

POLYNOMIAL KNOTS AND THEIR SPACES

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Doctor of Philosophy

by

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**INDIAN INSTITUTE OF SCIENCE EDUCATION AND
RESEARCH PUNE**

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Dedicated to
My Parents and Teachers

Certificate

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Date: October 24, 2016

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Abstract

The main focus of this thesis is to study the topology of some spaces associated with polynomial knots and determining the least polynomial degree in which a given knot can be represented. A polynomial knot is an embedding of $\mathbb{R}$ in $\mathbb{R}^3$ whose component functions are real polynomials. The image of a polynomial knot is a long knot. Polynomial knots were mainly studied by Vassiliev (1990–1996), Shastri (1992) and Mishra-Prabhakar (1994–2009). Vassiliev looked at the topology of the space $\mathcal{V}_d$ consisting of polynomial knots whose component functions are monic polynomials of degree d with no constant term, whereas Shastri, Mishra and Prabhakar focused on finding concrete polynomial representation of a given knot. In this thesis, we have studied polynomial knots from both the perspectives. We have generalized the space $\mathcal{V}_d$ giving rise to some interesting spaces and explored the topology (path components and the homotopy type) of those spaces. Furthermore, we have studied the homotopy type of the space of all polynomial knots with respect to some natural topology on it. On the other side, we have focused on the polynomial representations of the knots up to six crossings. The knots 0_1 , 3_1 , 4_1 and 5_1 were known to have representations in their minimal degree. We have found the polynomial representations of the knots 5_2 , 6_1 , 6_2 , 6_3 , $3_1\#3_1$ and $3_1\#3_1^*$ in degree 7, where the representations of the knots $3_1\#3_1$ and $3_1\#3_1^*$ are in their minimal degree. We have shown that it is almost impossible to represent the knots 5_2 , 6_1 , 6_2 and 6_3 in degree less than 7.

Chapter 1

Introduction

A polynomial knot is a smooth embedding $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ which is given by $t \mapsto (f(t), g(t), h(t))$, where f , g and h are polynomials with real coefficients. For example, the map $t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$ is a polynomial knot representing the trefoil knot (see [4]). A *Mathematica* plot of this polynomial knot is given in Figure 1.1.

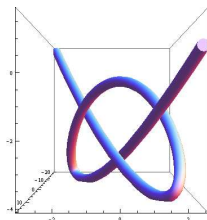


Figure 1.1: A polynomial knot

Classically, knots are embeddings of S^1 in S^3 . However, polynomial knots are long knots, i.e. their ends are open. A polynomial knot can be extended uniquely to an embedding of S^1 in S^3 , which gives rise to a knot in the classical sense. Polynomial knots were mainly studied by the following two groups of mathematicians:

- (1) Vassiliev (1990–1996), Durfee (2006) and O’shea (2006).
- (2) Shastri (1992), Mishra (1994–2009) and Prabhakar (2006–2009).

These groups had different motivations. The first group was motivated to study the topology of the space of knots, while the second group was motivated to settle Abhyankar's conjecture (see Conjecture 1.1) regarding the non-rectifiable embeddings of $\mathbb{C}$ in $\mathbb{C}^3$.

To study the cohomology of the space of knots, Vassiliev [40, 41] introduced the space $\mathcal{V}_d$ for $d \geq 3$. The space $\mathcal{V}_d$ is the interior of the set of all smooth embeddings in the space $\mathcal{W}_d$ of all maps of the type

$$t \mapsto \left(t^d + a_{d-1}t^{d-1} + \cdots + a_1t, t^d + b_{d-1}t^{d-1} + \cdots + b_1t, t^d + c_{d-1}t^{d-1} + \cdots + c_1t \right).$$

The space $\mathcal{W}_d$ inherits the natural topology from the usual topology of $\mathbb{R}^{3d-3}$. It was noted that the space $\mathcal{V}_3$ is contractible and the space $\mathcal{V}_4$ is homology equivalent to S^1 (see [41]). To compute the cohomology of the space of knots, Vassiliev considered the space of all smooth immersions of S^1 in S^3 . In this space, he took the discriminant, i.e. the complement of the space of knots in the space of smooth immersions. Thereafter, he constructed a resolution of the discriminant via the finite dimensional spaces $\mathcal{W}_d$. This resolution has a natural filtration which gives a spectral sequence that can be used to obtain cohomology classes of the space of knots (see [40]). It is not known whether these classes generate the whole cohomology of the space of knots. However, these cohomology classes can be used to construct a set of knot invariants. These invariants are called the Vassiliev invariants (see [42]). It is conjectured that the Vassiliev invariants form a complete set of knot invariants.

In [1], Durfee and O'Shea generalized the space $\mathcal{V}_d$, for $d \geq 3$, to the space $\mathcal{K}_d$ which is the space of all polynomial knots of the type

$$t \mapsto \left(a_0 + a_1t + \cdots + a_dt^d, b_0 + b_1t + \cdots + b_dt^d, c_0 + c_1t + \cdots + c_dt^d \right).$$

They proved that if two polynomial knots are path equivalent in $\mathcal{K}_d$, then

they are topologically equivalent, i.e. their extensions as embeddings of S^1 in S^3 are ambient isotopic. The same is true for the space $\mathcal{V}_d$, since it is a subspace of the space $\mathcal{K}_d$. However, it is felt that the converse may not be true. There is no known example of polynomial knots in $\mathcal{K}_d$ (or in $\mathcal{V}_d$) which are topologically equivalent but not path equivalent in it.

A polynomial knot in $\mathcal{K}_d$, for $d \geq 3$, can be transformed to a polynomial knot $t \mapsto (f(t), g(t), h(t))$ such that $\deg(f) \leq d - 2$, $\deg(g) \leq d - 1$ and $\deg(h) \leq d$. This can be done by composing the knot with a suitable orientation preserving linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. The new knot obtained upon such transformation is topologically equivalent to the original one. Let $\mathcal{O}_d$ denote the space of all polynomial knots $t \mapsto (f(t), g(t), h(t))$ with $\deg(f) \leq d - 2$, $\deg(g) \leq d - 1$ and $\deg(h) \leq d$. Note that the topology of the space $\mathcal{O}_d$ inherits from the usual topology of $\mathbb{R}^{3d}$. Further, composing a polynomial knot in $\mathcal{O}_d$ with a suitable orientation preserving polynomial automorphism $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, one can get a polynomial knot $t \mapsto (u(t), v(t), w(t))$ such that $\deg(u) < \deg(v) < \deg(w) \leq d$. Let $\mathcal{P}_d$ denote the space of all polynomial knots $t \mapsto (u(t), v(t), w(t))$ with the condition $\deg(u) < \deg(v) < \deg(w) \leq d$ on the component polynomials. Suppose $t \mapsto (u(t), v(t), w(t))$ be a polynomial knot in $\mathcal{P}_d$, then there exists an $\epsilon > 0$ such that the polynomial knot $t \mapsto (\epsilon t^{d-2} + u(t), \epsilon t^{d-1} + v(t), \epsilon t^d + w(t))$ is topologically equivalent to the original one. The resulting polynomial knot obtained in this way has degree sequence $(d - 2, d - 1, d)$ (see Definition 2.2.6). Let $\mathcal{Q}_d$ denote the space of all polynomial knots $t \mapsto (x(t), y(t), z(t))$ with $\deg(x) = d - 2$, $\deg(y) = d - 1$ and $\deg(z) = d$.

We have pointed out that the path equivalence in the spaces $\mathcal{K}_d$ and $\mathcal{V}_d$ implies the topological equivalence. This might make one believe that this is the case for any space of polynomial knots. We observe that it is not true in general. For example, two polynomial knots in $\mathcal{O}_d$ (or in $\mathcal{P}_d$) are

path equivalent even though they are not topologically equivalent. Using the theory of real semialgebraic sets we ensure that the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ have only finitely many path components. In fact, we prove that the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$ are path connected. However, the space $\mathcal{Q}_d$ behaves in a similar manner as the spaces $\mathcal{K}_d$ and $\mathcal{V}_d$ do. We rigorously show that if two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are topologically equivalent. This means that two topologically different knots cannot belong to the same path component of the space $\mathcal{Q}_d$. The converse of this is not true. Unlike the spaces $\mathcal{K}_d$ and $\mathcal{V}_d$, here we have examples of topologically equivalent knots which lie in different path components of the space $\mathcal{Q}_d$. In this thesis, we study the topology of the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$. We discuss the homotopy type of these spaces (or of the path components of the spaces). Furthermore, we study the space of all polynomial knots with some natural topology on it and discuss its homotopy type. We also prove that every polynomial knot is isotopic to some linear knot (an unknot) by a smooth isotopy through polynomial knots.

We now discuss the polynomial knots from the perspective of the second group. As it is stated earlier, this group was motivated by the conjecture of Abhyankar, which states as follows:

Conjecture 1.1 (Abhyankar [39]). *There exist polynomial embeddings of $\mathbb{C}$ in $\mathbb{C}^3$ which are non-rectifiable.*

A rectifiable embedding of $\mathbb{C}$ in $\mathbb{C}^3$ is a polynomial embedding which is equivalent to the standard embedding $z \mapsto (0, 0, z)$ by a polynomial automorphism of $\mathbb{C}^3$. To find an example of a non-rectifiable embedding, Shastri [4] started to give examples of polynomial embeddings of $\mathbb{R}$ in $\mathbb{R}^3$ which represent the nontrivial knots and can be extended to the embeddings of $\mathbb{C}$ in $\mathbb{C}^3$. However, it could not be proved that the embeddings he obtained are non-rectifiable or not. He found concrete polynomial representations of

the trefoil knot and the figure-eight knot. In fact, he proved that *every long knot has a polynomial representation and the polynomial embedding is in fact an embedding of $\mathbb{C}$ in $\mathbb{C}^3$* . Later, Mishra and Ranjan [5, 32, 33] gave polynomial representations of the torus knots, strongly invertible knots and strongly negative amphicheiral knots. Furthermore, Mishra and Prabhakar [19–21, 31] found minimal degree sequence for 2-bridge knots and some types of torus knots. They also gave polynomial representations of knots up to eight crossings (see [22]). Wright [24] gave an algorithmic method to find a polynomial representation of any knot.

These constructions suggest that one should look for a polynomial representation that involves the polynomials of least possible degree. In this connection the notion of *the degree sequence, minimal degree sequence and minimal polynomial degree* have been introduced (see [9, 19–21, 31, 34]). For instance, the representations of the knots 3_1 and 4_1 given by Shastri [4] and McFeron [9] are in their minimal degrees 5 and 6 respectively. Also, Auerbach [28] gave a polynomial representation of the knot 5_1 in degree 7, which is its minimal polynomial degree. Finding the minimal polynomial degree for a given knot-type is difficult in general. To make a claim one can make use of some numerical knot invariants such as *the crossing number, bridge index and superbridge index*. For example, we can conclude that the knot 5_1 has no polynomial representation in degree less than 7, since its superbridge index is 4 (see [1, 26]). In [1], Durfee and O'Shea asked that: is there any five crossing knot in degree 6? We have partially answered this question. The only possible five crossing knot which could be represented in degree 6 is the knot 5_2 . In this thesis, we show that it is almost impossible to represent the knot 5_2 in degree 6. The knots 5_2 , 6_1 , 6_2 and 6_3 were known to have representations in degree much higher than 7 (see [3, 17, 22]). We have found polynomial representations of these knots together with the representations

of the knots $3_1 \# 3_1$ and $3_1 \# 3_1^*$, all in degree 7. In this thesis, we present all these polynomial representations along with their 3D plots obtained using *Mathematica*. Finally, we note that all the knots up to six crossings (except the knots 5_2 , 6_1 , 6_2 and 6_3) have representations in their minimal degree. Regarding the knots 5_2 , 6_1 , 6_2 and 6_3 , we conjecture that they cannot be represented in degree less than 7.

The thesis is organized as follows: Chapter 2 includes the basic terminologies and known results which are required in this thesis. This chapter is divided into four sections. In Section 2.1, we define a knot in the classical sense as well as a long knot which is an embedding of $\mathbb{R}$ in $\mathbb{R}^3$. We also define the equivalence of two knots/two long knots. In Section 2.2, we give the definition of a polynomial knot and discuss some important results regarding the polynomial knots. In Section 2.3, we introduce real semialgebraic sets and the theorem of *Tarski-Seidenberg* which will be required in Chapter 3. In Section 2.4, we discuss the *Thom's first isotopy lemma* which will be used to prove that the path equivalence in $\mathcal{Q}_d$ implies the topological equivalence.

We divide Chapter 3 into two sections. In Section 3.1, we introduce some interesting spaces of polynomial knots of degree at most d , for $d \geq 3$. We denote these spaces by $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$. In Section 3.2, we check some elementary facts related to these spaces. Using the theory of real semialgebraic sets, we prove that the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ are homeomorphic to some semialgebraic subsets of $\mathbb{R}^{3d}$ and hence they have only finitely many path components (see Theorem 3.2.11, Corollary 3.2.12 and Corollary 3.2.13).

In Chapter 4, we discuss the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ in detail. This chapter is divided into five sections. In Section 4.1, we prove the following theorem:

Theorem 4.1.1. *The space $\mathcal{O}_d$, for $d \geq 3$, is homotopy equivalent to S^2 .*

This theorem also proves that the space $\mathcal{O}_d$, for $d \geq 3$, is path connected.

Thus, any two polynomial knots in $\mathcal{O}_d$ are path equivalent (that is, they belong to the same path component of the space $\mathcal{O}_d$) even though they are not topologically equivalent.

In Section 4.2, we discuss the homotopy type of the space of all polynomial knots. Note that the set $\mathcal{P}$ of all polynomial knots can be written as a union of the spaces $\mathcal{O}_d$ for $d \geq 3$. Thus, it has inductive limit topology which comes from its stratification as a union of the spaces $\mathcal{O}_d$ for $d \geq 3$. In this section, we prove the following theorem:

Theorem 4.2.1. *The space $\mathcal{P}$ is homotopy equivalent to S^2 .*

In [35], Mishra proved that *if two polynomial knots are topologically equivalent then there exists a polynomial isotopy which connects them*. We show that the converse of this result is false. In fact, we prove the following:

Theorem 4.2.4. *Every polynomial knot is isotopic to some linear knot (an unknot) by a smooth isotopy through polynomial knots.*

The isotopy in Theorem 4.2.4 rolls a given polynomial knot to the infinity so that the knot gets unraveled up to become an unknot. In general, the same thing happens for any long (non-compact) knot. More precisely, every long knot as a smooth embedding of $\mathbb{R}$ in $\mathbb{R}^3$ is isotopic to some unknot by a smooth isotopy through long knots. However, this does not happen if the knots are compact (that is, the smooth embeddings of S^1 in S^3). Otherwise, by the isotopy extension theorem (see [23]), every knot will be ambient isotopic to some unknot, which is a contradiction.

Corollary 4.2.5. *Every polynomial knot can be connected to a linear knot by a smooth path in the space of all polynomial knots.*

In Section 4.3, we prove the following results related to $\mathcal{P}_d$, for $d \geq 3$.

Proposition 4.3.1. *Every polynomial knot in $\mathcal{P}_d$ can be connected to a polynomial knot in $\mathcal{Q}_d$ by a path in $\mathcal{P}_d$.*

Theorem 4.3.4. *The space $\mathcal{P}_d$, for $d \geq 3$, is path connected.*

Using Theorem 4.3.4, one can say that any two polynomial knots in $\mathcal{P}_d$ are path equivalent even though they are not topologically equivalent.

In Section 4.4, we discuss the path equivalence in $\mathcal{Q}_d$ for $d \geq 3$. Two polynomial knots are said to be path equivalent in $\mathcal{Q}_d$ if they belong to its same path component. Similarly, the path equivalence can be defined for the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$. Since the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$ are path connected, any two polynomial knots in $\mathcal{O}_d$ (or, in $\mathcal{P}_d$) are path equivalent in it. Using some techniques from differential topology and the *Thom's first isotopy lemma*, we prove the following:

Theorem 4.4.4. *If two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are topologically equivalent.*

The converse of Theorem 4.4.4 is false. This fact will be clear from the next proposition.

Proposition 4.4.5. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $t \mapsto (f(t), g(t), h(t))$ be a polynomial knot in $\mathcal{Q}_d$, and let ψ be its mirror image which is given by $t \mapsto (f(t), g(t), -h(t))$. Then the polynomial knots ϕ and ψ do not belong to the same path component of the space $\mathcal{Q}_d$.*

Corollary 4.4.6. *Let $t \mapsto (f(t), g(t), h(t))$ be a polynomial knot in $\mathcal{Q}_d$. Then there are eight distinct path components in $\mathcal{Q}_d$ each of which contains exactly one of the knot $t \mapsto ((e_1 f(t), e_2 g(t), e_3 h(t))$ for $(e_1, e_2, e_3) \in \{-1, 1\}^3$.*

Using this corollary, we can say that there exist polynomial knots which are topologically equivalent but not path equivalent in $\mathcal{Q}_d$ for $d \geq 3$.

Corollary 4.4.7. *The number of path components of the space $\mathcal{Q}_d$ are in multiples of eight.*

Corollary 4.4.8. *If there are n distinct knot-types (up to mirror images) which can be represented in $\mathcal{Q}_d$, then it has at least $8n$ path components.*

In Chapter 5, we discuss polynomial representations of knots in their minimal degree. This chapter is divided into three sections. In Section 5.1, we list all the knots which can be represented in degrees 5, 6 and 7. In this section, we prove the following propositions:

Proposition 5.1.1. *The unknot is the only knot that can be represented by a polynomial knot in degree $d \leq 4$.*

Proposition 5.1.2. *The unknot, the trefoil knot and its mirror image are the only knots that can be represented in degree 5.*

Note that if we have a polynomial knot in degree 6, then its crossing number must be less than or equal to 6 (by Proposition 2.2.4). We have a polynomial representation of the knot 4_1 in degree 6 (see [9]). Also, since the knots 0_1 and 3_1 have polynomial representations in degree less than 6 (see [4]), they must have polynomial representations in degree 6 as well. However, the knots 5_1 , $3_1 \# 3_1$ and $3_1 \# 3_1^*$ cannot be represented in degree 6, since their superbridge index is 4 (see Proposition 2.2.4). Regarding the knot 5_2 , we have the following:

Theorem 5.1.4. *For a generic choice of a regular projection $t \mapsto (f(t), g(t))$ of the knot 5_2 with $\deg(f) = 4$ and $\deg(g) = 5$, there does not exist any polynomial h of degree 6 such that a polynomial map $t \mapsto (f(t), g(t), h(t))$ represents the knot 5_2 .*

This theorem says that it is almost impossible to represent the knot 5_2

in degree 6. The same is true for the knots 6_1 , 6_2 and 6_3 . Therefore, we conjecture the following:

Conjecture 5.1. *Each of the knots 5_2 , 6_1 , 6_2 and 6_3 cannot be represented by a polynomial knot in degree 6.*

All the knots up to six crossings (including the eight crossing knot 8_{19}) have polynomial representations in degree 7 (see Section 5.2). Besides these knots, no knot is known to have a polynomial representation in degree 7. Other possible knots in degree 7 are the knots up to ten crossings having superbridge index at most 4.

In Section 5.2, we prove the following theorem:

Theorem 5.2.1. *Let $t \mapsto (f(t), g(t))$ be a regular projection of a long knot $\kappa : \mathbb{R} \rightarrow \mathbb{R}^3$ having n double points, where f and g are real polynomials of degrees n and $n + 1$ respectively. Suppose the crossing data of κ is such that there are r changes from over/under crossings to under/over crossings as we move along the knot. Then there exists a polynomial h with degree $d \leq \min\{n + 2, r\}$ such that the polynomial map $t \mapsto (f(t), g(t), h(t))$ is an embedding which is ambient isotopic to κ .*

Furthermore, in this section, we include the polynomial representations of the knots 3_1 , 4_1 , 5_1 and 8_{19} which are given by Shastri [4], McFeron [9], Auerbach [28] and Mishra-Prabhakar [22] respectively. The representations of the knots 3_1 and 4_1 are respectively in degrees 5 and 6, which are the minimal degrees for these knots. The knots 5_1 and 8_{19} are represented in degree 7, which is their minimal degree (since the superbridge index of these polynomial knots is 4). We have found polynomial representations of the knots 5_2 , 6_1 , 6_2 , 6_3 , $3_1 \# 3_1$ and $3_1 \# 3_1^*$ in degree 7, which will be included in this section. Among these representations, the representations of the knots $3_1 \# 3_1$ and $3_1 \# 3_1^*$ are in their minimal degree.

In Section 5.3, we discuss about the path components in $\mathcal{Q}_d$ for $d \leq 7$. It is easy to see that the space $\mathcal{Q}_3$ has exactly eight path components and all of them are contractible (see Proposition 5.3.1). Also, in this section, we prove the following theorems:

Theorem 5.3.2. *The space $\mathcal{Q}_4$ has exactly eight path components.*

Theorem 5.3.8. *The path components of the space $\mathcal{Q}_4$ are contractible.*

Proposition 5.3.10. *The space $\mathcal{Q}_5$ has exactly eight path components corresponding to the trefoil knot and its mirror image. Moreover, all these path components are contractible.*

Furthermore, we estimate some lower bound on the number of path components of the spaces $\mathcal{Q}_5$, $\mathcal{Q}_6$ and $\mathcal{Q}_7$. Regarding this, we observed that they have at least 16, 24 and 88 path components respectively.

In Chapter 6, we summarize the thesis and discuss some problems and observations that we encountered during the work.

Chapter 2

Preliminaries

This chapter contains some basic definitions and known results that serve as a prerequisite material for this thesis. In Section 2.1, we define a knot in the classical sense, a long knot and the equivalence of knots. In Section 2.2, we give the definition of a polynomial knot and discuss some important results related to the polynomial knots. In Section 2.3, we briefly introduce real semialgebraic sets and the theorem of Tarski-Seidenberg which is utilized to prove that the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$, introduced in Chapter 3, have only finitely many path components. In Section 2.4, we discuss some differential topology and in particular Thom's first isotopy lemma to be used later in Chapter 4 to prove an important theorem in this thesis.

2.1 Knots and their equivalences

Definition 2.1.1. *A knot is a continuous embedding of S^1 in $\mathbb{R}^3$.*

Remark 2.1. *The Euclidean space $\mathbb{R}^3$ can be embedded in S^3 , so we may think of knots as embeddings of S^1 in S^3 .*

Note that any two knots are homeomorphic to each other. In fact, by

the definition itself, all the knots are homeomorphic to S^1 . It is interesting to know if two knots are embedded in the same way or not. That is, can we transform one to the other by a continuous deformation of the space? A continuous deformation of the space is called a homeotopy (an ambient isotopy) and the knots which can be transformed from one to the other by such deformation are called ambient isotopic.

Definition 2.1.2. *A homeotopy (ambient isotopy) of S^3 is a continuous map $H : I \times S^3 \rightarrow S^3$ such that:*

- (1) *the map $H_0 = H(0, \cdot)$ is the identity map of S^3 , and*
- (2) *the map $H_s = H(s, \cdot)$ is a self-homeomorphism of S^3 for all $s \in I$.*

Definition 2.1.3. *Two knots $\phi : S^1 \rightarrow S^3$ and $\psi : S^1 \rightarrow S^3$ are said to be ambient isotopic if there is a homeotopy $H : I \times S^3 \rightarrow S^3$ of the ambient space S^3 such that $\psi = H_1 \circ \phi$.*

Remark 2.2. *Two knots being ambient isotopic is an equivalence relation; therefore, it is often said that the knots are equivalent if they are ambient isotopic. An equivalence class under this equivalence is called as a knot-type or an ambient isotopy class. This equivalence class is simply called as a knot if there is no confusion between a knot as an embedding and a knot as an ambient isotopy class.*

At this point, we note that there exist knots which have some pathological behavior as in Figure 2.1. Such knots are called wild knots (see [30]). Knots

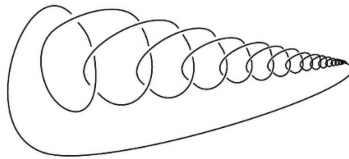


Figure 2.1: A wild knot

which are not wild are called tame knots. More precise definition is given below.

Definition 2.1.4. *A tame knot is a knot $\kappa : S^1 \rightarrow S^3$ which can be extended to a continuous embedding $\bar{\kappa} : S^1 \times B^2 \rightarrow S^3$, where B^2 is the open unit disc in $\mathbb{R}^2$ centered at origin.*

Remark 2.3. *A knot is tame if and only if it is ambient isotopic to a polygonal knot, a knot which is a union of a finite number of straight line segments.*

It is known that every piecewise C^1 embedding of S^1 in S^3 is a tame knot (see [1, 30]). In particular, all smooth embeddings (that is, C^∞ embeddings) of S^1 in S^3 are tame knots. In this thesis, we will only discuss the tame knots. Thus, a word ‘*knot*’, would always mean a ‘*tame knot*’.

Definition 2.1.5. *A long knot is a proper smooth embedding $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ such that the map $t \mapsto \|\phi(t)\|$ is strictly monotone outside some closed interval of the real line.*

Definition 2.1.6. *Two long knots $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ are said to be isotopic by an isotopy through long knots if there is a continuous map $F : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ such that*

- (1) *the map $F_s = F(s, \cdot)$ is a long knot for all $s \in I$, and*
- (2) *$F_0 = \phi$ and $F_1 = \psi$.*

Every long knot $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ can be extended to an unique embedding $\tilde{\phi} : S^1 \rightarrow S^3$ by means of the stereographic projection from the north pole of S^3 . Note that the extended knot is a tame knot, since it is piecewise smooth. On the other hand, every ambient isotopy class of tame knots contains a smooth knot $\psi : S^1 \rightarrow S^3$ which fixes the north poles and has a nonzero derivative at the north pole. The restriction $\hat{\psi} : \mathbb{R} \rightarrow \mathbb{R}^3$ of ψ is a long knot.

This shows that there is a bijection between the ambient isotopy classes of tame knots and the ambient isotopy classes of long knots.

Definition 2.1.7. *Two long knots $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ are said to be topologically equivalent if their extensions $\tilde{\phi} : S^1 \rightarrow S^3$ and $\tilde{\psi} : S^1 \rightarrow S^3$ are ambient isotopic.*

2.2 Polynomial knots

Definition 2.2.1. *A map $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ is a polynomial map if its component functions are polynomials with real coefficients.*

Definition 2.2.2. *A polynomial knot is a polynomial map $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ which is a smooth embedding.*

It is easy to note that a polynomial knot is a long knot and its extension as an embedding of S^1 in S^3 is a tame knot.

Definition 2.2.3. *We say that a polynomial knot $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ represents a knot-type $[\kappa]$ (an ambient isotopy class of κ) if its extension $\tilde{\phi} : S^1 \rightarrow S^3$ is ambient isotopic to κ . The knot ϕ is called a polynomial representation of the knot-type $[\kappa]$.*

Theorem 2.2.1 ([4]). *Every knot-type has a polynomial representation.*

Definition 2.2.4. *Two polynomial knots ϕ and ψ are polynomially isotopic if there is an isotopy $F : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ such that*

- (1) *the map $F_s = F(s, \cdot)$ is a polynomial knot for all $s \in I$, and*
- (2) *$F_0 = \phi$ and $F_1 = \psi$.*

Remark 2.4. *If two polynomial knots are polynomially isotopic, then we often say that they are isotopic by an isotopy through polynomial knots.*

Theorem 2.2.2 ([35]). *If two polynomial knots represent the same knot-type, then they are polynomially isotopic.*

Definition 2.2.5. *A polynomial automorphism of Euclidean space $\mathbb{R}^n$ is a bijective map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ whose component functions are real polynomials in n variables and it is such that its inverse is also of the same kind.*

Note that a map $S : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial automorphism if and only if it is a linear polynomial. Also, it is easy to see that an affine transformation (a composition of an invertible linear transformation and a translation) of $\mathbb{R}^n$ is a polynomial automorphism.

