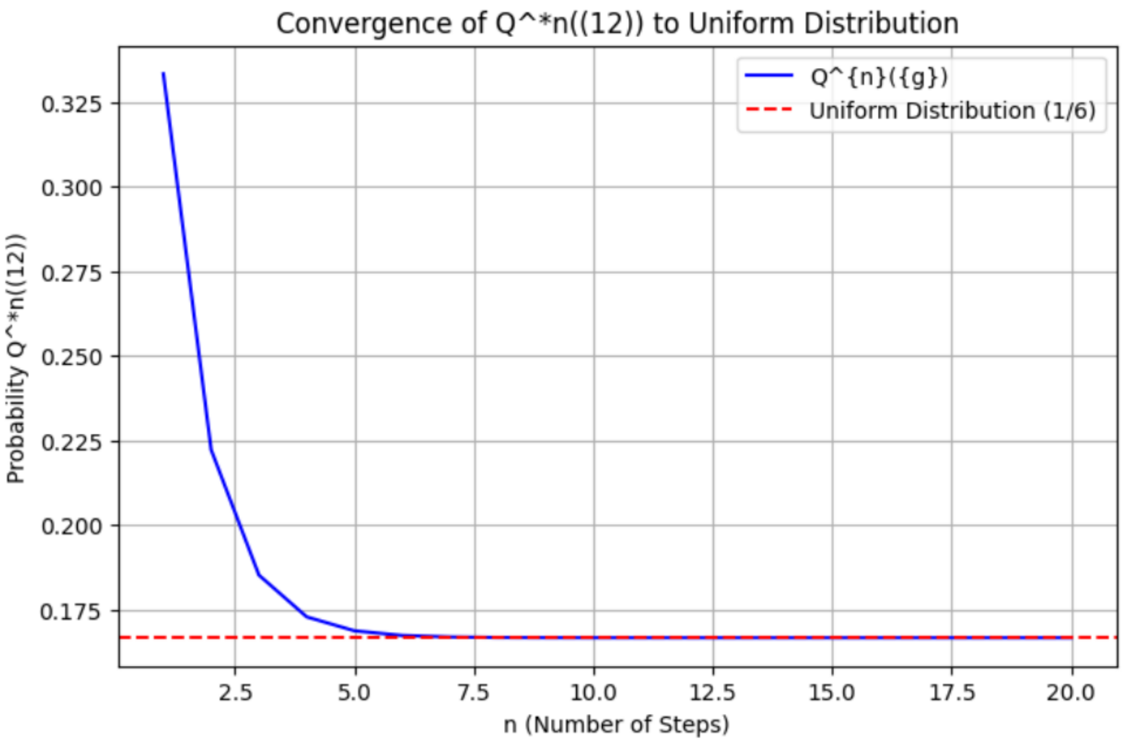
Then,

$$\lim_{n \to \infty} Q^{n*}(g) = \frac{1}{6}.$$

Hence Q^{n*} converges to the uniform distribution as $n \to \infty$.

We also proved this graphically:


Convergence of Q^*n(I) to Uniform Distribution
Q^{n}({g})
Uniform Distribution (1/6)
Probability Q^*n(I)
0.325
0.300
0.275
0.250
0.225
0.200
0.175
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
n (Number of Steps)


Convergence of Q^*n((12)) to Uniform Distribution
Q^{n}({g})
Uniform Distribution (1/6)
Probability Q^*n((12))
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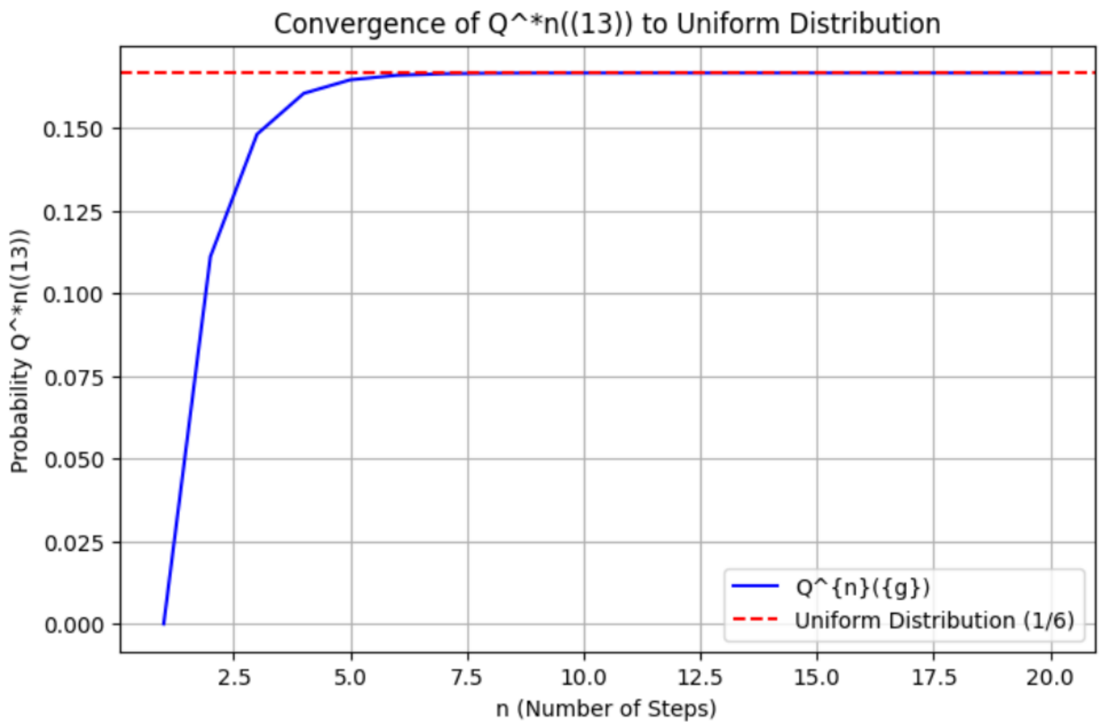
0.300
0.275
0.250
0.225
0.200
0.175
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
n (Number of Steps)


Convergence of Q^*n((13)) to Uniform Distribution
Probability Q^*n((13))
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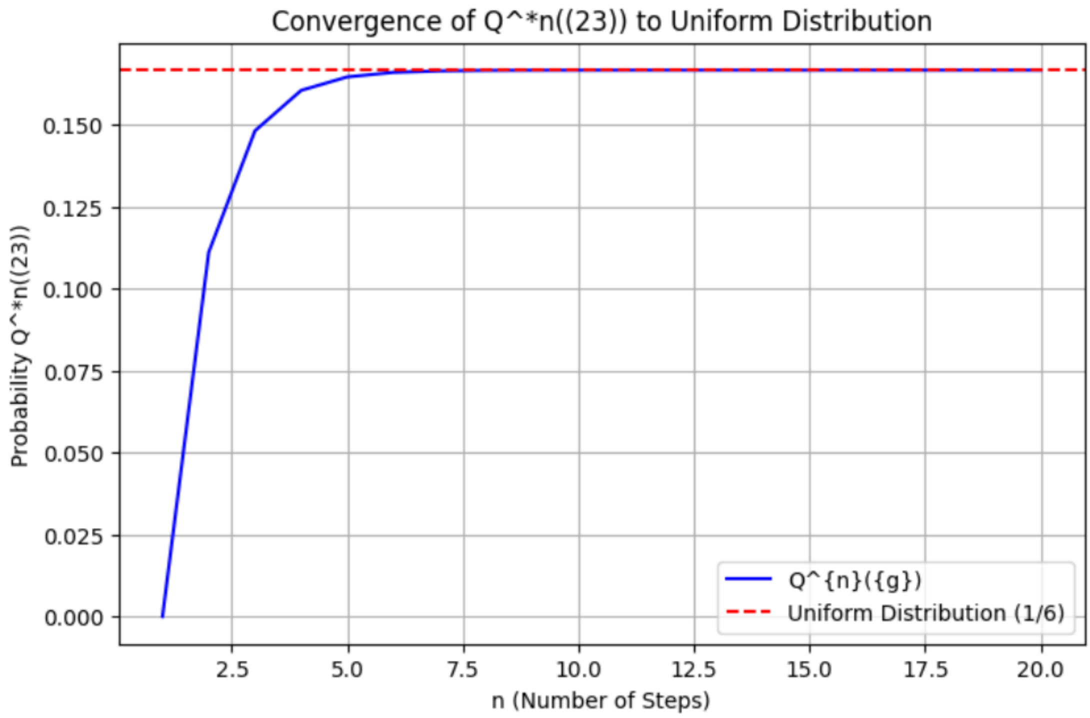
0.025
0.050
0.075
0.100
0.125
0.150
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
n (Number of Steps)
Q^{n}({g})
Uniform Distribution (1/6)


Convergence of Q^*n((23)) to Uniform Distribution
Probability Q^*n((23))
0.000
0.025
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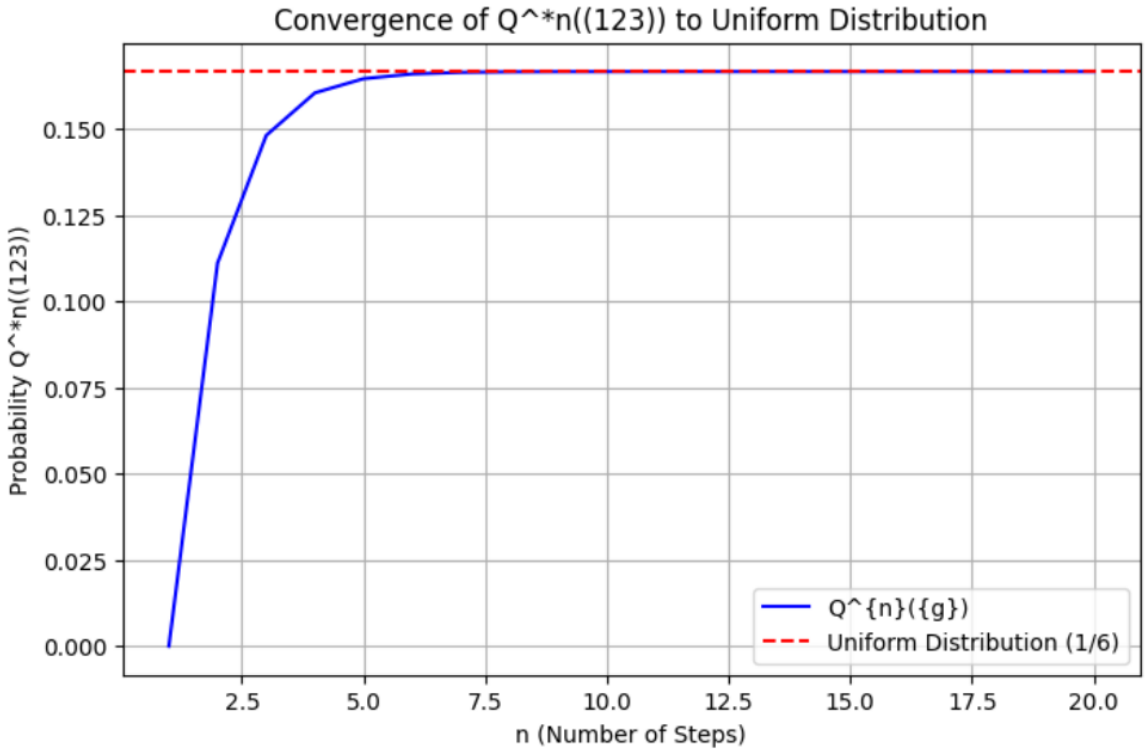
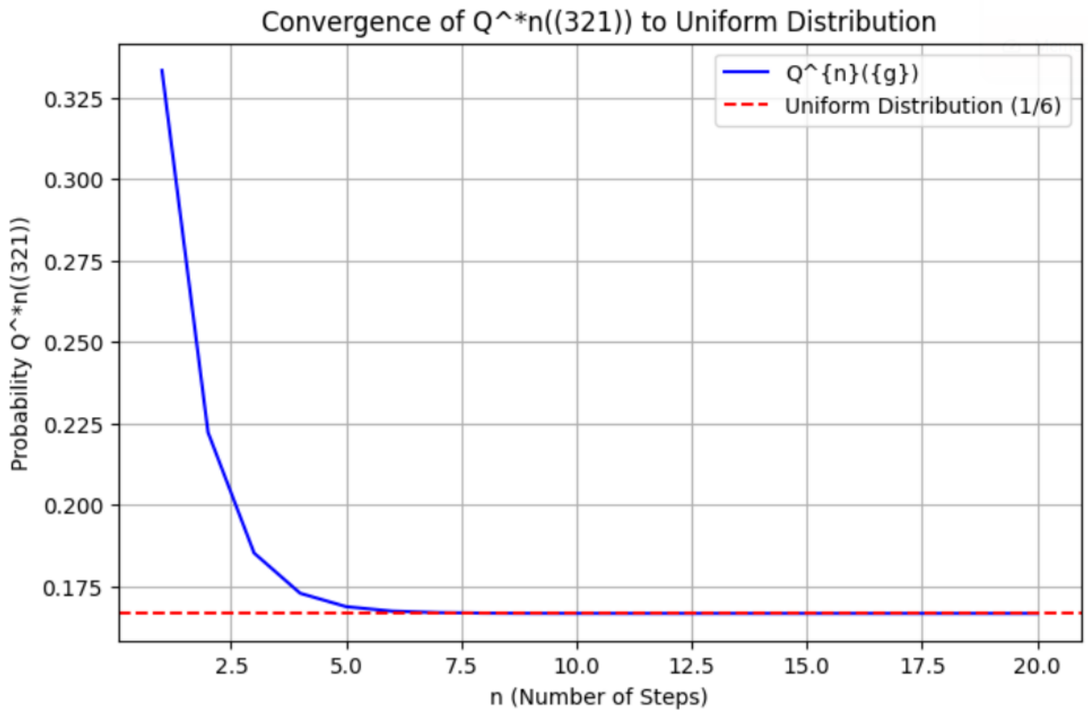
17.5
20.0
n (Number of Steps)
Q^{n}({g})
Uniform Distribution (1/6)





We also found that the threshold at which $Q^{n*}(g)$ is within 10^{-6} of the uniform distribution for all elements is $N^* = 14$. (Refer to Appendix 9 for the code used.)

A natural way to measure the difference between two probability distributions Q_1 and Q_2 on G is through the **total variation distance**.

Definition 5.1.1: (Refer to [3]) The total variation distance between two probability distributions Q_1 and Q_2 on a group G is given by

$$\|Q_1 - Q_2\|_{\text{TV}} = \frac{1}{2} \sum_{g \in G} |Q_1(g) - Q_2(g)|.$$

This is also represented as $d_{\text{TV}}(Q_1, Q_2)$.

Theorem 5.1.2: (Refer to [3]) Let G be a finite group, and let $f : G \to G$ be a bijective function. Suppose Q is a probability distribution on G. Then, the total variation distance satisfies

$$\|Q - U\| = \|Qf^{-1} - U\|,$$

where the probability distribution Qf^{-1} is defined as $Qf^{-1}(g) = Q(f^{-1}(g))$.

This theorem is crucial in random walks on groups and Markov chain mixing, as it shows that the Variation distance between Q and U is invariant under relabeling of states.

5.2 Strong Uniform Time

Definition 5.2.1: (Refer to [3]) Let G be a finite group, and G^∞ the set of all G-valued infinite sequences

$$\mathbf{g} = (g_1, g_2, \dots).$$

A stopping rule $\hat{T}$ is a function $\hat{T} : G^\infty \to \{1, 2, 3, \dots, \infty\}$ such that if $\hat{T}(\mathbf{g}) = j$, then $\hat{T}(\hat{\mathbf{g}}) = j$ for all $\hat{\mathbf{g}}$ with $\hat{g}_i = g_i$ for all $i < j$.

Definition 5.2.2: (Refer to [3]) Let Q be a distribution on G, and let (X_k) be the associated random walk. Given a stopping rule T, the random time $T = \hat{T}(X_1, X_2, \dots)$ is called a stopping time. Call T a strong uniform time (for U) if for each $k < \infty$:

(a) $P(T = k, X_k = g)$ does not depend on g.

Remark. (i) Note that (a) is equivalent to:

(b) $P(X_k = g \mid T = k) = \frac{1}{|G|}, \quad g \in G,$

and to:

(c) $P(X_k = g \mid T \leq k) = \frac{1}{|G|}, \quad g \in G.$

So, the time is strongly uniform if, given that stopping after k steps occurs, the value of the product is uniform on G.

Theorem 5.2.3: (Refer to [3]) Let T be the first time that the binary matrix formed from inverse shuffling has distinct rows. Then T is a strong uniform time.

To complete this analysis, the chance that $T > k$ must be computed. We make use of the familiar question of the birthday problem (Refer to 2.1.1) and its well-known answer to compute this.

Lemma 5.2.4 (Upper bound lemma): (Refer to [3]) Let Q be a probability distribution on a finite group G. Let T be a strong uniform time for Q. Then

$$d(k) \equiv \|Q^{k*} - U\| \leq P(T > k), \quad \forall \quad k \geq 0.$$

The proof follows by conditioning on the stopping time T, splitting the probability into cases where T has occurred and where it has not.

Theorem 5.2.5: (Refer to [3]) For the "top-in-at-random" shuffle:

(a) $d(n \log n + cn) \leq e^{-c}; \quad \forall \quad c \geq 0, \quad n \geq 2.$

(b) $d(n \log n - c_n n) \to 1 \quad \text{as} \quad n \to \infty; \quad \forall \quad c_n \to \infty.$

The first part is proven by comparing the given problem with the famous "coupon-collector's problem" (Refer to 2.1.1) and the second part is proven using Chebyshev's inequality (Refer to 2.1.1). The lower bounds are obtained by guessing some set A for which $|Q^{k*}(A) - U(A)|$ should be large, and using the obvious inequality:

$$d(k) = \|Q^{k*} - U\| \geq |Q^{k*}(A) - U(A)|.$$

This gives a sharp sense to the assertion that $n \log n$ shuffles suffice. This is a particular case of a general cut-off phenomenon, which occurs in all shuffling models we have been able to analyze; there is a critical number k_0 of shuffles such that $d(k_0 + o(k_0)) \approx 0$ but $d(k_0 - o(k_0)) \approx 1$.

Consider simple random walk on the integers mod n. Here, each step is equally likely to move 1 to the right or 1 to the left. This random walk has step distribution Q on $\mathbb{Z}_n$:

$$Q(1) = Q(-1) = \frac{1}{2}.$$

The following theorem shows that the number of steps k required to become uniform is slightly more than n^2.

Theorem 5.2.6: (Refer to [3]) Let $n \geq 3$ be an odd integer. For the simple random walk on the integers mod n with Q as defined above, for $k \geq n^2$,

$$d(k) \leq 6e^{-\frac{ak}{n^2}} \text{ , with } a = \frac{4\pi^2}{3}.$$

We prove this by constructing a strong uniform time for the simple random walk on the integers mod n.

Let G be a finite group and Q a probability distribution on G. The following result shows that Q^{k*} converges to the uniform distribution geometrically fast, provided Q is not concentrated on a subgroup or a translate of a subgroup.

Theorem 5.2.7: (Refer to [3]) Suppose for some k_0 and $0 < c < 1$,

$$Q^{k_0*}(g) \geq cU(g) \quad \text{for all } g \in G.$$

Then,

$$d(k) \leq (1-c)^{\left[\frac{k}{k_0}\right]} \quad \text{for all } k \geq k_0.$$

We prove this by constructing another process that behaves like the original random walk but easily exhibits a strong uniform time.

But the converse of **Theorem 5.2.7** is not true.

We prove this by finding a counterexample. Consider a left translate of a subgroup of S_3. Let $H = \{I, (12)\}$. Then $(H, \cdot)$ is a subgroup of $(S_3, \cdot)$. Consider $X = (13)\cdot\{I, (12)\} = \{(13), (123)\}$. Let $Q((13)) = \frac{1}{2}$, $Q((123)) = \frac{1}{2}$, and $Q(\text{anything else}) = 0$. Here, we can see Q^{n*} converging to the uniform distribution geometrically fast.

To prove this, let $p(n)$ be the statement:

- If n is odd:

$$Q^{n*}(g) = \begin{cases} \frac{\frac{2^{n-1}-1}{3}}{2^n} = \frac{1}{6} - \frac{1}{6(2^{n-1})}, & \text{if } Q(g) = 0 \\ \frac{\frac{2^{n-1}-1}{3}+1}{2^n} = \frac{1}{6} + \frac{1}{3(2^{n-1})}, & \text{if } Q(g) = \frac{1}{2} \end{cases}$$

and

- If n is even:

$$Q^{n*}(g) = \begin{cases} \frac{\frac{2^{n-1}-2}{3}}{2^n} = \frac{1}{6} - \frac{1}{3(2^{n-1})}, & \text{if } Q(g) = \frac{1}{2} \\ \frac{\frac{2^{n-1}-2}{3}+1}{2^n} = \frac{1}{6} + \frac{1}{6(2^{n-1})}, & \text{if } Q(g) = 0 \end{cases}$$

We can observe that $p(5)$ is true. Assume that $p(k)$ is true.

Then, if k is odd,

for $g \in G$ such that $Q(g) = \frac{1}{2}$,

$$Q^{k*}(g) = \frac{\frac{2^{k-1}-1}{3}}{2^k}.$$

Then,

$$Q^{k+1*}(g) = \frac{1}{2}\frac{\frac{2^{k-1}-1}{3}}{2^k} + \frac{1}{2}\frac{\frac{2^{k-1}-1}{3}}{2^k} = \frac{\frac{2^k-2}{3}}{2^{k+1}}.$$

For $g \in G$ such that $Q(g) = 0$,

$$Q^{k*}(g) = \frac{\frac{2^{k-1}-1}{3}}{2^k}.$$

Then,

$$Q^{k+1*}(g) = \frac{1}{2}\frac{\frac{2^{k-1}-1}{3}+1}{2^k} + \frac{1}{2}\frac{\frac{2^{k-1}-1}{3}}{2^k} = \frac{\frac{2^k-2}{3}+1}{2^{k+1}}.$$

If k is even,

for $g \in G$ such that $Q(g) = \frac{1}{2}$,

$$Q^{k*}(g) = \frac{\frac{2^{k-1}-2}{3}}{2^k}.$$

Then,

$$Q^{k+1*}(g) = \frac{1}{2}\frac{\frac{2^{k-1}-2}{3}+1}{2^k} + \frac{1}{2}\frac{\frac{2^{k-1}-2}{3}+1}{2^k} = \frac{\frac{2^k-1}{3}+1}{2^{k+1}}.$$

For $g \in G$ such that $Q(g) = 0$,

$$Q^{k*}(g) = \frac{\frac{2^{k-1}-2}{3}+1}{2^k}.$$

Then,

$$Q^{k+1*}(g) = \frac{1}{2}\frac{\frac{2^{k-1}-2}{3}+1}{2^k} + \frac{1}{2}\frac{\frac{2^{k-1}-1}{3}}{2^k} = \frac{\frac{2^k-1}{3}}{2^{k+1}}.$$

Hence $p(k+1)$ is true.

Thus, by induction, $p(n)$ holds for all $n \in \mathbb{N}$.

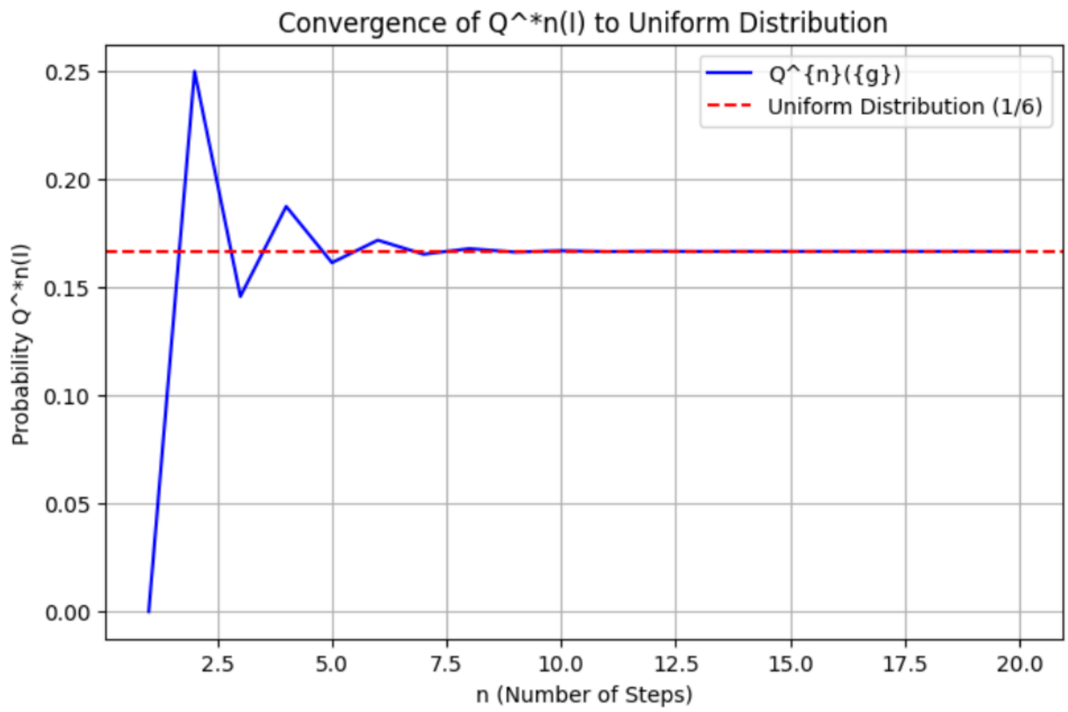
$$\lim_{n\to\infty} \frac{1}{(2^{n-1})} = 0.$$

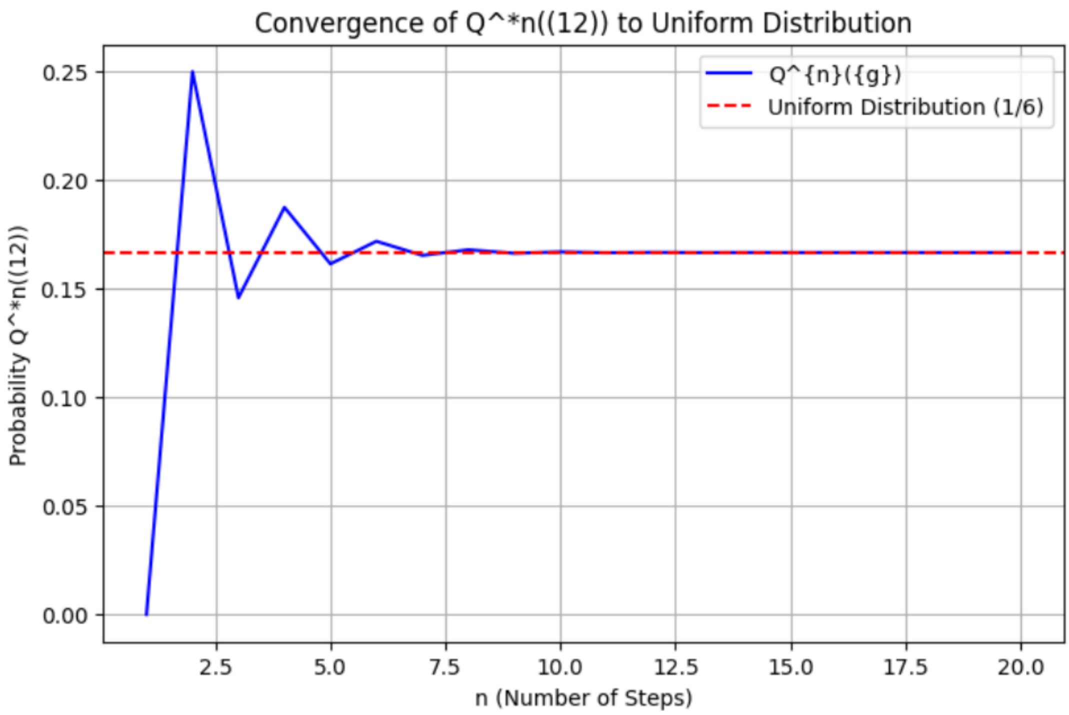
Then,

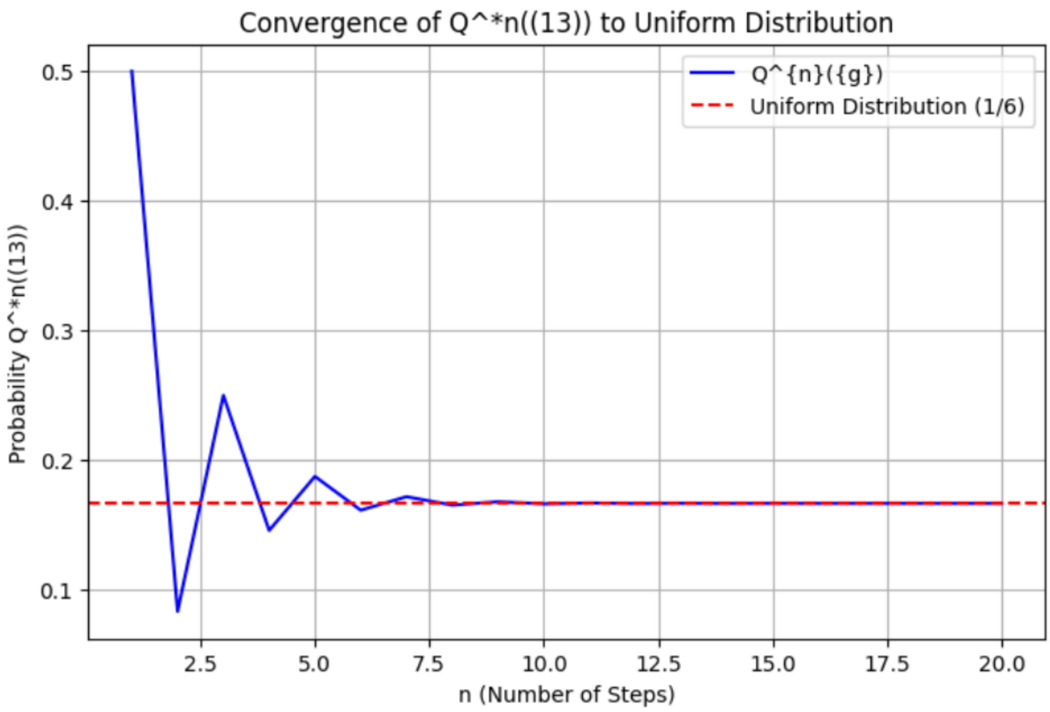
$$\lim_{n\to\infty} Q^{n*}(g) = \frac{1}{6}.$$

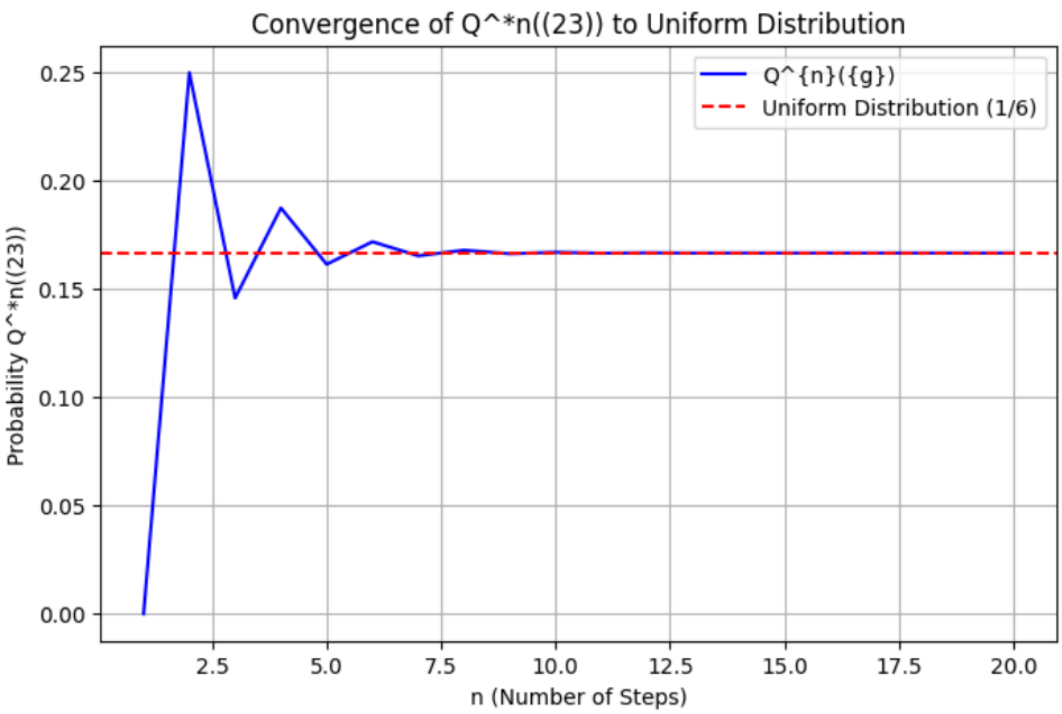
Then, Q^{n*} converges to the uniform distribution geometrically fast.