Remark 2.5. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial map, and suppose $S : \mathbb{R} \rightarrow \mathbb{R}$ and $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be polynomial automorphisms. Then the map ϕ is a smooth embedding if and only if the composition $T \circ \phi \circ S$ is a smooth embedding.*

Remark 2.6. *Let α and β be real numbers such that $\alpha \neq 0$, and suppose $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial knot. Then the map $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $t \mapsto \phi(\alpha t + \beta)$ is also a polynomial knot.*

Remark 2.7. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial knot and $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an affine transformation. Then the composition $T \circ \phi$ is also a polynomial knot.*

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial knot, and suppose $S : \mathbb{R} \rightarrow \mathbb{R}$ and $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be orientation preserving polynomial automorphisms. Then the polynomial knots ϕ and $T \circ \phi \circ S$ are topologically equivalent. In particular, the same is true if S and T are orientation preserving affine transformations.

If a polynomial knot $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $t \mapsto (f(t), g(t), h(t))$ represents a knot κ , then the knot $t \mapsto (f(t), g(t), -h(t))$ represents a mirror image κ^* of κ . In general, if $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an affine transformation, then the polynomial knot $S \circ \phi$ represents either κ or κ^* depending on whether S is orientation preserving or not.

Definition 2.2.6. A polynomial map $t \mapsto (f(t), g(t), h(t))$ is said to have degree sequence (d_1, d_2, d_3) if $\deg(f) = d_1$, $\deg(g) = d_2$ and $\deg(h) = d_3$.

Definition 2.2.7. The degree of a polynomial map $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ is the maximum of the degrees of its component polynomials.

Proposition 2.2.3 ([1]). Let $\{\alpha_s \mid s \in I\}$ be a family of polynomial knots depending continuously on parameter $s \in I$. If the degree of α_s is independent of s , then there exists $r_0 \gg 0$ such that for any $s \in I$ and any $|t| \geq r_0$, the vector $\alpha'_s(t) \in \mathbb{R}^3$ intersects transversely to the sphere of radius $|\alpha_s(t)|$ about the origin. Moreover, for any $s \in I$, the angle between the vectors $\alpha_s(t)$ and $\alpha'_s(t)$ approaches 0 as $t \rightarrow +\infty$ and it approaches π as $t \rightarrow -\infty$.

Proof. For $s \in I$, let α_s be defined as

$$\alpha_s(t) = \mathbf{a}_d(s)t^d + \mathbf{a}_{d-1}(s)t^{d-1} + \cdots + \mathbf{a}_1(s)t + \mathbf{a}_0(s),$$

where $\mathbf{a}_i$'s are continuous functions from I to $\mathbb{R}^3$. Note that $\mathbf{a}_d(s) \neq 0$ for all $s \in I$, since the degree of α_s is independent of s . For any $s \in I$ and any $|t| \gg 0$, the cosine of the angle $\theta_s(t)$ between the vectors $\alpha_s(t)$ and $\alpha'_s(t)$ is given by

$$\frac{\alpha_s(t)}{\|\alpha_s(t)\|} \cdot \frac{\alpha'_s(t)}{\|\alpha'_s(t)\|},$$

which becomes

$$\pm \frac{\mathbf{a}_d(s) + \frac{1}{t}\mathbf{a}_{d-1}(s) + \cdots + \frac{1}{t^d}\mathbf{a}_0(s)}{\|\mathbf{a}_d(s) + \frac{1}{t}\mathbf{a}_{d-1}(s) + \cdots + \frac{1}{t^d}\mathbf{a}_0(s)\|} \cdot \frac{d\mathbf{a}_d(s) + \frac{d-1}{t}\mathbf{a}_{d-1}(s) + \cdots + \frac{1}{t^{d-1}}\mathbf{a}_1(s)}{\|d\mathbf{a}_d(s) + \frac{d-1}{t}\mathbf{a}_{d-1}(s) + \cdots + \frac{1}{t^{d-1}}\mathbf{a}_1(s)\|} \quad (2.1)$$

accordingly as t is positive or negative. Thus, we get

$$|\cos(\theta_s(t))| \geq \frac{\left| d \|\mathbf{a}_d(s)\|^2 + \frac{1}{t} f_1(s) + \frac{1}{t^2} f_2(s) + \cdots + \frac{1}{t^{2d-1}} f_{2d-1}(s) \right|}{d \|\mathbf{a}_d(s)\|^2 + \frac{1}{|t|} h_1(s) + \frac{1}{|t^2|} h_2(s) + \cdots + \frac{1}{|t^{2d-1}|} h_{2d-1}(s)} \quad (2.2)$$

$$\geq \frac{\left| d \|\mathbf{a}_d(s)\|^2 - \left| \frac{1}{t} f_1(s) + \frac{1}{t^2} f_2(s) + \cdots + \frac{1}{t^{2d-1}} f_{2d-1}(s) \right| \right|}{d \|\mathbf{a}_d(s)\|^2 + \frac{1}{|t|} h_1(s) + \frac{1}{|t^2|} h_2(s) + \cdots + \frac{1}{|t^{2d-1}|} h_{2d-1}(s)} \quad (2.3)$$

for all $|t| \gg 0$ and for all $s \in I$, where f_i 's and h_i 's are given by

$$f_i(s) = \sum_{j,k} N_{ijk} \mathbf{a}_j(s) \cdot \mathbf{a}_k(s) \quad \text{and}$$

$$h_i(s) = \sum_{j,k} M_{ijk} \|\mathbf{a}_j(s)\| \|\mathbf{a}_k(s)\|.$$

Note that N_{ijk} 's and M_{ijk} 's are some nonnegative integers. Since $\mathbf{a}_d$ is continuous and $\mathbf{a}_d(s) \neq 0$ for all $s \in I$, there exist some positive real numbers m and M such that

$$m \leq \|\mathbf{a}_d(s)\| \leq M \quad (2.4)$$

for all $s \in I$. Note that f_i 's and h_i 's are continuous and bounded real valued functions, so for some $r_0 \gg 0$, we have

$$\left| \frac{1}{t} f_1(s) + \frac{1}{t^2} f_2(s) + \cdots + \frac{1}{t^{2d-1}} f_{2d-1}(s) \right| \leq \frac{dm^2}{2} \quad \text{and} \quad (2.5)$$

$$\frac{1}{|t|} h_1(s) + \frac{1}{|t^2|} h_2(s) + \cdots + \frac{1}{|t^{2d-1}|} h_{2d-1}(s) \leq dM^2 \quad (2.6)$$

for all $|t| \geq r_0$ and for all $s \in I$. Using (2.4)–(2.6) in (2.3), we get

$$|\cos(\theta_s(t))| \geq \frac{dm^2 - \frac{dm^2}{2}}{dM^2 + dM^2} \geq \frac{m^2}{4M^2} > 0$$

for all $|t| \geq r_0$ and for all $s \in I$. This proves the first statement of the

proposition. Also, for any $s \in I$, the expression (2.1) approaches

$$\pm \frac{\mathbf{a}_d(s) \cdot d\mathbf{a}_d(s)}{\|\mathbf{a}_d(s)\| \|d\mathbf{a}_d(s)\|} = \pm 1$$

accordingly as $t \rightarrow \pm\infty$. In other words, for any $s \in I$, the angle $\theta_s(t)$ between the vectors $\alpha_s(t)$ and $\alpha'_s(t)$ approaches 0 as $t \rightarrow +\infty$ and it approaches π as $t \rightarrow -\infty$. $\square$

Given a knot-type, there can be many polynomial knots representing it. The degrees of these polynomial knots may be different. The least among these degrees is called the *polynomial degree* of that knot-type. More precise definition is given below.

Definition 2.2.8. *The polynomial degree of a knot-type $[\kappa]$ is defined by $d[\kappa] = \min \{ \deg(\phi) \mid \phi \text{ is a polynomial knot representing the knot-type } [\kappa] \}$, where $\deg(\phi)$ is the degree of a polynomial knot ϕ .*

Example 2.1 ([4]). The polynomial knot $t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$ is a representation of the trefoil knot. It can be easily verified that the trefoil knot has polynomial degree 5.

Note that the polynomial degree of a knot-type is a knot invariant and it is same for the knots which are mirror images of each other. There are some relations between the polynomial degree and other knot invariants: namely, the crossing number, the bridge index [12] and the superbridge index [26].

Definition 2.2.9. *The crossing number of a knot-type $[\kappa]$ is defined by $c[\kappa] = \min \{ c(D) \mid D \text{ is a knot diagram of the knot-type } [\kappa] \}$, where $c(D)$ is the number of crossings in a knot diagram D .*

Definition 2.2.10. *The bridge index $b[\kappa]$ of a knot-type $[\kappa]$ is defined by $b[\kappa] = \min_{\phi \in [\kappa]} \min_{v \in S^2} (\# \text{ of local maxima of } \phi \text{ in the direction of } v)$, where the first minimum is taken over all embeddings $\phi : S^1 \rightarrow S^3$ which are*

ambient isotopic to κ and the second minimum is taken over all possible directions in $\mathbb{R}^3$.

Definition 2.2.11. *The superbridge index of a knot-type $[\kappa]$ is defined by $sb[\kappa] = \min_{\phi \in [\kappa]} \max_{v \in S^2} (\# \text{ of local maxima of } \phi \text{ in the direction of } v)$.*

Example 2.2. The crossing number of the trefoil knot is 3. Its bridge index and superbridge index are 2 and 3 respectively. The crossing number, bridge index and superbridge index of the figure-eight knot are 4, 2 and 3 respectively.

Proposition 2.2.4 ([1]). *Let $\kappa : S^1 \rightarrow S^3$ be a knot, then we have*

$$(1) \ 2c[\kappa] \leq (d[\kappa] - 2)(d[\kappa] - 3),$$

$$(2) \ 2b[\kappa] \leq d[\kappa] - 1 \quad \text{and}$$

$$(3) \ 2sb[\kappa] \leq d[\kappa] + 1.$$

2.3 Real semialgebraic sets

A set of common zeros of a finite number of polynomials is an algebraic set. In algebraic geometry, we study these sets. Over the field of real numbers, the notion of algebraic sets can be extended to a larger class of sets known as semialgebraic sets.

Definition 2.3.1. *A semialgebraic subset of $\mathbb{R}^n$ is a union of finite number of sets of the type*

$$\{x \in \mathbb{R}^n \mid P(x) = 0 \text{ and } Q_1(x) > 0 \text{ and } \dots \text{ and } Q_r(x) > 0\},$$

where $P, Q_1, \dots, Q_r \in \mathbb{R}[x_1, x_2, \dots, x_n]$ and r is a non-negative integer.

For example, the region $A = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0 \text{ and } x^2 + y^2 \leq 25 \text{ and } y \geq x^4 - 5x^2 + 4\}$ of the upper half plane which is bounded by the curves $y = 0$, $x^2 + y^2 = 25$ and $y = x^4 - 5x^2 + 4$ is a semialgebraic subset of $\mathbb{R}^2$. Also, the region $B = \{(x, y, z) \in \mathbb{R}^3 \mid 5z > 9x^2 + 16y^2\}$ bounded by the paraboloid $5z = 9x^2 + 16y^2$ is a semialgebraic subset of $\mathbb{R}^3$.

It can be easily verified that complement of a semialgebraic set is semialgebraic. Also, finite unions and finite intersections of semialgebraic sets are semialgebraic. A detailed study of semialgebraic sets and the following results can be found in [18] and [29].

Theorem 2.3.1 (Tarski-Seidenberg). *Let A be a semialgebraic subset of $\mathbb{R}^{n+1}$ and suppose $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be the projection onto the space of the first n coordinates. Then $\pi(A)$ is a semialgebraic subset of $\mathbb{R}^n$.*

Corollary 2.3.2. *If A is a semialgebraic subset of $\mathbb{R}^{n+k}$, then its image by the projection onto the space of the first n coordinates is a semialgebraic subset of $\mathbb{R}^n$.*

Theorem 2.3.3. *A semialgebraic set has only finitely many connected components and all they are semialgebraic.*

Proposition 2.3.4. *A connected semialgebraic set is path connected.*

2.4 Thom's first isotopy lemma

Let M be a smooth manifold of dimension n and let X be an r -dimensional smooth submanifold of it. Let $x \in X$, then the tangent space TX_x of X at x is a point in the Grassmannian of r -planes in TM_x . The convergence of a sequence $\{TX_{x_i}\}$ of tangent spaces, we mean the convergence in the standard topology of the Grassmannian.

Definition 2.4.1. A pair (X, Y) of smooth submanifolds of M is said to satisfy the first Whitney condition at $y \in Y$ if the following holds: If $\{x_i\}$ be a sequence of points in X which converges to y and it is such that the sequence $\{TX_{x_i}\}$ converges to an r -plane $\tau \subseteq TM_y$, then $TY_y \subseteq \tau$.

Definition 2.4.2. We say that the pair (X, Y) satisfies the first Whitney condition if it satisfies this condition at every point in Y .

Example 2.3 ([13]). Let x, y and z denote the coordinates in $\mathbb{C}^3$. Let X be the set $\{(x, y, z) \in \mathbb{C}^3 \mid x^2y - z^2 = 0\}$ with the y -axis deleted, and suppose Y be the y -axis. Then (X, Y) satisfies the first Whitney condition at all points of Y except the origin.

If $x, y \in \mathbb{R}^n$ and $x \neq y$, then the secant line $(\overline{x, y})$ is a line passing through the origin which is parallel to the line joining x and y .

Definition 2.4.3. A pair (X, Y) of smooth submanifolds of $\mathbb{R}^n$ satisfies the second Whitney condition at $y \in Y$ if the following holds: Let $\{x_i\} \subseteq X$ and $\{y_i\} \subseteq Y$ be sequences which converge to y and they are such that $x_i \neq y_i$ for all $i \in \mathbb{N}$. Suppose the tangent spaces TX_{x_i} converge to an r -plane $\tau \subseteq \mathbb{R}^n$ and the secant lines $(\overline{x_i, y_i})$ converge to a line $l \subseteq \mathbb{R}^n$. Then $l \subseteq \tau$.

Lemma 2.4.1 ([16]). Let (X', Y') be another pair of smooth submanifolds of $\mathbb{R}^n$, and let $y' \in Y'$. Suppose U and U' be neighborhoods of y and y' in $\mathbb{R}^n$ respectively, and let $\phi : U \rightarrow U'$ be a smooth diffeomorphism such that $\phi(U \cap X) = U' \cap X'$, $\phi(U \cap Y) = U' \cap Y'$ and $\phi(y) = y'$. Then the pair (X, Y) satisfies the second Whitney condition at y if and only if the pair (X', Y') satisfies the this condition at y' .

Definition 2.4.4. Let X and Y be smooth submanifolds of an n -dimensional smooth manifold M . Then the pair (X, Y) is said to satisfy the second Whitney condition at $y \in Y$ if there is a smooth coordinate chart $\phi : U \rightarrow \mathbb{R}^n$ of

M at y such that the pair $(\phi(U \cap X), \phi(U \cap Y))$ satisfies the second Whitney condition at $\phi(y)$.

In view of Lemma 2.4.1, note that Definition 2.4.4 is independent of the choice of the coordinate chart at y .

Definition 2.4.5. We say that the pair (X, Y) satisfies the second Whitney condition if it satisfies this condition at every point in Y .

Proposition 2.4.2 ([16]). If a pair (X, Y) satisfies the second Whitney condition at y , then it satisfies the first Whitney condition at that point.

Definition 2.4.6. Let A be a subset of a smooth manifold M . A Whitney pre-stratification of A is a collection $\mathcal{S}$ of pairwise disjoint smooth submanifolds (called strata of A) of M which satisfies the following conditions:

- (1) $A = \bigcup_{X \in \mathcal{S}} X$;
- (2) $\mathcal{S}$ is locally finite;
- (3) $\mathcal{S}$ satisfies the condition of frontier (that is, for each stratum X in $\mathcal{S}$, its frontier $(\bar{X} \setminus X) \cap A$ is a union of strata in $\mathcal{S}$); and
- (4) each pair (X, Y) of strata in $\mathcal{S}$ satisfies the second Whitney condition.

Proposition 2.4.3 (Thom's first isotopy lemma [36]). Let M and P be smooth manifolds, $f : M \rightarrow P$ be a smooth mapping, and A be a closed subset of M which admits a Whitney pre-stratification $\mathcal{S}$. Suppose $f|_A : A \rightarrow P$ is a proper map and its restriction to any stratum X in $\mathcal{S}$ is a submersion. Then $f|_X : X \rightarrow P$ is a locally trivial fibration for each stratum X in $\mathcal{S}$.

A detailed proof of Thom's first isotopy lemma can be found in [16]. Note that if P is contractible CW-complex (in particular, if $P = [0, 1]$), then the locally trivial fibration $f|_X : X \rightarrow P$, for $X \in \mathcal{S}$, becomes a trivial fiber bundle over P .

Chapter 3

Spaces of Polynomial Knots

In this chapter, we study some interesting spaces of polynomial knots. Let $\mathcal{A}_d$, for $d \geq 3$, be the space of all polynomial maps $t \mapsto (f(t), g(t), h(t))$ such that $\deg(f) \leq d-2$, $\deg(g) \leq d-1$ and $\deg(h) \leq d$. In Section 3.1, we introduce three important spaces of polynomial knots which are subspaces of the space $\mathcal{A}_d$. We formally define these spaces in this section and denote them by $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$. These spaces are the main objects of study in this thesis. In Section 3.2, we check some of the elementary properties associated to these spaces. In particular, we prove that the space $\mathcal{Q}_d$ is open and dense in $\mathcal{A}_d$. We show that the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$ are dense in $\mathcal{A}_d$, but they are not open. Later, using the theory of real semialgebraic sets, we prove that the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ are homeomorphic to some semialgebraic subsets of $\mathbb{R}^{3d}$ and hence they have only finitely many path components.

3.1 The spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$

Recall that a polynomial knot is a smooth embedding of $\mathbb{R}$ in $\mathbb{R}^3$ whose component functions are real polynomials. Let us denote the set of all polynomial knots by $\mathcal{P}$, and let $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$, for $d \geq 3$, be defined as follows:

$$\mathcal{O}_d = \{(f, g, h) \in \mathcal{P} \mid \deg(f) \leq d-2, \deg(g) \leq d-1 \text{ and } \deg(h) \leq d\},$$

$$\mathcal{P}_d = \{(f, g, h) \in \mathcal{P} \mid \deg(f) < \deg(g) < \deg(h) \leq d\}, \quad \text{and}$$

$$\mathcal{Q}_d = \{(f, g, h) \in \mathcal{P} \mid \deg(f) = d-2, \deg(g) = d-1 \text{ and } \deg(h) = d\}.$$

It can be easily seen that $\mathcal{Q}_d \subseteq \mathcal{P}_d \subseteq \mathcal{O}_d$ for all $d \geq 3$.

Remark 3.1. *A polynomial knot of degree d (see Definition 2.2.7), for $d \geq 3$, can be transformed to a polynomial knot in $\mathcal{O}_d$ by an orientation preserving linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.*

Remark 3.2. *Let $t \mapsto (f(t), g(t), h(t))$ be a polynomial knot in $\mathcal{O}_d$. For a sufficiently small $\epsilon > 0$, by adding ϵt^{d-2} , ϵt^{d-1} and ϵt^d in the respective components, the degrees of f , g and h can be made to be $d-2$, $d-1$ and d respectively without changing the topological type of the knot.*

Remark 3.3. *A polynomial knot of degree d , for $d \geq 3$, can be transformed to some polynomial knots in $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ without changing the topological type of the knot.*

At this point, we would like to provide some natural topology to the sets $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$. Note that for $d \geq 3$, the set $\mathcal{A}_d$ of all polynomial maps $t \mapsto (f(t), g(t), h(t))$ with $\deg(f) \leq d-2$, $\deg(g) \leq d-1$ and $\deg(h) \leq d$ can be identified with $\mathbb{R}^{3d}$. In fact, we have a bijection $\eta : \mathcal{A}_d \rightarrow \mathbb{R}^{3d}$ given by

$$(f, g, h) \mapsto (a_0, a_1, \dots, a_{d-2}, b_0, b_1, \dots, b_{d-1}, c_0, c_1, \dots, c_d), \quad (3.1)$$

where a_i , b_i and c_i , for $i \in \{0, 1, \dots, d\}$, are coefficients of t^i in the polyno-

mials f , g and h respectively. Let us define a metric ρ on $\mathcal{A}_d$ by

$$\rho(\phi, \psi) = \xi(\eta(\phi), \eta(\psi)), \quad (3.2)$$

where ξ denotes the Euclidean distance in $\mathbb{R}^{3d}$. With this metric, the set $\mathcal{A}_d$ becomes a metric space and hence a topological space. The map η is a homeomorphism between the spaces $\mathcal{A}_d$ and $\mathbb{R}^{3d}$.

Let $\mathcal{B}_d$ be the set of all polynomial maps $t \mapsto (f(t), g(t), h(t))$ with $\deg(f) < \deg(g) < \deg(h) \leq d$, and let $\mathcal{C}_d$ be the set of all polynomial maps $t \mapsto (x(t), y(t), z(t))$ such that $\deg(x) = d-2$, $\deg(y) = d-1$ and $\deg(z) = d$. Note that the sets $\mathcal{B}_d$, $\mathcal{C}_d$, $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ are subsets of the space $\mathcal{A}_d$, hence they can be given the subspace topology. The spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ are the spaces of polynomial knots of degree at most d .

Remark 3.4. Note that $\mathcal{C}_3 = \mathcal{Q}_3$, $\mathcal{C}_d \not\subseteq \mathcal{C}_{d+1}$ and $\mathcal{Q}_d \not\subseteq \mathcal{Q}_{d+1}$ for all $d \geq 3$. Moreover, for the collection $\mathfrak{F} = \{\mathcal{A}_d, \mathcal{B}_d, \mathcal{C}_d, \mathcal{O}_d, \mathcal{P}_d, \mathcal{Q}_d \mid d \geq 3\}$, with respect to the set inclusion as partial order, we have the Hasse diagram as given in the figure below (see Figure 3.1).

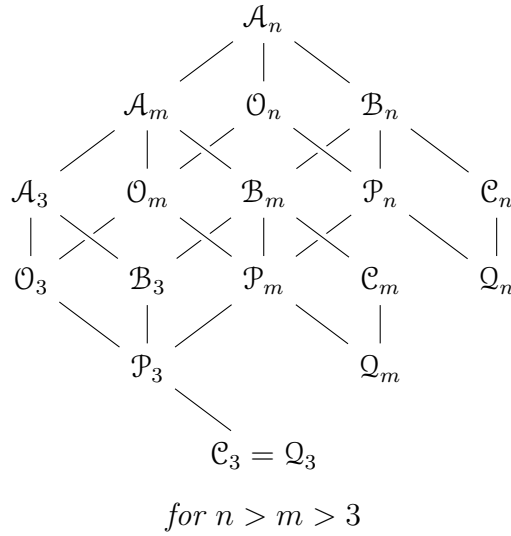


Figure 3.1: Partial order in $\mathfrak{F}$

3.2 Some elementary properties

Proposition 3.2.1. *The space $\mathcal{C}_d$, for $d \geq 3$, is open and dense in $\mathcal{A}_d$.*

Proof. The set $\eta(\mathcal{C}_d)$ can be written as follows:

$$\eta(\mathcal{C}_d) = \left\{ (a_0, a_1, \dots, a_{d-2}, b_0, b_1, \dots, b_{d-1}, c_0, c_1, \dots, c_d) \in \mathbb{R}^{3d} \mid a_{d-2}b_{d-1}c_d \neq 0 \right\},$$

where η is the homeomorphism as defined in (3.1). The set $\eta(\mathcal{C}_d)$ is an open and dense subset of $\eta(\mathcal{A}_d) = \mathbb{R}^{3d}$, and hence $\mathcal{C}_d$ is open and dense in $\mathcal{A}_d$. $\square$

We now prove that the space $\mathcal{Q}_d$ is open in $\mathcal{A}_d$. To prove this, we require some lemmas which are given below.

Lemma 3.2.2. *For any nonzero real numbers a and b , and an even integer $n \geq 2$, we have the following:*

$$(1) \quad 0 < \frac{1}{a^n + a^{n-1}b + \dots + b^n} \leq \max \left\{ \frac{1}{a^n}, \frac{1}{b^n} \right\}.$$

$$(2) \quad 0 < \frac{a^k + a^{k-1}b + \dots + b^k}{a^n + a^{n-1}b + \dots + b^n} \leq \max \left\{ \frac{1}{a^{n-k}}, \frac{1}{b^{n-k}} \right\}, \text{ for an even integer } k \in \{1, 2, \dots, n\}.$$

$$(3) \quad \left| \frac{a^k + a^{k-1}b + \dots + b^k}{a^n + a^{n-1}b + \dots + b^n} \right| < \min \left\{ \frac{1}{|a^{n-k}|}, \frac{1}{|b^{n-k}|} \right\}, \text{ for an odd integer } k \in \{1, 2, \dots, n\}.$$

Proof. Without loss of generality we assume that $|a| \geq |b|$.

(1) Since $|a| \geq |b|$, we have $a + b \geq 0$ if $a > 0$ and $a + b \leq 0$ if $a < 0$. An odd power of a and an even power of b occur in each term of the expression $y = a^{n-1} + a^{n-3}b^2 + \dots + ab^{n-2}$, so we have $y \geq 0$ if $a > 0$ and $y \leq 0$ if $a < 0$. Thus, in either case $(a + b)(a^{n-1} + a^{n-3}b^2 + \dots + ab^{n-2}) = (a + b)y \geq 0$ and hence we have

$$a^n + a^{n-1}b + \dots + b^n = (a + b)(a^{n-1} + a^{n-3}b^2 + \dots + ab^{n-2}) + b^n \geq b^n > 0.$$

In other words, we have $0 < \frac{1}{a^n + a^{n-1}b + \dots + b^n} \leq \frac{1}{b^n} = \max \left\{ \frac{1}{a^n}, \frac{1}{b^n} \right\}$.

(2) Let an even integer $k \in \{1, 2, \dots, n\}$ be given. Note that the inequality is trivially true for $k = n$. So assume that $k < n$. By the first part, we note

that $a^k + a^{k-1}b + \dots + b^k > 0$. Now consider the following expression:

$$x_k = \frac{a^n + a^{n-1}b + \dots + b^n}{a^k + a^{k-1}b + \dots + b^k} \quad (3.3)$$

$$= \frac{a^n + a^{n-1}b + \dots + a^{k+1}b^{n-k-1}}{a^k + a^{k-1}b + \dots + b^k} + b^{n-k} \quad (3.4)$$

$$= \frac{(a+b)(a^{n-1} + a^{n-3}b^2 + \dots + a^{k+1}b^{n-k-2})}{a^k + a^{k-1}b + \dots + b^k} + b^{n-k}. \quad (3.5)$$

An odd power of a and an even power of b occur in each term of the expression $y_k = a^{n-1} + a^{n-3}b^2 + \dots + a^{k+1}b^{n-k-2}$, so we have $y_k \geq 0$ if $a > 0$ and $y_k \leq 0$ if $a < 0$. Recall that $a + b \geq 0$ if $a > 0$ and $a + b \leq 0$ if $a < 0$. Hence in either case, the numerator of the first term in (3.5) is non-negative. Since $a^k + a^{k-1}b + \dots + b^k > 0$, the first term in (3.5) is non-negative. Thus, we have $x_k \geq b^{n-k} > 0$ and hence $0 < \frac{1}{x_k} \leq \frac{1}{b^{n-k}} = \max \left\{ \frac{1}{a^{n-k}}, \frac{1}{b^{n-k}} \right\}$.