We also proved this graphically:


Convergence of Q^*n(I) to Uniform Distribution
Q^{n}({g})
Uniform Distribution (1/6)
Probability Q^*n(I)
0.00
0.05
0.10
0.15
0.20
0.25
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
n (Number of Steps)


Convergence of Q^*n((12)) to Uniform Distribution
Q^{n}({g})
Uniform Distribution (1/6)
Probability Q^*n((12))
0.00
0.05
0.10
0.15
0.20
0.25
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
n (Number of Steps)


Convergence of Q^*n((13)) to Uniform Distribution
Q^{n}({g})
Uniform Distribution (1/6)
Probability Q^*n((13))
0.5
0.4
0.3
0.2
0.1
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
n (Number of Steps)


Convergence of Q^*n((23)) to Uniform Distribution
Q^{n}({g})
Uniform Distribution (1/6)
Probability Q^*n((23))
0.25
0.20
0.15
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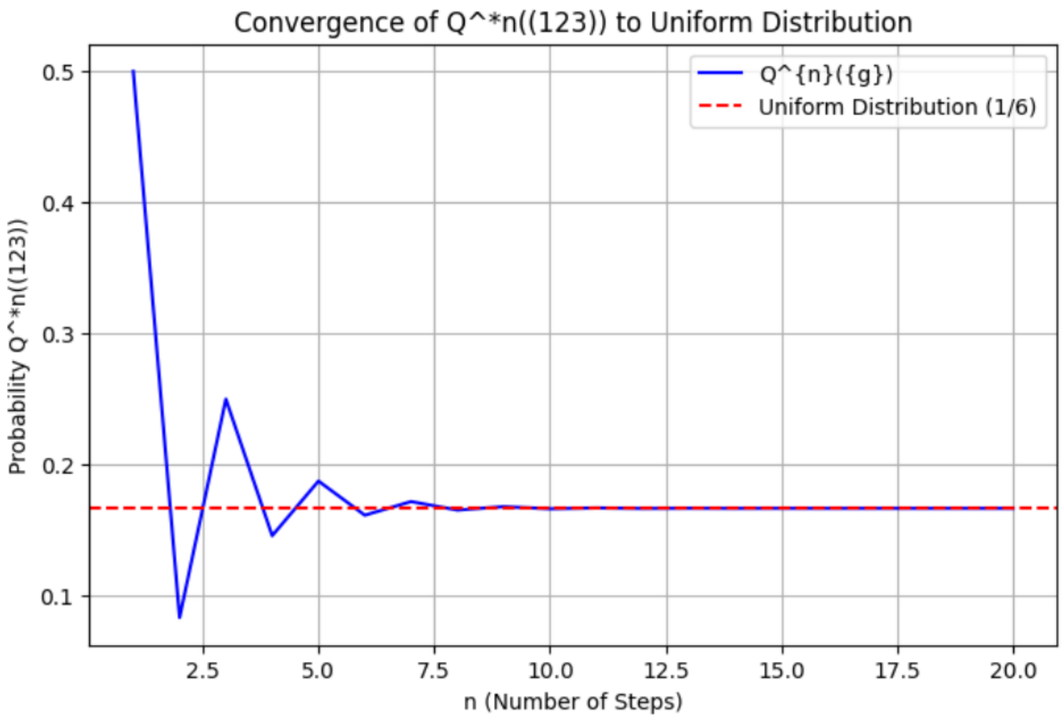
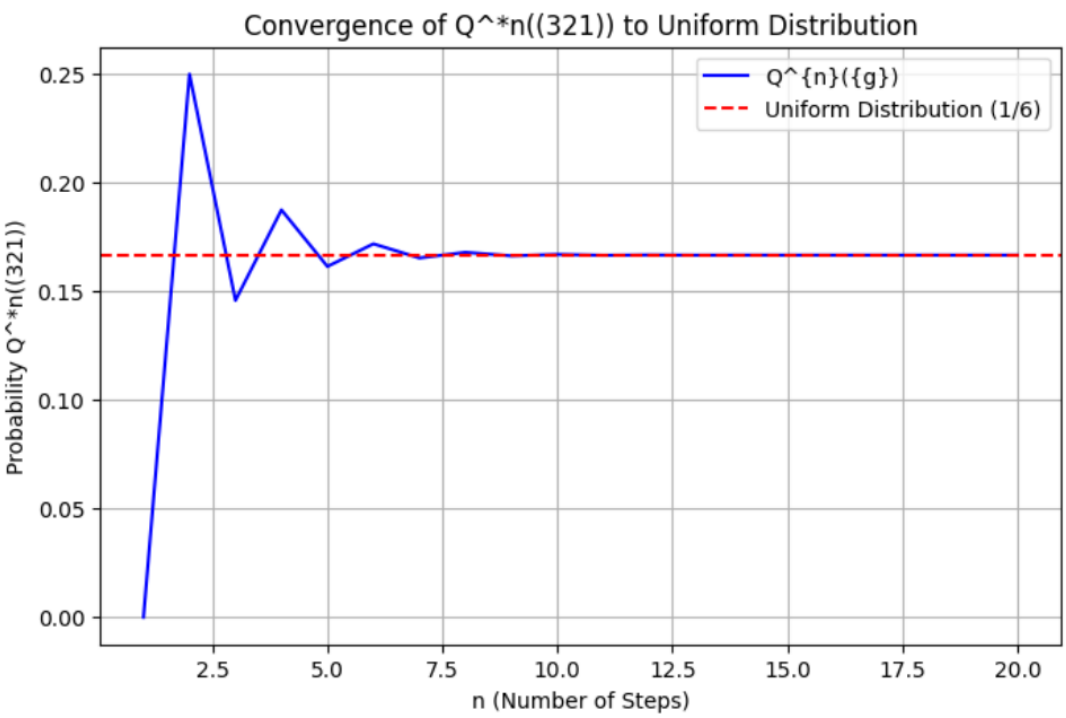
0.00
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
n (Number of Steps)





We also found that the threshold at which $Q^{n*}(g)$ is within 10^{-6} of the uniform distribution for all elements is $N^* = 22$.(Refer to Appendix 9 for the code used.)

Now, we consider a random walk on $\mathbb{Z}_n$ arising in random number generation. Random number generators often work by recursively computing

$$X_k = aX_{k-1} + b \mod n,$$

where a, b and n are chosen carefully.

Theorem 5.2.8: (Refer to [3]) Let Q_k be the probability distribution of X_k, defined by

$$X_k = 2X_{k-1} + b_k \mod n, \quad b_k = 0, +1 \text{ with probability } \frac{1}{3}$$

(deterministic doubling),

with $n = 2^l - 1$. Let $d(k) = \|Q_k - U\|$. Then,

$$d(c \log l) \to 0 \quad \text{as} \quad l \to \infty, \quad \text{for} \quad c > \frac{1}{\log 3}.$$

To prove this, we begin by defining a U^* which is very close to uniformly distributed mod $2^l - 1$ and finding a stopping time T that has a distribution at least as close to uniform as U^*. An appropriate modification of the upper bound lemma will complete the proof.

Chapter 6

Results of Lovász

This chapter explores a formula given by Lovász that expresses the eigenvalues of a graph G (i.e., the eigenvalues of its adjacency matrix A_G) in terms of the characters of a transitive subgroup of the automorphism group Γ of G.

Consider a graph G with the vertex set $V(G) = \{x_1, \dots, x_n\}$ that no loops or multiple edges.

Now, consider a transitive group of automorphisms Γ of G. Let $\chi_1, \chi_2, \dots, \chi_t$ be the irreducible characters of Γ, $n_1, \dots, n_t$ be the dimensions of the corresponding irreducible representations. For each $\gamma \in \Gamma$, let $p_{\gamma k}$ be the number of walks of length k connecting a point to its image.

Assume first Γ is regular, then $|\Gamma| = n$. Let $\gamma \mapsto P_\gamma$ be the regular representation of Γ . Then the fact that γ is an automorphism of G is reflected by

$$AP_\gamma = P_\gamma A.$$

Let v be an eigenvector of A, with eigenvalue λ. Then:

$$A(P_\gamma v) = P_\gamma(Av) = \lambda(P_\gamma v)$$

i.e., $P_\gamma v$ is an eigenvector of A with the same eigenvalue.

Let M_λ be the eigensubspace of A belonging to λ; then, by the above, M_λ is invariant under $\Gamma' = \{P_\gamma : \gamma \in \Gamma\}$.

Let $\chi_1, \dots, \chi_t$ be the irreducible characters of Γ. Refine the decomposition $\bigoplus M_\lambda$ into a decomposition into irreducible Γ-invariant subspaces $N_{ij}, i = 1, \dots, t, j = 1, \dots, n_i$, where the character belonging to the representation N_{ij} on is χ_i. Note that each element of N_{ij} is an eigenvector of A with the same eigenvalue λ_{ij}.

Let $b_{ij_1}, \dots, b_{ij_{n_i}}$ be any orthogonal normalized basis of N_{ij}; then

$$\chi_i(\gamma) = \mathrm{Sp}(P_\gamma | N_{ij}) = \sum_{p=1}^{n_i} b_{ijp} P_\gamma b_{ijp}$$

and so,

$$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\lambda_{ij}^k\chi_i(\gamma)=\sum_{i=1}^{t}\sum_{j=1}^{n_i}\sum_{p=1}^{n_i}b_{ijp}P_\gamma(\lambda_{ij}^k b_{ijp})=\sum_{i=1}^{t}\sum_{j=1}^{n_i}\sum_{p=1}^{n_i}b_{ijp}P_\gamma A^k b_{ijp}=\mathrm{Sp}(P_\gamma A^k)=p_{\gamma k}.$$

Also, since λ_{ij}^k and $p_{\gamma k}$ are real,

$$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\lambda_{ij}^k\overline{\chi_i(\gamma)}=p_{\gamma k}.$$

So,

$$\begin{aligned}\sum_{\gamma\in\Gamma}p_{\gamma k}\chi_i(\gamma)&=\sum_{\gamma\in\Gamma}\sum_{\mu=1}^{t}\sum_{j=1}^{n_\mu}\lambda_{ij}^k\overline{\chi_\mu(\gamma)}\chi_i(\gamma)\\&=\sum_{\mu=1}^{t}\sum_{j=1}^{n_\mu}\lambda_{ij}^k\sum_{\gamma\in\Gamma}\overline{\chi_\mu(\gamma)}\chi_i(\gamma)=n\sum_{j=1}^{n_i}\lambda_{ij}^k.\end{aligned}$$

Set

$$f_{ik}=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}p_{\gamma k}\chi_i(\gamma). \tag{6.1}$$

According to Lovász, the numbers λ_{ij} can be determined from the numbers f_{ik} in several ways, either expressing their elementary symmetric polynomials by the f_{ik} or as the roots of the polynomial

$$\det\begin{bmatrix}1 & f_{i1} & \dots & f_{i,n_i-1} & f_{in_i}\\ 0 & 1 & \dots & f_{i,n_i-2} & f_{i,n_i-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & f_{i1}\\ \frac{1}{n_i!} & \frac{x}{(n_i-1)!} & \dots & \frac{x^{n_i-1}}{1!} & x^{n_i}\end{bmatrix} \tag{6.2}$$

We found that this holds only for $n_i=2$.

For $n_i=3$, the characteristic equation is:

$$\prod_{j=1}^{3}(x-\lambda_{ij})=0$$

or

$$x^3-\sum_{j=1}^{3}\lambda_{ij}x^2+\sum_{p\neq q}\lambda_{ip}\lambda_{iq}x-\prod_{j=1}^{3}\lambda_{ij}=0.$$

If λ_{ij} can be expressed as the roots of the following polynomial:

$$\det\begin{bmatrix}1 & a & b & c\\ 0 & 1 & a & b\\ 0 & 0 & 1 & a\\ \frac{1}{6} & \frac{x}{2} & x^2 & x^3\end{bmatrix},$$

then

$$a = \sum_{j=1}^{3} \lambda_{ij} = f_{i1},$$

$$b = \sum_{j=1}^{3} \lambda_{ij}^2 = f_{i2},$$

and

$$c = 2ab - a^3 + 6\prod_{j=1}^{3} \lambda_{ij} = 2f_{i1}f_{i2} - f_{i1}^3 + 2f_{i3} - 3f_{i1}f_{i2} + f_{i1}^3 = 2f_{i3} - f_{i1}f_{i2}.$$

Thus, λ_{ij} can be expressed as the roots of the following polynomial:

$$\det \begin{bmatrix} 1 & f_{i1} & f_{i2} & 2f_{i3} - f_{i1}f_{i2} \\ 0 & 1 & f_{i1} & f_{i2} \\ 0 & 0 & 1 & f_{i1} \\ \frac{1}{6} & \frac{x}{2} & x^2 & x^3 \end{bmatrix},$$

i.e., the roots of the polynomial

$$x^3 - f_{i1}x^2 + \frac{f_{i1}^2 - f_{i2}}{2}x - \frac{f_{i1}^3 - 3f_{i1}f_{i2} + 2f_{i3}}{6}.$$

Assume now Γ is not regular but transitive. Define a graph G' by

$$V(G') = \Gamma, \quad (\gamma_1, \gamma_2) \in E(G') \Leftrightarrow (x_1\gamma_1, x_1\gamma_2) \in E(G)$$

(this is the lexicographic product (Refer to 2.2.1) of G and the null graph on $s = \frac{n}{|\Gamma|}$ points).

Let A' denote the adjacency matrix of G'. Then, A' is given by the Kronecker product (Refer to 2.2.2) of A and the matrix

$$J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

of size $s = \frac{n}{|\Gamma|}$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G, then the spectrum of A' consists of the products of λ_i with the eigenvalues of J, specifically, $(s-1)n$ zeroes and the values $s\lambda_1, \ldots, s\lambda_n$.

Since Γ acts regularly on G', we can determine the spectrum of G'. From this, we obtain the spectrum of G by throwing out $(s-1)n$ zeroes and dividing the rest by s.

So we get the following *rule to calculate the eigenvalues of* G:

Theorem 6.1.1: (Refer to [6]) Let Γ be a transitive group of automorphisms of the (finite) graph G and let, for each $\gamma \in \Gamma$, p_γ denote the number of points adjacent to their γ-image. Let $\chi_1, \ldots, \chi_t$ be the irreducible characters of Γ of dimensions $n_1, \ldots, n_t$, respectively. Write down each root of (6.2) (where f_{ik} is determined by (6.1)) n_i times for $i = 1, \dots, t$. Remove $|\Gamma| - |G|$ zeros from this sequence. The remaining numbers are the eigenvalues of G.

If Γ is commutative this rule simplifies. In this case, obviously, Γ is regular . Let $x_i \sim x_j$ denote that x_i, x_j are adjacent. We have

$$p_\gamma = \begin{cases} n & \text{if } x_1\gamma \sim x_1, \\ 0 & \text{otherwise;} \end{cases}$$

since if $x_1\gamma \sim x_1$ then

$$x_i\gamma = (x_1\gamma_i)\gamma = (x_1\gamma)\gamma_i \sim x_1\gamma_i = x_i.$$

Here, $n_i = 1$. So we get one eigenvalue associated with each character χ_i, namely

$$\lambda_i = \sum_{x_1\gamma \sim x_1} \chi_i(\gamma). \tag{6.3}$$

Also,

$$v_i = \begin{pmatrix} \chi_i(\gamma_1) \\ \vdots \\ \chi_i(\gamma_n) \end{pmatrix}$$

is an eigenvector (γ_i is defined by $x_1\gamma_i = x_i$).

To prove that v_i is an eigenvector of λ_i we need to show that:

$$Av_i = \lambda_i v_i$$

$$(Av_i)_r = \sum_{x_r \sim x_j} \chi_i(\gamma_j).$$

where

$$A_{ij} = \begin{cases} 1, & \text{if } x_i \sim x_j \\ 0, & \text{otherwise} \end{cases}$$

$$(\lambda_i v_i)_r = \lambda_i \chi_i(\gamma_r) = \left(\sum_{x_1\gamma \sim x_1} \chi_i(\gamma) \right) \chi_i(\gamma_r).$$

Since χ_i is a character, it satisfies the homomorphism property:

$$\chi_i(\gamma)\chi_i(\gamma_r) = \chi_i(\gamma\gamma_r).$$

Thus,

$$(\lambda_i v_i)_r = \sum_{x_1\gamma\sim x_1} \chi_i(\gamma\gamma_r).$$

Due to the transitivity of Γ, we can rewrite the sum as:

$$\sum_{x_1\gamma\gamma_r\sim x_1\gamma_r} \chi_i(\gamma\gamma_r) = \sum_{x_1\gamma\gamma_r\sim x_r} \chi_i(\gamma\gamma_r).$$

This simplifies to:

$$\sum_{x_r\sim x_j} \chi_i(\gamma_j) = (Av_i)_r.$$

Therefore, we have shown that:

$$Av_i = \lambda_i v_i.$$

For example, consider graphs G with $V(G) = \{1, \varepsilon, \ldots, \varepsilon^{n-1}\}$ where $\varepsilon = e^{2\pi i/n}$. Such a graph is called *cyclic* if the rotation (i.e., the multiplication by ε) is an *automorphism* of it. G admits the "rotations" as automorphisms, i.e., the mappings

$$\varphi_v : \varepsilon^k \to \varepsilon^{k+v}.$$

These mappings form a cyclic group of order n; the characters are, as known, given by

$$\chi_t(\varphi_v) = \varepsilon^{tv}.$$

Let 1 be adjacent to $\varepsilon^{a_1}, \ldots, \varepsilon^{a_m}$ in G (clearly, $n - a_i$ is also among $a_1, \ldots, a_m$ for $i = 1, \ldots, m$). Then by (6.3), the eigenvalues of G are

$$\chi_t(\varphi_v) = \varepsilon^{tv}.$$

Suppose the vertex 1 is adjacent to $\varepsilon^{a_1}, \ldots, \varepsilon^{a_m}$ in G. Then, for each $i = 1, \ldots, m$, the value $n - a_i$ is also among $a_1, \ldots, a_m$. By applying equation (6.3), the eigenvalues of G can be determined as follows:

$$\lambda_t = \sum_{i=1}^{m} \varepsilon^{ta_i}, \quad (t = 0, \ldots, n-1).$$

$u_t = (\epsilon^{ta_1}, \epsilon^{ta_2}, \cdots \epsilon^{ta_m})^\top$ is an eigenvector with eigenvalue λ_t.

To calculate the eigenvalues associated with characters of degree-two representations, we need to compute $p_{\gamma 1}$ and $p_{\gamma 2}$.

$$p_{\gamma 2} = \mathrm{Sp}(P_\gamma A^2)$$

$$= \sum_{i=1}^{n} (P_\gamma A^2)_{i,i}$$

$$= \sum_i \sum_{j,k} P_{\gamma(i),j} A_{j,k} A_{k,i}$$

$$= \sum_i \sum_k A_{\gamma^{-1}(i),k} A_{k,i}$$

$$= \sum_i \sum_k A_{i,k} A_{k,\gamma(i)}$$

$$= \sum_i \sum_{k \sim i} A_{k,\gamma(i)}$$

$$= n \sum_{k \sim 1} A_{k,\gamma(1)}$$

(Since if $x_1 \sim \gamma(x_1)$, then

$$\gamma(x_i) = \gamma(\gamma_i(x_1)) = \gamma_i(\gamma(x_1)) \sim \gamma_i(x_1) = x_i.$$

Here, $x_1 = 1$.)

i.e. To calculate $p_{\gamma 2}$ for a particular $\gamma = v$, we should first determine the number of vertices that are neighbors of both v and 1, and then multiply that count by the total number of vertices.