(3) Let $k \in \{1, 2, \dots, n\}$ be an odd integer. If $a^k + a^{k-1}b + \dots + b^k = 0$, then the inequality is trivially true. So assume that $a^k + a^{k-1}b + \dots + b^k \neq 0$. Now consider the following expression:

$$x_k = \frac{a^n + a^{n-1}b + \dots + b^n}{a^k + a^{k-1}b + \dots + b^k} \quad (3.6)$$

$$= a^{n-k} + \frac{a^{n-k-1}b^{k+1} + a^{n-k-2}b^{k+2} + \dots + b^n}{a^k + a^{k-1}b + \dots + b^k} \quad (3.7)$$

$$= a^{n-k} + \frac{(a+b)(a^{n-k-2}b^{k+1} + a^{n-k-4}b^{k+3} + \dots + ab^{n-2}) + b^n}{(a+b)(a^{k-1} + a^{k-3}b^2 + \dots + b^{k-1})}. \quad (3.8)$$

Note that an odd power of a and an even power of b occur in each term of the expression $a^{n-k-2}b^{k+1} + a^{n-k-4}b^{k+3} + \dots + ab^{n-2}$, so by a similar argument as in the second part, $(a+b)(a^{n-k-2}b^{k+1} + a^{n-k-4}b^{k+3} + \dots + ab^{n-2})$ is non-negative, and hence the numerator of the second term in (3.8) is positive. Also a and b occur with even powers in each term of the expression $a^{k-1} + a^{k-3}b^2 + \dots + b^{k-1}$, so it is positive. This shows that the sign of the second

term in (3.8) is same as the sign of $a + b$ and hence it is same as the sign of a . This shows that $x_k > a^{n-k} > 0$ if $a > 0$ and $x_k < a^{n-k} < 0$ if $a < 0$ and hence we have $\frac{1}{|x_k|} < \frac{1}{|a^{n-k}|} = \min \left\{ \frac{1}{|a^{n-k}|}, \frac{1}{|b^{n-k}|} \right\}$. $\square$

Lemma 3.2.3. *Let $n \geq 2$ be a fixed even integer, and for $i \in \{0, 1, \dots, n\}$, let $\{\alpha_{ij}\}_{j=1}^\infty \subseteq \mathbb{R}$ be a sequence which converges to α_i . Suppose $\alpha_n \neq 0$, and let $\{s_j\} \subseteq \mathbb{R}$ and $\{t_j\} \subseteq \mathbb{R}$ be sequences such that $s_j \rightarrow \pm\infty$ and $t_j \rightarrow \pm\infty$ as $j \rightarrow \infty$. Then*

$$\lim_{j \rightarrow \infty} \left(\alpha_{0j} + \alpha_{1j}(s_j + t_j) + \dots + \alpha_{nj}(s_j^n + s_j^{n-1}t_j + \dots + t_j^n) \right) = \pm\infty$$

depending on whether α_n is positive or negative.

Proof. Since $s_j \rightarrow \pm\infty$ and $t_j \rightarrow \pm\infty$ as $j \rightarrow \infty$, we can choose $N \in \mathbb{N}$ such that $s_j \neq 0$ and $t_j \neq 0$ for all $j \geq N$. For each $j \geq N$, let

$$z_j = \frac{\alpha_{0j} + \alpha_{1j}(s_j + t_j) + \dots + \alpha_{nj}(s_j^n + s_j^{n-1}t_j + \dots + t_j^n)}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n}. \quad (3.9)$$

The expression for z_j can be rewritten as follows:

$$\begin{aligned} z_j &= \alpha_{0j} \left(\frac{1}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n} \right) + \alpha_{1j} \left(\frac{s_j + t_j}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n} \right) \\ &\quad + \dots + \alpha_{n-1,j} \left(\frac{s_j^{n-1} + s_j^{n-2}t_j + \dots + t_j^{n-1}}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n} \right) + \alpha_{nj}. \end{aligned} \quad (3.10)$$

By Lemma 3.2.2, we have

$$0 \leq \frac{1}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n} \leq \max \left\{ \frac{1}{s_j^n}, \frac{1}{t_j^n} \right\} \quad \text{and} \quad (3.11)$$

$$\left| \frac{s_j^i + s_j^{i-1}t_j + \dots + t_j^i}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n} \right| \leq \max \left\{ \frac{1}{|s_j^{n-i}|}, \frac{1}{|t_j^{n-i}|} \right\}, \quad (3.12)$$

for all $i \in \{0, 1, \dots, n-1\}$ and for all $j \geq N$. Also, for any $i \in \{0, 1, \dots, n-1\}$, note that $\frac{1}{s_j^{n-i}} \rightarrow 0$ and $\frac{1}{t_j^{n-i}} \rightarrow 0$ as $j \rightarrow \infty$, so by (3.11) and (3.12), we have the following:

$$\lim_{j \rightarrow \infty} \frac{1}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n} = 0 \quad \text{and} \quad (3.13)$$

$$\lim_{j \rightarrow \infty} \frac{s_j^i + s_j^{i-1}t_j + \dots + t_j^i}{s_j^n + s_j^{n-1}t_j + \dots + t_j^n} = 0. \quad (3.14)$$

For each $i \in \{0, 1, \dots, n\}$, since $\alpha_{ij} \rightarrow \alpha_i$ as $j \rightarrow \infty$, so by (3.10), (3.13) and (3.14), we have $z_j \rightarrow \alpha_n$ as $j \rightarrow \infty$. Also, by Lemma 3.2.2, $s_j^n + s_j^{n-1}t_j + \dots + t_j^n \geq \min\{s_j^n, t_j^n\}$ for all $j \geq N$, so we have $s_j^n + s_j^{n-1}t_j + \dots + t_j^n \rightarrow \infty$ as $j \rightarrow \infty$. Therefore, by (3.9), we get the required result. $\square$

Lemma 3.2.4. *Let $\gamma(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_k t^k$ be a polynomial in a variable t , and let $\Gamma(s, t) = \alpha_1 + \alpha_2(s + t) + \dots + \alpha_k(s^{k-1} + s^{k-2}t + \dots + t^{k-1})$ be a polynomial in variables s and t . Then $(s_0, t_0) \in \mathbb{R}^2$ is a zero of Γ if and only if either $\gamma(s_0) = \gamma(t_0)$ or $\gamma'(t_0) = 0$ accordingly as $s_0 \neq t_0$ or $s_0 = t_0$.*

Proof. It is easy to check the following:

$$\Gamma(s, t) = \frac{\gamma(s) - \gamma(t)}{s - t} \quad \text{for } s \neq t, \text{ and} \quad (3.15)$$

$$\Gamma(t, t) = \gamma'(t) \quad \text{for } t \in \mathbb{R}. \quad (3.16)$$

Let $(s_0, t_0) \in \mathbb{R}^2$ be a zero of Γ . If $s_0 \neq t_0$, then by (3.15), $\gamma(s_0) = \gamma(t_0)$. If $s_0 = t_0$, then by (3.16), $\gamma'(t_0) = 0$.

If $\gamma(s_0) = \gamma(t_0)$ for $(s_0, t_0) \in \mathbb{R}^2$ with $s_0 \neq t_0$, then by (3.15), the point $(s, t) = (s_0, t_0)$ is a zero of Γ . If $\gamma'(t_0) = 0$ for some $t_0 \in \mathbb{R}$, then by (3.16), the point $(s, t) = (t_0, t_0)$ is a zero of Γ . $\square$

Lemma 3.2.5. *Let $f(t) = a_0 + a_1 t + \dots + a_{d-2} t^{d-2}$, $g(t) = b_0 + b_1 t +$*

$\cdots + b_{d-1}t^{d-1}$ and $h(t) = c_0 + c_1t + \cdots + c_d t^d$ be real polynomials. Then the polynomial map $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $t \mapsto (f(t), g(t), h(t))$ is a smooth embedding if and only if the polynomials

$$F(s, t) = a_1 + a_2(s + t) + \cdots + a_{d-2} \left(s^{d-3} + s^{d-4}t + \cdots + t^{d-3} \right),$$

$$G(s, t) = b_1 + b_2(s + t) + \cdots + b_{d-1} \left(s^{d-2} + s^{d-3}t + \cdots + t^{d-2} \right) \quad \text{and}$$

$$H(s, t) = c_1 + c_2(s + t) + \cdots + c_d \left(s^{d-1} + s^{d-2}t + \cdots + t^{d-1} \right)$$

do not have a common zero.

Proof. Suppose that a point $(s_0, t_0) \in \mathbb{R}^2$ is a common zero of the polynomials F , G and H . If $s_0 \neq t_0$, then by Lemma 3.2.4, $f(s_0) = f(t_0)$, $g(s_0) = g(t_0)$ and $h(s_0) = h(t_0)$, and hence $\phi(s_0) = \phi(t_0)$. If $s_0 = t_0$, then again by Lemma 3.2.4, $f'(t_0) = g'(t_0) = h'(t_0) = 0$; that is, $\phi'(t_0) = 0$. Thus, in either case, the map ϕ is not a smooth embedding.

To prove the converse, assume that ϕ is not a smooth embedding. Then $\phi(s_0) = \phi(t_0)$ for some $s_0 \neq t_0$ or $\phi'(u_0) = 0$ for some $u_0 \in \mathbb{R}$. In other words, $f(s_0) = f(t_0)$, $g(s_0) = g(t_0)$ and $h(s_0) = h(t_0)$ for some $s_0 \neq t_0$, or $f'(u_0) = g'(u_0) = h'(u_0) = 0$ for some $u_0 \in \mathbb{R}$. In any case, the polynomials F , G and H have a common zero by Lemma 3.2.4. $\square$

Theorem 3.2.6. *The space $\mathcal{Q}_d$, for $d \geq 3$, is open in $\mathcal{C}_d$.*

Proof. Since $\mathcal{Q}_3 = \mathcal{C}_3$, the theorem is trivially true for $d = 3$. Let us assume that $d \geq 4$. We show that $\mathcal{C}_d \setminus \mathcal{Q}_d$ is closed in $\mathcal{C}_d$. Suppose $\{\phi_j\}_{j=1}^\infty$ be a sequence of points in $\mathcal{C}_d \setminus \mathcal{Q}_d$ which converges to a point ϕ in $\mathcal{C}_d$. We have to prove that $\phi \in \mathcal{C}_d \setminus \mathcal{Q}_d$. For $j \in \mathbb{N}$, let ϕ_j be given by $t \mapsto (f_j(t), g_j(t), h_j(t))$, and its component polynomials be given by $f_j(t) = a_{0j} + a_{1j}t + \cdots + a_{d-2,j}t^{d-2}$, $g_j(t) = b_{0j} + b_{1j}t + \cdots + b_{d-1,j}t^{d-1}$ and $h_j(t) = c_{0j} + c_{1j}t + \cdots + c_{dj}t^d$. Also, suppose ϕ be given by $t \mapsto$

$(f(t), g(t), h(t))$, and its component polynomials f , g and h be given by $f(t) = a_0 + a_1t + \cdots + a_{d-2}t^{d-2}$, $g(t) = b_0 + b_1t + \cdots + b_{d-1}t^{d-1}$ and $h(t) = c_0 + c_1t + \cdots + c_d t^d$. For $j \in \mathbb{N}$, since ϕ_j is not an embedding, so by Lemma 3.2.5, we can choose $(s_j, t_j) \in \mathbb{R}^2$ such that

$$a_{1j} + a_{2j}(s_j + t_j) + \cdots + a_{d-2,j}(s_j^{d-3} + s_j^{d-4}t_j + \cdots + t_j^{d-3}) = 0, \quad (3.17)$$

$$b_{1j} + b_{2j}(s_j + t_j) + \cdots + b_{d-1,j}(s_j^{d-2} + s_j^{d-3}t_j + \cdots + t_j^{d-2}) = 0, \quad (3.18)$$

$$c_{1j} + c_{2j}(s_j + t_j) + \cdots + c_{dj}(s_j^{d-1} + s_j^{d-2}t_j + \cdots + t_j^{d-1}) = 0. \quad (3.19)$$

We claim that the sequence $\{(s_j, t_j)\}$ is bounded. Suppose on the contrary that it is unbounded, then at least one of the sequence $\{s_j\}$ or $\{t_j\}$ is unbounded. We may assume that the sequence $\{s_j\}$ is unbounded, so it has a subsequence which diverges to $\pm\infty$. Again, without loss of generality, we may assume that the sequence $\{s_j\}$ itself diverges to $\pm\infty$. There are two cases according to which either $\{t_j\}$ is bounded or not.

(1) If the sequence $\{t_j\}$ is bounded: Note that, for each $i \in \{0, 1, \dots, d\}$, we have $c_{ij} \rightarrow c_i$ as $j \rightarrow \infty$ (where $c_d \neq 0$). Also, since the sequence $\{s_j\}$ diverges to $\pm\infty$, we have the following:

$$\lim_{j \rightarrow \infty} (c_{1j} + c_{2j}(s_j + t_j) + \cdots + c_{dj}(s_j^{d-1} + s_j^{d-2}t_j + \cdots + t_j^{d-1})) = \pm\infty.$$

This is a contradiction, since the right hand side in (3.19) is zero for all $j \in \mathbb{N}$.

(2) If the sequence $\{t_j\}$ is unbounded: In this case, the sequence $\{t_j\}$ has a subsequence which diverges to $\pm\infty$. We may assume that the sequence $\{t_j\}$ itself diverges to $\pm\infty$. Note that, for each $i \in \{0, 1, \dots, d\}$, we have $b_{ij} \rightarrow b_i$ and $c_{ij} \rightarrow c_i$ as $j \rightarrow \infty$ (where $b_{d-1} \neq 0$ and $c_d \neq 0$). Also, the

sequence $\{s_j\}$ diverges to $\pm\infty$, so by Lemma 3.2.3, we have

$$\lim_{j \rightarrow \infty} \left(b_{1j} + b_{2j}(s_j + t_j) + \cdots + b_{j,d-1} \left(s_j^{d-2} + s_j^{d-3}t_j + \cdots + t_j^{d-2} \right) \right) = \pm\infty \text{ or}$$

$$\lim_{j \rightarrow \infty} \left(c_{1j} + c_{2j}(s_j + t_j) + \cdots + c_{dj} \left(s_j^{d-1} + s_j^{d-2}t_j + \cdots + t_j^{d-1} \right) \right) = \pm\infty.$$

This is again a contradiction, since the right hand sides in (3.18) and (3.19) are zero for all $j \in \mathbb{N}$.

The cases (1) and (2) both together show that the sequence $\{(s_j, t_j)\}_{j=1}^\infty$ is bounded and hence it has a subsequence which is convergent. We may assume that the sequence $\{(s_j, t_j)\}_{j=1}^\infty$ itself is convergent, say converges to a point $(s_0, t_0) \in \mathbb{R}^2$. Also, since the sequence

$$\{(a_{0j}, a_{1j}, \dots, a_{d-2,j}, b_{0j}, b_{1j}, \dots, b_{d-1,j}, c_{0j}, c_{1j}, \dots, c_{dj})\}_{j=1}^\infty$$

converges to $(a_0, a_1, \dots, a_{d-2}, b_0, b_1, \dots, b_{d-1}, c_0, c_1, \dots, c_d)$, the left hand sides in (3.17), (3.18) and (3.19) converges respectively to

$$a_1 + a_2(s_0 + t_0) + \cdots + a_{d-2} \left(s_0^{d-3} + s_0^{d-4}t_0 + \cdots + t_0^{d-3} \right), \quad (3.20)$$

$$b_1 + b_2(s_0 + t_0) + \cdots + b_{d-1} \left(s_0^{d-2} + s_0^{d-3}t_0 + \cdots + t_0^{d-2} \right) \quad \text{and} \quad (3.21)$$

$$c_1 + c_2(s_0 + t_0) + \cdots + c_d \left(s_0^{d-1} + s_0^{d-2}t_0 + \cdots + t_0^{d-1} \right). \quad (3.22)$$

Note that the right hand sides in (3.17), (3.18) and (3.19) are zero, and hence so are the expressions (3.20), (3.21) and (3.22). Therefore, by Lemma 3.2.5, ϕ is not an embedding, and hence it is an element of the space $\mathcal{C}_d \setminus \mathcal{Q}_d$. $\square$

Now using Proposition 3.2.1, we have:

Corollary 3.2.7. *The space $\mathcal{Q}_d$, for $d \geq 3$, is open in $\mathcal{A}_d$.*

The examples given below witness that the space $\mathcal{O}_d$, for $d \geq 3$, is not

open in $\mathcal{A}_d$, and the space $\mathcal{P}_d$ is not open in both the spaces $\mathcal{A}_d$ and $\mathcal{B}_d$.

Example 3.1. Let ϕ be an element of $\mathcal{O}_d$ given by $t \mapsto (1, 1, t)$. For any $\epsilon > 0$, an open ball $V(\phi, \epsilon) = \{\tau \in \mathcal{A}_d \mid \rho(\phi, \tau) < \epsilon\}$ (where ρ is the metric as defined in (3.2)) contains an element $\phi_\epsilon \in \mathcal{A}_d$ given by

$$t \mapsto \left(1, 1, t - (\epsilon/2)t^2\right).$$

This map is not a smooth embedding, since the derivative of ϕ_ϵ at $t = 1/\epsilon$ is zero. This shows that ϕ is not an interior point of $\mathcal{O}_d$.

Example 3.2. We have an element $\psi \in \mathcal{P}_d$ given by $t \mapsto (1, t, t^2)$. For any $\epsilon > 0$, an open ball $V(\psi, \epsilon)$ contains an element $\psi_\epsilon \in \mathcal{B}_d$ given by

$$t \mapsto \left(1, t - (\epsilon/2)t^{d-1}, t^2 - (\epsilon/2)t^d\right).$$

Note that the map ψ_ϵ has same value at the points $t = 0$ and $t = (2/\epsilon)^{\frac{1}{d-2}}$, so it is not an embedding. This shows that ψ is not an interior point of $\mathcal{P}_d$ with respect to both the spaces $\mathcal{A}_d$ and $\mathcal{B}_d$.

Theorem 3.2.8. *The space $\mathcal{Q}_d$, for $d \geq 3$, is dense in $\mathcal{C}_d$.*

Proof. Note that $\mathcal{Q}_3 = \mathcal{C}_3$, so the theorem is trivially true for $d = 3$. Let us assume that $d \geq 4$. In order to prove that $\mathcal{Q}_d$ is dense in $\mathcal{C}_d$, we have to show that for any $\phi \in \mathcal{C}_d$ and any $\epsilon > 0$, there is a polynomial knot $\psi \in \mathcal{Q}_d$ which belongs to the ϵ -neighborhood $V(\phi, \epsilon)$ of ϕ . Suppose an arbitrary $\phi \in \mathcal{C}_d$ and an $\epsilon > 0$ be given. Let ϕ be given by $t \mapsto (f(t), g(t), h(t))$, and let

$$m_1 = \min \{ |h'(t)| : t \in \mathbb{R}, g'(t) = 0 \text{ and } h'(t) \neq 0 \}.$$

Choose a positive real number r such that $r < \min\{m_1, \epsilon/2\}$, and define a curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ by $\alpha(t) = (g(t), \hat{h}(t))$, where $\hat{h}(t) = h(t) + \frac{r}{2}t$. Let g and

$\hat{h}$ be given by $g(t) = b_0 + b_1t + \cdots + b_{d-1}t^{d-1}$ and $\hat{h}(t) = c_0 + c_1t + \cdots + c_d t^d$. Note that $\alpha'(t) \neq 0$ for all $t \in \mathbb{R}$, so by Lemma 3.2.4, the self-intersections of the curve α are exactly the common zeros of the following polynomials:

$$G(s, t) = b_1 + b_2(s + t) + \cdots + b_{d-1} \left(s^{d-2} + s^{d-3}t + \cdots + t^{d-2} \right) \quad \text{and}$$

$$H(s, t) = c_1 + c_2(s + t) + \cdots + c_d \left(s^{d-1} + s^{d-2}t + \cdots + t^{d-1} \right).$$

We claim that the polynomials G and H do not have a non-constant common factor. Suppose on the contrary that they have a non-constant common factor, say Ψ , and let ζ be its leading term. Then ζ is a non-constant common factor of the polynomials $\mu(s, t) = s^{d-2} + s^{d-3}t + \cdots + t^{d-2}$ and $\nu(s, t) = s^{d-1} + s^{d-2}t + \cdots + t^{d-1}$. Let μ_1, ν_1 and ζ_1 be polynomials given by $\mu_1(t) = \mu(1, t) = 1 + t + \cdots + t^{d-2}$, $\nu_1(t) = \nu(1, t) = 1 + t + \cdots + t^{d-1}$ and $\zeta_1(t) = \zeta(1, t)$. It is obvious that the polynomials ζ and ζ_1 have same degree. Thus, ζ_1 must be a non-constant polynomial. Note that $t^{d-1} - 1 = \mu_1(t)(t - 1)$ and $t^d - 1 = \nu_1(t)(t - 1)$. Since ζ_1 is a common factor of the polynomials μ_1 and ν_1 , every root of ζ_1 is a complex $(d - 1)^{\text{th}}$ as well as d^{th} root of unity other than 1. This is a contradiction, since any complex $(d - 1)^{\text{th}}$ root of unity other than 1 cannot be equal to a complex d^{th} root of unity. This proves the claim.

By Bézout's theorem, the polynomials G and H have only finitely many common zeros, and hence the curve α has only finitely many self-intersections. Let $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$ be the pairs of parameters at which the self-intersections occur in α . Now choose a positive real number u such that $u < \min \{m_2, \epsilon/2\}$, where

$$m_2 = \min \left\{ \left| \frac{f(s_i) - f(t_i)}{s_i - t_i} \right| : i = 1, 2, \dots, k \text{ and } \frac{f(s_i) - f(t_i)}{s_i - t_i} \neq 0 \right\}.$$

Let us define $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}$ by $t \mapsto f(t) + \frac{u}{2}t$. Then it can be easily seen that

$\hat{f}(s_i) \neq \hat{f}(t_i)$ for all $i = 1, 2, \dots, k$. This shows that the map $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ defined by $t \mapsto (\hat{f}(t), g(t), \hat{h}(t))$ has no self-intersections. Note that $\psi'(t) \neq 0$ for all $t \in \mathbb{R}$ (since $\alpha'(t) \neq 0$ for $t \in \mathbb{R}$). Also, it is easy to see that ψ has degree sequence $(d-2, d-1, d)$. Thus, it is an element of the space $\mathcal{Q}_d$. We now estimate $\rho(\phi, \psi)$ as follows:

$$\begin{aligned} \rho(\phi, \psi) &\leq \rho(\phi, (f, g, \hat{h})) + \rho((f, g, \hat{h}), \psi) \\ &\leq r/2 + u/2 \\ &\leq \epsilon/4 + \epsilon/4 \\ &\leq \epsilon/2. \end{aligned}$$

Thus, we have $\psi \in V(\phi, \epsilon)$. This proves the theorem. $\square$

Again by Proposition 3.2.1, we have following:

Corollary 3.2.9. *The space $\mathcal{Q}_d$, for $d \geq 3$, is dense in $\mathcal{A}_d$.*

Corollary 3.2.10. *The spaces $\mathcal{O}_d$ and $\mathcal{P}_d$, for $d \geq 3$, are dense in $\mathcal{A}_d$.*

Now, we would like to investigate more on these spaces.

Theorem 3.2.11. *The space $\mathcal{O}_d$, for $d \geq 3$, is homeomorphic to a semialgebraic subset of $\mathbb{R}^{3d}$.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ denote the real tuples $(a_0, a_1, \dots, a_{d-2})$, $(b_0, b_1, \dots, b_{d-1})$ and $(c_0, c_1, \dots, c_d)$ respectively, and let $\phi_{\mathbf{abc}} : \mathbb{R} \rightarrow \mathbb{R}^3$ be given by

$$t \mapsto (a_0 + a_1 t + \dots + a_{d-2} t^{d-2}, b_0 + b_1 t + \dots + b_{d-1} t^{d-1}, c_0 + c_1 t + \dots + c_d t^d).$$

Then we have $\eta(\phi_{\mathbf{abc}}) = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{R}^{3d}$ (where η is the homeomorphism as defined in (3.1)). Thus, we can rewrite the set $\eta(\mathcal{A}_d \setminus \mathcal{O}_d)$

as follows:

$$\eta(\mathcal{A}_d \setminus \mathcal{O}_d) = \left\{ (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{R}^{3d} \mid \phi_{\mathbf{abc}} \text{ is not a smooth embedding} \right\}.$$

For $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{R}^{3d}$, let $F_{\mathbf{a}}$, $G_{\mathbf{b}}$ and $H_{\mathbf{c}}$ be polynomials which are defined by

$$\begin{aligned} F_{\mathbf{a}}(s, t) &= a_1 + a_2(s + t) + \cdots + a_{d-2} \left(s^{d-3} + s^{d-4}t + \cdots + t^{d-3} \right), \\ G_{\mathbf{b}}(s, t) &= b_1 + b_2(s + t) + \cdots + b_{d-1} \left(s^{d-2} + s^{d-3}t + \cdots + t^{d-2} \right) \quad \text{and} \\ H_{\mathbf{c}}(s, t) &= c_1 + c_2(s + t) + \cdots + c_d \left(s^{d-1} + s^{d-2}t + \cdots + t^{d-1} \right). \end{aligned}$$

Then by Lemma 3.2.5, $F_{\mathbf{a}}(s, t) = G_{\mathbf{b}}(s, t) = H_{\mathbf{c}}(s, t) = 0$ for some $(s, t) \in \mathbb{R}^2$ if and only if the polynomial map $\phi_{\mathbf{abc}}$ is not a smooth embedding. This shows that the set $\eta(\mathcal{A}_d \setminus \mathcal{O}_d)$ is the image of the set

$$U_d = \left\{ (\mathbf{a}, \mathbf{b}, \mathbf{c}, s, t) \in \mathbb{R}^{3d+2} \mid F_{\mathbf{a}}(s, t) = G_{\mathbf{b}}(s, t) = H_{\mathbf{c}}(s, t) = 0 \right\}$$

under the projection $\pi : \mathbb{R}^{3d+2} \rightarrow \mathbb{R}^{3d}$ onto the space of the first $3d$ coordinates. Note that U_d is a semialgebraic subset of $\mathbb{R}^{3d+2}$, so by Corollary 2.3.2, the set $\eta(\mathcal{A}_d \setminus \mathcal{O}_d)$ is a semialgebraic subset of $\mathbb{R}^{3d}$ and hence so is its complement $\eta(\mathcal{O}_d)$. $\square$

Corollary 3.2.12. *The spaces $\mathcal{P}_d$ and $\mathcal{Q}_d$, for $d \geq 3$, are homeomorphic to some semialgebraic subsets of $\mathbb{R}^{3d}$.*

Proof. We have $\eta(\mathcal{P}_d) = \eta(\mathcal{B}_d) \cap \eta(\mathcal{O}_d)$ and $\eta(\mathcal{Q}_d) = \eta(\mathcal{C}_d) \cap \eta(\mathcal{O}_d)$. It is easy to note that the set $\eta(\mathcal{C}_d)$ is semialgebraic. Also by Theorem 3.2.11, the set $\eta(\mathcal{O}_d)$ is semialgebraic. Now it is enough to prove that the set $\eta(\mathcal{B}_d)$ is semialgebraic. For $0 \leq p < q < r \leq d$, let

$$W_{pqr} = \left\{ (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{R}^{3d} \mid \phi_{\mathbf{abc}} \text{ has degree sequence } (p, q, r) \right\}.$$

Then it can be easily seen that W_{pqr} is a semialgebraic subset of $\mathbb{R}^{3d}$ for all p, q and r such that $0 \leq p < q < r \leq d$. Note that $\eta(\mathcal{B}_d) = \bigcup_{0 \leq p < q < r \leq d} W_{pqr}$. Thus, the set $\eta(\mathcal{B}_d)$ is also a semialgebraic subset of $\mathbb{R}^{3d}$. $\square$

Corollary 3.2.13. *The spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$, for $d \geq 3$, have only finitely many path components.*

Proof. By Theorem 3.2.11 and Corollary 3.2.12, the sets $\eta(\mathcal{O}_d)$, $\eta(\mathcal{P}_d)$ and $\eta(\mathcal{Q}_d)$ are semialgebraic subsets of $\mathbb{R}^{3d}$, so by Theorem 2.3.3 and Proposition 2.3.4, they have only finitely many path components. The same is true for the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$, since η is a homeomorphism between the spaces $\mathcal{A}_d$ and $\mathbb{R}^{3d}$. $\square$

Chapter 4

Topology of the Spaces

In Chapter 3, we noticed that the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ have only finitely many path components. In this chapter, we focus on the number of path components of these spaces and their homotopy type. In Section 4.1, we show that the space $\mathcal{O}_d$ has the same homotopy type as S^2 . This also proves that the space $\mathcal{O}_d$ is path connected. In Section 4.2, we discuss the space $\mathcal{P}$ of all polynomial knots with some natural topology on it and find its homotopy type. Furthermore, we prove that every polynomial knot is isotopic to some polynomial unknot by a smooth isotopy through polynomial knots. In Section 4.3, we show that the space $\mathcal{P}_d$ is path connected. The last section (i.e. Section 4.4) is devoted to the space $\mathcal{Q}_d$, where we show that this space is not path connected. In fact, we prove that the path equivalence in $\mathcal{Q}_d$ implies the topological equivalence. If there are n distinct knot-types (up to mirror images) which can be represented in $\mathcal{Q}_d$, then it has at least $8n$ path components. We also look at the homotopy type of the path components of $\mathcal{Q}_d$ for some small values of d .