Similarly to calculate the eigenvalues associated with characters of degree-three representations, we need to compute $p_{\gamma 1}$, $p_{\gamma 2}$, and $p_{\gamma 3}$

$$p_{\gamma 3} = \mathrm{Sp}(P_\gamma A^3)$$

$$= \sum_{i=1}^{n} (P_\gamma A^3)_{i,i}$$

$$= \sum_i \sum_{j,k,l} P_{\gamma(i),j} A_{j,k} A_{k,l} A_{l,i}$$

$$= \sum_i \sum_{k,l} A_{\gamma^{-1}(i),k} A_{k,l} A_{l,i}$$

$$= \sum_i \sum_{k,l} A_{i,k} A_{k,l} A_{l,\gamma(i)}$$

$$= \sum_i \sum_l \sum_{k \sim l} A_{i,k} A_{l,\gamma(i)}$$

$$= n \sum_l \sum_{k \sim l} A_{1,k} A_{l,\gamma(1)}$$

i.e. To calculate $p_{\gamma 3}$ for a particular $\gamma = v$, we should first determine the number of vertices that are neighbors of both v and each neighbor of 1, and then multiply that count by the total number of vertices.

To find an eigenvector of an eigenvalue associated with a character of a degree-two or degree-three representation, we make use of matrix coefficients.

Definition 6.1.2: (Matrix Coefficient) The function $\pi_{e_i,e_j} : G \to \mathbb{C}$ is defined as:

$$\pi_{e_i,e_j}(x) = \text{(i,j)-entry of } \pi(x).$$

This means that the values of $\pi_{e_i,e_j}(x)$ are the matrix entries of the representation $\pi(x)$ at position (i,j). These functions span a space that is orthogonal to subspaces corresponding to other irreducible representations.

The key reason is **Schur's orthogonality relations**, which state that for two irreducible representations π and σ of G:

$$\sum_{x \in G} \pi_{e_i,e_j}(x) \overline{\sigma_{f_k,f_l}(x)} = 0, \quad \text{if } \pi \ncong \sigma.$$

This implies that matrix coefficients from different irreducible representations are perpendicular in $L^2(G)$.

For any vectors $v, w \in V_\pi$, the matrix coefficient function is given by:

$$\pi_{v,w}(x) = \langle \pi(x) w, v \rangle.$$

This describes how the right regular representation of G acts on functions:

$$\rho_R(g) f(x) = f(xg), \quad \forall g, x \in G, \quad f : G \to \mathbb{C}.$$

Applying this to matrix coefficients, we get:

$$\rho_R(g) \pi_{v,w}(x) = \pi_{v,w}(xg) = \langle \pi(x)\pi(g) w, v \rangle = \pi_{v,\pi(g)w}(x).$$

This means that the space spanned by the functions $\pi_{v,w}$ is invariant under the right regular representation.

Let $\{e_1, e_2, \ldots, e_d\}$ be an orthonormal basis for V_π, where d is the dimension of π. Then, we define subspaces:

$$V_i = \{\pi_{e_i,w} : w \in V_\pi\}.$$

Each V_i is a subrepresentation isomorphic to π. The span of all matrix coefficients:

$$\text{Span}\{\pi_{v,w} : v, w \in V_\pi\}$$

is a direct sum of d copies of π, forming a d^2-dimensional space:

$$W = \pi \oplus \pi \oplus \cdots \oplus \pi \quad (d \text{ times}).$$

Thus, any eigenvector corresponding to an eigenvalue associated with a character of a representation of degrees two or three will be a linear combination of its matrix coefficients. Since the adjacency matrix is symmetric, the Spectral Theorem guarantees that we can find an orthonormal basis of eigenvectors.

Chapter 7

Cayley graphs of Dihedral groups and S_4

This chapter focuses on calculating the eigenvalues and eigenvectors of the Cayley graph of $\mathcal{D}_{2n}$ generated by $\{r, r^{-1}, s\}$ and the Cayley graph of S_4 generated by the top-in-at-random shuffle, excluding the identity permutation, using results from Lovász.

7.1 Cayley graphs of Dihedral groups

The dihedral group of order $2n$, denoted by $\mathcal{D}_{2n}$, represents the symmetry group of a regular polygon with n sides. It can be defined as a group generated by r and s with the following relations:

$$r^n = s^2 = srsr = 1.$$

A concrete description of $\mathcal{D}_{2n}$ is:

$$\mathcal{D}_{2n} = \langle r, s \mid r^n, s^2, srsr \rangle = \{r^j, sr^j \mid 0 \leq j < n\}.$$

The structure of conjugacy classes in $\mathcal{D}_{2n}$ depends on whether n is even or odd.
If n is even, i.e. $n = 2m$, there are $m + 3 = \frac{n}{2} + 3$ conjugacy classes:

- $\{1\}$
- $\{r^m\}$,
- $\{r^j, r^{2m-j} \mid 1 \leq j \leq m-1\}$,
- $\{sr^j \mid j \text{ even},\ 0 \leq j \leq 2m-2\}$,
- $\{sr^j \mid j \text{ odd},\ 1 \leq j \leq 2m-1\}$

If n is odd, i.e. $n = 2m+1$, there are $m + 2 = \frac{n+3}{2}$ conjugacy classes:

- $\{1\}$
- $\{r^j, r^{2m+1-j} \mid 1 \leq j \leq m\}$
- $\{sr^j \mid 0 \leq j \leq 2m\}$

For $n \geq 2$, the commutator subgroup (Refer to 2.3.1) $[\mathcal{D}_{2n}, \mathcal{D}_{2n}]$ is the cyclic subgroup generated by r^2.

The quotient (Refer to 2.3.1) $\mathcal{D}_{2n}/[\mathcal{D}_{2n}, \mathcal{D}_{2n}]$, also known as the abelianization (Refer to 2.3.1) of $\mathcal{D}_{2n}$, is described as follows:

- For even $n \geq 2$, $\mathcal{D}_{2n}/[\mathcal{D}_{2n}, \mathcal{D}_{2n}]$ is the Klein-4 group ($\mathbb{Z}_2 \times \mathbb{Z}_2$) generated by cosets $\overline{r}$ and $\overline{s}$.
- For odd $n \geq 3$, $\mathcal{D}_{2n}/[\mathcal{D}_{2n}, \mathcal{D}_{2n}]$ is the order two group $\{\overline{1}, \overline{s}\}$.

Since $\mathcal{D}_{2n}$ has a cyclic (hence abelian) subgroup $\langle r \rangle$ of order n, every irreducible representation of $\mathcal{D}_{2n}$ is either of degree one or degree two.

Let a_n denote the number of distinct degree one representations of $\mathcal{D}_{2n}$, and b_n the number of non-isomorphic degree two representations. Then:

(I) If n is even:

$$a_n = 4, \quad b_n = \frac{n}{2} - 1.$$

(II) If n is odd:

$$a_n = 2, \quad b_n = \frac{n-1}{2}.$$

7.1.1 Spectrum of Cayley graph of D_8 generated by {r, r^{-1}, s}

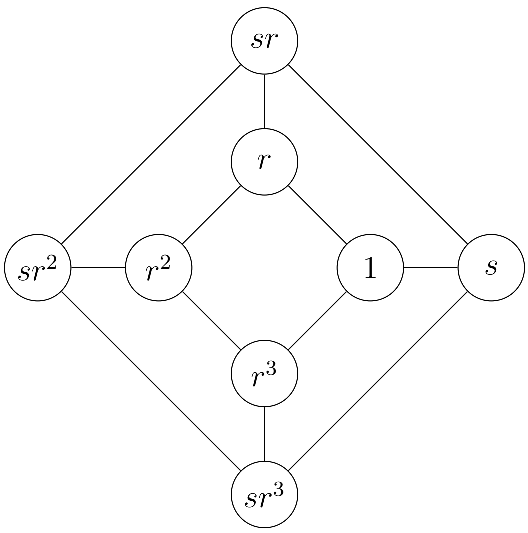


Figure 7.1: Cayley graph of $\mathcal{D}_8$ generated by $\{r, r^{-1}, s\}$.

Here, the adjacency relations are $x \sim rx$, $x \sim r^{-1}x$, and $x \sim sx$ for every $x \in \mathcal{D}_8$.

Here $n = 4$ (even) and hence $\mathcal{D}_8/[\mathcal{D}_8, \mathcal{D}_8]$ is the Klein's-4 group generated by cosets r and s. Hence $\mathcal{D}_8$ has four degree-one irreducible representations which assign values ± 1

to r and s. $\mathcal{D}_8$ has 5 conjugacy classes ($\frac{n}{2}+3$ conjugacy classes). So, it has a degree-two irreducible representation. The characters of the degree-two irreducible representations, π, can be calculated by considering

$$\pi(r^k) = \begin{pmatrix} \cos\frac{k\pi}{2} & -\sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & \cos\frac{k\pi}{2} \end{pmatrix}$$

for $k \in \{0,1,2,3\}$ and

$$\pi_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponding to the counterclockwise rotation by angle $\pi/4$ around $(0,0)$ and reflection along the x-axis, respectively. This completes the character table of $\mathcal{D}_8$ as given below.

	$[1]$	$[r^2]$	$[r]$	$[s]$	$[sr]$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
π	2	-2	0	0	0

γ	1	r	r^2	r^3	s	sr	sr^2	sr^3
$p_{\gamma 1}$	0	8	0	8	8	0	0	0
$p_{\gamma 2}$	24	0	16	0	0	16	0	16

The eigenvalues associated with characters of degree one representations are:

(i)

$$\chi_1(r) + \chi_1(r^{-1}) + \chi_1(s) = 3$$

This is a trivial eigenvalue, since the degree of every vertex is equal to 3.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$$

is an eigenvector with this eigenvalue.

(ii)

$$\chi_2(r) + \chi_2(r^{-1}) + \chi_2(s) = 1$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}^\top$$

is an eigenvector with this eigenvalue.

(iii)

$$\chi_3(r) + \chi_3(r^{-1}) + \chi_3(s) = -1$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^\top$$

is an eigenvector with this eigenvalue.

(iv)

$$\chi_4(r) + \chi_4(r^{-1}) + \chi_4(s) = -3$$

$$\begin{bmatrix}1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\end{bmatrix}^\top$$

is an eigenvector with this eigenvalue.

(v)

Let λ_{51} and λ_{52} be the eigenvalues associated with the character of the degree two representation π. Then,

$$8(\lambda_{51} + \lambda_{52}) = 8(\pi(r) + \pi(r^{-1}) + \pi(s)) = 0$$
$$\implies f_{51} = \lambda_{51} + \lambda_{52} = 0$$

and

$$8(\lambda_{51}^2 + \lambda_{52}^2) = 24\pi(1) + 16\pi(r^2) = 16$$
$$\implies f_{52} = \lambda_{51}^2 + \lambda_{52}^2 = 2$$

Then we get

$$\lambda_{51} = 1 \quad \text{and} \quad \lambda_{52} = -1.$$

The matrix coefficients of π are :

$$\begin{bmatrix}\pi_{e_1,e_1}(1)\\ \pi_{e_1,e_1}(r)\\ \pi_{e_1,e_1}(r^2)\\ \pi_{e_1,e_1}(r^3)\\ \pi_{e_1,e_1}(s)\\ \pi_{e_1,e_1}(sr)\\ \pi_{e_1,e_1}(sr^2)\\ \pi_{e_1,e_1}(sr^3)\end{bmatrix} = \begin{bmatrix}1\\0\\-1\\0\\1\\0\\-1\\0\end{bmatrix}, \quad \begin{bmatrix}\pi_{e_1,e_2}(1)\\ \pi_{e_1,e_2}(r)\\ \pi_{e_1,e_2}(r^2)\\ \pi_{e_1,e_2}(r^3)\\ \pi_{e_1,e_2}(s)\\ \pi_{e_1,e_2}(sr)\\ \pi_{e_1,e_2}(sr^2)\\ \pi_{e_1,e_2}(sr^3)\end{bmatrix} = \begin{bmatrix}0\\-1\\0\\1\\0\\-1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}\pi_{e_2,e_1}(1)\\ \pi_{e_2,e_1}(r)\\ \pi_{e_2,e_1}(r^2)\\ \pi_{e_2,e_1}(r^3)\\ \pi_{e_2,e_1}(s)\\ \pi_{e_2,e_1}(sr)\\ \pi_{e_2,e_1}(sr^2)\\ \pi_{e_2,e_1}(sr^3)\end{bmatrix} = \begin{bmatrix}0\\1\\0\\-1\\0\\-1\\0\\1\end{bmatrix}, \text{ and } \begin{bmatrix}\pi_{e_2,e_2}(1)\\ \pi_{e_2,e_2}(r)\\ \pi_{e_2,e_2}(r^2)\\ \pi_{e_2,e_2}(r^3)\\ \pi_{e_2,e_2}(s)\\ \pi_{e_2,e_2}(sr)\\ \pi_{e_2,e_2}(sr^2)\\ \pi_{e_2,e_2}(sr^3)\end{bmatrix} = \begin{bmatrix}1\\0\\-1\\0\\-1\\0\\1\\0\end{bmatrix}.$$

Here we can observe that

$$\begin{bmatrix}1 & 0 & -1 & 0 & 1 & 0 & -1 & 0\end{bmatrix}^\top \text{ and } \begin{bmatrix}0 & -1 & 0 & 1 & 0 & -1 & 0 & 1\end{bmatrix}^\top$$

are eigenvectors of eigenvalue 1 which are orthogonal to

$$\begin{bmatrix}1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\end{bmatrix}^\top$$

.

Similarly,

$$\begin{bmatrix}0 & 1 & 0 & -1 & 0 & -1 & 0 & 1\end{bmatrix}^\top \text{ and } \begin{bmatrix}1 & 0 & -1 & 0 & -1 & 0 & 1 & 0\end{bmatrix}^\top$$

are eigenvectors of eigenvalue -1 which are orthogonal to

$$\begin{bmatrix}1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\end{bmatrix}^\top$$

The table below lists the eigenvalues of the Cayley graph of $\mathcal{D}_8$ generated by $\{r, r^{-1}, s\}$ and corresponding orthogonal eigenvectors.

Eigenvalue	**Corresponding Orthogonal Eigenvectors**
3	$\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}^\top$
1	$\begin{bmatrix}1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\end{bmatrix}^\top$, $\begin{bmatrix}1 & 0 & -1 & 0 & 1 & 0 & -1 & 0\end{bmatrix}^\top$, $\begin{bmatrix}0 & -1 & 0 & 1 & 0 & -1 & 0 & 1\end{bmatrix}^\top$
-1	$\begin{bmatrix}1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\end{bmatrix}^\top$, $\begin{bmatrix}0 & 1 & 0 & -1 & 0 & -1 & 0 & 1\end{bmatrix}^\top$, $\begin{bmatrix}1 & 0 & -1 & 0 & -1 & 0 & 1 & 0\end{bmatrix}^\top$
-3	$\begin{bmatrix}1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\end{bmatrix}^\top$

7.1.2 Spectrum of Cayley graph of D_{10} generated by {r, r^{-1}, s}

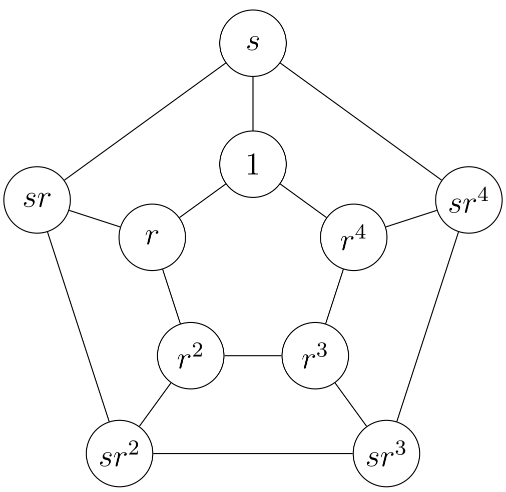


Figure 7.2: Cayley graph of $\mathcal{D}_{10}$ generated by $\{r, r^{-1}, s\}$.

Here, the adjacency relations are $x \sim rx$, $x \sim r^{-1}x$, and $x \sim sx$ for every $x \in \mathcal{D}_{10}$.

Here $n = 5$ (odd) and hence $\mathcal{D}_{10}/[\mathcal{D}_{10}, \mathcal{D}_{10}]$ is the is the order two group $\{\overline{1}, \overline{s}\}$. Hence $\mathcal{D}_{10}$ has two degree-one irreducible representations which assign values ± 1 to s and 1 to r . $\mathcal{D}_{10}$ has 4 conjugacy classes ($\frac{n+3}{2}$ conjugacy classes). So, it has two degree-two irreducible representations. The characters of one of the two degree-two irreducible representations, π_1, can be calculated by considering

$$\pi_1(r^k) = \begin{pmatrix} \cos\frac{2k\pi}{5} & -\sin\frac{2k\pi}{5} \\ \sin\frac{2k\pi}{5} & \cos\frac{2k\pi}{5} \end{pmatrix}$$

for $k \in \{0, 1, 2, 3, 4\}$ and

$$\pi_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponding to the counterclockwise rotation by angle $\pi/5$ around $(0, 0)$ and reflection along the x-axis, respectively.

The character of the other degree-two representation π_2 can be calculated by considering

$$\pi_2(r^k) = \begin{pmatrix} \cos\frac{4k\pi}{5} & -\sin\frac{4k\pi}{5} \\ \sin\frac{4k\pi}{5} & \cos\frac{4k\pi}{5} \end{pmatrix}$$

for $k \in \{0, 1, 2, 3, 4\}$ and

$$\pi_2(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponding to the counterclockwise rotation by angle $2\pi/5$ around $(0, 0)$ and reflection along the x-axis, respectively. This completes the character table of $\mathcal{D}_{10}$ as given below.

The character table of $\mathcal{D}_{10}$ is:

	$[1]$	$[r]$	$[r^2]$	$[s]$
χ_1	1	1	1	1
χ	1	1	1	-1
π_1	2	$\xi + \overline{\xi}$	$\xi^2 + \overline{\xi}^2$	0
π_2	2	$\xi^2 + \overline{\xi}^2$	$\xi + \overline{\xi}$	0

Here, $\xi = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

Let $\theta = \frac{2\pi}{5}$, then $\xi = \cos\theta + i\sin\theta$.

γ	1	r	r^2	r^3	r^4	s	sr	sr^2	sr^3	sr^4
$p_{\gamma 1}$	0	10	0	0	10	10	0	0	0	0
$p_{\gamma 2}$	30	0	10	10	0	0	20	0	0	20

The eigenvalues associated with characters of degree one representations are:

(i)

$$\chi_1(r) + \chi_1(r^{-1}) + \chi_1(s) = 3$$

This is a trivial eigenvalue, since the degree of every vertex is equal to 3.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$$

is an eigenvector with this eigenvalue.

(ii)

$$\chi(r) + \chi(r^{-1}) + \chi(s) = 1$$

$$\begin{bmatrix}1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1\end{bmatrix}^\top$$

is an eigenvector with this eigenvalue.

(iii)

Let λ_{31} and λ_{32} be the eigenvalues associated with the character of the degree two representation π_1. Then,

$$\begin{aligned}10(\lambda_{31} + \lambda_{32}) = 10(\pi_1(r) + \pi_1(r^{-1}) + \pi_1(s)) &= 20(\xi + \overline{\xi})\\ \implies \lambda_{31} + \lambda_{32} &= 2(\xi + \overline{\xi})\end{aligned}$$

and

$$\begin{aligned}10(\lambda_{31}^2 + \lambda_{32}^2) = 30\pi_1(1) + 10\pi_1(r^2) + 10\pi_1(r^3) &= 60 + 20(\xi^2 + \overline{\xi}^2)\\ \implies \lambda_{31}^2 + \lambda_{32}^2 &= 6 + 2(\xi^2 + \overline{\xi}^2)\end{aligned}$$

Then we get

$$\lambda_{31} = \xi + \overline{\xi} + 1 \quad \text{and} \quad \lambda_{32} = \xi + \overline{\xi} - 1.$$

The matrix coefficients of π_1 are :

$$\begin{bmatrix}\pi_{1_{e_1,e_1}}(1)\\ \pi_{1_{e_1,e_1}}(r)\\ \pi_{1_{e_1,e_1}}(r^2)\\ \pi_{1_{e_1,e_1}}(r^3)\\ \pi_{1_{e_1,e_1}}(r^4)\\ \pi_{1_{e_1,e_1}}(s)\\ \pi_{1_{e_1,e_1}}(sr)\\ \pi_{1_{e_1,e_1}}(sr^2)\\ \pi_{1_{e_1,e_1}}(sr^3)\\ \pi_{1_{e_1,e_1}}(sr^4)\end{bmatrix} = \begin{bmatrix}1\\ \cos\theta\\ \cos 2\theta\\ \cos 2\theta\\ \cos\theta\\ 1\\ \cos\theta\\ \cos 2\theta\\ \cos 2\theta\\ \cos\theta\end{bmatrix}, \quad \begin{bmatrix}\pi_{1_{e_1,e_2}}(1)\\ \pi_{1_{e_1,e_2}}(r)\\ \pi_{1_{e_1,e_2}}(r^2)\\ \pi_{1_{e_1,e_2}}(r^3)\\ \pi_{1_{e_1,e_2}}(r^4)\\ \pi_{1_{e_1,e_2}}(s)\\ \pi_{1_{e_1,e_2}}(sr)\\ \pi_{1_{e_1,e_2}}(sr^2)\\ \pi_{1_{e_1,e_2}}(sr^3)\\ \pi_{1_{e_1,e_2}}(sr^4)\end{bmatrix} = \begin{bmatrix}0\\ -\sin\theta\\ -\sin 2\theta\\ \sin 2\theta\\ \sin\theta\\ 0\\ -\sin\theta\\ -\sin 2\theta\\ \sin 2\theta\\ \sin\theta\end{bmatrix},$$

$$\begin{bmatrix}\pi_{1_{e_2,e_1}}(1)\\ \pi_{1_{e_2,e_1}}(r)\\ \pi_{1_{e_2,e_1}}(r^2)\\ \pi_{1_{e_2,e_1}}(r^3)\\ \pi_{1_{e_2,e_1}}(r^4)\\ \pi_{1_{e_2,e_1}}(s)\\ \pi_{1_{e_2,e_1}}(sr)\\ \pi_{1_{e_2,e_1}}(sr^2)\\ \pi_{1_{e_2,e_1}}(sr^3)\\ \pi_{1_{e_2,e_1}}(sr^4)\end{bmatrix} = \begin{bmatrix}0\\ \sin\theta\\ \sin 2\theta\\ -\sin 2\theta\\ -\sin\theta\\ 0\\ -\sin\theta\\ -\sin 2\theta\\ \sin 2\theta\\ \sin\theta\end{bmatrix}, \text{ and } \begin{bmatrix}\pi_{1_{e_2,e_2}}(1)\\ \pi_{1_{e_2,e_2}}(r)\\ \pi_{1_{e_2,e_2}}(r^2)\\ \pi_{1_{e_2,e_2}}(r^3)\\ \pi_{1_{e_2,e_2}}(r^4)\\ \pi_{1_{e_2,e_2}}(s)\\ \pi_{1_{e_2,e_2}}(sr)\\ \pi_{1_{e_2,e_2}}(sr^2)\\ \pi_{1_{e_2,e_2}}(sr^3)\\ \pi_{1_{e_2,e_2}}(sr^4)\end{bmatrix} = \begin{bmatrix}1\\ \cos\theta\\ \cos 2\theta\\ \cos 2\theta\\ \cos\theta\\ -1\\ -\cos\theta\\ -\cos 2\theta\\ -\cos 2\theta\\ -\cos\theta\end{bmatrix}.$$

Here we can observe that

$$\begin{bmatrix}1 & \cos\theta & \cos 2\theta & \cos 2\theta & \cos\theta & 1 & \cos\theta & \cos 2\theta & \cos 2\theta & \cos\theta\end{bmatrix}^\top$$

$$\text{and } \begin{bmatrix}0 & -\sin\theta & -\sin 2\theta & \sin 2\theta & \sin\theta & 0 & -\sin\theta & -\sin 2\theta & \sin 2\theta & \sin\theta\end{bmatrix}^\top$$

are eigenvectors of eigenvalue $\xi + \overline{\xi} + 1$.

Similarly,

$$\begin{bmatrix}0 & \sin\theta & \sin 2\theta & -\sin 2\theta & -\sin\theta & 0 & -\sin\theta & -\sin 2\theta & \sin 2\theta & \sin\theta\end{bmatrix}^\top$$

$$\text{and } \begin{bmatrix}1 & \cos\theta & \cos 2\theta & \cos 2\theta & \cos\theta & -1 & -\cos\theta & -\cos 2\theta & -\cos 2\theta & -\cos\theta\end{bmatrix}^\top$$

are eigenvectors of eigenvalue $\xi + \overline{\xi} - 1$.

(iv)

Let λ_{41} and λ_{42} be the eigenvalues associated with the character of the degree two representation π_2. Then,

$$\begin{aligned}10(\lambda_{41} + \lambda_{42}) = 10(\pi_2(r) + \pi_2(r^{-1}) + \pi_2(s)) = 20(\xi^2 + \overline{\xi}^2)\\ \implies \lambda_{41} + \lambda_{42} = 2(\xi^2 + \overline{\xi}^2)\end{aligned}$$

and

$$\begin{aligned}10(\lambda_{41}^2 + \lambda_{42}^2) = 30\pi_2(1) + 10\pi_2(r^2) + 10\pi_2(r^3) = 60 + 20(\xi + \overline{\xi})\\ \implies \lambda_{41}^2 + \lambda_{42}^2 = 6 + 2(\xi + \overline{\xi})\end{aligned}$$

Then we get

$$\lambda_{41} = \xi^2 + \overline{\xi}^2 + 1 \quad \text{and} \quad \lambda_{42} = \xi^2 + \overline{\xi}^2 - 1.$$

The matrix coefficients of π_2 are :

$$\begin{bmatrix}\pi_{2_{e_1,e_1}}(1)\\ \pi_{2_{e_1,e_1}}(r)\\ \pi_{2_{e_1,e_1}}(r^2)\\ \pi_{2_{e_1,e_1}}(r^3)\\ \pi_{2_{e_1,e_1}}(r^4)\\ \pi_{2_{e_1,e_1}}(s)\\ \pi_{2_{e_1,e_1}}(sr)\\ \pi_{2_{e_1,e_1}}(sr^2)\\ \pi_{2_{e_1,e_1}}(sr^3)\\ \pi_{2_{e_1,e_1}}(sr^4)\end{bmatrix} = \begin{bmatrix}1\\ \cos 2\theta\\ \cos\theta\\ \cos\theta\\ \cos 2\theta\\ 1\\ \cos 2\theta\\ \cos\theta\\ \cos\theta\\ \cos 2\theta\end{bmatrix}, \quad \begin{bmatrix}\pi_{2_{e_1,e_2}}(1)\\ \pi_{2_{e_1,e_2}}(r)\\ \pi_{2_{e_1,e_2}}(r^2)\\ \pi_{2_{e_1,e_2}}(r^3)\\ \pi_{2_{e_1,e_2}}(r^4)\\ \pi_{2_{e_1,e_2}}(s)\\ \pi_{2_{e_1,e_2}}(sr)\\ \pi_{2_{e_1,e_2}}(sr^2)\\ \pi_{2_{e_1,e_2}}(sr^3)\\ \pi_{2_{e_1,e_2}}(sr^4)\end{bmatrix} = \begin{bmatrix}0\\ -\sin 2\theta\\ \sin\theta\\ -\sin\theta\\ \sin 2\theta\\ 0\\ -\sin 2\theta\\ \sin\theta\\ -\sin\theta\\ \sin 2\theta\end{bmatrix},$$

$$
\begin{bmatrix} \pi_{2_{e_2,e_1}}(1) \\ \pi_{2_{e_2,e_1}}(r) \\ \pi_{2_{e_2,e_1}}(r^2) \\ \pi_{2_{e_2,e_1}}(r^3) \\ \pi_{2_{e_2,e_1}}(r^4) \\ \pi_{2_{e_2,e_1}}(s) \\ \pi_{2_{e_2,e_1}}(sr) \\ \pi_{2_{e_2,e_1}}(sr^2) \\ \pi_{2_{e_2,e_1}}(sr^3) \\ \pi_{2_{e_2,e_1}}(sr^4) \end{bmatrix} = \begin{bmatrix} 0 \\ \sin 2\theta \\ -\sin\theta \\ \sin\theta \\ -\sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin\theta \\ -\sin\theta \\ \sin 2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \pi_{2_{e_2,e_2}}(1) \\ \pi_{2_{e_2,e_2}}(r) \\ \pi_{2_{e_2,e_2}}(r^2) \\ \pi_{2_{e_2,e_2}}(r^3) \\ \pi_{2_{e_2,e_2}}(r^4) \\ \pi_{2_{e_2,e_2}}(s) \\ \pi_{2_{e_2,e_2}}(sr) \\ \pi_{2_{e_2,e_2}}(sr^2) \\ \pi_{2_{e_2,e_2}}(sr^3) \\ \pi_{2_{e_2,e_2}}(sr^4) \end{bmatrix} = \begin{bmatrix} 1 \\ \cos 2\theta \\ \cos\theta \\ \cos\theta \\ \cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos\theta \\ -\cos\theta \\ -\cos 2\theta \end{bmatrix}.
$$

Here we can observe that

$$
\begin{bmatrix} 1 & \cos 2\theta & \cos\theta & \cos\theta & \cos 2\theta & 1 & \cos 2\theta & \cos\theta & \cos\theta & \cos 2\theta \end{bmatrix}^\top
$$
$$
\text{and } \begin{bmatrix} 0 & -\sin 2\theta & \sin\theta & -\sin\theta & \sin 2\theta & 0 & -\sin 2\theta & \sin\theta & -\sin\theta & \sin 2\theta \end{bmatrix}^\top
$$

are eigenvectors of eigenvalue $\xi^2 + \overline{\xi}^2 + 1$.

Similarly,

$$
\begin{bmatrix} 0 & \sin 2\theta & -\sin\theta & \sin\theta & -\sin 2\theta & 0 & -\sin 2\theta & \sin\theta & -\sin\theta & \sin 2\theta \end{bmatrix}^\top
$$
$$
\text{and } \begin{bmatrix} 1 & \cos 2\theta & \cos\theta & \cos\theta & \cos 2\theta & -1 & -\cos 2\theta & -\cos\theta & -\cos\theta & -\cos 2\theta \end{bmatrix}^\top
$$

are eigenvectors of eigenvalue $\xi^2 + \overline{\xi}^2 - 1$.

The table below lists the eigenvalues of the Cayley graph of $\mathcal{D}_{10}$ generated by $\{r, r^{-1}, s\}$ and corresponding orthogonal eigenvectors.

Eigenvalue	**Corresponding Orthogonal Eigenvectors**
3	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$
1	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}^\top$

$\xi+\bar{\xi}+1$	$\begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 2\theta \\ \cos\theta \\ 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 2\theta \\ \cos\theta \end{bmatrix}, \begin{bmatrix} 0 \\ -\sin\theta \\ -\sin 2\theta \\ \sin 2\theta \\ \sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix}$
$\xi+\bar{\xi}-1$	$\begin{bmatrix} 0 \\ \sin\theta \\ \sin 2\theta \\ -\sin 2\theta \\ -\sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 2\theta \\ \cos\theta \\ -1 \\ -\cos\theta \\ -\cos 2\theta \\ -\cos 2\theta \\ -\cos\theta \end{bmatrix}$
$\xi^2+\bar{\xi}^2+1$	$\begin{bmatrix} 1 \\ \cos 2\theta \\ \cos\theta \\ \cos\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos\theta \\ \cos\theta \\ \cos 2\theta \end{bmatrix}, \begin{bmatrix} 0 \\ -\sin 2\theta \\ \sin\theta \\ -\sin\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin\theta \\ -\sin\theta \\ \sin 2\theta \end{bmatrix}$
$\xi^2+\bar{\xi}^2-1$	$\begin{bmatrix} 0 \\ \sin 2\theta \\ -\sin\theta \\ \sin\theta \\ -\sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin\theta \\ -\sin\theta \\ \sin 2\theta \end{bmatrix}, \begin{bmatrix} 1 \\ \cos 2\theta \\ \cos\theta \\ \cos\theta \\ \cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos\theta \\ -\cos\theta \\ -\cos 2\theta \end{bmatrix}$

7.1.3 Spectrum of Cayley graph of D_{12} generated by $\{\mathbf{r}, \mathbf{r}^{-1}, \mathbf{s}\}$

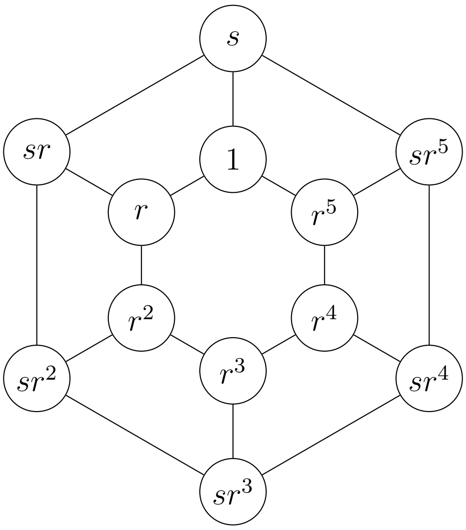


Figure 7.3: The Cayley graph of $\mathcal{D}_{12}$ generated by $\{r, r^{-1}, s\}$.

Here, the adjacency relations are $x \sim rx$, $x \sim r^{-1}x$, and $x \sim sx$ for every $x \in \mathcal{D}_{12}$.

Here $n = 6$ (even) and hence $\mathcal{D}_{12}/[\mathcal{D}_{12}, \mathcal{D}_{12}]$ is the Klein's-4 group generated by cosets r and s. Hence $\mathcal{D}_{12}$ has four degree-one irreducible representations which assign values ± 1 to r and s. $\mathcal{D}_{12}$ has 6 conjugacy classes ($\frac{n}{2} + 3$ conjugacy classes). So, it has two degree-two irreducible representations. The characters of one of the two degree-two irreducible representations, π_1, can be calculated by considering

$$\pi_1(r^k) = \begin{pmatrix} \cos\frac{k\pi}{3} & -\sin\frac{k\pi}{3} \\ \sin\frac{k\pi}{3} & \cos\frac{k\pi}{3} \end{pmatrix}$$

for $k \in \{0, 1, 2, 3, 4, 5\}$ and

$$\pi_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponding to the counterclockwise rotation by angle $\pi/6$ around $(0, 0)$ and reflection along the x-axis, respectively.