4.1 Homotopy type of the space $\mathcal{O}_d$

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial map (that is, its component functions are real polynomials). Let us denote its first, second and the third components by f_ϕ , g_ϕ and h_ϕ respectively, i.e. $\phi(t) = (f_\phi(t), g_\phi(t), h_\phi(t))$. Also, let $a_{\phi i}$, $b_{\phi i}$ and $c_{\phi i}$ be the coefficients of t^i in the polynomials f_ϕ , g_ϕ and h_ϕ respectively. We will use these notations throughout the thesis.

Theorem 4.1.1. *The space $\mathcal{O}_d$, for $d \geq 3$, is homotopy equivalent to S^2 .*

Proof. For any $\phi \in \mathcal{O}_d$, we have $(a_{\phi 1})^2 + (b_{\phi 1})^2 + (c_{\phi 1})^2 \neq 0$ (otherwise $\phi'(0) = 0$). Let us define maps $\Gamma_d : \mathcal{O}_d \rightarrow S^2$ and $\Lambda_d : S^2 \rightarrow \mathcal{O}_d$ by

$$\Gamma_d(\phi) = \frac{1}{\sqrt{(a_{\phi 1})^2 + (b_{\phi 1})^2 + (c_{\phi 1})^2}} (a_{\phi 1}, b_{\phi 1}, c_{\phi 1})$$

and $\Lambda_d(x) = \phi_x$, where for $x = (x_1, x_2, x_3) \in S^2$, $\phi_x(t) = (x_1 t, x_2 t, x_3 t)$ for $t \in \mathbb{R}$. Note that the maps Γ_d and Λ_d are continuous. The map $\Gamma_d \circ \Lambda_d$ is the identity map of S^2 . We need to prove that the map $\Lambda_d \circ \Gamma_d$ is homotopy equivalent to the identity map of $\mathcal{O}_d$. For $\phi \in \mathcal{O}_d$, let $\mu_\phi : I \rightarrow \mathbb{R}^+$ be given by

$$\mu_\phi(s) = s + \frac{1-s}{\sqrt{(a_{\phi 1})^2 + (b_{\phi 1})^2 + (c_{\phi 1})^2}}.$$

Define $H_d : I \times \mathcal{O}_d \rightarrow \mathcal{O}_d$ by $H_d(s, \phi) = H_{s\phi}$, where

$$H_{s\phi}(t) = (a_{\phi 0}, b_{\phi 0}, c_{\phi 0})s + \mu_\phi(s) \left((a_{\phi 1}, b_{\phi 1}, c_{\phi 1})t + \sum_{i=2}^d (a_{\phi i}, b_{\phi i}, c_{\phi i})s^{i-1}t^i \right)$$

for $t \in \mathbb{R}$. Note that, for each $(s, \phi) \in (0, 1] \times \mathcal{O}_d$, $H'_{s\phi}(t) = \mu_\phi(s)\phi'(st) \neq 0$ for all $t \in \mathbb{R}$ and

$$H_{s\phi}(u) - H_{s\phi}(v) = \frac{\mu_\phi(s)(\phi(su) - \phi(sv))}{s} \neq 0$$

for all $u \neq v$. Thus, for each $(s, \phi) \in (0, 1] \times \mathcal{O}_d$, the map $H_{s\phi}$ is a smooth embedding. Also, for $s = 0$ and $\phi \in \mathcal{O}_d$, $H_{s\phi}$ is a linear map and hence it is smooth embedding. This shows that the image of H_d is in fact contained in $\mathcal{O}_d$. Note that H_d is a continuous map. Furthermore, for each $\phi \in \mathcal{O}_d$, we have $H_d(0, \phi) = \Lambda_d(\Gamma_d(\phi))$ and $H_d(1, \phi) = \phi$. This shows that the map $\Lambda_d \circ \Gamma_d$ is homotopy equivalent to the identity map of $\mathcal{O}_d$. $\square$

Corollary 4.1.2. *The space $\mathcal{O}_d$, for $d \geq 3$, is path connected.*

4.2 The space of all polynomial knots

In Chapter 3, we have introduced the set $\mathcal{P}$ of all polynomial knots. It is easy to note that $\mathcal{P} = \bigcup_{d \geq 3} \mathcal{O}_d$. We would like to give the inductive limit topology on $\mathcal{P}$; that is, a set $U \subseteq \mathcal{P}$ is open in $\mathcal{P}$ if and only if the set $U \cap \mathcal{O}_d$ is open in $\mathcal{O}_d$ for all $d \geq 3$. With this topology on $\mathcal{P}$, we have the following:

Theorem 4.2.1. *The space $\mathcal{P}$ is homotopy equivalent to S^2 .*

To prove this theorem, we require a result by Whitehead (see [15, §18.5]) which is states as follows:

Proposition 4.2.2 (Whitehead [15]). *Let $\{X_i\}$ and $\{Y_i\}$ be sequences of topological spaces with the inductive limits X and Y respectively. Suppose one of the space X or Y is compact and regular. Then the inductive limit of the sequence $\{X_i \times Y_i\}$ is the product space $X \times Y$.*

Proof of Theorem 4.2.1. Let us define maps $\Gamma : \mathcal{P} \rightarrow S^2$ and $\Lambda : S^2 \rightarrow \mathcal{P}$ by $\Gamma(\phi) = \Gamma_d(\phi)$ if $\phi \in \mathcal{O}_d$ and $\Lambda(x) = \Lambda_3(x)$ if $x \in S^2$ (where the maps Γ_d and Λ_3 are as defined in Theorem 4.1.1). Note that Γ is well defined; that is, for $n > d \geq 3$, we have $\Gamma_n(\phi) = \Gamma_d(\phi)$ for all $\phi \in \mathcal{O}_d$. To check the continuity of the maps Γ and Λ , consider an open set U in S^2 and an open set V in $\mathcal{P}$. It

is easy to check the following:

$$\Gamma^{-1}(U) \cap \mathcal{O}_d = \Gamma_d^{-1}(U) \quad \text{and} \quad \Lambda^{-1}(V) = \Lambda_3^{-1}(V \cap \mathcal{O}_3) \quad (4.1)$$

for all $d \geq 3$. Since the maps Γ_d and Λ_3 are continuous, the set $\Gamma_d^{-1}(U)$ is open in $\mathcal{O}_d$ for all $d \geq 3$ and $\Lambda_3^{-1}(V \cap \mathcal{O}_3)$ is open in S^2 . Thus, by (4.1), $\Gamma^{-1}(U)$ is open in $\mathcal{P}$ and $\Lambda^{-1}(V)$ is open in S^2 . This proves that the maps Γ and Λ are continuous. Note that the map $\Gamma \circ \Lambda = \Gamma_3 \circ \Lambda_3$ is the identity map of S^2 . Now we need to show that the map $\Lambda \circ \Gamma$ is homotopy equivalent to the identity map of $\mathcal{P}$.

Define $H : I \times \mathcal{P} \rightarrow \mathcal{P}$ by $H(s, \phi) = H_d(s, \phi)$ if $(s, \phi) \in I \times \mathcal{O}_d$ (where the map H_d is as defined in Theorem 4.1.1). The map H is well defined; that is, for $n > d \geq 3$, we have $H_n(s, \phi) = H_d(s, \phi)$ for all $(s, \phi) \in I \times \mathcal{O}_d$. Let V be an open set in $\mathcal{P}$. Then we have

$$H^{-1}(V) \cap I \times \mathcal{O}_d = H_d^{-1}(V \cap \mathcal{O}_d) \quad (4.2)$$

for all $d \geq 3$. Since H_d is continuous, the set $H_d^{-1}(V \cap \mathcal{O}_d)$ is open in $I \times \mathcal{O}_d$ for all $d \geq 3$. Thus, by (4.2), the set $H^{-1}(V)$ is open in $I \times \mathcal{P}$ with respect to the inductive limit topology which comes from its stratification as $I \times \mathcal{P} = \bigcup_{d \geq 3} I \times \mathcal{O}_d$. By Proposition 4.2.2, the inductive limit topology on $I \times \mathcal{P}$ is same as the product topology on it. Therefore, the set $H^{-1}(V)$ is open in $I \times \mathcal{P}$ with respect to its product topology. This shows that the map H is continuous. Note that, for each $d \geq 3$ and for each $\phi \in \mathcal{O}_d$, $H_d(0, \phi) = \Lambda_3(\Gamma_d(\phi))$ and $H_d(1, \phi) = \phi$. Therefore, we have $H(0, \phi) = \Lambda(\Gamma(\phi))$ and $H(1, \phi) = \phi$ for all $\phi \in \mathcal{P}$. This shows that the map $\Lambda \circ \Gamma$ is homotopy equivalent to the identity map of $\mathcal{P}$. $\square$

Corollary 4.2.3. *The space $\mathcal{P}$ is path connected.*

It is known that if two polynomial knots are topologically equivalent, then they are polynomially isotopic (see Theorem 2.2.2). In the following theorem, we prove that the converse is false.

Theorem 4.2.4. *Every polynomial knot is isotopic to some linear knot (an unknot) by a smooth isotopy through polynomial knots.*

Proof. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be any polynomial knot, and let it be given by

$$t \mapsto (a_0 + a_1t + \cdots + a_d t^d, b_0 + b_1t + \cdots + b_d t^d, c_0 + c_1t + \cdots + c_d t^d),$$

where d is the degree of ϕ . Define $F : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ by $F(s, t) = F_s(t)$, where

$$F_s(t) = (a_0, b_0, c_0)(1 - s) + (a_1, b_1, c_1)t + \sum_{i=2}^d (a_i, b_i, c_i)(1 - s)^{i-1}t^i.$$

It is easy to note that the map F is smooth, since its components are polynomials in two variables. For each $s \in [0, 1]$, we have $F'_s(t) = \phi'(t - st) \neq 0$ for all $t \in \mathbb{R}$ and

$$F_s(u) - F_s(v) = \frac{\phi(u - su) - \phi(v - sv)}{1 - s} \neq 0$$

for all $u \neq v$. This shows that F_s is a polynomial knot for all $s \in [0, 1]$. Also, for $s = 1$, F_s is a polynomial knot (since it is a linear map). Let $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ be given by $t \mapsto (a_1t, b_1t, c_1t)$. Then we have $F(0, t) = \phi(t)$ and $F(1, t) = \psi(t)$ for all $t \in \mathbb{R}$. This shows that F is a smooth isotopy through polynomial knots which connects ϕ and ψ . $\square$

The isotopy in Theorem 4.2.4 rolls a given polynomial knot to the infinity and unravels it up to make it into an unknot. In general, the same thing happens for any long knot. More precisely, every long knot is isotopic to some unknot by a smooth isotopy through long knots. However, this does

not happen if the knots are compact (that is, the smooth embeddings of S^1 in S^3). Otherwise by the isotopy extension theorem (see [23]), every knot would be ambient isotopic to some unknot, which is certainly not the case.

Corollary 4.2.5. *Every polynomial knot can be connected to a linear knot by a smooth path in the space of all polynomial knots.*

Proof. For a polynomial knot ϕ , define $\alpha : I \rightarrow \mathcal{P}$ by $\alpha(s) = F_s$ (where F_s , for $s \in I$, is as defined in Theorem 4.2.4). Then α becomes a smooth path in $\mathcal{P}$ which connects ϕ to some linear knot. $\square$

Example 4.1. We have a polynomial representation $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ of the figure-eight knot (the knot 4_1) given by

$$\begin{aligned} t \mapsto & (0, 0, 0.5)t^6 + (0, 1, -0.5)t^5 + (1, -1.5, -19.1)t^4 + (8.5, -29.1, 11.3)t^3 \\ & + (-31.9, 32.4, 187.2)t^2 + (-164, 160.5, -35.8)t + (51.8, -50.3, 0). \end{aligned}$$

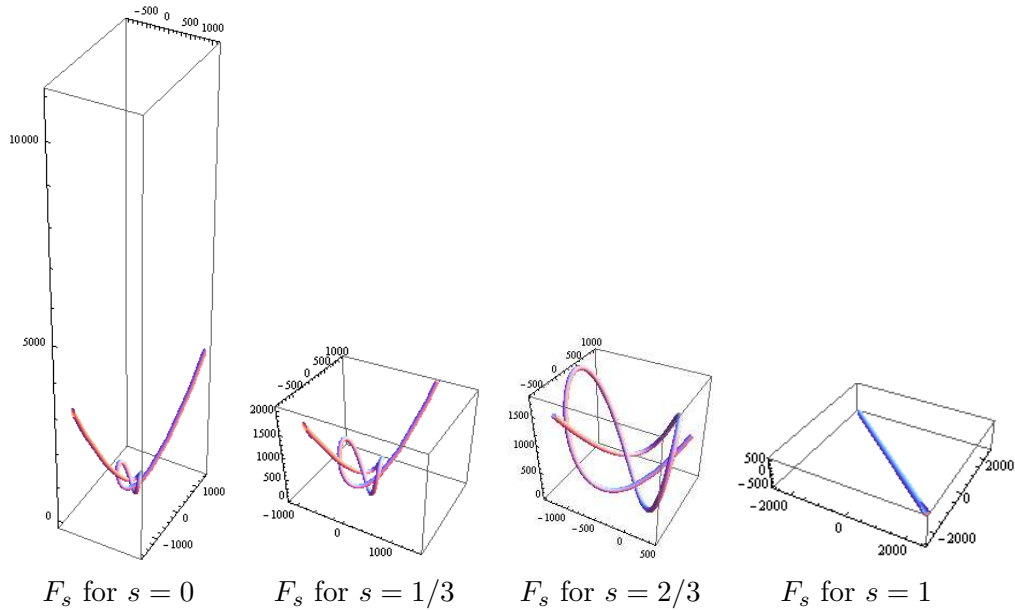


Figure 4.1: Depiction of the path $F : I \rightarrow \mathcal{O}_d$ connecting a figure-eight knot ϕ to an unknot ψ .

By Corollary 4.2.5, we have a path $F : I \rightarrow \mathcal{P}$ given by $F(s) = F_s$, where

$$\begin{aligned} F_s(t) = & (0, 0, 0.5)(1-s)^5 t^6 + (0, 1, -0.5)(1-s)^4 t^5 + (1, -1.5, -19.1)(1-s)^3 t^4 \\ & + (8.5, -29.1, 11.3)(1-s)^2 t^3 + (-31.9, 32.4, 187.2)(1-s)t^2 \\ & + (-164, 160.5, -35.8)t + (51.8, -50.3, 0)(1-s) \end{aligned}$$

for $t \in \mathbb{R}$. This path joins the polynomial knot $F_0 = \phi$ (which represents the figure-eight knot) to the unknot F_1 given by $t \mapsto (-164t, 160.5t, -35.8t)$. The path F rolls the polynomial knot ϕ to the infinity so that the knot gets unraveled up to become an unknot (see Figure 4.1). Note that the image of the map F lies in $\mathcal{Q}_6$. Thus, it is also a path in this space connecting a figure-eight knot to a linear knot. Also, there exists such path in $\mathcal{P}_6$ (that is, the path which connects a figure-eight knot to an unknot). However, there does not exist a path in $\mathcal{Q}_6$ which connects a figure-eight knot to an unknot. In fact, in Section 4.4, we show that two topologically different knots cannot lie in the same path component of the space $\mathcal{Q}_d$ for any $d \geq 3$.

4.3 The space $\mathcal{P}_d$

In this section, we prove that the space $\mathcal{P}_d$ (as defined in Section 3.1), for $d \geq 3$, is path connected by explicitly constructing a path from a given element in $\mathcal{P}_d$ to a fixed element in $\mathcal{P}_d$. For that, we first construct a path from an element in $\mathcal{P}_d$ to some element in $\mathcal{Q}_d$ (note that $\mathcal{Q}_d \subseteq \mathcal{P}_d$) and then produce a path from this new element to a fixed element in $\mathcal{P}_d$.

Proposition 4.3.1. *Every polynomial knot in $\mathcal{P}_d$ can be connected to a polynomial knot in $\mathcal{Q}_d$ by a path in $\mathcal{P}_d$.*

Proof. Let ϕ be an element of the space $\mathcal{P}_d$. Since the set $\eta(\mathcal{Q}_d)$ is dense in $\mathbb{R}^{3d}$ and it has only finitely many path components (see Corollary 3.2.9

and Corollary 3.2.13), the element $\eta(\phi)$ must belong to the closure of some path component, say U , of $\eta(\mathcal{Q}_d)$. The set $\eta(\mathcal{Q}_d)$ is a semialgebraic subset of $\mathbb{R}^{3d}$, so by Theorem 2.3.3, U is also a semialgebraic subset of $\mathbb{R}^{3d}$. This implies that the set $U \cup \{\eta(\phi)\}$ is a semialgebraic subset of $\mathbb{R}^{3d}$, since a one point set is always semialgebraic. Note that $U \cup \{\eta(\phi)\}$ is connected. Therefore, by Proposition 2.3.4, this set is path connected and hence so is the set $\eta^{-1}(U) \cup \{\phi\}$. In other words, there is path in $\eta^{-1}(U) \cup \{\phi\} \subseteq \mathcal{P}_d$ from ϕ to an element $\psi \in \eta^{-1}(U) \subseteq \mathcal{Q}_d$. $\square$

Corollary 4.3.2. *Every polynomial knot in $\mathcal{O}_d$ can be connected to a polynomial knot in $\mathcal{Q}_d$ by a path in $\mathcal{O}_d$.*

Corollary 4.3.3. *Every polynomial knot in $\mathcal{O}_d$ can be connected to a polynomial knot in $\mathcal{P}_d$ by a path in $\mathcal{O}_d$.*

Theorem 4.3.4. *The space $\mathcal{P}_d$, for $d \geq 3$, is path connected.*

Proof. Let ϕ be an element of the space $\mathcal{P}_d$. By Proposition 4.3.1, there is a path in $\mathcal{P}_d$ from ϕ to some element $\psi \in \mathcal{Q}_d$. Note that $\mathcal{Q}_d$ is open in $\mathcal{A}_d$, so we have a path in $\mathcal{Q}_d$ from ψ to an element $\sigma \in \mathcal{Q}_d$ in which the coefficient $b_{\sigma 1}$ of t in the second component is nonzero. Define $\Lambda : I \rightarrow \mathcal{Q}_d$ by

$$\Lambda(s) = \left(f_\sigma - a_{\sigma 0}s, g_\sigma - b_{\sigma 0}s, h_\sigma - \frac{c_{\sigma 1}g_\sigma - b_{\sigma 0}c_{\sigma 1} + b_{\sigma 1}c_{\sigma 0}}{b_{\sigma 1}}s \right).$$

Suppose $\tau = \Lambda(1)$, then we have $a_{\tau 0} = b_{\tau 0} = c_{\tau 0} = c_{\tau 1} = 0$ and $b_{\tau 1} = b_{\sigma 1} \neq 0$. It is clear that Λ is a path in $\mathcal{Q}_d$ from σ and τ .

Since $\mathcal{Q}_d$ is open in $\mathcal{A}_d$, we can find a path in $\mathcal{Q}_d$ from τ to an element $\omega \in \mathcal{Q}_d$ which differs from τ only in the coefficient of t^2 in the third component and that coefficient is nonzero in ω . For $s \in I$, suppose f_s, g_s and h_s be

polynomials which are given by

$$f_s(t) = a_{\omega 1} s t + a_{\omega 2} s^2 t^2 + \cdots + a_{\omega, d-2} s^{d-2} t^{d-2},$$

$$g_s(t) = b_{\omega 1} t + b_{\omega 2} s t^2 + \cdots + b_{\omega, d-1} s^{d-2} t^{d-1} \quad \text{and}$$

$$h_s(t) = c_{\omega 2} t^2 + c_{\omega 3} s t^3 + \cdots + c_{\omega d} s^{d-2} t^d.$$

Define $\Gamma : I \rightarrow \mathcal{P}_d$ by $\Gamma(s) = \Gamma_s$, where $\Gamma_s(t) = (f_s(t), g_s(t), h_s(t))$ for $t \in \mathbb{R}$. Since $a_{\omega 0} = a_{\tau 0} = 0$, $b_{\omega 0} = b_{\tau 0} = 0$, $c_{\omega 0} = c_{\tau 0} = 0$ and $c_{\omega 1} = c_{\tau 1} = 0$, we have $\Gamma(1) = (f_1, g_1, h_1) = \omega$. Also, for each $s \in (0, 1]$ and each $t \in \mathbb{R}$, we have the following:

$$f_s(t) = f_1(st), \quad s g_s(t) = g_1(st) \quad \text{and} \quad s^2 h_s(t) = h_1(st), \quad \text{and} \quad (4.3)$$

$$f'_s(t) = s f'_1(st), \quad g'_s(t) = g'_1(st) \quad \text{and} \quad s h'_s(t) = h'_1(st). \quad (4.4)$$

Thus, for each $s \in (0, 1]$, we must have $\Gamma_s(u) \neq \Gamma_s(v)$ for all $u \neq v$ and $\Gamma'_s(t) = (f'_s(t), g'_s(t), h'_s(t)) \neq 0$ for all $t \in \mathbb{R}$ (otherwise by (4.3) and (4.4), we would have $\omega(su) = \omega(sv)$ or $\omega'(st) = (f'_1(st), g'_1(st), h'_1(st)) = 0$, which is a contradiction to the fact that ω is an embedding). Therefore, for each $s \in (0, 1]$, Γ_s is a smooth embedding and hence it is an element in $\mathcal{P}_d$. Also, for $s = 0$, $\Gamma_s \in \mathcal{P}_d$ (since $b_{\omega 1} = b_{\tau 1} \neq 0$ and $c_{\omega 2} \neq 0$). Hence the image of Γ is in fact contained in $\mathcal{P}_d$. It is clear that Γ is continuous. Let $\mu : \mathbb{R} \rightarrow \mathbb{R}^3$ be given by $t \mapsto (0, b_{\omega 1} t, c_{\omega 2} t^2)$. Then we have $\Gamma(0) = (f_0, g_0, h_0) = \mu$. This shows that Γ is a path in $\mathcal{P}_d$ from μ to ω .

Let ϕ_0 be a fixed element in $\mathcal{P}_d$ given by $t \mapsto (0, t, t^2)$. Define $\Omega : I \rightarrow \mathcal{P}_d$ by $\Omega(s) = \Omega_s$, where

$$\Omega_s(t) = (s(1-s)t, (b_{\omega 1} s + 1-s)t + s(1-s)t^2, (c_{\omega 2} s + 1-s)t^2 + s(1-s)t^3)$$

for $t \in \mathbb{R}$. It can be easily seen that Ω is path in $\mathcal{P}_d$ from ϕ_0 to μ .

By combining the paths (which we have constructed above) between the consecutive elements in the list $\phi, \psi, \sigma, \tau, \omega, \mu$ and ϕ_0 , we can obtain a path in $\mathcal{P}_d$ from ϕ to ϕ_0 . $\square$

4.4 Path equivalence in the space $\mathcal{Q}_d$

Definition 4.4.1. *Two polynomial knots are said to be path equivalent in $\mathcal{Q}_d$ if they belong to its same path component.*

Similarly, the path equivalence can be defined for the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$. Note that the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$, for $d \geq 3$, are path connected. Thus, any two polynomial knots in $\mathcal{O}_d$ (or, in $\mathcal{P}_d$) are path equivalent.

Proposition 4.4.1. *Two polynomial knots are path equivalent in $\mathcal{Q}_d$ if and only if they are isotopic by an isotopy of polynomial knots in $\mathcal{Q}_d$.*

Proof. Suppose ϕ and ψ be polynomial knots in $\mathcal{Q}_d$, and let $F : I \rightarrow \mathcal{Q}_d$ be a path from ϕ to ψ . For $s \in I$, let $F_s = F(s)$ and let it be given by

$$F_s(t) = \left(\alpha_0(s) + \alpha_1(s)t + \cdots + \alpha_{d-2}(s)t^{d-2}, \beta_0(s) + \beta_1(s)t + \cdots + \beta_{d-1}(s)t^{d-1}, \right. \\ \left. \gamma_0(s) + \gamma_1(s)t + \cdots + \gamma_d(s)t^d \right)$$

for $t \in \mathbb{R}$, where α_i 's, β_i 's and γ_i 's are real valued functions. Define a map $\Gamma : I \rightarrow \mathbb{R}^{3d}$ by

$$s \mapsto (\alpha_0(s), \alpha_1(s), \dots, \alpha_{d-2}(s), \beta_0(s), \beta_1(s), \dots, \beta_{d-1}(s), \gamma_0(s), \gamma_1(s), \dots, \gamma_d(s)).$$

It is clear that $\Gamma = \eta \circ F$ (where η is the homeomorphism as defined in (3.1)). Therefore, the map Γ is continuous and hence so are the component

functions α_i 's, β_i 's and γ_i 's. This implies that the maps

$$\begin{aligned} (s, t) &\xrightarrow{\Omega_1} \alpha_0(s) + \alpha_1(s)t + \cdots + \alpha_{d-2}(s)t^{d-2}, \\ (s, t) &\xrightarrow{\Omega_2} \beta_0(s) + \beta_1(s)t + \cdots + \beta_{d-1}(s)t^{d-1} \quad \text{and} \\ (s, t) &\xrightarrow{\Omega_3} \gamma_0(s) + \gamma_1(s)t + \cdots + \gamma_d(s)t^d \end{aligned}$$

are continuous as maps from $I \times \mathbb{R}$ to $\mathbb{R}$. Define $G : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ by $G(s, t) = F_s(t)$. It is easy to note that G is continuous, since its component functions Ω_1, Ω_2 and Ω_3 are continuous. Note that $G(0, t) = F_0(t) = \phi(t)$ and $G(1, t) = F_1(t) = \psi(t)$ for $t \in \mathbb{R}$. This shows that G is an isotopy of polynomial knots in $\mathcal{Q}_d$ connecting ϕ and ψ .