The character of the other degree-two representation π_2 can be calculated by considering

$$\pi_2(r^k) = \begin{pmatrix} \cos\frac{2k\pi}{3} & -\sin\frac{2k\pi}{3} \\ \sin\frac{2k\pi}{3} & \cos\frac{2k\pi}{3} \end{pmatrix}$$

for $k \in \{0, 1, 2, 3, 4, 5\}$ and

$$\pi_2(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponding to the counterclockwise rotation by angle $\pi/3$ around $(0, 0)$ and reflection along the x-axis, respectively.. This completes the character table of $\mathcal{D}_{12}$ as given below.

	$[1]$	$[r]$	$[r^2]$	$[r^3]$	$[s]$	$[sr]$
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	1	-1	-1	1
π_1	2	1	-1	-2	0	0
π_2	2	-1	-1	2	0	0

Here, $\xi = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

Let $\theta = \frac{\pi}{3}$, then $\xi = \cos\theta + i\sin\theta$.

γ	1	r	r^2	r^3	r^4	r^5	s	sr	sr^2	sr^3	sr^4	sr^5
$p_{\gamma 1}$	0	12	0	0	0	12	12	0	0	0	0	0
$p_{\gamma 2}$	36	0	12	0	12	0	0	24	0	0	0	24

The eigenvalues associated with characters of degree one representations are:

(i)

$$\chi_1(r) + \chi_1(r^{-1}) + \chi_1(s) = 3$$

This is a trivial eigenvalue, since the degree of every vertex is equal to 3.

$$\begin{bmatrix} 1 & 1 & ..... & 1 \end{bmatrix}^\top \text{(a } 12 \times 1 \text{ matrix with all entries 1.)}$$

is an eigenvector with this eigenvalue.

(ii)

$$\chi_2(r) + \chi_2(r^{-1}) + \chi_2(s) = -1$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

is an eigenvector with this eigenvalue.

(iii)

$$\chi_3(r) + \chi_3(r^{-1}) + \chi_3(s) = 1$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

is an eigenvector with this eigenvalue.

(iv)

$$\chi_4(r) + \chi_4(r^{-1}) + \chi_4(s) = -3$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

is an eigenvector with this eigenvalue.

(v)

Let λ_{51} and λ_{52} be the eigenvalues associated with the characters of the degree two representation π_1. Then,

$$12(\lambda_{51} + \lambda_{52}) = 12(\pi_1(r) + \pi_1(r^{-1}) + \pi_1(s)) = 24$$
$$\implies \lambda_{51} + \lambda_{52} = 2$$

and

$$12(\lambda_{51}^2 + \lambda_{52}^2) = 36\pi_1(1) + 12\pi_1(r^2) + 12\pi_1(r^4) = 48$$

$$\implies \lambda_{51}^2 + \lambda_{52}^2 = 4$$

Then we get

$$\lambda_{51} = 2 \quad \text{and} \quad \lambda_{52} = 0.$$

The matrix coefficients of π_1 are :

$$\begin{bmatrix} {\pi_1}_{e_1,e_1}(1) \\ {\pi_1}_{e_1,e_1}(r) \\ {\pi_1}_{e_1,e_1}(r^2) \\ {\pi_1}_{e_1,e_1}(r^3) \\ {\pi_1}_{e_1,e_1}(r^4) \\ {\pi_1}_{e_1,e_1}(r^5) \\ {\pi_1}_{e_1,e_1}(s) \\ {\pi_1}_{e_1,e_1}(sr) \\ {\pi_1}_{e_1,e_1}(sr^2) \\ {\pi_1}_{e_1,e_1}(sr^3) \\ {\pi_1}_{e_1,e_1}(sr^4) \\ {\pi_1}_{e_1,e_1}(sr^5) \end{bmatrix} = \begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \\ 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \end{bmatrix}, \quad \begin{bmatrix} {\pi_1}_{e_1,e_2}(1) \\ {\pi_1}_{e_1,e_2}(r) \\ {\pi_1}_{e_1,e_2}(r^2) \\ {\pi_1}_{e_1,e_2}(r^3) \\ {\pi_1}_{e_1,e_2}(r^4) \\ {\pi_1}_{e_1,e_2}(r^5) \\ {\pi_1}_{e_1,e_2}(s) \\ {\pi_1}_{e_1,e_2}(sr) \\ {\pi_1}_{e_1,e_2}(sr^2) \\ {\pi_1}_{e_1,e_2}(sr^3) \\ {\pi_1}_{e_1,e_2}(sr^4) \\ {\pi_1}_{e_1,e_2}(sr^5) \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix},$$

$$\begin{bmatrix} {\pi_1}_{e_2,e_1}(1) \\ {\pi_1}_{e_2,e_1}(r) \\ {\pi_1}_{e_2,e_1}(r^2) \\ {\pi_1}_{e_2,e_1}(r^3) \\ {\pi_1}_{e_2,e_1}(r^4) \\ {\pi_1}_{e_2,e_1}(r^5) \\ {\pi_1}_{e_2,e_1}(s) \\ {\pi_1}_{e_2,e_1}(sr) \\ {\pi_1}_{e_2,e_1}(sr^2) \\ {\pi_1}_{e_2,e_1}(sr^3) \\ {\pi_1}_{e_2,e_1}(sr^4) \\ {\pi_1}_{e_2,e_1}(sr^5) \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\theta \\ \sin 2\theta \\ \sin 3\theta \\ -\sin 2\theta \\ -\sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix}, \text{ and } \begin{bmatrix} {\pi_1}_{e_2,e_2}(1) \\ {\pi_1}_{e_2,e_2}(r) \\ {\pi_1}_{e_2,e_2}(r^2) \\ {\pi_1}_{e_2,e_2}(r^3) \\ {\pi_1}_{e_2,e_2}(r^4) \\ {\pi_1}_{e_2,e_2}(r^5) \\ {\pi_1}_{e_2,e_2}(s) \\ {\pi_1}_{e_2,e_2}(sr) \\ {\pi_1}_{e_2,e_2}(sr^2) \\ {\pi_1}_{e_2,e_2}(sr^3) \\ {\pi_1}_{e_2,e_2}(sr^4) \\ {\pi_1}_{e_2,e_2}(sr^5) \end{bmatrix} = \begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \\ -1 \\ -\cos\theta \\ -\cos 2\theta \\ -\cos 3\theta \\ -\cos 2\theta \\ -\cos\theta \end{bmatrix}.$$

Here we can observe that

$$\begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \\ 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix}$$

are eigenvectors of eigenvalue 2 .

Similarly,

$$\begin{bmatrix} 0 \\ \sin\theta \\ \sin 2\theta \\ \sin 3\theta \\ -\sin 2\theta \\ -\sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \\ -1 \\ -\cos\theta \\ -\cos 2\theta \\ -\cos 3\theta \\ -\cos 2\theta \\ -\cos\theta \end{bmatrix}$$

are eigenvectors of eigenvalue 0 .

(vi)

Let λ_{61} and λ_{62} be the eigenvalues associated with the characters of the degree two representation π_2. Then,

$$12(\lambda_{61}+\lambda_{62}) = 12(\pi_2(r)+\pi_2(r^{-1})+\pi_2(s)) = -24$$
$$\implies \lambda_{61}+\lambda_{62} = -2$$

and

$$12(\lambda_{61}^2+\lambda_{62}^2) = 36\pi_2(1)+12\pi_2(r^2)+12\pi_2(r^4) = 48$$
$$\implies \lambda_{61}^2+\lambda_{62}^2 = 4$$

Then we get

$$\lambda_{61} = 0 \quad \text{and} \quad \lambda_{62} = -2.$$

The matrix coefficients of π_2 are :

$$\begin{bmatrix} \pi_{2_{e_1,e_1}}(1) \\ \pi_{2_{e_1,e_1}}(r) \\ \pi_{2_{e_1,e_1}}(r^2) \\ \pi_{2_{e_1,e_1}}(r^3) \\ \pi_{2_{e_1,e_1}}(r^4) \\ \pi_{2_{e_1,e_1}}(r^5) \\ \pi_{2_{e_1,e_1}}(s) \\ \pi_{2_{e_1,e_1}}(sr) \\ \pi_{2_{e_1,e_1}}(sr^2) \\ \pi_{2_{e_1,e_1}}(sr^3) \\ \pi_{2_{e_1,e_1}}(sr^4) \\ \pi_{2_{e_1,e_1}}(sr^5) \end{bmatrix} = \begin{bmatrix} 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \end{bmatrix}, \quad \begin{bmatrix} \pi_{2_{e_1,e_2}}(1) \\ \pi_{2_{e_1,e_2}}(r) \\ \pi_{2_{e_1,e_2}}(r^2) \\ \pi_{2_{e_1,e_2}}(r^3) \\ \pi_{2_{e_1,e_2}}(r^4) \\ \pi_{2_{e_1,e_2}}(r^5) \\ \pi_{2_{e_1,e_2}}(s) \\ \pi_{2_{e_1,e_2}}(sr) \\ \pi_{2_{e_1,e_2}}(sr^2) \\ \pi_{2_{e_1,e_2}}(sr^3) \\ \pi_{2_{e_1,e_2}}(sr^4) \\ \pi_{2_{e_1,e_2}}(sr^5) \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \end{bmatrix},$$

$$\begin{bmatrix} \pi_{2_{e_2,e_1}}(1) \\ \pi_{2_{e_2,e_1}}(r) \\ \pi_{2_{e_2,e_1}}(r^2) \\ \pi_{2_{e_2,e_1}}(r^3) \\ \pi_{2_{e_2,e_1}}(r^4) \\ \pi_{2_{e_2,e_1}}(r^5) \\ \pi_{2_{e_2,e_1}}(s) \\ \pi_{2_{e_2,e_1}}(sr) \\ \pi_{2_{e_2,e_1}}(sr^2) \\ \pi_{2_{e_2,e_1}}(sr^3) \\ \pi_{2_{e_2,e_1}}(sr^4) \\ \pi_{2_{e_2,e_1}}(sr^5) \end{bmatrix} = \begin{bmatrix} 0 \\ \sin 2\theta \\ -\sin 2\theta \\ 0 \\ \sin 2\theta \\ -\sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \end{bmatrix}, \text{ and } \begin{bmatrix} \pi_{2_{e_2,e_2}}(1) \\ \pi_{2_{e_2,e_2}}(r) \\ \pi_{2_{e_2,e_2}}(r^2) \\ \pi_{2_{e_2,e_2}}(r^3) \\ \pi_{2_{e_2,e_2}}(r^4) \\ \pi_{2_{e_2,e_2}}(r^5) \\ \pi_{2_{e_2,e_2}}(s) \\ \pi_{2_{e_2,e_2}}(sr) \\ \pi_{2_{e_2,e_2}}(sr^2) \\ \pi_{2_{e_2,e_2}}(sr^3) \\ \pi_{2_{e_2,e_2}}(sr^4) \\ \pi_{2_{e_2,e_2}}(sr^5) \end{bmatrix} = \begin{bmatrix} 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos 2\theta \end{bmatrix}.$$

Here we can observe that

$$\begin{bmatrix} 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \end{bmatrix}$$

are eigenvectors of eigenvalue 0 .

Similarly,

$$\begin{bmatrix} 0 \\ \sin 2\theta \\ -\sin 2\theta \\ 0 \\ \sin 2\theta \\ -\sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos 2\theta \end{bmatrix}$$

are eigenvectors of eigenvalue -2.

The table below lists the eigenvalues of the Cayley graph of $\mathcal{D}_{12}$ generated by $\{r, r^{-1}, s\}$

and corresponding orthogonal eigenvectors.

Eigenvalue	**Corresponding Orthogonal Eigenvectors**
3	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
-1	$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$

-3	$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$
2	$\begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \\ 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \end{bmatrix}, \begin{bmatrix} 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix}$
0	$\begin{bmatrix} 0 \\ \sin\theta \\ \sin 2\theta \\ \sin 3\theta \\ -\sin 2\theta \\ -\sin\theta \\ 0 \\ -\sin\theta \\ -\sin 2\theta \\ -\sin 3\theta \\ \sin 2\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \cos 2\theta \\ \cos\theta \\ -1 \\ -\cos\theta- \\ -\cos 2\theta \\ -\cos 3\theta \\ -\cos 2\theta \\ -\cos\theta \end{bmatrix}, \begin{bmatrix} 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \end{bmatrix}, \begin{bmatrix} 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \end{bmatrix}$

-2	$\begin{bmatrix} 0 \\ \sin 2\theta \\ -\sin 2\theta \\ 0 \\ \sin 2\theta \\ -\sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \\ 0 \\ -\sin 2\theta \\ \sin 2\theta \end{bmatrix}, \begin{bmatrix} 1 \\ \cos 2\theta \\ \cos 2\theta \\ 1 \\ \cos 2\theta \\ \cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos 2\theta \\ -1 \\ -\cos 2\theta \\ -\cos 2\theta \end{bmatrix}$

7.1.4 Spectrum of Cayley graph of D_{2n} generated by {r, r^{-1}, s}

Now we consider the eigenvalues of the adjacency matrix of Cayley graph of D_{2n} generated by $\{r, s, r^{-1}\}$. The Cayley graph of $\mathcal{D}_{2n}$ generated by $\{r, r^{-1}, s\}$ consists of an inner cycle and an outer cycle, connected by edges. The inner cycle represents the rotational elements $\{1, r, r^2, \ldots, r^{n-1}\}$, while the outer cycle represents the reflection elements $\{s, sr, sr^2, \ldots, sr^{n-1}\}$. The inner cycle is connected to the outer cycle by edges, where each rotational element r^k is connected to its reflected counterpart sr^k.

As previously discussed, for degree one representations of dihedral groups, we get one eigenvalue associated with each character χ_i, namely

$$\sum_{x_1\gamma\sim x_1} \chi_i(\gamma).$$

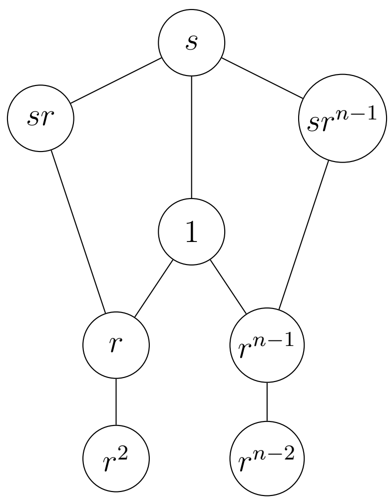


Figure 7.4: Representation of a part of the Cayley graph of D_{2n} generated by $\{r, r^{-1}, s\}$, showing the vertex 1 and all vertices at a distance of at most 2 from it.

γ	1	r	r^2	r^{n-2}	r^{n-1}	s	sr	sr^{n-1}
$p_{\gamma 1}$	0	2n	0	0	2n	2n	0	0
$p_{\gamma 2}$	6n	0	2n	2n	0	0	4n	4n

Theorem 7.2.4.1:(Refer to [7])Let G be a finite group. Suppose (π, Y) is a finite degree representation of G such that:

1. $Y = \bigoplus_{1\le j\le m} Y_j$.
2. G acts on the set of spaces $\{Y_j \mid 1 \le j \le m\}$ transitively via the maps $\pi(g)$, $g \in G$, i.e.,
   - For every $1 \le j \le m$ and $g \in G$, there exists j' such that $\pi(g)(Y_j) = Y_{j'}$.
   - Given any $1 \le j, j' \le m$, there exists $g \in G$ such that $\pi(g)(Y_j) = Y_{j'}$.

Let $W := Y_1$ and let H be the stabilizer of W, i.e., the subgroup $\{g \in G : \pi(g)W = W\}$. For every $h \in H$, let $\sigma(h) := \pi(h)$ be a map $W \to W$ (thus (σ, W) is the restriction of the representation π to H on W).

Then $(\pi, Y) \cong (\operatorname{Ind}_H^G \sigma, V)$ as representations of G.

The formula for the character of the induced representation $\operatorname{Ind}_H^G \sigma$ is given by

$$\chi_{\operatorname{Ind}_H^G \sigma}(g) = \frac{1}{|H|} \sum_{x\in G: xgx^{-1}\in H} \chi_\sigma(xgx^{-1}).$$

Let $G = D_{2n}$ be the dihedral group of order $2n$, and let $H = \langle r\rangle = \{r^j : 0 \le j \le n-1\}$ be the cyclic subgroup generated by r. Then it has exactly n mutually inequivalent degree one representations over $\mathbb{C}$, say ψ_j, $0 \le j \le n-1$, defined by

$$\psi_j(r^\ell) = \xi^{j\ell} \quad \text{for all } 0 \le \ell, j \le n-1$$

where $\xi = e^{2\pi i/n}$. Let $\theta = \frac{2\pi}{n}$, then $\xi = \cos\theta + i\sin\theta$. Since G is the disjoint union of H and sH, H is a normal subgroup of G.

Now, we compute xgx^{-1} for every $g \in G$.