To prove the converse, suppose ϕ and ψ be polynomial knots in $\mathcal{Q}_d$, and let $H : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ be an isotopy of polynomial knots in $\mathcal{Q}_d$ which connects ϕ and ψ (that is, $H(0, t) = \phi(t)$ and $H(1, t) = \psi(t)$ for $t \in \mathbb{R}$). Let H be given by

$$(s, t) \mapsto \left(\mu_0(s) + \mu_1(s)t + \cdots + \mu_{d-2}(s)t^{d-2}, \sigma_0(s) + \sigma_1(s)t + \cdots + \sigma_{d-1}(s)t^{d-1}, \right. \\ \left. \tau_0(s) + \tau_1(s)t + \cdots + \tau_d(s)t^d \right),$$

where μ_i 's, σ_i 's and τ_i 's are real valued functions. Note that for any $t \in \mathbb{R}$, the maps

$$\begin{aligned} s &\xrightarrow{\Omega_{1t}} \mu_0(s) + \mu_1(s)t + \cdots + \mu_{d-2}(s)t^{d-2}, \\ s &\xrightarrow{\Omega_{2t}} \sigma_0(s) + \sigma_1(s)t + \cdots + \sigma_{d-1}(s)t^{d-1} \quad \text{and} \\ s &\xrightarrow{\Omega_{3t}} \tau_0(s) + \tau_1(s)t + \cdots + \tau_d(s)t^d \end{aligned}$$

are continuous as maps from I to $\mathbb{R}$. It is easy to check that: (a) μ_i 's are linear combinations of the maps $\Omega_{10}, \Omega_{11}, \dots, \Omega_{1,d-2}$, (b) σ_i 's are linear combi-

nations of the maps $\Omega_{20}, \Omega_{21}, \dots, \Omega_{2,d-1}$, and (c) τ_i 's are linear combinations of the maps $\Omega_{30}, \Omega_{31}, \dots, \Omega_{3d}$. Therefore, the maps μ_i 's, σ_i 's and τ_i 's are continuous and hence the map $\Lambda : I \rightarrow \mathbb{R}^{3d}$ defined by

$$s \mapsto (\mu_0(s), \mu_1(s), \dots, \mu_{d-2}(s), \sigma_0(s), \sigma_1(s), \dots, \sigma_{d-1}(s), \tau_0(s), \tau_1(s), \dots, \tau_d(s))$$

is also continuous. Define $\Psi : I \rightarrow \mathcal{Q}_d$ by $\Psi(s) = H_s$, where $H_s(t) = H(s, t)$ for $t \in \mathbb{R}$. Since $\Psi = \eta^{-1} \circ \Lambda$, the map Ψ is continuous. Also, note that $\Psi(0) = H_0 = \phi$ and $\Psi(1) = H_1 = \psi$. This shows that the map Ψ is a path in $\mathcal{Q}_d$ which connects ϕ and ψ . $\square$

Corollary 4.4.2. *Any two polynomial knots in $\mathcal{O}_d$ are isotopic by an isotopy of polynomial knots in $\mathcal{O}_d$.*

Corollary 4.4.3. *Any two polynomial knots in $\mathcal{P}_d$ are isotopic by an isotopy of polynomial knots in $\mathcal{P}_d$.*

Theorem 4.4.4. *If two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are topologically equivalent.*

Proof. Let ϕ and ψ be two polynomial knots which belong to the same path component of the space $\mathcal{Q}_d$. Since $\mathcal{Q}_d$ is open in $\mathcal{A}_d$, we can choose a smooth path $\alpha : I \rightarrow \mathcal{Q}_d$ such that $\alpha(0) = \phi$ and $\alpha(1) = \psi$. Define $F : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ by $F(s, t) = \alpha_s(t)$, where $\alpha_s = \alpha(s)$ for $s \in I$. This map is smooth and it is an isotopy of polynomial knots in $\mathcal{Q}_d$ connecting ϕ and ψ . Let $H : I \times \mathbb{R} \rightarrow I \times \mathbb{R}^3$ be the track of the isotopy F , i.e. $H(s, t) = (s, F(s, t)) = (s, \alpha_s(t))$. Note that H is a level preserving map and it is a smooth embedding of $I \times \mathbb{R}$ in $I \times \mathbb{R}^3$. Thus, its image $H(I \times \mathbb{R})$ is a smooth submanifold of $I \times \mathbb{R}^3$. One can identify $I \times \mathbb{R}^3$ as a smooth submanifold of $I \times S^3$ by the embedding $1 \times \Gamma : I \times \mathbb{R}^3 \rightarrow I \times S^3 \setminus \{N\}$, where 1 is the identity mapping of I , and Γ is the inverse of the stereographic projection from the north pole N of S^3 . Let

K be the image of $(1 \times \Gamma) \circ H$ and let $\bar{K}$ be its closure in $I \times S^3$. It is obvious that the collection $\mathcal{S} = \{I \times S^3 \setminus \bar{K}, K, I \times \{N\}\}$ is a collection of pairwise disjoint smooth submanifolds of $I \times S^3$ which covers it. It is easy to check that it satisfies the condition of frontier (see Definition 2.4.6). Note that $I \times S^3 \setminus \bar{K}$ is an open submanifold of $I \times S^3$, so its dimension is 4. Therefore, it is trivial that the pairs $(I \times S^3 \setminus \bar{K}, K)$ and $(I \times S^3 \setminus \bar{K}, I \times \{N\})$ satisfy the second Whitney condition (see Definition 2.4.4).

Claim: *The pair $(K, I \times \{N\})$ satisfies the second Whitney condition.*

Proof of the claim. Let $y = (u, N)$ be an element in $I \times \{N\}$. Suppose $\Phi : U \rightarrow \mathbb{R}^4$ be a smooth chart of $I \times S^3$ at y . We need to show that the pair $(\Phi(U \cap K), \Phi(U \cap I \times \{N\}))$ satisfies the second Whitney condition at $\Phi(y)$. Let $(x_i) \subseteq \Phi(U \cap K)$ and $(y_i) \subseteq \Phi(U \cap I \times \{N\})$ be sequences both converging to $\Phi(y)$ and they are such that: (a) $x_i \neq y_i$ for all $i \in \mathbb{N}$, (b) the tangent spaces $\tau_i = T(\Phi(U \cap K))_{x_i}$ converge to a plane $\tau \subseteq \mathbb{R}^4$, and (c) the secant lines $l_i = \overline{(x_i, y_i)}$ converge to a line $l \subseteq \mathbb{R}^4$. We can assume that $U = V \times W$ for some connected open neighborhoods $V \subseteq I$ of u and $W \subseteq S^3$ of N , and $\Phi(s, x) = (s, \Psi(x))$ for $(s, x) \in V \times W$, where Ψ is the restriction of the stereographic projection from the south pole of S^3 . For $i \in \mathbb{N}$, let

$$x_i = \Phi \circ (1 \times \Gamma) \circ H(s_i, t_i) = (s_i, \Psi \circ \Gamma \circ F(s_i, t_i)) = \left(s_i, \|F(s_i, t_i)\|^{-2} F(s_i, t_i)\right)$$

for some $(s_i, t_i) \in I \times \mathbb{R}$, and suppose

$$y_i = \Phi(u_i, N) = (u_i, \Psi(N)) = (u_i, 0)$$

for some $u_i \in I$. Note that for $i \in \mathbb{N}$, the vector

$$\lambda_i = x_i - y_i = \left(s_i - u_i, \|F(s_i, t_i)\|^{-2} F(s_i, t_i)\right)$$

spans the secant line l_i . Also, for $i \in \mathbb{N}$, it can be easily seen that the vectors

$$\begin{aligned}
 v_i &= \left(1, \frac{\partial}{\partial s} \Psi \circ \Gamma \circ F(s, t) \right) \Big|_{(s_i, t_i)} \\
 &= \left(1, \frac{\partial}{\partial s} \left(\frac{F(s, t)}{\|F(s, t)\|^2} \right) \right) \Big|_{(s_i, t_i)} \\
 &= \left(1, \frac{\frac{\partial}{\partial s} F(s, t)}{\|F(s, t)\|^2} - \frac{\frac{\partial}{\partial s} \|F(s, t)\|^2}{\|F(s, t)\|^4} F(s, t) \right) \Big|_{(s_i, t_i)} \quad \text{and} \\
 w_i &= \left(0, \frac{\partial}{\partial t} \Psi \circ \Gamma \circ F(s, t) \right) \Big|_{(s_i, t_i)} \\
 &= \left(0, \frac{\partial}{\partial t} \left(\frac{F(s, t)}{\|F(s, t)\|^2} \right) \right) \Big|_{(s_i, t_i)} \\
 &= \left(0, \frac{\frac{\partial}{\partial t} F(s, t)}{\|F(s, t)\|^2} - \frac{\frac{\partial}{\partial t} \|F(s, t)\|^2}{\|F(s, t)\|^4} F(s, t) \right) \Big|_{(s_i, t_i)}
 \end{aligned}$$

both together span the tangent space τ_i . After normalizing the vectors λ_i , v_i and w_i , we get the unit vectors as follows:

$$\begin{aligned}
 \hat{\lambda}_i &= \frac{\lambda_i}{\|\lambda_i\|} \\
 &= \left(\frac{s_i - u_i}{\sqrt{(s_i - u_i)^2 + \|F(s_i, t_i)\|^2}}^{-2}, \frac{F(s_i, t_i)}{\|F(s_i, t_i)\|^2 \sqrt{(s_i - u_i)^2 + \|F(s_i, t_i)\|^2}} \right), \\
 \hat{v}_i &= \frac{v_i}{\|v_i\|} \\
 &= \frac{\|F(s, t)\|^2}{\sqrt{\|F(s, t)\|^4 + \left\| \frac{\partial}{\partial s} F(s, t) \right\|^2}} \left(1, \frac{\frac{\partial}{\partial s} F(s, t)}{\|F(s, t)\|^2} - \frac{\frac{\partial}{\partial s} \|F(s, t)\|^2}{\|F(s, t)\|^4} F(s, t) \right) \Big|_{(s_i, t_i)}
 \end{aligned}$$

and

$$\hat{w}_i = \frac{w_i}{\|w_i\|}$$

$$\begin{aligned}
&= \left(0, \frac{\frac{\partial}{\partial t} F(s, t)}{\left\| \frac{\partial}{\partial t} F(s, t) \right\|} - \frac{2F(s, t) \cdot \frac{\partial}{\partial t} F(s, t)}{\|F(s, t)\|^2 \left\| \frac{\partial}{\partial t} F(s, t) \right\|} F(s, t) \right) \Big|_{(s_i, t_i)} \\
&= \left(0, \frac{\frac{\partial}{\partial t} F(s, t)}{\left\| \frac{\partial}{\partial t} F(s, t) \right\|} - \frac{2 \cos \theta_i}{\|F(s, t)\|} F(s, t) \right) \Big|_{(s_i, t_i)},
\end{aligned}$$

where θ_i is the angle between the vectors $F(s_i, t_i)$ and $\frac{\partial F}{\partial t}(s_i, t_i)$. Note that for $i \in \mathbb{N}$, the vector $\hat{\lambda}_i$ forms a basis for the secant line l_i , and the vectors $\hat{v}_i$ and $\hat{w}_i$ together form a basis for the tangent space τ_i . Since $|t_i| \rightarrow \infty$ as $i \rightarrow \infty$, there exists a subsequence of the sequence $\{t_i\}$ which diverges to $\pm\infty$. We may assume that the sequence $\{t_i\}$ itself diverges to $\pm\infty$. Note that the sequences $\{\|F(s_i, t_i)\|^{-1} F(s_i, t_i)\}$, $\{\|\frac{\partial F}{\partial t}(s_i, t_i)\|^{-1} \frac{\partial F}{\partial t}(s_i, t_i)\}$ and $\{\hat{\lambda}_i\}$ are bounded, so they have convergent subsequences. Again, we may assume that these sequences themselves are convergent. It is easy to check that

$$\lim_{i \rightarrow \infty} \left[\frac{\frac{\partial}{\partial s} F(s, t)}{\|F(s, t)\|^2} - \frac{\frac{\partial}{\partial s} \|F(s, t)\|^2}{\|F(s, t)\|^4} F(s, t) \right]_{(s_i, t_i)} = 0. \quad (4.5)$$

Note that the sequence $\{\hat{\lambda}_i\}$ is convergent, hence so is the sequence $\{\|\lambda_i\|^{-1} \|F(s_i, t_i)\|^{-1}\}$ (since it is the sequence of norms of the second component of $\hat{\lambda}_i$). By Proposition 2.2.3, the sequence $\{\theta_i\}$ converges to 0 or π depending on $\{t_i\}$ diverges to ∞ or $-\infty$, so we have

$$\lim_{i \rightarrow \infty} \left[\frac{\frac{\partial}{\partial t} F(s, t)}{\left\| \frac{\partial}{\partial t} F(s, t) \right\| \cos \theta_i} - \frac{F(s, t)}{\|F(s, t)\|} \right]_{(s_i, t_i)} = 0. \quad (4.6)$$

For $i \in \mathbb{N}$, let

$$z_i = \frac{(s_i - u_i) \|v_i\|}{\|\lambda_i\|} \hat{v}_i - \frac{1}{\|\lambda_i\| \|F(s_i, t_i)\| \cos \theta_i} \hat{w}_i.$$

Using (4.5) and (4.6), we compute the limit of z_i as follows:

$$\begin{aligned}
\lim_{i \rightarrow \infty} z_i &= \lim_{i \rightarrow \infty} \left(\frac{(s_i - u_i) \|v_i\|}{\|\lambda_i\|} \hat{v}_i - \frac{1}{\|\lambda_i\| \|F(s_i, t_i)\| \cos \theta_i} \hat{w}_i \right) \\
&= \lim_{i \rightarrow \infty} \left(\frac{s_i - u_i}{\|\lambda_i\|}, \frac{s_i - u_i}{\|\lambda_i\|} \left[\frac{\frac{\partial}{\partial s} F(s, t)}{\|F(s, t)\|^2} - \frac{\frac{\partial}{\partial s} \|F(s, t)\|^2}{\|F(s, t)\|^4} F(s, t) \right]_{(s_i, t_i)} \right. \\
&\quad \left. - \frac{1}{\|\lambda_i\| \|F(s_i, t_i)\|} \left[\frac{\frac{\partial}{\partial t} F(s, t)}{\left\| \frac{\partial}{\partial t} F(s, t) \right\| \cos \theta_i} - \frac{2F(s, t)}{\|F(s, t)\|} \right]_{(s_i, t_i)} \right) \\
&= \lim_{i \rightarrow \infty} \left(\frac{s_i - u_i}{\|\lambda_i\|}, \frac{-1}{\|\lambda_i\| \|F(s_i, t_i)\|} \left[\frac{\frac{\partial}{\partial t} F(s, t)}{\left\| \frac{\partial}{\partial t} F(s, t) \right\| \cos \theta_i} - \frac{2F(s, t)}{\|F(s, t)\|} \right]_{(s_i, t_i)} \right) \\
&= \lim_{i \rightarrow \infty} \hat{\lambda}_i.
\end{aligned}$$

Note that $\hat{\lambda}_i \in l_i$ and $z_i \in \tau_i$, so the nonzero vector $\hat{\lambda} = \lim_{i \rightarrow \infty} \hat{\lambda}_i$ (which is same as $\lim_{i \rightarrow \infty} z_i$) lies in both l and τ . Thus, we have $l \subseteq \tau$. This shows that the pair $(K, I \times \{N\})$ satisfies the second Whitney condition.

The collection $\mathcal{S}$ becomes a Whitney pre-stratification for $I \times S^3$ (see Definition 2.4.6). Let $\pi : I \times S^3 \rightarrow I$ be the projection onto the first coordinate. This map is proper. It is easy to check that the map π when restricted to any stratum in $\mathcal{S}$ is a submersion. Therefore, by Thom's first isotopy lemma (see Proposition 2.4.3), the map π when restricted to any stratum in $\mathcal{S}$ is a trivial fiber bundle. In particular, $\pi|_{I \times S^3 \setminus \bar{K}} : I \times S^3 \setminus \bar{K} \rightarrow I$ is a trivial bundle over I and thus its fibers $\pi^{-1}(0) = \{0\} \times S^3 \setminus \tilde{\alpha}_0(S^1)$ and $\pi^{-1}(1) = \{1\} \times S^3 \setminus \tilde{\alpha}_1(S^1)$ are homeomorphic (where $\tilde{\alpha}_0 : S^1 \rightarrow S^3$ and $\tilde{\alpha}_1 : S^1 \rightarrow S^3$ are the extensions of α_0 and α_1 respectively). In other words, the complements of the knots $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ are homeomorphic. Note that the complements of the knots in S^3 are homeomorphic if and only if they are ambient isotopic (see [8]). Therefore, the knots $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ are ambient isotopic, and hence the knots $\phi = \alpha_0$ and $\psi = \alpha_1$ are topologically equivalent. $\square$

The converse of Theorem 4.4.4 is not true. This fact will be clear from the next proposition and its corollaries.

Proposition 4.4.5. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $t \mapsto (f(t), g(t), h(t))$ be a polynomial knot in $\mathcal{Q}_d$, and let ψ be its mirror image which is given by $t \mapsto (f(t), g(t), -h(t))$. Then the polynomial knots ϕ and ψ do not belong to the same path component of the space $\mathcal{Q}_d$.*

Proof. Suppose on the contrary that the polynomial knots ϕ and ψ belong to the same path component of the space $\mathcal{Q}_d$, and let $\Phi : I \rightarrow \mathcal{Q}_d$ be a path from ϕ to ψ . For $s \in I$, let $\Phi_s = \Phi(s)$ and let it be given by

$$\Phi_s(t) = \left(\alpha_0(s) + \alpha_1(s)t + \cdots + \alpha_{d-2}(s)t^{d-2}, \beta_0(s) + \beta_1(s)t + \cdots + \beta_{d-1}(s)t^{d-1}, \right. \\ \left. \gamma_0(s) + \gamma_1(s)t + \cdots + \gamma_d(s)t^d \right)$$

for $t \in \mathbb{R}$. It is easy to see that the maps α_i 's, β_i 's and γ_i 's are continuous. Let f , g and h be given by

$$\begin{aligned} f(t) &= a_0 + a_1t + \cdots + a_{d-2}t^{d-2}, \\ g(t) &= b_0 + b_1t + \cdots + b_{d-1}t^{d-1} \quad \text{and} \\ h(t) &= c_0 + c_1t + \cdots + c_d t^d. \end{aligned}$$

Since $\Phi_0 = \phi$ and $\Phi_1 = \psi$, we have $\alpha_i(0) = \alpha_i(1) = a_i$, $\beta_i(0) = \beta_i(1) = b_i$, $\gamma_i(0) = c_i$ and $\gamma_i(1) = -c_i$ for each i . In particular, $\gamma_d(0) = c_d$ and $\gamma_d(1) = -c_d$. Note that $c_d \neq 0$ and γ_d is a continuous map, so by the intermediate value theorem, $\gamma_d(u) = 0$ for some $u \in (0, 1)$. Therefore, the third component of Φ_u has degree less than d . Hence Φ_u is not an element of the space $\mathcal{Q}_d$. This is a contradiction, since Φ was a path in $\mathcal{Q}_d$. $\square$

Corollary 4.4.6. *Let $t \mapsto (f(t), g(t), h(t))$ be a polynomial knot in $\mathcal{Q}_d$. Then*

there are eight distinct path components in $\mathcal{Q}_d$ each of which contains exactly one of the knot $t \mapsto ((e_1f(t), e_2g(t), e_3h(t)))$ for $(e_1, e_2, e_3) \in \{-1, 1\}^3$.

Corollary 4.4.7. *The number of path components of the space $\mathcal{Q}_d$ are in multiples of eight.*

Note that by Theorem 4.4.4, two polynomial knots which are not topologically equivalent cannot belong to the same path component in $\mathcal{Q}_d$. Therefore, we have the following:

Corollary 4.4.8. *If there are n distinct knot-types (up to mirror images) which can be represented in $\mathcal{Q}_d$, then it has at least $8n$ path components.*

Proposition 4.4.9. *Let ϕ be a polynomial knot in $\mathcal{Q}_d$. Suppose $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ be given by $t \mapsto \phi(\alpha t + \beta)$, where $\alpha > 0$ and β be any fixed real numbers. Then both ϕ and ψ belong to the same path component of the space $\mathcal{Q}_d$.*

Proof. Define $F : I \rightarrow \mathcal{Q}_d$ by $F(s) = F_s$, where

$$F_s(t) = \phi((1 - s + \alpha s)t + \beta s)$$

for $t \in \mathbb{R}$. By Remark 2.6, for each $s \in I$, F_s is an element of the space $\mathcal{Q}_d$. Thus, the image of F is in fact contained in $\mathcal{Q}_d$. Note that $F_0 = \phi$ and $F_1 = \psi$. Hence F is a path in $\mathcal{Q}_d$ which connects ϕ and ψ . $\square$

Proposition 4.4.10. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $t \mapsto (f(t), g(t), h(t))$ be a polynomial knot in $\mathcal{Q}_d$. For each $i \in \{1, 2, 3\}$, let $\alpha_i > 0$, β_i and γ_i be any fixed real numbers, and suppose $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial knot given by $t \mapsto (\alpha_1 f(t) + \gamma_1, \alpha_2 g(t) + \beta_1 f(t) + \gamma_2, \alpha_3 h(t) + \beta_2 f(t) + \beta_3 g(t) + \gamma_3)$. Then both ϕ and ψ belong to the same path component of the space $\mathcal{Q}_d$.*

Proof. For $s \in I$, define $T_s : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$\begin{aligned} T_s(x, y, z) = & ((1 - s + \alpha_1 s)x + \gamma_1 s, \beta_1 s x + (1 - s + \alpha_2 s)y + \gamma_2 s, \\ & \beta_2 s x + \beta_3 s y + (1 - s + \alpha_3 s)z + \gamma_3 s). \end{aligned}$$

It is easy to see that for any $s \in I$, T_s is an affine transformation. Let $H_s = T_s \circ \phi$, then by Remark 2.7, $H_s \in \mathcal{Q}_d$ for all $s \in I$. Note that $H_0 = \phi$ and $H_1 = \psi$. This shows that the map $H : I \rightarrow \mathcal{Q}_d$ given by $H(s) = H_s$ is a path in $\mathcal{Q}_d$ which connects ϕ and ψ . $\square$

Corollary 4.4.11. *Every path component of the space $\mathcal{Q}_d$ is unbounded.*

Proof. Let U be a path component of the space $\mathcal{Q}_d$, and let ϕ be a polynomial knot belonging to it. In Proposition 4.4.10 by taking α_i 's arbitrarily large, and β_i 's and γ_i 's to be zero, the distance of ψ from the zero map $0 : \mathbb{R} \rightarrow \mathbb{R}^3$ (given by $0(t) = (0, 0, 0)$) can be made arbitrarily large. Thus, there are polynomial knots at arbitrarily large distances from the origin which belong to U . In other words, each path component of the space $\mathcal{Q}_d$ is unbounded. $\square$

Corollary 4.4.12. *Every neighborhood of the zero map $0 \in \mathcal{A}_d$ intersects each path component of the space $\mathcal{Q}_d$.*

Proof. Let U be a path component of the space $\mathcal{Q}_d$, and let $\phi \in U$. In Proposition 4.4.10 by taking α_i 's arbitrarily small, and β_i 's and γ_i 's to be zero, the distance of ψ from the zero map can be made arbitrarily small. Hence, there are polynomial knots in U which are arbitrarily close to the origin. In other words, every neighborhood of the zero map intersects each path component of the space $\mathcal{Q}_d$. $\square$

Chapter 5

Polynomial Representations of Knots

In this chapter, we discuss polynomial representations of knots in their minimal degree. It is known that the knots 0_1 , 3_1 and 4_1 have polynomial representations in degree less than or equal to 6 (see [4, 9]). By Proposition 2.2.4, the other possible knots in degree 6 must be the five and six crossing knots which have superbridge index less than or equal to 3. Since the knots 5_1 , $3_1\#3_1$ and $3_1\#3_1^*$ have superbridge index 4, they cannot be represented in degree 6. Thus, the five and six crossing knots which could have a representation in degree 6 are the knots 5_2 , 6_1 , 6_2 and 6_3 . In Section 5.1, we show that it is almost impossible to represent these knots in degree 6. Thus, we conjecture that they cannot be represented in degree 6. In [4, 9, 22, 28], the knots 3_1 , 4_1 , 5_1 and 8_{19} have been represented in degrees 5, 6, 7 and 7 respectively, which are the minimal degrees for these knots. In Section 5.2, we present polynomial representations of the knots 5_2 , 6_1 , 6_2 , 6_3 , $3_1\#3_1$ and $3_1\#3_1^*$. These representations are in degree 7. Among these, the representations of the knots $3_1\#3_1$ and $3_1\#3_1^*$ are in their minimal degree, since their superbridge index is 4. There are no known examples of knots other than 0_1 ,

$3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3, 3_1\#3_1, 3_1\#3_1^*$ and 8_{19} in degree 7. In Section 5.3, we find the exact number of path components of the spaces $\mathcal{Q}_3$ and $\mathcal{Q}_4$. We prove that the path components of these spaces are contractible. We also find the exact number of the path components of the space $\mathcal{Q}_5$ corresponding to the trefoil knot and its mirror image and show that these path components of the space $\mathcal{Q}_5$ are contractible. At the end of Section 5.3, we give some lower bound on the number of path components for $\mathcal{Q}_5, \mathcal{Q}_6$ and $\mathcal{Q}_7$.

5.1 Polynomial knots in degree $d \leq 7$

In connection with the polynomial representations of knots, we have the following question:

Question 5.1. *Given a positive integer d , what are the knots that can be represented in degree d ?*

For example, by Proposition 2.2.4, the unknot, the trefoil knot and its mirror image are the only knots that can be represented in degree 5. In general, Question 5.1 is difficult to answer. In this section, we partially answer this question for $d \leq 7$.

Proposition 5.1.1. *The unknot is the only knot that can be represented by a polynomial knot in degree $d \leq 4$.*

Proof. Let κ be a knot which has a polynomial representation in degree $d \leq 4$. By Proposition 2.2.4, we have $c[\kappa] \leq \frac{(d-2)(d-3)}{2} \leq \frac{(4-2)(4-3)}{2} = 1$. Thus, the knot κ must be an unknot. $\square$

Proposition 5.1.2. *The unknot, the trefoil knot and its mirror image are the only knots that can be represented in degree 5.*

Proof. Let κ be a knot which has a polynomial representation in degree 5.

By Proposition 2.2.4, we have $c[\kappa] \leq \frac{(5-2)(5-3)}{2} = 3$. Thus, the knot κ is either an unknot or a trefoil knot or its mirror image. $\square$

The knots which have polynomial representation in degree 5 naturally have their representation in degree 6 as well. By Proposition 2.2.4, a knot which has a representation in degree 6, the crossing number and the superbridge index must be at most 6 and 3 respectively. We have a polynomial representation of the figure-eight knot in degree 6 (see [9]). However, the knots 5_1 , $3_1 \# 3_1$ and $3_1 \# 3_1^*$ cannot be represented in degree 6, since their superbridge index is 4. For the knot 5_2 , we have the following:

Proposition 5.1.3. *There exist polynomials f and g of degrees 4 and 5 respectively such that the map $t \mapsto (f(t), g(t))$ represents a regular projection of the knot 5_2 .*

Proof. Consider a plane curve C given by the parametric equation $t \mapsto (t^4, t^5)$. This curve has an isolated singularity at the origin. For such plane curves, there are two important numbers that remain invariant under any formal isomorphisms of plane curves. The first one is the Milnor number μ and the other is the δ invariant (see [25]). For a single component plane curve, they satisfy the relation $2\delta = \mu$.

In the present case, it turns out that the δ invariant is equal to $\frac{(5-1)(4-1)}{2} = 6$. The δ invariant of a plane curve which is singular at the origin measures the number of double points that can be created in a neighborhood of the origin. Note that, we have $\delta \geq 5$. Using a result of Pecker [10] we can deform the curve C into a new curve $\tilde{C}$ given by $t \mapsto (a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0, b_5t^5 + b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0)$ such that $\tilde{C}$ has 5 real nodes and 1 imaginary node. By continuity argument, we can choose the coefficients a_i 's and b_i 's such that the nodes occur in the order they are in the regular projection of the given knot.