Case 1: $g \in H$
Since H is normal in G, conjugating any element of H by an arbitrary $x \in G$ results in an element of H, i.e.,

$$xgx^{-1} \in H, \quad \forall x \in G.$$

Case 2: $g \notin H$
If $g \notin H$, then it must be of the form $g = sr^j$ for some j. Conjugating g by any $x \in G$ results in an element that is not in H, i.e.,

$$xgx^{-1} \notin H, \quad \forall x \in G.$$

Then,

$$\chi_{\mathrm{Ind}_H^G \psi_j}(sr^l) = 0,\ \forall\ 0 \leq l \leq n-1.$$

and

$$\chi_{\mathrm{Ind}_H^G \psi_j}(r^l) = \frac{1}{n}\sum_{x\in G: xgx^{-1}\in H} \psi_j(xr^lx^{-1}) = \frac{1}{n}[n\psi_j(r^l)+n\psi_j(r^{-l})] = \xi^{jl}+\xi^{-jl},\ \ \forall\, 0 \leq l \leq n-1.$$

The complete list of isomorphism classes of degree two irreducible representations of $\mathcal{D}_{2n}$ that are induced from degree one representations of $H = \langle r \rangle$ is as follows:

- **Case 1)** n is even: $\quad \mathrm{Ind}_H^G \psi_j$; $1 \leq j \leq \frac{n}{2} - 1$.
- **Case 2)** n is odd: $\quad \mathrm{Ind}_H^G \psi_j$; $1 \leq j \leq \frac{n-1}{2}$.

Let π_a be a degree two representation of D_{2n}. The characters of π_a, can be calculated by considering

$$\pi_a(r^k) = \begin{pmatrix} \cos\frac{2a\pi}{n} & -\sin\frac{2a\pi}{n} \\ \sin\frac{2a\pi}{n} & \cos\frac{2a\pi}{n} \end{pmatrix} = \begin{pmatrix} \cos a\theta & -\sin a\theta \\ \sin a\theta & \cos a\theta \end{pmatrix}$$

for $a \in \{0, 1, 2, ...n\}$ and

$$\pi_a(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponding to the counterclockwise rotation by angle $a\pi/n$ around $(0, 0)$ and reflection along the x-axis, respectively. Let λ_{a1} and λ_{a2} be the eigenvalues associated with the character of the degree two representation π_a.

Then,

$$2n(\lambda_{a1} + \lambda_{a2}) = 2n(\chi_{\pi_a}(r) + \chi_{\pi_a}(r^{-1}) + \chi_{\pi_a}(s))$$

$$\implies f_{a,1} = \lambda_{a1} + \lambda_{a2} = \chi_{\pi_a}(r) + \chi_{\pi_a}(r^{-1}) + \chi_{\pi_a}(s) = 2(\xi^a + \xi^{-a})$$

and

$$2n(\lambda_{a1}^2 + \lambda_{a2}^2) = 6n\chi_{\pi_a}(1) + 2n\chi_{\pi_a}(r^2) + 2n\chi_{\pi_a}(r^{-2}) + 4n\chi_{\pi_a}(sr) + 4n\chi_{\pi_a}(sr^{-1})$$

$$\implies f_{a,2} = \lambda_{a1}^2 + \lambda_{a2}^2 = 3\chi_{\pi_a}(1) + \chi_{\pi_a}(r^2) + \chi_{\pi_a}(r^{-2}) = 3(2) + 2(\xi^{2a} + \xi^{-2a})$$

$$\implies f_{a,2} = 6 + 2(\xi^{2a} + \xi^{-2a})$$

Then, λ_{a1} and λ_{a2} are the roots of the polynomial

$$\det\begin{bmatrix}1 & f_{a,1} & f_{a,2}\\ 0 & 1 & f_{a,1}\\ \frac{1}{2} & x & x^2\end{bmatrix} = \det\begin{bmatrix}1 & 2(\xi^a+\xi^{-a}) & 6+2(\xi^{2a}+\xi^{-2a})\\ 0 & 1 & 2(\xi^a+\xi^{-a})\\ \frac{1}{2} & x & x^2\end{bmatrix}$$

Then,

$$\lambda_{a1} = \xi^a + \xi^{-a} + 1, \quad \lambda_{a2} = \xi^a + \xi^{-a} - 1,$$

each with multiplicity 2.

Any eigenvector corresponding to an eigenvalue associated with a character of a degree-two representation will be a linear combination of its matrix coefficients.

The matrix coefficients of π_a are of the following form:

$$(\pi_{a_{11}})_j = \cos{(j-1)a\theta}, \quad (\pi_{a_{12}})_j = -\sin{(j-1)a\theta}$$

$$(\pi_{a_{21}})_j = \begin{cases}\sin{(j-1)a\theta}, & \text{if } j \leq n\\ -\sin{(j-1)a\theta}, & \text{otherwise}\end{cases}$$

$$(\pi_{a_{22}})_i = \begin{cases}\cos{(j-1)a\theta}, & \text{if } j \leq n\\ -\cos{(j-1)a\theta}, & \text{otherwise}\end{cases}$$

Here we can observe that

- $$\begin{aligned}(A_{\mathcal{D}_{2n}}\pi_{a_{11}})_j &= \cos(ja\theta) + \cos((j-2)a\theta) + \cos((j-1)a\theta)\\ &= 2\cos((j-1)a\theta)\cos(a\theta) + \cos((j-1)a\theta)\\ &= (2\cos(a\theta)+1)\cos((j-1)a\theta) = (\xi^a+\overline{\xi}^a+1)\cos((j-1)a\theta) = ((\xi^a+\overline{\xi}^a+1)\pi_{a_{11}})_j\end{aligned}$$

- $$\begin{aligned}(A_{\mathcal{D}_{2n}}\pi_{a_{12}})_j &= -\sin(ja\theta) - \sin((j-2)a\theta) - \sin((j-1)a\theta)\\ &= -2\sin((j-1)a\theta)\cos(a\theta) - \sin((j-1)a\theta)\\ &= (2\cos(a\theta)+1)(-\sin((j-1)a\theta)) = (\xi^a+\overline{\xi}^a+1)(-\sin((j-1)a\theta)) = ((\xi^a+\overline{\xi}^a+1)\pi_{a_{12}})_j\end{aligned}$$

- – If $j \leq n$,

  $$\begin{aligned}(A_{\mathcal{D}_{2n}}\pi_{a_{21}})_j &= \sin(ja\theta) + \sin((j-2)a\theta) - \sin((j-1)a\theta)\\ &= 2\sin((j-1)a\theta)\cos(a\theta) - \sin((j-1)a\theta)\\ &= (2\cos(a\theta)-1)(\sin((j-1)a\theta)) = (\xi^a+\overline{\xi}^a-1)(\sin((j-1)a\theta)) = ((\xi^a+\overline{\xi}^a-1)\pi_{a_{21}})_j\end{aligned}$$

  – If $j > n$,

  $$\begin{aligned}(A_{\mathcal{D}_{2n}}\pi_{a_{21}})_j &= -\sin(ja\theta) - \sin((j-2)a\theta) + \sin((j-1)a\theta)\\ &= -2\sin((j-1)a\theta)\cos(a\theta) + \sin((j-1)a\theta)\\ &= (2\cos(a\theta)-1)(-\sin((j-1)a\theta)) = (\xi^a+\overline{\xi}^a-1)(-\sin((j-1)a\theta)) = ((\xi^a+\overline{\xi}^a-1)\pi_{a_{21}})_j\end{aligned}$$

- – If $j \leq n$,

$$(A_{\mathcal{D}_{2n}}\pi_{a_{22}})_j = \cos(ja\theta) + \cos((j-2)a\theta) - \cos((j-1)a\theta)$$

$$= 2\cos((j-1)a\theta)\cos(a\theta) - \cos((j-1)a\theta)$$

$$= (2\cos(a\theta)-1)\cos((j-1)a\theta) = (\xi^a+\overline{\xi}^a-1)\cos((j-1)a\theta) = ((\xi^a+\overline{\xi}^a-1)\pi_{a_{22}})_j$$

  – If $j > n$,

$$(A_{\mathcal{D}_{2n}}\pi_{a_{22}})_j = -\cos(ja\theta) - \cos((j-2)a\theta) + \cos((j-1)a\theta)$$

$$= -2\cos((j-1)a\theta)\cos(a\theta) + \cos((j-1)a\theta)$$

$$= (2\cos(a\theta)-1)(-\cos((j-1)a\theta)) = (\xi^a+\overline{\xi}^a-1)(-\cos((j-1)a\theta)) = ((\xi^a+\overline{\xi}^a-1)\pi_{a_{22}})_j$$

Hence

$$\begin{bmatrix}\pi_{a_{11}}(1) & \pi_{a_{11}}(r) & .... & \pi_{a_{11}}(r^{n-1}) & \pi_{a_{11}}(s) & ...... & \pi_{a_{11}}(sr^{n-1})\end{bmatrix}^\top$$

and

$$\begin{bmatrix}\pi_{a_{12}}(1) & \pi_{a_{12}}(r) & .... & \pi_{a_{12}}(r^{n-1}) & \pi_{a_{12}}(s) & ...... & \pi_{a_{12}}(sr^{n-1})\end{bmatrix}^\top$$

are eigenvectors of eigenvalue $\xi^a + \overline{\xi}^a + 1$.

and

$$\begin{bmatrix}\pi_{a_{21}}(1) & \pi_{a_{21}}(r) & .... & \pi_{a_{21}}(r^{n-1}) & \pi_{a_{21}}(s) & ...... & \pi_{a_{21}}(sr^{n-1})\end{bmatrix}^\top$$

and

$$\begin{bmatrix}\pi_{a_{22}}(1) & \pi_{a_{22}}(r) & .... & \pi_{a_{22}}(r^{n-1}) & \pi_{a_{22}}(s) & ...... & \pi_{a_{22}}(sr^{n-1})\end{bmatrix}^\top$$

are eigenvectors of eigenvalue $\xi^a + \overline{\xi}^a - 1$.

Hence, the eigenvalues of the Cayley graph of D_{2n} generated by $\{r, r^{-1}, s\}$ are:

Case 1: n even

$$3, 1, -1, 3, \quad \{\xi^a + \xi^{-a} + 1 \mid 1 \leq a \leq \frac{n}{2} - 1\}, \quad \{\xi^a + \xi^{-a} + 1 \mid 1 \leq a \leq \frac{n}{2} - 1\},$$

$$\{\xi^a + \xi^{-a} - 1 \mid 1 \leq a \leq \frac{n}{2} - 1\}, \text{ and } \quad \{\xi^a + \xi^{-a} - 1 \mid 1 \leq a \leq \frac{n}{2} - 1\}.$$

Case 2: n odd

$$3, 1, \quad \{\xi^a + \xi^{-a} + 1 \mid 1 \leq a \leq \frac{n-1}{2}\}, \quad \{\xi^a + \xi^{-a} + 1 \mid 1 \leq a \leq \frac{n-1}{2}\},$$

$$\{\xi^a + \xi^{-a} - 1 \mid 1 \leq a \leq \frac{n-1}{2}\}, \text{ and } \quad \{\xi^a + \xi^{-a} - 1 \mid 1 \leq a \leq \frac{n-1}{2}\}.$$

We have also found eigenvectors of each of these eigenvalues.

7.2 Cayley graphs of S_4

This section presents the calculation of the eigenvalues of the Cayley graph of S_4 generated by the top-in-at-random shuffle, excluding the identity permutation. Specifically, we computed the eigenvalues of the Cayley graph of S_4 generated by $\{(12), (321), (4321), (123), (1234)\}$, using the results of Lovász.

The *symmetric group* S_n is the group of all permutations of the set $\{1, 2, \ldots, n\}$, with composition as the group operation.

In the permutation group S_n, the conjugacy classes are classified by cycle types, i.e., how an element decomposes as a product of disjoint cycles of lengths $\ell_1, \ell_2, \ldots, \ell_k$.

Thus, there is a bijective correspondence between partitions of n as a sum of positive integers and conjugacy classes in S_n.

There are five partitions of 4 given by:

$$4 = \begin{cases} 1+1+1+1, \\ 1+1+2, \\ 1+3, \\ 4, \\ 2+2. \end{cases}$$

The four mutually inequivalent irreducible representations of S_4 are 1, sgn, st, and $st\otimes$sgn.

- The Trivial Representation 1
  - Every element of S_4 is mapped to the identity transformation.
  - This is a 1-dimensional representation.
- The Sign Representation sgn
  - Each permutation $\sigma \in S_4$ is mapped to ± 1, depending on whether it is even or odd.
  - This is also a 1-dimensional representation.
- The Standard Representation st
  - This is a 3-dimensional representation obtained from the natural action of S_4 on $\mathbb{R}^4$ by permutation of coordinates, while restricting to the subspace where the sum of coordinates is 0 (i.e., the space orthogonal to $(1, 1, 1, 1)$).
  - It is not equivalent to the permutation representation but is a subrepresentation of it.
- The Tensor Product $st \otimes \mathrm{sgn}$
  - This is the "twist" of the standard representation by the sign representation.
  - If $st(\sigma)$ is a matrix representing a permutation σ, then $st \otimes \mathrm{sgn}$ is given by multiplying $st(\sigma)$ by $\mathrm{sgn}(\sigma)$.

– This is another 3-dimensional representation.

The fifth irreducible representation, say π, must be two-dimensional, since

$$1^2 + 1^2 + 3^2 + 3^2 = 20$$

and

$$|S_4| = 24.$$

Its character can be calculated using the Column Orthogonality of Characters (Refer to 2.3.2).

Then, we get the character table of S_4 as given below:

	[1]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
χ_1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_{st}	3	1	0	-1	-1
$\chi_{st\otimes\mathrm{sgn}}$	3	-1	0	1	-1
χ_π	2	0	-1	0	2

$$p_{\gamma 1} = \begin{cases} 24, & \text{if } \gamma \in \{(12), (321), (123), (4321), (1234)\} \\ 0, & \text{otherwise} \end{cases}$$

γ	$p_{\gamma 2}$
[1]	120
(12), (24), (124), (421), (4321), (1234), ((12)(34)), ((14)(23))	0
(13), (14), (23), (34), ((13)(24))	48
(321), (123), (143), (234), (243), (134), (1423), (1243), (1342), (1324)	24

γ	$p_{\gamma 3}$
[1], (34), (24), ((13)(24))	48
(12), (4321), (1234)	264
(13), (14)	72
(23), (1423), (1243), (1342), (1324), ((14)(23))	96
((12)(34))	144

It's not necessary to calculate $p_{\gamma 3}$ for those γ for which $\chi_{st}(\gamma) = \chi_{st\otimes\mathrm{sgn}}(\gamma) = 0$. The eigenvalues associated with characters of degree one representations are:

(i)

$$\chi_1((12)) + \chi_1((321)) + \chi_1((4321)) + \chi_1((123)) + \chi_1((1234)) = 5$$

This is a trivial eigenvalue, since the degree of every vertex is equal to 5.

$$\begin{bmatrix}1 & 1 & .... & 1\end{bmatrix}^\top \text{ (a } 24 \times 1 \text{ matrix with all entries 1.)}$$

is an eigenvector with this eigenvalue.

(ii)

$$\chi_{\text{sgn}}((12)) + \chi_{\text{sgn}}((321)) + \chi_{\text{sgn}}((4321)) + \chi_{\text{sgn}}((123)) + \chi_{\text{sgn}}((1234)) = -1$$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector with this eigenvalue.

(iii)

Let λ_{31}, λ_{32}, and λ_{33} be the eigenvalues associated with the character of the degree three representation χ_{st}. Then,

$$24(\lambda_{31} + \lambda_{32} + \lambda_{33}) = 24\chi_{st}((12)) + 48\chi_{st}((123)) + 48\chi_{st}((1234))) = -24$$

$$\implies \lambda_{31} + \lambda_{32} + \lambda_{33} = -1$$

$$\implies f_{31} = -1$$

and

$$24(\lambda_{31}^2 + \lambda_{32}^2 + \lambda_{33}^2) = 120\chi_{st}([1]) + 4 \times 48\chi_{st}((12)) + 4 \times 24\chi_{st}((1234)) + 48\chi_{st}((12)(34))$$

$$\implies \lambda_{31}^2 + \lambda_{32}^2 + \lambda_{33}^2 = 5\chi_{st}([1]) + 8\chi_{st}((12)) + 4\chi_{st}((1234)) + 2\chi_{st}((12)(34)) = 17$$

$$\implies f_{32} = 17$$

$$\begin{aligned}24(\lambda_{31}^3+\lambda_{32}^3+\lambda_{33}^3) &= 48\chi_{st}([1]) + (48\times 2 + 264 + 72\times 2)\chi_{st}((12)) \\ &+ (264\times 2 + 96\times 4)\chi_{st}((1234)) + (96+144+48)\chi_{st}((12)(34))\end{aligned}$$

$$\implies \lambda_{31}^3+\lambda_{32}^3+\lambda_{33}^3 = 2\chi_{st}([1]) + 25\chi_{st}((12)) + 38\chi_{st}((1234)) + 12\chi_{st}((12)(34)) = -19$$

$$\implies f_{33} = -19$$

$$2f_{33} - f_{31}f_{32} = -21$$

Then, λ_{31}, λ_{32}, and λ_{33} are the roots of the polynomial:

$$\det\begin{bmatrix}1 & -1 & 17 & -21\\ 0 & 1 & -1 & 17\\ 0 & 0 & 1 & -1\\ \frac{1}{6} & \frac{x}{2} & x^2 & x^3\end{bmatrix}.$$

i.e. λ_{31}, λ_{32}, and λ_{33} are the roots of the polynomial:

$$x^3 + x^2 - 8x - 2.$$

Then,
$\lambda_{31} = -\frac{1}{3} + \frac{10}{3}\sin\left(-\frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{69}}{2}\right) + \frac{\pi}{6}\right)$,

$$\lambda_{32} = -\frac{1}{3} + \frac{10}{3}\sin\left(\frac{1}{3}\arctan\left(\frac{3\sqrt{69}}{2}\right) + \frac{\pi}{6}\right), \text{ and}$$

$\lambda_{33} = -\frac{10}{3}\cos\left(\frac{\tan^{-1}\left(\frac{3\sqrt{69}}{2}\right)}{3}\right) - \frac{1}{3}$.
Approximated to five decimal places, λ_{31} =-0.24436, λ_{32}=2.50790, and λ_{33}= -3.26354.

(iv)

Let λ_{41}, λ_{42}, and λ_{43} be the eigenvalues associated with the character of the degree three representation $\chi_{st\otimes\mathrm{sgn}}$. Then,

$$24(\lambda_{41}+\lambda_{42}+\lambda_{43}) = 24\chi_{st\otimes\mathrm{sgn}}((12)) + 48\chi_{st\otimes\mathrm{sgn}}((1234)) + 48\chi_{st\otimes\mathrm{sgn}}((1234)) = 24$$

$$\implies \lambda_{41}+\lambda_{42}+\lambda_{43} = 1$$

$$\implies f_{41} = 1.$$

$$\begin{aligned}24(\lambda_{41}^2+\lambda_{42}^2+\lambda_{43}^2) &= 120\chi_{st\otimes\mathrm{sgn}}([1]) + (4\times 48)\chi_{st\otimes\mathrm{sgn}}((12)) \\ &+ (4\times 24)\chi_{st\otimes\mathrm{sgn}}((1234)) + 48\chi_{st\otimes\mathrm{sgn}}((12)(34)) \\ \implies \lambda_{41}^2+\lambda_{42}^2+\lambda_{43}^2 &= 5\chi_{st\otimes\mathrm{sgn}}([1]) + 8\chi_{st\otimes\mathrm{sgn}}((12)) \\ &+ 4\chi_{st\otimes\mathrm{sgn}}((123)) + 2\chi_{st\otimes\mathrm{sgn}}((12)(34)) = 9\end{aligned}$$

$$\implies f_{42} = 9.$$

$$
\begin{aligned}
24(\lambda_{41}^3+\lambda_{42}^3+\lambda_{43}^3) &= 48\chi_{st\otimes\mathrm{sgn}}([1]) + (48\times 2+264+2\times 72+96)\chi_{st\otimes\mathrm{sgn}}((12)) \\
&\quad + (2\times 264+4\times 96)\chi_{st\otimes\mathrm{sgn}}((1234)) \\
&\quad + (48+96+144)\chi_{st\otimes\mathrm{sgn}}((12)(34)) \\
\implies \lambda_{41}^3+\lambda_{42}^3+\lambda_{43}^3 &= 2\chi_{st\otimes\mathrm{sgn}}([1]) + 25\chi_{st\otimes\mathrm{sgn}}((12)) \\
&\quad + 38\chi_{st\otimes\mathrm{sgn}}((1234)) + 12\chi_{st\otimes\mathrm{sgn}}((12)(34)) = 7 \\
\implies f_{43} &= 7.
\end{aligned}
$$

$$2f_{43} - f_{41}f_{42} = 5$$

Then λ_{41}, λ_{42}, and λ_{43} are the roots of the polynomial:

$$\det\begin{bmatrix} 1 & 1 & 9 & 5 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 1 \\ \frac{1}{6} & \frac{x}{2} & x^2 & x^3 \end{bmatrix}.$$

i.e. λ_{41}, λ_{42}, and λ_{43} are the roots of the polynomial:

$$x^3 - x^2 - 4x + 2.$$

Then,
$\lambda_{41} = \frac{2\sqrt{13}\sin\left(-\tan^{-1}\left(\frac{3\sqrt{237}}{8}\right)/3+\frac{\pi}{6}\right)}{3} + \frac{1}{3}$,
$\lambda_{42} = \frac{1}{3} + \frac{2\sqrt{13}\sin\left(\tan^{-1}\left(\frac{3\sqrt{237}}{8}\right)/3+\frac{\pi}{6}\right)}{3}$, and
$\lambda_{43} = \frac{-2\sqrt{13}\cos\left(\tan^{-1}\left(\frac{3\sqrt{237}}{8}\right)/3\right)}{3} + \frac{1}{3}$.
Approximated to five decimal places,

$$\lambda_{41} = 0.47068, \quad \lambda_{42} = 2.34290, \text{ and } \quad \lambda_{43} = -1.81360.$$

(v)

Let λ_{51} and λ_{52} be the eigenvalues associated with the character of the degree two representation χ_π. Then,

$$24(\lambda_{51}+\lambda_{52}) = 24(\chi_\pi((12)) + \chi_\pi((321)) + \chi_\pi((4321)) + \chi_\pi((123)) + \chi_\pi((1234))) = -10$$

$$
\begin{aligned}
24(\lambda_{51}+\lambda_{52}) = 24(\chi_\pi((12)) + 2\chi_\pi((123)) + 2\chi_\pi((1234))) = -48 \\
\implies \lambda_{51}+\lambda_{52} = -2
\end{aligned}
$$

and

$$24(\lambda_{51}^2+\lambda_{52}^2) = 120(\chi_\pi([1]) + 24(\chi_\pi((321)) + 24(\chi_\pi((123)) + 24(\chi_\pi((143)) + 24(\chi_\pi((234))$$

$$+24(\chi_\pi((243)) + 24(\chi_\pi((134)) + 48(\chi_\pi((13)(24))$$

$$\implies 24(\lambda_{51}^2 + \lambda_{52}^2) = 120(\chi_\pi([1]) + (24 \times 6)(\chi_\pi((123)) + 48(\chi_\pi((12)(34))$$

$$\implies \lambda_{51}^2 + \lambda_{52}^2 = 5(\chi_\pi([1]) + 6(\chi_\pi((123)) + 2(\chi_\pi((12)(34) = 10 - 6 + 2 = 6$$

Then, we get

$$\lambda_{51} = -1 + \sqrt{2} \quad \text{and} \quad \lambda_{52} = -1 - \sqrt{2}.$$

Thus, the eigenvalues of the Cayley graph of S_4 generated by $\{(12), (321), (4321), (123), (1234)\}$ are:

$5, -1, -0.24436, -0.24436, -0.24436, 2.50790, 2.50790, 2.50790, -3.26354, -3.26354,$
$-3.26354, 0.47068, 0.47068, 0.47068, 2.34290, 2.34290, 2.34290, -1.81360, -1.81360, -1.81360,$
$-1 + \sqrt{2}, -1 + \sqrt{2}, -1 - \sqrt{2}, -1 - \sqrt{2}.$

Calculation of eigenvectors of eigenvalues associated with the character of the degree two and degree three representations as linear combinations of their matrix coefficients requires finding the adjacency matrix of the Cayley graph of S_4 generated by $\{(12), (321), (4321), (123), (1234)\}$.

Chapter 8

Conclusion

This thesis incorporates fundamental results from spectral graph theory, Cayley graphs, Markov chains, and probabilistic methods in card shuffling, culminating in explicit spectral computations for Cayley graphs of dihedral groups and the symmetric group S_4. In particular, we analyze how group structure influences the spectral properties of these graphs.

We verify the convergence of repeated convolutions to the uniform distribution in specific cases. Additionally, we show that the converse of Theorem 5.2.7 (Refer to 5.2) from Aldous and Diaconis does not hold in general. To demonstrate this, we construct an example where Q^{k*} converges to the uniform distribution geometrically fast, given that Q, a probability distribution on a group G, is not concentrated on a subgroup or a translate of a subgroup of G. Furthermore, we identify a limitation in Lovász's formula for computing the eigenvalues of adjacency matrices: it holds only when $n_i = 2$. To address this, we derive a new formula for the case $n_i = 3$.

Through detailed calculations, we obtain the full spectrum and eigenvectors of Cayley graphs of dihedral groups generated by $\{r, r^{-1}, s\}$ (Refer to 7.1). Additionally, we compute the eigenvalues of the Cayley graph of S_4 generated by the top-in-at-random shuffle (Refer to 7.2). The final step - determining the eigenvectors corresponding to the characters of the degree-two and degree-three representations - requires an explicit construction of the adjacency matrix for the Cayley graph of S_4 with generators $\{(12), (321), (4321), (123), (1234)\}$.

The results obtained highlight the powerful interplay between spectral graph theory and group representation theory. The methods developed in this thesis provide a foundation for further exploration of the spectral properties of other families of Cayley graphs and their applications in Markov chain mixing times and random walks. Future research could extend these ideas to other transitive graphs, higher-dimensional representations, and non-trivial automorphism groups beyond the symmetric and dihedral cases.

Chapter 9

Appendix

The following Python code is used to graphically demonstrate that, for a random walk on S_3 with Q concentrated on $\{I, (12), (321)\}$, the convolution powers Q^{*n} converge to the uniform distribution as $n \to \infty$.

Python Code:

```
import matplotlib.pyplot as plt

# Define the elements of S3
S3_elements = ['I', '(12)', '(13)', '(23)', '(123)', '(321)']

# Define the initial probabilities
initial_prob = {'I': 1/3, '(12)': 1/3, '(321)': 1/3, '(13)': 0, '
    (23)': 0, '(123)': 0}

# Function to calculate Q^n*(g) for given n based on the inductive
    formula
def q_n_star(g, n, initial_prob):
    if initial_prob[g] == 1/3:
        return 1/6 + 1 / (2 * (3 ** n))
    else:
        return 1/6 - 1 / (2 * (3 ** n))

# Store probabilities for each element in S3 over 100 convolutions
n_steps = 20
distribution_history = {g: [] for g in S3_elements}

for n in range(1, n_steps + 1):
    for g in S3_elements:
        distribution_history[g].append(q_n_star(g, n, initial_prob)
            )
```

```
# Generate individual plots for each element
for g in S3_elements:
    plt.figure(figsize=(8, 5))
    plt.plot(range(1, n_steps + 1), distribution_history[g], label=
        'Q^{n}({g})', color='blue')
    plt.xlabel('n (Number of Steps)')
    plt.ylabel(f'Probability Q^*n({g})')
    plt.title(f'Convergence of Q^*n({g}) to Uniform Distribution')
    plt.axhline(y=1/6, color='red', linestyle='--', label='Uniform
        Distribution (1/6)')
    plt.legend()
    plt.grid()
    plt.show()
```

The following Python code is used to find the threshold at which $Q^{n*}(g)$ is within 10^{-6} of the uniform distribution for all elements of S_3 with Q concentrated on $\{I, (12), (321)\}$.

Python Code:

```
import matplotlib.pyplot as plt

# Define the elements of S3
S3_elements = ['I', '(12)', '(13)', '(23)', '(123)', '(321)']

# Define the initial probabilities
initial_prob = {'I': 1/3, '(12)': 1/3, '(321)': 1/3, '(13)': 0, '
    (23)': 0, '(123)': 0}

# Function to calculate Q^n*(g) for given n based on the inductive
    formula
def q_n_star(g, n, initial_prob):
    if g in initial_prob and initial_prob[g] == 1/3:
        return 1/6 + 1 / (2 * (3 ** n))
    else:
        return 1/6 - 1 / (2 * (3 ** n))

# Define the convergence threshold
epsilon = 1e-6
max_steps = 1000  # Increased limit to ensure accurate convergence

# Track deviations
deviation_history = []
N_star = None
stable_count = 0  # Counter to check if we remain below epsilon for
     consecutive steps

for n in range(1, max_steps + 1):
    max_deviation = max(abs(q_n_star(g, n, initial_prob) - 1/6) for
         g in S3_elements)
    deviation_history.append(max_deviation)
```

```
    if max_deviation < epsilon:
        stable_count += 1
        if stable_count >= 3:  # Ensure stability for at least 3
            consecutive steps
            N_star = n
            break
    else:
        stable_count = 0  # Reset if deviation goes above threshold
# Print the threshold N*
if N_star is not None:
print(f"Q^N* converges to uniform distribution within 10^-6 at N* =
    {N_star}")

else:
    print(f"Q^N* did not converge within {max_steps} steps.")

# Plot the maximum deviation to visualize convergence
plt.figure(figsize=(8, 5))
plt.plot(range(1, len(deviation_history) + 1), deviation_history,
   label="Max Deviation", color='blue')



plt.axhline(
    y=epsilon,
    color='red',
    linestyle='--',
    label=r"Convergence Threshold ($\epsilon = 10^{-6}$)"
)

plt.xlabel('n (Number of Steps)')
plt.ylabel('Max |Q^n(g) - 1/6|')
plt.title('Convergence of Q^n(g) to Uniform Distribution')
plt.yscale('log')  # Log scale to better visualize small deviations
plt.legend()
plt.grid()
plt.show()
```

The following Python code is used to graphically demonstrate that, for a random walk on S_3 with Q concentrated on $\{(13), (123)\}$, the convolution powers Q^{*n} converge to the uniform distribution as $n \to \infty$.

Python Code:

```
import matplotlib.pyplot as plt

# Define elements of S3
S3_elements = ['I', '(12)', '(321)', '(23)', '(13)', '(123)']

# Initial probabilities as per the problem description
initial_prob = {'(13)': 0.5, '(123)': 0.5, 'I': 0, '(12)': 0, '
    (321)': 0, '(23)': 0}

# Function to compute Q^n*(g) based on n being odd or even
def q_n_star(g, n):
    # Base case: initial probabilities for n=1
    if n == 1:
        return initial_prob[g]
    # Recursive case based on whether n is odd or even
    if initial_prob[g] == 0.5:
        if n % 2 == 0:  # n is even
            return 1/6 - 1 / (3 * (2 ** n))
        else:           # n is odd
            return 1/6 + 1 / (3 * (2 ** (n - 1)))
    else:               # initial_prob[g] == 0
        if n % 2 == 0:  # n is even
            return 1/6 + 1 / (6 * (2 ** (n - 1)))
        else:           # n is odd
            return 1/6 - 1 / (6 * (2 ** n))

# Number of steps to simulate
n_steps = 20

# Store probabilities for each element over n_steps
distribution_history = {g: [] for g in S3_elements}

# Calculate probabilities over n steps
for n in range(1, n_steps + 1):
    for g in S3_elements:
        distribution_history[g].append(q_n_star(g, n))

# Plotting each element's probability convergence without negative
    y-axis
for g in S3_elements:
    plt.figure(figsize=(8, 5))
    plt.plot(range(1, n_steps + 1), distribution_history[g], label=
        'Q^{n}({g})', color='blue')
    plt.xlabel('n (Number of Steps)')
    plt.ylabel(f'Probability Q^*n({g})')
    plt.title(f'Convergence of Q^*n({g}) to Uniform Distribution')
    plt.axhline(y=1/6, color='red', linestyle='--', label='Uniform
        Distribution (1/6)')
    plt.legend()
    plt.grid()
    plt.show()
```

The following Python code is used to find the threshold at which $Q^{n*}(g)$ is within 10^{-6} of the uniform distribution for all elements of S_3 with Q concentrated on $\{(13), (123)\}$.

Python Code:

```
import matplotlib.pyplot as plt

# Define elements of S3
S3_elements = ['I', '(12)', '(321)', '(23)', '(13)', '(123)']

# Initial probabilities as per the problem description
initial_prob = {'(13)': 0.5, '(123)': 0.5, 'I': 0, '(12)': 0, '
    (321)': 0, '(23)': 0}

# Function to compute Q^n*(g) based on n being odd or even
def q_n_star(g, n):
    # Get the initial probability, defaulting to 0 if g is not in
       initial_prob
    p0 = initial_prob.get(g, 0)

    # Base case: initial probabilities for n=1
    if n == 1:
        return p0

    # Recursive case based on whether n is odd or even
    if p0 == 0.5:
        if n % 2 == 0:  # n is even
            return 1/6 - 1 / (3 * (2 ** n))
        else:           # n is odd
            return 1/6 + 1 / (3 * (2 ** (n - 1)))
    else:               # initial probability is 0
        if n % 2 == 0:  # n is even
            return 1/6 + 1 / (6 * (2 ** (n - 1)))
        else:           # n is odd
            return 1/6 - 1 / (6 * (2 ** n))

# Define the convergence threshold
epsilon = 1e-6
max_steps = 1000  # Increased limit to ensure accurate convergence

# Track deviations
deviation_history = []
N_star = None
stable_count = 0  # Counter to check if we remain below epsilon for
    consecutive steps

for n in range(1, max_steps + 1):
    max_deviation = max(abs(q_n_star(g, n) - 1/6) for g in
       S3_elements)
    deviation_history.append(max_deviation)
```

```
    if max_deviation < epsilon:
        stable_count += 1
        if stable_count >= 3:  # Ensure stability for at least 3
            consecutive steps
            N_star = n
            break
    else:
        stable_count = 0  # Reset if deviation goes above threshold


# Print the threshold N*
if N_star is not None:
 print(f"Q^N* converges to uniform distribution within 10^-6 at N*
    = {N_star}")

else:
    print(f"Q^N* did not converge within {max_steps} steps.")

# Plot the maximum deviation to visualize convergence
plt.figure(figsize=(8, 5))
plt.plot(range(1, len(deviation_history) + 1), deviation_history,
   label="Max Deviation", color='blue')
plt.axhline(y=epsilon, color='red', linestyle='--', label=r"
   Convergence Threshold ($\varepsilon = 10^{-6}$)")

plt.xlabel('n (Number of Steps)')
plt.ylabel('Max |Q^n(g) - 1/6|')
plt.title('Convergence of Q^n(g) to Uniform Distribution')
plt.yscale('log')  # Log scale to better visualize small deviations
plt.legend()
plt.grid()
plt.show()
```

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