In fact, we have the curve $t \mapsto (2(t-2)(t+4)(t^2-11), t(t^2-6)(t^2-16))$,

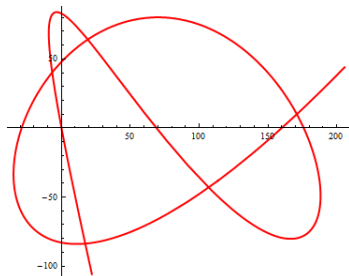


Figure 5.1: Projection of the knot 5_2

which represents a regular projection of the knot 5_2 as in Figure 5.1. $\square$

For a regular projection $t \mapsto (f(t), g(t))$ of the knot 5_2 with $\deg(f) = 4$ and $\deg(g) = 5$, we would like to find a polynomial h of least possible degree such that $t \mapsto (f(t), g(t), h(t))$ represents the knot 5_2 . In this direction, we have the following theorem:

Theorem 5.1.4. *For a generic choice of a regular projection $t \mapsto (f(t), g(t))$ of the knot 5_2 with $\deg(f) = 4$ and $\deg(g) = 5$, there does not exist any polynomial h of degree 6 such that a polynomial map $t \mapsto (f(t), g(t), h(t))$ represents the knot 5_2 .*

Proof. Let $t \mapsto (x(t), y(t))$ be a regular projection of the knot 5_2 such that $\deg(x) = 4$ and $\deg(y) = 5$ (note that, by Proposition 5.1.3, such a regular projection exists). Suppose $x(t) = \alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0$ and $y(t) = \beta_5 t^5 + \beta_4 t^4 + \beta_3 t^3 + \beta_2 t^2 + \beta_1 t + \beta_0$. By Bézout's theorem, the projection $t \mapsto (x(t), y(t))$ has either 5 or 6 double points. Let us consider the following two cases:

Case (1) If the projection has 5 crossings: It is easy to see that there exists an open set U in $\mathbb{R}^{11}$ such that $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \in U$

and $U \subseteq V$, where

$$V = \left\{ (a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^{11} \mid \text{a map } t \mapsto (a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0, b_5t^5 + b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0) \text{ is a regular projection of the knot } 5_2 \text{ with 5 double points} \right\}.$$

Let $(a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, b_5)$ be an element in U , and suppose $f(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ and $g(t) = b_5t^5 + b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0$. We want a polynomial $h(t) = c_6t^6 + c_5t^5 + c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$ such that $t \mapsto (f(t), g(t), h(t))$ is a smooth embedding which represents the knot 5_2 . By composing this embedding with a suitable affine transformation, we can assume that the coefficients c_0 , c_4 and c_5 are zero. Thus, we can take $h(t) = c_6t^6 + c_3t^3 + c_2t^2 + c_1t$. For $i \in \{1, 2, \dots, 5\}$, let (s_i, t_i) with $s_i < t_i$ and $s_1 < s_2 < \dots < s_5$ be the pairs of parametric values at which the crossings occur in the curve $t \mapsto (f(t), g(t))$. Since we want alternatively over and under crossings, we should have $h(t_i) - h(s_i)$ is positive if i is odd (that is, the i^{th} crossing is over crossing) and it is negative if i is even (that is, the i^{th} crossing is under crossing). In other words, we want to find the coefficients c_1 , c_2 , c_3 and c_6 such that $h(t_i) - h(s_i) = (-1)^{i+1}r_i$ for some positive real numbers r_1, r_2, r_3, r_4 and r_5 . This gives us a system of five linear equations in four unknowns as follows:

$$c_6(t_1^6 - s_1^6) + c_3(t_1^3 - s_1^3) + c_2(t_1^2 - s_1^2) + c_1(t_1 - s_1) = r_1 \quad (5.1)$$

$$c_6(t_2^6 - s_2^6) + c_3(t_2^3 - s_2^3) + c_2(t_2^2 - s_2^2) + c_1(t_2 - s_2) = -r_2 \quad (5.2)$$

$$c_6(t_3^6 - s_3^6) + c_3(t_3^3 - s_3^3) + c_2(t_3^2 - s_3^2) + c_1(t_3 - s_3) = r_3 \quad (5.3)$$

$$c_6(t_4^6 - s_4^6) + c_3(t_4^3 - s_4^3) + c_2(t_4^2 - s_4^2) + c_1(t_4 - s_4) = -r_4 \quad (5.4)$$

$$c_6(t_5^6 - s_5^6) + c_3(t_5^3 - s_5^3) + c_2(t_5^2 - s_5^2) + c_1(t_5 - s_5) = r_5. \quad (5.5)$$

The rank of the coefficient matrix

$$A = \begin{bmatrix} t_1^6 - s_1^6 & t_1^3 - s_1^3 & t_1^2 - s_1^2 & t_1 - s_1 \\ t_2^6 - s_2^6 & t_2^3 - s_2^3 & t_2^2 - s_2^2 & t_2 - s_2 \\ t_3^6 - s_3^6 & t_3^3 - s_3^3 & t_3^2 - s_3^2 & t_3 - s_3 \\ t_4^6 - s_4^6 & t_4^3 - s_4^3 & t_4^2 - s_4^2 & t_4 - s_4 \\ t_5^6 - s_5^6 & t_5^3 - s_5^3 & t_5^2 - s_5^2 & t_5 - s_5 \end{bmatrix}$$

of the system of linear equations (5.1)–(5.5) is at most 4. The system has a solution if and only if the rank of A is equal to the rank of the augmented matrix $\tilde{A}$ which is given by

$$\tilde{A} = \left[\begin{array}{cccc|c} t_1^6 - s_1^6 & t_1^3 - s_1^3 & t_1^2 - s_1^2 & t_1 - s_1 & r_1 \\ t_2^6 - s_2^6 & t_2^3 - s_2^3 & t_2^2 - s_2^2 & t_2 - s_2 & -r_2 \\ t_3^6 - s_3^6 & t_3^3 - s_3^3 & t_3^2 - s_3^2 & t_3 - s_3 & r_3 \\ t_4^6 - s_4^6 & t_4^3 - s_4^3 & t_4^2 - s_4^2 & t_4 - s_4 & -r_4 \\ t_5^6 - s_5^6 & t_5^3 - s_5^3 & t_5^2 - s_5^2 & t_5 - s_5 & r_5 \end{array} \right].$$

In other words, the system of linear equations (5.1)–(5.5) has no solution if $\det(\tilde{A}) \neq 0$. Note that, for any fixed $(r_1, r_2, r_3, r_4, r_5) \in (\mathbb{R}^+)^5$, the determinant of $\tilde{A}$ is an algebraic function of s_i 's and t_i 's. Since s_i 's and t_i 's are analytic functions of the coefficients a_k 's of f and the coefficients b_k 's of g , the determinant of $\tilde{A}$ is also an analytic function of a_k 's and b_k 's. Thus, for any fixed $(r_1, r_2, r_3, r_4, r_5) \in (\mathbb{R}^+)^5$,

$$W = \{(a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^{11} \mid \det(\tilde{A}) \neq 0\}$$

is an open and dense subset of $\mathbb{R}^{11}$, and hence the set $U \cap W$ is open and dense in U . Note that, for any choice of an element in $U \cap W$, the system of linear equations (5.1)–(5.5) has no solution. This shows that for a generic

choice of a regular projection $t \mapsto (f(t), g(t))$ of the knot 5_2 with $\deg(f) = 4$ and $\deg(g) = 5$ and having 5 crossings, there does not exist a polynomial h of degree 6 such that $t \mapsto (f(t), g(t), h(t))$ represents the knot 5_2 .

Case (2) If the projection has 6 crossings: By a similar argument as above we can show that for a generic choice of a regular projection $t \mapsto (f(t), g(t))$ of the knot 5_2 with $\deg(f) = 4$ and $\deg(g) = 5$ and having 6 crossings, there does not exist a polynomial h of degree 6 such that $t \mapsto (f(t), g(t), h(t))$ represents the knot 5_2 . $\square$

By Theorem 5.1.4, it is almost impossible to represent the knot 5_2 in degree 6. The same can be argued for the knots 6_1 , 6_2 and 6_3 .

Conjecture 5.1. *Each of the knots 5_2 , 6_1 , 6_2 and 6_3 cannot be represented by a polynomial knot in degree 6.*

The knots 0_1 , 3_1 , 4_1 , 5_1 , 5_2 , 6_1 , 6_2 , 6_3 , $3_1 \# 3_1$, $3_1 \# 3_1^*$ and 8_{19} have polynomial representations in degree less than or equal to 7 (see §5.2). Thus, each of them has a polynomial representation in degree 7 as well. Besides these knots, no knot is known to have a polynomial representation in degree 7. By Proposition 2.2.4, the other possible knots in degree 7 are the knots up to ten crossings having superbridge index less than or equal to 4. Let us summarize the discussion in Table 5.1.

Table 5.1: Polynomial knots of degree at most 7

Degree d	Knots which are known to have a representation in degree d	Other possible knots in degree d
$d \leq 4$	0_1	none
$d = 5$	0_1 and 3_1	none
$d = 6$	0_1 , 3_1 and 4_1	5_2 , 6_1 , 6_2 and 6_3
$d = 7$	0_1 , 3_1 , 4_1 , 5_1 , 5_2 , 6_1 , 6_2 , 6_3 , $3_1 \# 3_1$, $3_1 \# 3_1^*$ and 8_{19}	knots with $c[\kappa] \leq 10$ and $sb[\kappa] \leq 4$

5.2 Polynomial representations of knots up to six crossings

By Proposition 2.2.4, it is clear that in order to represent a knot with a certain number of crossings the degree of its polynomial representation has some lower bound. The knots with the same number of crossings may have different crossing patterns. The result below tells us how the degree relates to the nature of the crossings.

Theorem 5.2.1. *Let $t \mapsto (f(t), g(t))$ be a regular projection of a long knot $\kappa : \mathbb{R} \rightarrow \mathbb{R}^3$ having n double points, where f and g are real polynomials of degrees n and $n + 1$ respectively. Suppose the crossing data of κ is such that there are r changes from over/under crossings to under/over crossings as we move along the knot. Then there exists a polynomial h with degree $d \leq \min\{n + 2, r\}$ such that the polynomial map $t \mapsto (f(t), g(t), h(t))$ is an embedding which is ambient isotopic to κ .*

Proof. Let f and g be given by $f(t) = a_0 + a_1t + \cdots + a_nt^n$ and $g(t) = b_0 + b_1t + \cdots + b_{n+1}t^{n+1}$. The double points of the curve $t \mapsto (f(t), g(t))$ can be obtained by finding the real roots of the resultant Γ of the polynomials

$$F(s, t) = a_1 + a_2(s + t) + \cdots + a_n(s^{n-1} + s^{n-2}t + \cdots + t^{n-1}) \quad \text{and}$$

$$G(s, t) = b_1 + b_2(s + t) + \cdots + b_{n+1}(s^n + s^{n-1}t + \cdots + t^n).$$

Let $[a, b]$ be an interval that contains all the roots of Γ . Let us call these roots as the crossing points. We divide the interval $[a, b]$ into sub intervals $a = u_0 < u_1 < \cdots < u_r < u_{r+1} = b$ in such a way that the u_i 's are not from the crossing points and within any sub interval $[u_i, u_{i+1}]$ all the crossing points are either under crossing points or over crossing points. Let $h_1(t) = \pm \prod_{i=1}^r (t - u_i)$ according to which the last crossing is over crossing

or under crossing. The map h_1 is a polynomial of degree r that has opposite signs at under crossing points and over crossing points. Thus, the embedding $t \mapsto (f(t), g(t), h_1(t))$ represents the knot κ .

For $i \in \{1, 2, \dots, n\}$, let (s_i, t_i) with $s_i < t_i$ and $s_1 < s_2 < \dots < s_n$ be pairs of parametric values at which the projection has double points. Let $c_1 t + c_2 t^2 + \dots + c_{n+2} t^{n+2}$ be a polynomial of degree $n+2$, where c_i 's are unknowns which we have to find by solving the system

$$\begin{aligned} c_{n+2} (t_1^{n+2} - s_1^{n+2}) + c_{n+1} (t_1^{n+1} - s_1^{n+1}) + \dots + c_1 (t_1 - s_1) &= e_1 \\ c_{n+2} (t_2^{n+2} - s_2^{n+2}) + c_{n+1} (t_2^{n+1} - s_2^{n+1}) + \dots + c_1 (t_2 - s_2) &= e_2 \\ \vdots & \\ c_{n+2} (t_n^{n+2} - s_n^{n+2}) + c_{n+1} (t_n^{n+1} - s_n^{n+1}) + \dots + c_1 (t_n - s_n) &= e_n \end{aligned}$$

of n linear equations in $n+2$ unknowns. The numbers e_i 's are arbitrary but fixed non-zero real numbers and they are positive or negative according to which the crossing is under crossing or over crossing. The conditions $f(s_i) = f(t_i)$ and $g(s_i) = g(t_i)$ for $i \in \{1, 2, \dots, n\}$ imply that the matrix

$$M = \begin{bmatrix} t_1^{n+2} - s_1^{n+2} & t_1^{n+1} - s_1^{n+1} & \dots & t_1 - s_1 \\ t_2^{n+2} - s_2^{n+2} & t_2^{n+1} - s_2^{n+1} & \dots & t_2 - s_2 \\ \vdots & \vdots & & \vdots \\ t_n^{n+2} - s_n^{n+2} & t_n^{n+1} - s_n^{n+1} & \dots & t_n - s_n \end{bmatrix}$$

has rank n . Therefore, the system above has infinitely many solutions. Let $c_1 = d_1, c_2 = d_2, \dots, c_{n+2} = d_{n+2}$ be any one of the solution. With this solution, the polynomial $h_2(t) = d_1 t + d_2 t^2 + \dots + d_{n+2} t^{n+2}$ is such that the embedding $t \mapsto (f(t), g(t), h_2(t))$ represents the knot κ . $\square$

Let us discuss the following question.

Question 5.2. *Given a knot, what is its polynomial degree (that is, the least number d such that it has a polynomial representation in degree d)?*

We partially answer this question for knots up to six crossings. It is easy to note that the unknot has a polynomial representation given by $t \mapsto (0, 0, t)$ which is in its minimal degree. We now give the polynomial representations of the knots 3_1 , 4_1 , 5_1 and 8_{19} which were known before.

(1) Shastri [4] found a polynomial representation of the trefoil knot which is given by $t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$. This polynomial representation has degree 5. Since the trefoil knot has crossing number 3, it cannot be represented in degree less than 5 (see Proposition 2.2.4).

(2) McFeron [9] gave a polynomial representation of the figure-eight knot which is defined as $t \mapsto (f(t), g(t), h(t))$, where

$$\begin{aligned} f(t) &= -t^4 + 2.279283653t^3 + 5t^2 - 8.63068748t + 0.35140383, \\ g(t) &= t^5 - 5t^3 + 4t \quad \text{and} \\ h(t) &= (t - 1.379221313)(t - 2.05)(t + 0.2122155248)(t + 1.916737670) \\ &\quad \times (t + 2.06)(t + 10). \end{aligned}$$

This polynomial representation has degree 6. By Proposition 2.2.4, one can say that the figure-eight knot cannot be represented in degree less than 6.

(3) Auerbach [28] found a polynomial representation of the knot 5_1 which is given by $t \mapsto (u(t), v(t), w(t))$, where

$$\begin{aligned} u(t) &= (t^2 - 7t + 10)(t^2 + 5t + 4), \\ v(t) &= (t^2 - 5.2t)(t^2 - 9)(t + 4.7) \quad \text{and} \\ w(t) &= t^7 - 3.90763t^6 - 25.8835t^5 + 83.4739t^4 + 176.691t^3 \\ &\quad - 364.064t^2 - 331.888t + 321.285. \end{aligned}$$

(4) Mishra and Prabhakar [22] gave a polynomial representation of the knot 8_{19} which is defined as $t \mapsto (x(t), y(t), z(t))$, where

$$\begin{aligned} x(t) &= t^5 - 5.5t^3 + 4.5t, \\ y(t) &= t^6 - 7.35t^4 + 14t^2 - 7.8375 \quad \text{and} \\ z(t) &= t^7 - 8.13297t^5 + 18.5762t^3 - 10.4337t. \end{aligned}$$

Note that the polynomial representations (3) and (4) given above are in degree 7. Since the knots 5_1 and 8_{19} have superbridge index 4, they cannot have a representation in degree less than 7 (by Proposition 2.2.4).

The knots 5_2 , 6_1 , 6_2 and 6_3 were known to have representations in degree much higher than 7 (see [3, 17, 22]). We have found polynomial representations for these knots in degree 7 (see Appendix A). We have also produced representations of the composite knots $3_1 \# 3_1$ and $3_1 \# 3_1^*$ in degree 7 (see Appendix A). All these representations are given below:

(1) A polynomial representation (see Figure 5.2) of the knot 5_2 is given by $t \mapsto (f(t), g(t), h(t))$, where

$$\begin{aligned} f(t) &= 20(t - 10)(t - 17)(t + 15)(t + 21), \\ g(t) &= t(t^2 - 121)(t^2 - 400) \quad \text{and} \end{aligned}$$

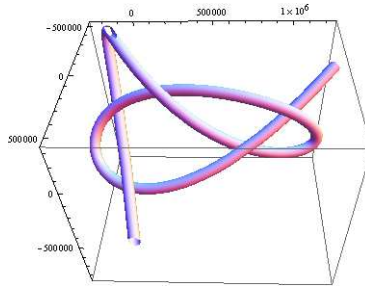


Figure 5.2: Representation of 5_2

$$h(t) = -0.005t(t^2 - 0.0343124t + 0.0107598)(t - 14.260128) \\ \times (t - 20.1133216)(t + 12.2430449)(t + 20.5785825).$$

(2) A polynomial representation (see Figure 5.3) of the knot 6_1 is given by $t \mapsto (u(t), v(t), w(t))$, where

$$u(t) = 60(t - 28)(t - 43.4)(t + 5)(t + 31.4)(t + 47.6), \\ v(t) = (t - 6)(t - 8)(t - 38)(t - 49)(t + 28)(t + 43.6) \quad \text{and} \\ w(t) = -0.07(t^2 - 54.4650519227t + 758.763745443)(t - 45.995024874) \\ \times (t^2 + 90.4819595699t + 2059.948386689)(t + 5.231021635) \\ \times (t + 19.036560084).$$

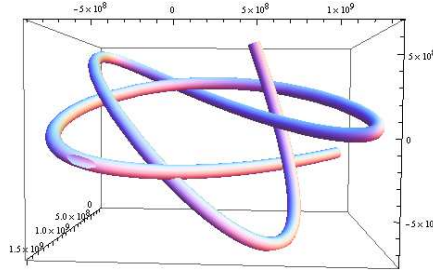
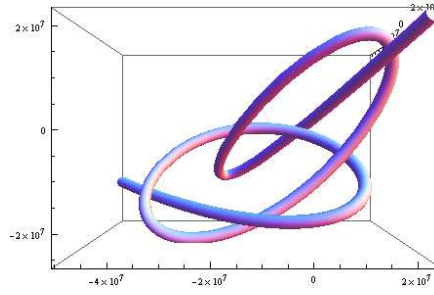


Figure 5.3: Representation of 6_1

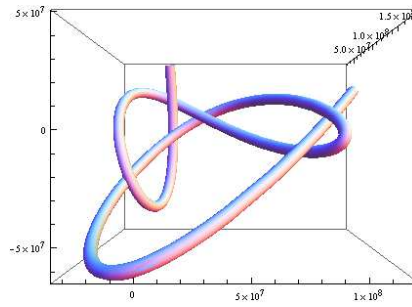
(3) A polynomial representation (see Figure 5.4) of the knot 6_2 is given by $t \mapsto (x(t), y(t), z(t))$, where

$$x(t) = 4(t^2 - 625)(t - 5)(t - 39)(t + 35), \\ y(t) = 0.1(t - 10)(t - 30)(t - 39)(t + 20)(t + 25)(t + 41) \quad \text{and} \\ z(t) = 0.005t(t^2 - 0.005429486t + 0.002423881)(t - 27.4156408) \\ \times (t - 39.8753791)(t + 28.436878)(t + 37.25572585).$$

Figure 5.4: Representation of 6_2

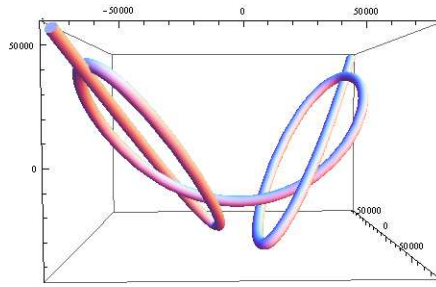
(4) A polynomial representation (see Figure 5.5) of the knot 6_3 is given by $t \mapsto (f(t), g(t), h(t))$, where

$$\begin{aligned}
 f(t) &= 15(t + 30)^2(t - 20)(t - 29)(t + 10), \\
 g(t) &= (t^2 - 400)(t - 6)(t - 32)(t + 4)(t + 30) \quad \text{and} \\
 h(t) &= -0.06(t^2 - 37.8892469397t + 376.737563885)(t - 33.329044815) \\
 &\quad \times (t^2 + 21.404400212t + 144.275534095) \\
 &\quad \times (t^2 + 61.56649851t + 955.985733648).
 \end{aligned}$$

Figure 5.5: Representation of 6_3

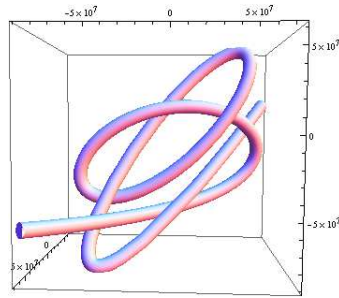
(5) A polynomial representation (see Figure 5.6) of the knot $3_1 \# 3_1$ is given by $t \mapsto (u(t), v(t), w(t))$, where

$$\begin{aligned}
u(t) &= 5t(t^2 - 17.5t + 77.3)(t^2 + 17.5t + 77.3), \\
v(t) &= (t^2 - 4.84)(t^2 - 53.29)(t^2 - 102.01) \quad \text{and} \\
w(t) &= -0.15t(t^2 - 68.11720396)(t^2 - 99.695462027)(t^2 + 0.025367747).
\end{aligned}$$

Figure 5.6: Representation of $3_1 \# 3_1$

(6) A polynomial representation (see Figure 5.7) of the knot $3_1 \# 3_1^*$ is given by $t \mapsto (x(t), y(t), z(t))$, where

$$\begin{aligned}
x(t) &= 30(t - 3.3)(t - 21.3)(t - 32.5)(t + 16.2)(t + 28), \\
y(t) &= (t - 6.8)(t - 23)(t - 34)(t + 12)(t + 21.7)(t + 33.1) \quad \text{and} \\
z(t) &= -0.03t(t^2 - 0.0082068t + 0.0043718)(t - 24.209735) \\
&\quad \times (t - 32.807367)(t + 15.257278)(t + 28.289226).
\end{aligned}$$

Figure 5.7: Representation of $3_1 \# 3_1^*$

The knots $3_1 \# 3_1$ and $3_1 \# 3_1^*$ have superbridge index 4, so by Proposition 2.2.4, they cannot be represented in degree less than 7. In other words, these knots have polynomial degree 7. The polynomial representations (1) to (4) above of the knots 5_2 , 6_1 , 6_2 and 6_3 are in degree 7. It is not known whether they can have a representation in degree 6 or not.

In Table 5.2, we give the polynomial degree of all knots up to six crossings including the eight crossing knot 8_{19} (recall that, the polynomial degree of a knot is the least number d such that the knot has a representation in degree d). The crossing number and the superbridge index are also mentioned.

Table 5.2: Polynomial degree of some knots

Sr. No.	Knot	Crossing number	Superbridge index	Polynomial degree
1.	0_1 (unknot)	0	1	1
2.	3_1 (trefoil knot)	3	3	5
3.	4_1 (figure-eight knot)	4	3	6
4.	5_1	5	4	7
5.	5_2	5	3 or 4	6 or 7
6.	6_1	6	3 or 4	6 or 7
7.	6_2	6	3 or 4	6 or 7
8.	6_3	6	3 or 4	6 or 7
9.	$3_1 \# 3_1$	6	4	7
10.	$3_1 \# 3_1^*$	6	4	7
11.	8_{19}	8	4	7

5.3 Path components in $\mathcal{Q}_d$ for $d \leq 7$

From Corollary 4.4.7, the total number of the path components of the space $\mathcal{Q}_d$, for $3 \leq d \leq 7$, are in multiples of eight. Note that we can represent the unknot in $\mathcal{Q}_3$ and $\mathcal{Q}_4$, so by Corollary 4.4.8, each of these spaces have at least eight path components. In fact, in this section, we prove that the

spaces $\mathcal{Q}_3$ and $\mathcal{Q}_4$, each has exactly eight path components. Furthermore, we estimate some lower bound on the number of path components of the spaces $\mathcal{Q}_5$, $\mathcal{Q}_6$ and $\mathcal{Q}_7$.

Proposition 5.3.1. *The space $\mathcal{Q}_3$ has exactly eight path components.*

Proof. For $e = (e_1, e_2, e_3)$ in $\{-1, 1\}^3$, consider the following set:

$$\mathcal{Q}_{3e} = \{ \phi \in \mathcal{A}_3 \mid e_1 a_{\phi 1} > 0, e_2 b_{\phi 2} > 0 \text{ and } e_3 c_{\phi 3} > 0 \}.$$

It is easy to see that $\mathcal{Q}_3 = \bigcup_e \mathcal{Q}_{3e}$. For any $e \in \{-1, 1\}^3$, the set $\mathcal{Q}_{3e}$ is path connected. Also, for $e \neq e'$, there is no path in $\mathcal{Q}_3$ from an element of the set $\mathcal{Q}_{3e}$ to an element of the set $\mathcal{Q}_{3e'}$. This shows that the sets $\mathcal{Q}_{3e}$, for $e \in \{-1, 1\}^3$, are nothing but the path components of the space $\mathcal{Q}_3$. $\square$

Theorem 5.3.2. *The space $\mathcal{Q}_4$ has exactly eight path components.*

For proving this theorem, we need the following lemmas.

Lemma 5.3.3. *A map $\phi \in \mathcal{A}_4$ given by $t \mapsto (t^2 + at, t^3 + bt, t^4 + ct)$ is an embedding if and only if $3a^2 + 4b > 0$ or $a^3 + 2ab + c \neq 0$.*

Proof. It is easy to verify that: a map $\phi \in \mathcal{A}_4$ given by $t \mapsto (t^2 + at, t^3 + bt, t^4 + ct)$ is a smooth embedding $\Leftrightarrow \phi(s) \neq \phi(t)$ for all $s \neq t$ and $\phi'(u) \neq 0$ for all $u \in \mathbb{R} \Leftrightarrow$ the equations

$$(s + t) + a = 0, \tag{5.6}$$

$$(s^2 + st + t^2) + b = 0 \quad \text{and} \tag{5.7}$$

$$(s^3 + s^2t + st^2 + t^3) + c = 0 \tag{5.8}$$

do not have a common real solution. Using (5.6) in (5.7), we get

$$t^2 + at + (a^2 + b) = 0. \quad (5.9)$$

This quadratic equation has the following solutions:

$$t_1 = \frac{-a - \sqrt{-3a^2 - 4b}}{2} \quad \text{and} \quad t_2 = \frac{-a + \sqrt{-3a^2 - 4b}}{2}.$$

In order to prove the lemma, it is sufficient to check the following three statements (which can be easily verified):

- (1) $3a^2 + 4b > 0 \Leftrightarrow$ the equation (5.9) has no real solution $\Leftrightarrow$ the equations (5.6) and (5.7) do not have a common real solution.
- (2) $3a^2 + 4b = 0$ and $a^3 + 2ab + c \neq 0 \Leftrightarrow 3a^2 + 4b = 0$ and $a^3 - 2c \neq 0 \Leftrightarrow t = -a/2$ is the only solution of the equation (5.9) and $a^3 - 2c \neq 0 \Leftrightarrow (s, t) = (-a/2, -a/2)$ is the only common real solution of the equations (5.6) and (5.7), but it is not a solution of the equation (5.8).
- (3) $3a^2 + 4b < 0$ and $a^3 + 2ab + c \neq 0 \Leftrightarrow t = t_1$ and $t = t_2$ are two distinct solutions of the equation (5.9) and $a^3 + 2ab + c \neq 0 \Leftrightarrow (s, t) = (t_1, t_2)$ and $(s, t) = (t_2, t_1)$ are the only common real solutions of the equations (5.6) and (5.7), but no one is a solution of the equation (5.8). $\square$

Lemma 5.3.4. *For $e_1, e_2, e_3 \in \{-1, 1\}$ and $a, b, c \in \mathbb{R}$, a polynomial map $\phi \in \mathcal{A}_4$ given by $t \mapsto (e_1 t^2 + at, e_2 t^3 + bt, e_3 t^4 + ct)$ is an embedding if and only if $3a^2 + 4e_2b > 0$ or $e_1 a^3 + 2e_1 e_2 ab + e_3 c \neq 0$.*

Proof. It is easy to note that the map ϕ is an embedding $\Leftrightarrow$ the map $\psi \in \mathcal{A}_4$ which is given by

$$t \mapsto (e_1(e_1 t^2 + at), e_2(e_2 t^3 + bt), e_3(e_3 t^4 + ct))$$

is an embedding. Note that the map ψ can be rewritten as $t \mapsto (t^2 + e_1 at, t^3 +$

$e_2bt, t^4 + e_3ct$). By Lemma 5.3.3, one can say that ψ (and hence ϕ) is an embedding $\Leftrightarrow 3a^2 + 4e_2b > 0$ or $e_1a^3 + 2e_1e_2ab + e_3c \neq 0$. $\square$

Lemma 5.3.5. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial knot in $\mathcal{Q}_4$. Then there exist $e_1, e_2, e_3 \in \{-1, 1\}$ and $a, b, c \in \mathbb{R}$ such that the polynomial knot ψ given by $t \mapsto (e_1t^2 + at, e_2t^3 + bt, e_3t^4 + ct)$ is path equivalent (in $\mathcal{Q}_4$) to the polynomial knot ϕ .*

Proof. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be a polynomial knot in $\mathcal{Q}_4$ and let it be given by $\phi(t) = (f(t), g(t), h(t))$, where $f(t) = a_2t^2 + a_1t + a_0$, $g(t) = b_3t^3 + b_2t^2 + b_1t + b_0$ and $h(t) = c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$. Note that a_2, b_3 and c_4 all are nonzero. Let us take $\alpha_1 = |a_2|^{-1}$, $\alpha_2 = |b_3|^{-1}$, $\alpha_3 = |c_4|^{-1}$, $\beta_1 = -b_2a_2^{-1}|b_3|^{-1}$, $\beta_2 = b_2c_3a_2^{-1}b_3^{-1}|c_4|^{-1} - c_2a_2^{-1}|c_4|^{-1}$, $\beta_3 = -c_3b_3^{-1}|c_4|^{-1}$, $\gamma_1 = -a_0|a_2|^{-1}$, $\gamma_2 = a_0b_2a_2^{-1}|b_3|^{-1} - b_0|b_3|^{-1}$ and $\gamma_3 = a_0c_2a_2^{-1}|c_4|^{-1} - a_0b_2c_3a_2^{-1}b_3^{-1}|c_4|^{-1} + b_0c_3b_3^{-1}|c_4|^{-1} - c_0|c_4|^{-1}$. Define $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$ by

$$t \mapsto (\alpha_1f(t) + \gamma_1, \alpha_2g(t) + \beta_1f(t) + \gamma_2, \alpha_3h(t) + \beta_2f(t) + \beta_3g(t) + \gamma_3).$$

By Proposition 4.4.10, the knots ϕ and ψ are path equivalent in $\mathcal{Q}_4$. Also, it can be easily checked that $\psi(t) = (e_1t^2 + at, e_2t^3 + bt, e_3t^4 + ct)$ for $t \in \mathbb{R}$, where $e_1 = a_2|a_2|^{-1}$, $e_2 = b_3|b_3|^{-1}$, $e_3 = c_4|c_4|^{-1}$, $a = a_1|a_2|^{-1}$, $b = b_1|b_3|^{-1} - a_1b_2a_2^{-1}|b_3|^{-1}$ and $c = a_1b_2c_3a_2^{-1}b_3^{-1}|c_4|^{-1} - a_1c_2a_2^{-1}|c_4|^{-1} - b_1c_3b_3^{-1}|c_4|^{-1} + c_1|c_4|^{-1}$. $\square$

Lemma 5.3.6. *For $e = (e_1, e_2, e_3) \in \{-1, 1\}^3$, the space*

$$\mathcal{N}_{4e} = \left\{ \phi \in \mathcal{Q}_4 \mid t \xrightarrow{\phi} (e_1t^2 + a_\phi t, e_2t^3 + b_\phi t, e_3t^4 + c_\phi t) \text{ for some } a_\phi, b_\phi, c_\phi \in \mathbb{R} \right\}$$

is path connected.

Proof. Suppose $e = (e_1, e_2, e_3)$ be an arbitrary element in the set $\{-1, 1\}^3$.

By Lemma 5.3.4, a map $\phi \in \mathcal{A}_4$ given by $t \mapsto (e_1 t^2 + a_\phi t, e_2 t^3 + b_\phi t, e_3 t^4 + c_\phi t)$ is an embedding if and only if $3a_\phi^2 + 4e_2 b_\phi > 0$ or $e_1 a_\phi^3 + 2e_1 e_2 a_\phi b_\phi + e_3 c_\phi \neq 0$. Therefore, it is easy to see that the set $\mathcal{N}_{4e}$ is the union of the following sets:

$$\begin{aligned}\mathcal{N}_{4e}^1 &= \left\{ \phi \in \mathcal{N}_{4e} \mid 3a_\phi^2 + 4e_2 b_\phi > 0 \right\}, \\ \mathcal{N}_{4e}^2 &= \left\{ \phi \in \mathcal{N}_{4e} \mid e_1 a_\phi^3 + 2e_1 e_2 a_\phi b_\phi + e_3 c_\phi > 0 \right\} \quad \text{and} \\ \mathcal{N}_{4e}^3 &= \left\{ \phi \in \mathcal{N}_{4e} \mid e_1 a_\phi^3 + 2e_1 e_2 a_\phi b_\phi + e_3 c_\phi < 0 \right\}.\end{aligned}$$

It is sufficient to show that every element of the sets $\mathcal{N}_{4e}^1$, $\mathcal{N}_{4e}^2$ and $\mathcal{N}_{4e}^3$ is connected by a path in $\mathcal{N}_{4e}$ to a fixed element $\phi_0 \in \mathcal{N}_{4e}^1$ which is given by $t \mapsto (e_1 t^2, e_2 t^3 + e_2 t, e_3 t^4)$. We do this in the following three steps:

(1) Let ϕ be an arbitrary element of the space $\mathcal{N}_{4e}^1$. Define $F : I \rightarrow \mathcal{A}_4$ by $F(s) = F_s$ for $s \in I$, where

$$F_s(t) = (e_1 t^2 + a_\phi s t, e_2 t^3 + (b_\phi s^2 + e_2 - e_2 s^2)t, e_3 t^4 + c_\phi s t)$$

for $t \in \mathbb{R}$. For $s \in (0, 1)$, we have $3(a_\phi s)^2 + 4e_2(b_\phi s^2 + e_2 - e_2 s^2) = s^2(3a_\phi^2 + 4e_2 b_\phi) + 4(1 - s^2) > 0$, since $\phi \in \mathcal{N}_{4e}^1$. This shows that the map F_s is an element of the space $\mathcal{N}_{4e}^1$. Also, we have $F_0 = \phi_0$ and $F_1 = \phi$. Hence F is a path in $\mathcal{N}_{4e}^1$ from ϕ_0 to ϕ .

(2) Let ψ be an arbitrary element of the space $\mathcal{N}_{4e}^2$. We choose an element $\psi_0 \in \mathcal{N}_{4e}^2$ given by $t \mapsto (e_1 t^2, e_2 t^3, e_3 t^4 + e_3 t)$. Define $G : I \rightarrow \mathcal{A}_4$ by $G(s) = G_s$, where

$$G_s(t) = (e_1 t^2 + a_\psi s t, e_2 t^3 + b_\psi s^2 t, e_3 t^4 + (c_\psi s^3 + e_3 - e_3 s^3)t)$$

for $t \in \mathbb{R}$. For $s \in (0, 1)$, the map G_s is an element of the space $\mathcal{N}_{4e}^2$, because $e_1(a_\psi s)^3 + 2e_1 e_2(a_\psi s)(b_\psi s^2) + e_3(c_\psi s^3 + e_3 - e_3 s^3) = s^3(e_1 a_\psi^3 + 2e_1 e_2 a_\psi b_\psi + e_3 c_\psi) + 1 - s^3 > 0$. Note that $G_0 = \psi_0$ and $G_1 = \psi$. Thus, the map G is a

path in $\mathcal{N}_{4e}^2$ from ψ_0 to ψ . Let $\Phi : I \rightarrow \mathcal{A}_4$ be a map defined as $\Phi(s) = \Phi_s$, where

$$\Phi_s(t) = (e_1t^2, e_2t^3 + e_2(1-s)t, e_3t^4 + e_3st)$$

for $t \in \mathbb{R}$. For $s \in (0, 1)$, the map Φ_s is an element of the space $\mathcal{N}_{4e}^1$. Note that $\Phi_0 = \phi_0$ and $\Phi_1 = \psi_0$. This shows that Φ is a path in $\mathcal{N}_{4e}$ from ϕ_0 to ψ_0 . The maps G and Φ both together give a path in $\mathcal{N}_{4e}$ from ϕ_0 to ψ .

(3) Suppose σ be an element in $\mathcal{N}_{4e}^3$. Let us take an element $\sigma_0 \in \mathcal{N}_{4e}^3$ given by $t \mapsto (e_1t^2, e_2t^3, e_3t^4 - e_3t)$. Define $H : I \rightarrow \mathcal{A}_4$ by $H(s) = H_s$, where

$$H_s(t) = (e_1t^2 + a_\sigma st, e_2t^3 + b_\sigma s^2t, e_3t^4 + (c_\sigma s^3 - e_3 + e_3s^3)t)$$

for $t \in \mathbb{R}$. For $s \in (0, 1)$, since $e_1(a_\sigma s)^3 + 2e_1e_2(a_\sigma s)(b_\sigma s^2) + e_3(c_\sigma s^3 - e_3 + e_3s^3) = s^3(e_1a_\sigma^3 + 2e_1e_2a_\sigma b_\sigma + e_3c_\sigma) - 1 + s^3 < 0$, so the map H_s is an element of the space $\mathcal{N}_{4e}^3$. Note that $H_0 = \sigma_0$ and $H_1 = \sigma$. This shows that H is a path in $\mathcal{N}_{4e}^3$ from σ_0 to σ . Let us define $\Psi : I \rightarrow \mathcal{A}_4$ by $\Psi(s) = \Psi_s$, where

$$\Psi_s(t) = (e_1t^2, e_2t^3 + e_2(1-s)t, e_3t^4 - e_3st)$$

for $t \in \mathbb{R}$. It is easy to note that $\Psi_0 = \phi_0$ and $\Psi_1 = \sigma_0$. For $s \in (0, 1)$, the map Ψ_s is an element of the space $\mathcal{N}_{4e}^1$. Hence Ψ is a path in $\mathcal{N}_{4e}$ from ϕ_0 to σ_0 . The maps H and Ψ both together give a path in $\mathcal{N}_{4e}$ from ϕ_0 to σ . $\square$

Proof of Theorem 5.3.2. For an element $e = (e_1, e_2, e_3)$ in $\{-1, 1\}^3$, consider the following set:

$$\mathcal{Q}_{4e} = \{ \phi \in \mathcal{Q}_4 \mid e_1a_{\phi 2} > 0, e_2b_{\phi 3} > 0 \text{ and } e_3c_{\phi 4} > 0 \}.$$

Note that $\mathcal{Q}_4 = \bigcup_e \mathcal{Q}_{4e}$. For $e \neq e'$, there is no path in $\mathcal{Q}_4$ from an element of the set $\mathcal{Q}_{4e}$ to an element of the set $\mathcal{Q}_{4e'}$. Now it is enough to prove that for each $e \in \{-1, 1\}^3$, the space $\mathcal{Q}_{4e}$ is path connected. Let $e = (e_1, e_2, e_3)$ be

an arbitrary element in the set $\{-1, 1\}^3$. From the proof of Lemma 5.3.5, it can be seen that every element in $\mathcal{Q}_{4e}$ is connected by a path in $\mathcal{Q}_{4e}$ to some element in $\mathcal{N}_{4e}$. By Lemma 5.3.6, the space $\mathcal{N}_{4e}$ is path connected, so the space $\mathcal{Q}_{4e}$ is also path connected. $\square$

It is easy to see that the path components of the space $\mathcal{Q}_3$ are contractible (for example, see proof of Theorem 5.3.1). Next, we prove that every path component of the space $\mathcal{Q}_4$ is contractible by showing that one of its one point subspaces is a *weak deformation retract* in the sense of the following:

Definition 5.3.1. *We say that a subset A of a topological space X is a weak deformation retract if there is a continuous map $F : I \times X \rightarrow X$ such that $F(0, x) = x$ and $F(1, x) \in A$ for all $x \in X$. The map F is called a weak deformation retraction.*

The notion of a weak deformation retract we defined here is different from the notion of a *weak deformation retract* as defined in the textbooks of algebraic topology (for example, see [2]).

Remark 5.1. *A topological space X is contractible if and only if a one point subspace is a weak deformation retract of it. Furthermore, a weak deformation retract of a topological space is homotopy equivalent to it.*

Lemma 5.3.7. *Let A be a weak deformation retract of a space X and B be a weak deformation retract of A . Then B is a weak deformation retract of the space X .*

Proof. Since A is a weak deformation retract of X , there is a continuous map $F : I \times X \rightarrow X$ such that $F(0, x) = x$ and $F(1, x) \in A$ for all $x \in X$. Also, since B is a weak deformation retract of A , there exists a continuous map $G : I \times A \rightarrow A$ such that $G(0, a) = a$ and $G(1, a) \in B$ for all $a \in A$. Let

$H : I \times X \rightarrow X$ be a map given by

$$H(u, x) = \begin{cases} F(2u, x) & \text{if } 0 \leq u \leq 1/2 \\ G(2u - 1, F(1, x)) & \text{if } 1/2 \leq u \leq 1. \end{cases}$$

For any $x \in X$, we have $F(2(1/2), x) = F(1, x) = G(0, F(1, x)) = G(2(1/2) - 1, F(1, x))$. That is, $F(2u, x) = G(2u - 1, F(1, x))$ for $u = 1/2$ and for all $x \in X$. Thus, the map H is well defined and it is continuous. Note that $H(0, x) = F(0, x) = x$ for all $x \in X$. Also for any $x \in X$, $H(1, x) = G(1, F(1, x)) = G(1, a) = b$ for some $a \in A$ and some $b \in B$. This shows that the space B is a weak deformation retract of the space X . $\square$

Remark 5.2. *Lemma 5.3.7 says that, being a weak deformation retract is a transitive relation. In particular, if a contractible subspace is a weak deformation retract of a space X , then the space X is contractible.*

Theorem 5.3.8. *The path components of the space $\mathcal{Q}_4$ are contractible.*

Proof. We need to prove that, for a given $e \in \{-1, 1\}^3$, the space

$$\mathcal{Q}_{4e} = \{ \phi \in \mathcal{Q}_4 \mid e_1 a_{\phi 2} > 0, e_2 b_{\phi 3} > 0 \text{ and } e_3 c_{\phi 4} > 0 \},$$

is contractible (see proof of Theorem 5.3.2). It is easy to see that the set

$$\mathcal{L}_{4e} = \{ \phi \in \mathcal{Q}_{4e} \mid a_{\phi 0} = b_{\phi 0} = c_{\phi 0} = 0 \}$$

is a weak deformation retract of the space $\mathcal{Q}_{4e}$. Define $F : I \times \mathcal{L}_{4e} \rightarrow \mathcal{L}_{4e}$ by $F(s, \phi) = F_{s\phi}$, where

$$F_{s\phi}(t) = \left(f_{\phi}(t), g_{\phi}(t) - \frac{b_{\phi 2}}{a_{\phi 2}} s f_{\phi}(t), h_{\phi}(t) + \frac{b_{\phi 2} c_{\phi 3} - b_{\phi 3} c_{\phi 2}}{a_{\phi 2} b_{\phi 3}} s f_{\phi}(t) - \frac{c_{\phi 3}}{b_{\phi 3}} s g_{\phi}(t) \right)$$

for $t \in \mathbb{R}$. The map F is continuous. Let us consider the following set:

$$\mathcal{M}_{4e} = \left\{ \phi \in \mathcal{Q}_{4e} \mid t \xrightarrow{\phi} (a_{\phi 2}t^2 + a_{\phi 1}t, b_{\phi 3}t^3 + b_{\phi 1}t, c_{\phi 4}t^4 + c_{\phi 1}t) \right\}.$$

We have $F(0, \phi) = \phi$ and $F(1, \phi) \in \mathcal{M}_{4e}$ for all $\phi \in \mathcal{L}_{4e}$. Thus, $\mathcal{M}_{4e}$ is a weak deformation retract of the space $\mathcal{L}_{4e}$. Define $H : I \times \mathcal{M}_{4e} \rightarrow \mathcal{M}_{4e}$ by $H(s, \phi) = H_{s\phi}$, where

$$H_{s\phi}(t) = \left(\left(1 - s + \frac{1}{|a_{\phi 2}|} s \right) f_{\phi}(t), \left(1 - s + \frac{1}{|b_{\phi 3}|} s \right) g_{\phi}(t), \left(1 - s + \frac{1}{|c_{\phi 4}|} s \right) h_{\phi}(t) \right)$$

for $t \in \mathbb{R}$. Note that H is continuous. Also, we have $H(0, \phi) = \phi$ and $H(1, \phi) \in \mathcal{N}_{4e}$ for all $\phi \in \mathcal{M}_{4e}$ (where $\mathcal{N}_{4e}$ is as defined in Lemma 5.3.6). Therefore, $\mathcal{N}_{4e}$ is a weak deformation retract of the space $\mathcal{M}_{4e}$. Let us define $\Gamma : I \times \mathcal{N}_{4e} \rightarrow \mathcal{N}_{4e}$ by $\Gamma(s, \phi) = \Gamma_{s\phi}$, where

$$\Gamma_{s\phi}(t) = \left(e_1 t^2 + a_{\phi} t, e_2 t^3 + (b_{\phi} + e_2 s |b_{\phi}| + e_2 s) t, e_3 t^4 + (c_{\phi} - 2e_1 e_3 s a_{\phi} |b_{\phi}| - 2e_1 e_3 s a_{\phi}) t \right)$$

for $t \in \mathbb{R}$. The map Γ is continuous. By Lemma 5.3.6, the space $\mathcal{N}_{4e}$ is a union of the following sets $\mathcal{N}_{4e}^1$, $\mathcal{N}_{4e}^2$ and $\mathcal{N}_{4e}^3$. One can check that: (a) $\Gamma_{s\phi} \in \mathcal{N}_{4e}^1$ for all $(s, \phi) \in (0, 1) \times \mathcal{N}_{4e}^1$; (b) $\Gamma_{s\phi} \in \mathcal{N}_{4e}^2$ for all $(s, \phi) \in (0, 1) \times \mathcal{N}_{4e}^2$; (c) $\Gamma_{s\phi} \in \mathcal{N}_{4e}^3$ for all $(s, \phi) \in (0, 1) \times \mathcal{N}_{4e}^3$; (d) $\Gamma_{0\phi} = \phi$ and $\Gamma_{1\phi} \in \mathcal{N}_{4e}^1$ for all $\phi \in \mathcal{N}_{4e}$. Thus, the image of Γ is in fact contained in $\mathcal{N}_{4e}$, and we have $\Gamma(0, \phi) = \phi$ and $\Gamma(1, \phi) \in \mathcal{N}_{4e}^1$ for all $\phi \in \mathcal{N}_{4e}$. This shows that the space $\mathcal{N}_{4e}^1$ is a weak deformation retract of the space $\mathcal{N}_{4e}$. Define $\Lambda : I \times \mathcal{N}_{4e}^1 \rightarrow \mathcal{N}_{4e}^1$ by $\Lambda(s, \phi) = \Lambda_{s\phi}$, where

$$\Lambda_{s\phi}(t) = \left(e_1 t^2 + (a_{\phi} - a_{\phi} s) t, e_2 t^3 + (b_{\phi} - 2b_{\phi} s + b_{\phi} s^2 + e_2 s) t, e_3 t^4 + (c_{\phi} - c_{\phi} s) t \right)$$

for $t \in \mathbb{R}$. Note that the map Λ is continuous. Let ϕ_0 be a fixed element in $\mathcal{Q}_{4e}$ which is given by $t \mapsto (e_1 t^2, e_2 t^3 + e_2 t, e_3 t^4)$. Note that $\Lambda(0, \phi) = \phi$ and

$\Lambda(1, \phi) = \phi_0$ for all $\phi \in \mathcal{N}_{4e}^1$. Therefore, the space $\mathcal{N}_{4e}^1$ is contractible, and hence by Lemma 5.3.7 and Remark 5.2, the space $\mathcal{Q}_{4e}$ is also contractible. $\square$

For $e = (e_1, e_2, e_3) \in \{-1, 1\}^3$, let

$$\mathcal{Q}_{5e} = \{ \phi \in \mathcal{Q}_5 \mid e_1 a_{\phi 3} > 0, e_2 b_{\phi 4} > 0 \text{ and } e_3 c_{\phi 5} > 0 \}.$$

Note that the space $\mathcal{Q}_5$ is disjoint union of the spaces $\mathcal{Q}_{5e}$ for $e \in \{-1, 1\}^3$.

Proposition 5.3.9. *For $e = (e_1, e_2, e_3) \in \{-1, 1\}^3$, let $p_e = e_1 e_2 e_3$. Then we have the following:*

- (1) *If $p_e = 1$, then the space $\mathcal{Q}_{5e}$ contains only one path component corresponding to the trefoil knot. Moreover, this path component is contractible. The space $\mathcal{Q}_{5e}$ does not contain any polynomial knot which represent the mirror image of the trefoil.*
- (2) *If $p_e = -1$, then the space $\mathcal{Q}_{5e}$ contains only one path component corresponding to the mirror image of the trefoil. Moreover, this path component is contractible. The space $\mathcal{Q}_{5e}$ does not contain any polynomial knot which represent the trefoil knot.*

Proof. Let $\mathcal{K}_{3,4,5}$ be the space of all polynomial knots of the type

$$t \mapsto (t^3 + a_2 t^2 + a_1 t, t^4 + b_3 t^3 + b_2 t^2 + b_1 t, t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t).$$

By Proposition 4-9 in [27], we have the following:

- (a) The space $\mathcal{K}_{3,4,5}$ contains only one path component which corresponds to the trefoil knot. Moreover, this path component is contractible.
- (b) The space $\mathcal{K}_{3,4,5}$ does not contain any polynomial knot which represent the mirror image of the trefoil.

For $e \in \{-1, 1\}^3$, let $F : \mathcal{Q}_{5e} \rightarrow \mathcal{K}_{3,4,5}$ and $H : \mathcal{K}_{3,4,5} \rightarrow \mathcal{Q}_{5e}$ be the maps given by

$$F(\phi) = \left(\frac{f_\phi - a_{\phi 0}}{a_{\phi 3}}, \frac{g_\phi - b_{\phi 0}}{b_{\phi 4}}, \frac{h_\phi - c_{\phi 0}}{b_{\phi 5}} \right) \quad \text{and} \quad H(\psi) = (e_1 f_\psi, e_2 g_\psi, e_3 h_\psi),$$

for $\phi \in \mathcal{Q}_{5e}$ and $\psi \in \mathcal{K}_{3,4,5}$. It is easy to note the following statements:

- (c) If $p_e = 1$, then a polynomial knot $\phi \in \mathcal{Q}_{5e}$ represents the trefoil knot if and only if the polynomial knot $F(\phi) \in \mathcal{K}_{3,4,5}$ represents the same knot. The space $\mathcal{Q}_{5e}$ does not contain any polynomial knot which represent the mirror image of the trefoil (since by the statement (b) above).
- (d) If $p_e = -1$, then a polynomial knot $\phi \in \mathcal{Q}_{5e}$ represents the mirror image the trefoil if and only if the polynomial knot $F(\phi) \in \mathcal{K}_{3,4,5}$ represents the trefoil knot. The space $\mathcal{Q}_{5e}$ does not contain any polynomial knot which represent the trefoil knot (since by the statement (b) above).

Note that the map $F \circ H$ is the identity map of $\mathcal{K}_{3,4,5}$. Also, it can be easily checked that the map $H \circ F$ is homotopy equivalent to the identity map of the space $\mathcal{Q}_{5e}$. Thus, we have the following:

- (e) The space $\mathcal{Q}_{5e}$ is homotopy equivalent to the space $\mathcal{K}_{3,4,5}$.

The statements (a), (c) and (e) together imply the statement (1) of the proposition, while the statements (a), (d) and (e) imply the statement (2) of the proposition. $\square$

Proposition 5.3.10. *The space $\mathcal{Q}_5$ has exactly eight path components corresponding to the trefoil knot and its mirror image. Moreover, all these path components are contractible.*

Proof. Since the space $\mathcal{Q}_5$ is disjoint union of the spaces $\mathcal{Q}_{5e}$, for $e \in \{-1, 1\}^3$, the result follows immediately from Proposition 5.3.9. $\square$

By Corollary 4.4.6, the space $\mathcal{Q}_5$ has at least eight path components corresponding to the unknot. Thus, we have an estimate as in Table 5.3.

Table 5.3: Number of path components of the space $\mathcal{Q}_5$

Sr. No.	Knot	# of path components of $\mathcal{Q}_5$ corresponding to a knot
1.	0_1	at least 8
2.	3_1 and 3_1^*	exactly 8 (all are contractible)
# of path components of $\mathcal{Q}_5$		at least 16

In Section 5.2, we gave the polynomial representations of the knots 0_1 , 3_1 and 4_1 . Note that all these representations are contained in $\mathcal{O}_6$. Thus, using Remark 3.2, we can get polynomial representations of these knots (and of their mirror images) which belong to $\mathcal{Q}_6$. Hence by Corollary 4.4.6 and Corollary 4.4.8, we have the estimate as in Table 5.4.

Table 5.4: Number of path components of the space $\mathcal{Q}_6$

Sr. No.	Knot	# of path components of $\mathcal{Q}_6$ corresponding to a knot
1.	0_1	at least 8
2.	3_1 and 3_1^*	at least 8
3.	4_1	at least 8
# of path components of $\mathcal{Q}_6$		at least 24

In Section 5.2, we gave the polynomial representations of the knots 0_1 , 3_1 , 4_1 , 5_1 , 5_2 , 6_1 , 6_2 , 6_3 , $3_1\#3_1$, $3_1\#3_1^*$ and 8_{19} which are contained in $\mathcal{O}_7$. Thus, by Remark 3.2, we can get representations of these knots (and of their mirror images) which belong to $\mathcal{Q}_7$. Hence by Corollary 4.4.6 and Corollary 4.4.8, we have the estimate as in Table 5.5.

Table 5.5: Number of path components of the space $\mathcal{Q}_7$

Sr. No.	Knot	# of path components of $\mathcal{Q}_7$ corresponding to a knot
1.	0_1	at least 8
2.	3_1 and 3_1^*	at least 8
3.	4_1	at least 8
4.	5_1 and 5_1^*	at least 8
5.	5_2 and 5_2^*	at least 8
6.	6_1 and 6_1^*	at least 8
7.	6_2 and 6_2^*	at least 8
8.	6_3	at least 8
9.	$3_1 \# 3_1$ and $3_1^* \# 3_1^*$	at least 8
10.	$3_1 \# 3_1^*$	at least 8
11.	8_{19} and 8_{19}^*	at least 8
# of path components of $\mathcal{Q}_7$		at least 88

Chapter 6

Conclusion

In this chapter, we summarize the thesis and discuss some problems and observations that we encountered during the work. We also share some thoughts in the direction of solving the problems that we discuss. This may extend the thesis work in the positive direction. We proceed chapter wise in the sequence.

In Chapter 3, we introduced the spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$, for $d \geq 3$, consisting of the polynomial knots of degree at most d with some extra conditions. We proved that these spaces are homeomorphic to some semialgebraic subsets of $\mathbb{R}^{3d}$ and hence they have only finitely many path components. *It will strengthen the result if we can make a precise number or at least have some estimate (in terms of d) on the number of path components.*

In Section 4.1, we discussed the topology of the space $\mathcal{O}_d$, for $d \geq 3$, and proved that this space has the same homotopy type as S^2 . This, in particular, says that the space $\mathcal{O}_d$ is path connected. In Section 4.3, we showed that the space $\mathcal{P}_d$, for $d \geq 3$, is path connected, but we could not infer its homotopy type. *It will be interesting to explore the homotopy type of this space.* Our impression is that this may be difficult in general, since we have to take care of the conditions imposed on the degrees of the component polynomials of

knots in $\mathcal{P}_d$ while constructing a suitable homotopy equivalence.

In Section 4.2, we discussed the space $\mathcal{P}$ of all polynomial knots with respect to the inductive limit topology that comes from the stratification $\mathcal{P} = \bigcup_{d \geq 3} \mathcal{O}_d$. We showed that this space has the same homotopy type as S^2 . In particular, the space $\mathcal{P}$ is path connected. *At this point, it is very natural to ask what happens if we have some other topology on the set $\mathcal{P}$ of all polynomial knots?* For example, we can identify the set $\mathcal{P}$ with a subset of $\mathbb{R}^\infty$, and hence it has topologies τ_b and τ_p which comes from the box topology and the product topology of $\mathbb{R}^\infty$ respectively. Also, we can make $\mathcal{P}$ into a metric space by extending the metric ρ (as defined in Section 3.1) on $\mathcal{O}_d$, for $d \geq 3$. Let us denote the corresponding topology by τ_m . It can be easily checked that the topologies τ_b , τ_p and τ_m are weaker (coarser) than the inductive limit topology that is defined in Section 4.2. Therefore, the maps Λ and H (as defined in the proof of Theorem 4.2.1) are also continuous with respect to each of the topologies τ_b , τ_p and τ_m . Furthermore, one can check that the map Γ is continuous with respect to all these topologies. *Hence by looking at the proof of Theorem 4.2.1, it becomes trivial that the space of all polynomial knots with respect any of the topologies τ_b , τ_p and τ_m also has the same homotopy type as S^2 .*

In [35], Mishra proved that if two polynomial knots are topologically equivalent, then there is an isotopy through polynomial knots which connects them. The converse of this statement is not true. In fact, in Section 4.2, we showed that every polynomial knot is isotopic to some linear knot (an unknot) by a smooth isotopy through polynomial knots. In other words, every polynomial knot can be connected to a linear knot by a smooth path in the space of all polynomial knots. The same is true for the space $\mathcal{O}_d$, for $d \geq 3$. More precisely, every polynomial knot in $\mathcal{O}_d$, can be connected to a linear knot by a smooth path in $\mathcal{O}_d$. This type of path rolls a given

polynomial knot to the infinity and unravels it up to make it into an unknot (see Theorem 4.2.4 and Example 4.1). Also, in this context, we have the following interesting question:

Question 6.1. *Given two polynomial knots in $\mathcal{O}_d$, does there exist a smooth path in $\mathcal{O}_d$ (or in $\mathcal{P}$) which connects them?*

For example, if ϕ and ψ are polynomial knots in $\mathcal{O}_d$ which represent the knots 3_1 and 4_1 respectively, then it is not known whether there exists a smooth path in $\mathcal{O}_d$ (or in $\mathcal{P}$) that connects these polynomial knots. We guess in general that the answer to the Question 6.1 is probably yes. However, in general, it will be difficult to find an explicit path in $\mathcal{O}_d$ which is smooth. One way to obtain a path in $\mathcal{O}_d$ is as follows: Let ϕ and ψ be polynomial knots in $\mathcal{O}_d$, then by Corollary 4.2.5, there exist smooth paths $F : I \rightarrow \mathcal{O}_d$ and $G : I \rightarrow \mathcal{O}_d$ such that $F(0) = \phi$ and $G(0) = \psi$, and the knots $\sigma = F(1)$ and $\tau = G(1)$ are linear. Also, is it easy to find a smooth path $H : I \rightarrow \mathcal{O}_d$ such that $H(0) = \sigma$ and $H(1) = \tau$. Thus, by concatenating the paths F , G and H (or the reverse paths if necessary) in the appropriate order, we can get a new path $\Omega : I \rightarrow \mathcal{O}_d$ such that $\Omega(0) = \phi$, $\Omega(1/3) = \sigma$, $\Omega(2/3) = \tau$ and $\Omega(1) = \psi$. The path Ω is smooth except possibly at $s = 1/3$ and $s = 2/3$, where it may have singularities. Note that ϕ and ψ may not be the interior points of $\mathcal{O}_d$. Therefore, it may be impossible to remove the singularities and get a smooth path in $\mathcal{O}_d$ which connects the given knots ϕ and ψ . Thus, to answer Question 6.1, we will have to try another way of finding a smooth path. Other question that goes parallel to the earlier one is the following:

Question 6.2. *Given two polynomial knots, does there exist a smooth isotopy through polynomial knots which connects them?*

In Section 4.4, we discussed the path equivalence in $\mathcal{Q}_d$, for $d \geq 3$. We proved that if two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are

topologically equivalent. In other words, two polynomial knots which represent two distinct knot-types cannot belong to the same path component of the space $\mathcal{Q}_d$. Furthermore, we observed that the number of path components in $\mathcal{Q}_d$ corresponding to a given knot-type are at least eight. *However, it is not known that for some $d \geq 3$, are there more than eight path components in $\mathcal{Q}_d$ corresponding to some knot-type? Also, it will be interesting to explore whether the path components of the interiors of the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$ are in one-to-one correspondence with the knot-types that can be represented there?*

In Section 5.1, we conjectured that each of the knots 5_2 , 6_1 , 6_2 and 6_3 cannot be represented by a polynomial knot in degree 6 (see Conjecture 5.1). One approach towards proving this conjecture is by looking at the path components of the space $\mathcal{Q}_6$. If one can show that all the path components of $\mathcal{Q}_6$ are occupied by the knots 0_1 , 3_1 and 4_1 , then there will be no place for the knots 5_2 , 6_1 , 6_2 and 6_3 in $\mathcal{Q}_6$ (since two topologically different knots cannot belong to the same path component of this space). This could be done by using some algorithms in real semialgebraic geometry (for example, see [38]). Another possible way to prove Conjecture 5.1 is by finding superbridge index of the knots 5_2 , 6_1 , 6_2 and 6_3 . In [6], Jeon and Jin proved that no knots other than 3_1 , 4_1 , 5_2 , 6_1 , 6_2 , 6_3 , 7_2 , 7_3 , 7_4 , 8_4 , 8_7 and 8_9 have superbridge index 3. Out of these knots, the knots 3_1 and 4_1 are known to have superbridge index 3. However, the superbridge indices are not known for the remaining knots in the list and it is conjectured that each one has superbridge index 4. If one could prove this fact, then the knots 5_2 , 6_1 , 6_2 and 6_3 each will have superbridge index 4, and thus by Proposition 2.2.4, none of them will have a polynomial representation in degree 6.

In Section 5.3, we showed that the spaces $\mathcal{Q}_3$ and $\mathcal{Q}_4$ have exactly eight path components, and all the path components of these spaces are contractible. We also showed that the space $\mathcal{Q}_5$ has exactly eight path com-

ponents corresponding to the trefoil knot and its mirror image. Moreover, all these path components are contractible. Note that the unknot also has a polynomial representation in $\mathcal{Q}_5$ (in fact, in each $\mathcal{Q}_d$ for $d \geq 3$), but the homotopy type of the path components of $\mathcal{Q}_5$ that correspond to the unknot cannot be predicted using the arguments known thus far. *This is a natural question to think about in future. In general, it will be interesting to find the exact number of path components of $\mathcal{Q}_d$, for $d \geq 5$, and explore their homotopy type.*

We may study some more spaces of polynomial knots and try to look into their topology. For instance, we can study the space of all polynomial knots $t \mapsto (f(t), g(t), h(t))$ such that $\deg(f) \leq d - 2$, $\deg(g) \leq d - 1$ and $\deg(h) = d$, or the space of all polynomial knots $t \mapsto (x(t), y(t), z(t))$ with $\deg(x) < \deg(y) < \deg(z) = d$.

Appendix A

Mathematica Programs

A.1 A representations of the knot 5_2

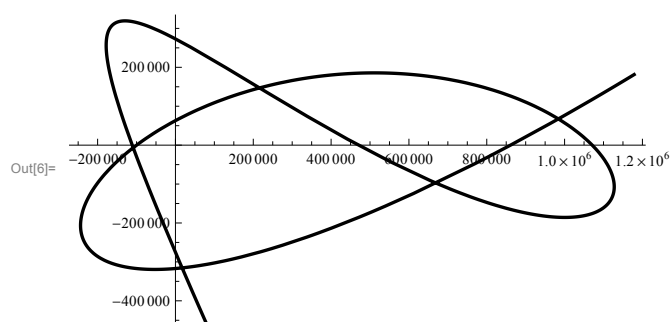
```

x[t_] := 20 (t+21) (t+15) (t-10) (t-17)
y[t_] := t (t^2 - 400) (t^2 - 121)

{{X}} = {Coefficient[x[t], {t, t^2, t^3, t^4}]} .
  {{1}, {t+s}, {t^2+t*s+s^2}, {t^3+t^2*s+t*s^2+s^3}};
{{Y}} = {Coefficient[y[t], {t, t^2, t^3, t^4, t^5}]} . {{1}, {t+s}, {t^2+t*s+s^2},
  {t^3+t^2*s+t*s^2+s^3}, {t^4+t^3*s+t^2*s^2+t*s^3+s^4}};

{res} = Resultant[{X}, {Y}, s];
ParametricPlot[{x[t], y[t]}, {t, -21.5, 20.7}, PlotStyle -> {Black, Thick}]
NRoots[res == 0, t]

```



Out[7]= t == -21.1177 || t == -20.5265 - 8.76032 i || t == -20.5265 + 8.76032 i ||
t == -20.0295 || t == -13.0399 || t == -9.36786 || t == 1.42386 ||
t == 8.18931 || t == 11.0978 || t == 17.0984 || t == 19.5147 || t == 20.2841

```

{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10} =
{-21.11773825223018, -20.029544134693722, -13.039942915734427,
-9.36785809566816, 1.4238551323911735, 8.189311078461055, 11.097757990448363,
17.09842176712548, 19.514665718414086, 20.284064224766038};
{a1, a2, a3, a4, a5} = {c1, c2, c3, c4, c5};
{b1, b2, b3, b4, b5} = {c8, c7, c6, c9, c10};

r1 = {a1^7 - b1^7, a1^6 - b1^6,
a1^5 - b1^5, a1^4 - b1^4, a1^3 - b1^3, a1^2 - b1^2, a1 - b1};
r2 = {a2^7 - b2^7, a2^6 - b2^6, a2^5 - b2^5, a2^4 - b2^4,
a2^3 - b2^3, a2^2 - b2^2, a2 - b2};
r3 = {a3^7 - b3^7, a3^6 - b3^6, a3^5 - b3^5, a3^4 - b3^4,
a3^3 - b3^3, a3^2 - b3^2, a3 - b3};
r4 = {a4^7 - b4^7, a4^6 - b4^6, a4^5 - b4^5, a4^4 - b4^4,
a4^3 - b4^3, a4^2 - b4^2, a4 - b4};
r5 = {a5^7 - b5^7, a5^6 - b5^6, a5^5 - b5^5, a5^4 - b5^4,
a5^3 - b5^3, a5^2 - b5^2, a5 - b5};

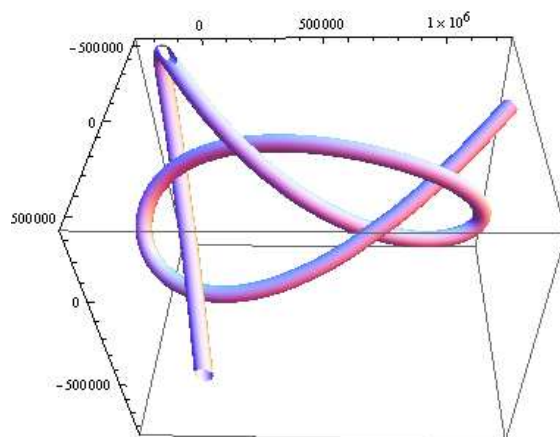
d = {{-100 000}, {100 000}, {-60 000}, {60 000}, {-60 000}}; m = {r1, r2, r3, r4, r5};
v = {{t^7, t^6, t^5, t^4, t^3, t^2, t}}; u = LinearSolve[m, d]; {{z}} = v.u;
Factor[z]

0.00536386 t (-20.1133 + 1. t) (-14.2601 + 1. t)
(12.243 + 1. t) (20.5786 + 1. t) (0.0107598 - 0.0343124 t + 1. t^2)

In[1]:= f[t_] := 20 (-17 + t) (-10 + t) (15 + t) (21 + t)
g[t_] := t (-400 + t^2) (-121 + t^2)
h[t_] := -0.005 t (-20.1133216 + t) (-14.260128 + t)
(12.2430449 + t) (20.5785825 + t) (0.0107598 - 0.0343124 t + t^2)

ParametricPlot3D[{f[t], g[t], h[t]}, {t, -24, 20.6}, PlotStyle -> {Tube[30 000]}]

```



A.2 A representation of the knot 6_1

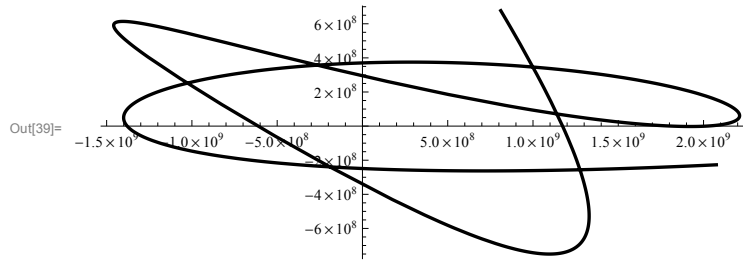
```

In[34]:= x[t_] := 100 (-43.4 + t) (-28 + t) (5 + t) (31.4 + t) (47.6 + t)
y[t_] := (-49 + t) (-38 + t) (-8 + t) (-6 + t) (28 + t) (43.6 + t)

{{X}} = {Coefficient[x[t], {t, t^2, t^3, t^4, t^5}]} . {{1}, {t + s}, {t^2 + t*s + s^2},
  {t^3 + t^2*s + t*s^2 + s^3}, {t^4 + t^3*s + t^2*s^2 + t*s^3 + s^4}};
{{Y}} = {Coefficient[y[t], {t, t^2, t^3, t^4, t^5, t^6}]} .
  {{1}, {t + s}, {t^2 + t*s + s^2}, {t^3 + t^2*s + t*s^2 + s^3},
  {t^4 + t^3*s + t^2*s^2 + t*s^3 + s^4},
  {t^5 + t^4*s + t^3*s^2 + t^2*s^3 + t*s^4 + s^5}};

{res} = Resultant[{X}, {Y}, s];
ParametricPlot[{x[t], y[t]}, {t, -45.4, 46.4}, PlotStyle -> {Black, Thick}]
NRroots[res == 0, t]

```



```

Out[39]= t == -52.165 - 22.4355 i || t == -52.165 + 22.4355 i || t == -44.6124 || t == -43.8297 ||
  t == -42.6396 || t == -30.3859 || t == -25.1773 || t == -19.2377 - 17.6343 i ||
  t == -19.2377 + 17.6343 i || t == -6.50511 || t == 1.34637 || t == 7.91131 - 8.93809 i ||
  t == 7.91131 + 8.93809 i || t == 23.4849 || t == 29.1728 || t == 33.0062 ||
  t == 43.0069 || t == 45.4301 || t == 47.1427 - 0.906303 i || t == 47.1427 + 0.906303 i

```

```

In[41]:= {c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12} =
  {-44.612370619788685, -43.82970755399989, -42.63955548439335, -30.38594129886014,
  -25.17730566355526, -6.505105662667016, 1.346365109682458, 23.484924352213127,
  29.172757565486513, 33.00624466339479, 43.0069479270461, 45.43009343522182};
{a1, a2, a3, a4, a5, a6} = {c1, c2, c3, c4, c5, c6};
{b1, b2, b3, b4, b5, b6} = {c8, c7, c12, c11, c10, c9};

r1 = {a1^7 - b1^7, a1^6 - b1^6,
  a1^5 - b1^5, a1^4 - b1^4, a1^3 - b1^3, a1^2 - b1^2, a1 - b1};
r2 = {a2^7 - b2^7, a2^6 - b2^6, a2^5 - b2^5, a2^4 - b2^4,
  a2^3 - b2^3, a2^2 - b2^2, a2 - b2};
r3 = {a3^7 - b3^7, a3^6 - b3^6, a3^5 - b3^5, a3^4 - b3^4,
  a3^3 - b3^3, a3^2 - b3^2, a3 - b3};

```

```

r4 = {a4^7 - b4^7, a4^6 - b4^6, a4^5 - b4^5, a4^4 - b4^4,
      a4^3 - b4^3, a4^2 - b4^2, a4 - b4};
r5 = {a5^7 - b5^7, a5^6 - b5^6, a5^5 - b5^5, a5^4 - b5^4,
      a5^3 - b5^3, a5^2 - b5^2, a5 - b5};
r6 = {a6^7 - b6^7, a6^6 - b6^6, a6^5 - b6^5, a6^4 - b6^4,
      a6^3 - b6^3, a6^2 - b6^2, a6 - b6};

n = {r1, r2, r3, r4, r5}; w = LinearSolve[Transpose[n], Transpose[{r6}]];
{{200 000 000, -200 000 000, 200 000 000, -130 000 000, 170 000 000}}.w

```

```
Out[51]= {{-3.97937 × 108}}
```

```

In[52]:= d = Transpose[{{200 000 000, -200 000 000,
      200 000 000, -130 000 000, 170 000 000, -397 937 039.8156442}}];
m = {r1, r2, r3, r4, r5, r6}; u = LinearSolve[m, d];
v = {{t^7, t^6, t^5, t^4, t^3, t^2, t}}; {{z}} = v.u;
Factor[1.5 z + x[t]]

```

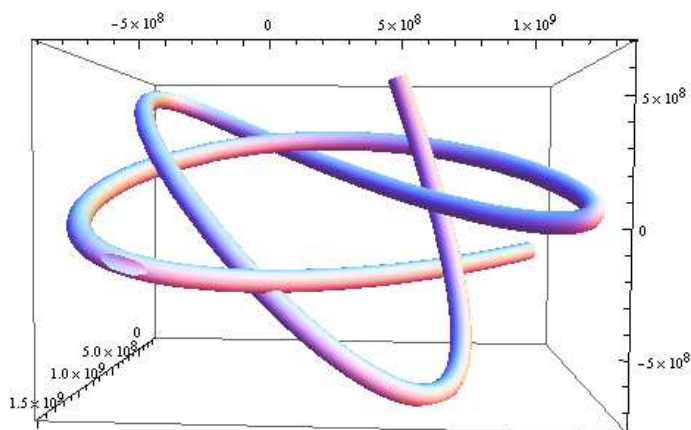
```
Out[55]= -0.126854 (-45.995 + 1. t) (5.23102 + 1. t) (19.0366 + 1. t)
      (758.764 - 54.4651 t + 1. t2) (2059.95 + 90.482 t + 1. t2)
```

```

In[56]:= f[t_] := 60 (-43.4 + t) (-28 + t) (5 + t) (31.4 + t) (47.6 + t)
g[t_] := (-49 + t) (-38 + t) (-8 + t) (-6 + t) (28 + t) (43.6 + t)
h[t_] := -0.07 (-45.995024874 + t) (5.231021635 + t) (19.036560084 + t)
      (758.763745443 - 54.4650519227 t + t2) (2059.948386689 + 90.4819595699 t + t2)

```

```
ParametricPlot3D[{f[t], g[t], h[t]}, {t, -45.4, 46.4}, PlotStyle -> {Tube[50 000 000]}]
```



A.3 A representation of the knot 6_2

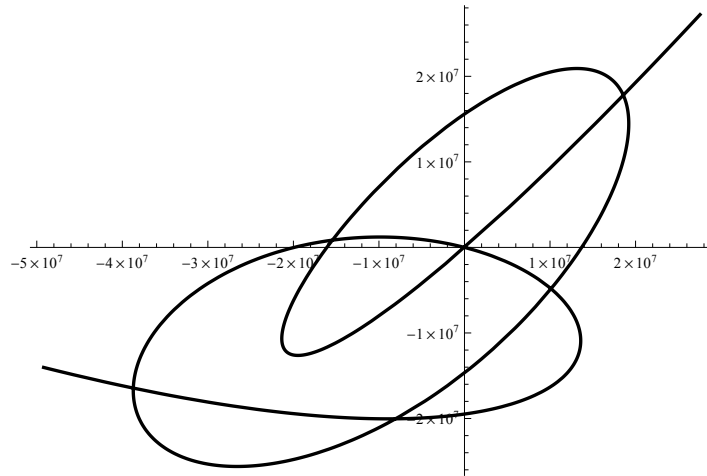
```

x[t_] := 4 (t+35) (t-5) (t-39) (t^2-625)
y[t_] := 0.1 (t+41) (t+25) (t+20) (t-10) (t-30) (t-39)

{{X}} = {Coefficient[x[t], {t, t^2, t^3, t^4, t^5}]} . {{1}, {t+s}, {t^2+t*s+s^2},
  {t^3+t^2*s+t*s^2+s^3}, {t^4+t^3*s+t^2*s^2+t*s^3+s^4}};
{{Y}} = {Coefficient[y[t], {t, t^2, t^3, t^4, t^5, t^6}]} .
  {{1}, {t+s}, {t^2+t*s+s^2}, {t^3+t^2*s+t*s^2+s^3},
  {t^4+t^3*s+t^2*s^2+t*s^3+s^4},
  {t^5+t^4*s+t^3*s^2+t^2*s^3+t*s^4+s^5}};
{res} = Resultant[{X}, {Y}, s];

ParametricPlot[{x[t], y[t]}, {t, -39, 41.3}, PlotStyle -> {Black, Thick}]
NRroots[res == 0, t]

```



```

t = -38.3941 || t = -37.5234 - 0.459148 i || t = -37.5234 + 0.459148 i ||
t = -35.9793 || t = -28.1103 || t = -25. || t = -21.1594 || t = -10.9137 ||
t = -4.69515 - 3.84229 i || t = -4.69515 + 3.84229 i || t = 2.61871 || t = 8.44009 ||
t = 16.8484 || t = 29.7593 || t = 35.5074 - 2.95225 i || t = 35.5074 + 2.95225 i ||
t = 35.8207 - 4.46467 i || t = 35.8207 + 4.46467 i || t = 39. || t = 40.6711

```

```

{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12} =
{-38.394059300190634, -35.97930636830068, -28.110345818548993, -25.00000000000564,
-21.159364062658817, -10.913696119313041, 2.6187143001569955, 8.440093883189082,
16.8484499437119, 29.759340223727147, 35.507400000000004, 35.820700000000004,
35.820700000000004, 39.000000000000004, 40.671100000000004};
{a1, a2, a3, a4, a5, a6} = {c1, c2, c3, c4, c5, c9};
{b1, b2, b3, b4, b5, b6} = {c6, c7, c8, c11, c10, c12};

q1 = {a1^7 - b1^7, a1^6 - b1^6,
a1^5 - b1^5, a1^4 - b1^4, a1^3 - b1^3, a1^2 - b1^2, a1 - b1};

```

```

q2 = {a2^7 - b2^7, a2^6 - b2^6, a2^5 - b2^5, a2^4 - b2^4,
      a2^3 - b2^3, a2^2 - b2^2, a2 - b2};
q3 = {a3^7 - b3^7, a3^6 - b3^6, a3^5 - b3^5, a3^4 - b3^4,
      a3^3 - b3^3, a3^2 - b3^2, a3 - b3};
q4 = {a4^7 - b4^7, a4^6 - b4^6, a4^5 - b4^5, a4^4 - b4^4,
      a4^3 - b4^3, a4^2 - b4^2, a4 - b4};
q5 = {a5^7 - b5^7, a5^6 - b5^6, a5^5 - b5^5, a5^4 - b5^4,
      a5^3 - b5^3, a5^2 - b5^2, a5 - b5};
q6 = {a6^7 - b6^7, a6^6 - b6^6, a6^5 - b6^5, a6^4 - b6^4,
      a6^3 - b6^3, a6^2 - b6^2, a6 - b6};

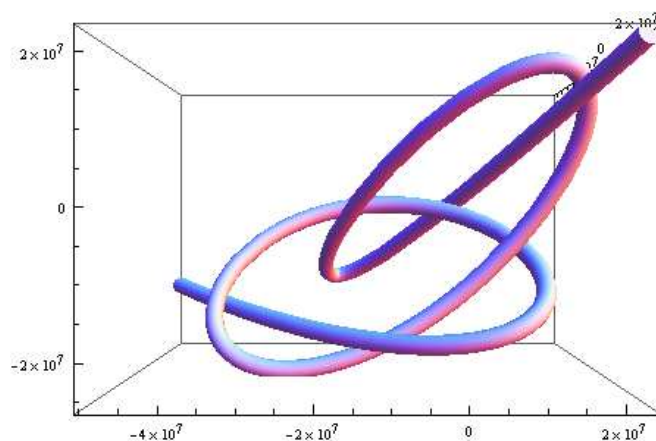
m = {q1, q2, q3, q4, q5}; w = LinearSolve[Transpose[m], Transpose[{q6}]] ;
{{10^7, -10^7, 4*10^6, -4*10^6, 4*10^6}}.w
{{4.5428*10^6}}

d = Transpose[{{10^7, -10^7, 4*10^6, -4*10^6, 4*10^6, 4542795.994769199}}];
n = {q1, q2, q3, q4, q5, q6}; u = LinearSolve[n, d];
v = {{t^7, t^6, t^5, t^4, t^3, t^2, t}}; {{z}} = v.u;
Factor[z]
-0.00468148 t (-39.8754 + 1. t) (-27.4156 + 1. t)
(28.4369 + 1. t) (37.2557 + 1. t) (0.00242388 - 0.00542949 t + 1. t^2)

f[t_] := 4 (-39 + t) (-5 + t) (35 + t) (-625 + t^2)
g[t_] := 0.1 (-39 + t) (-30 + t) (-10 + t) (20 + t) (25 + t) (41 + t)
h[t_] := 0.005 t (-39.8753791 + t) (-27.4156408 + t)
(28.436878 + t) (37.25572585 + t) (0.002423881 - 0.005429486 t + t^2)

ParametricPlot3D[{f[t], g[t], h[t]}, {t, -39, 41.3}, PlotStyle -> {Tube[1.5*10^6]}]

```



A.4 A representation of the knot 6_3

```

x[t_] := 20 (-29 + t) (-20 + t) (10 + t) (30 + t)^2
y[t_] := (-32 + t) (-6 + t) (4 + t) (30 + t) (-400 + t^2)

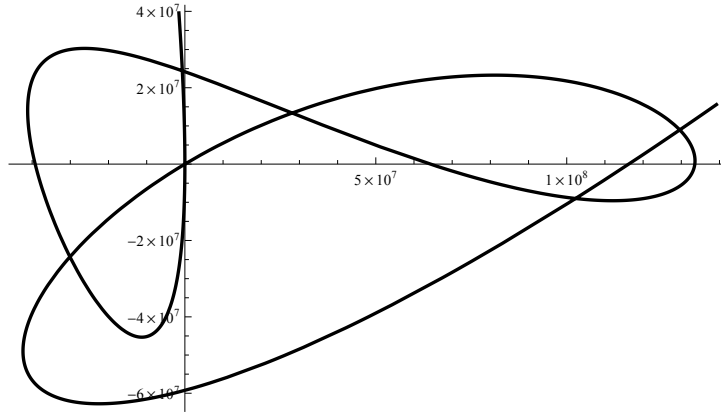
{{X}} = {Coefficient[x[t], {t, t^2, t^3, t^4, t^5}]} . {{1}, {t + s}, {t^2 + t*s + s^2},
  {t^3 + t^2*s + t*s^2 + s^3}, {t^4 + t^3*s + t^2*s^2 + t*s^3 + s^4}};
{{Y}} = {Coefficient[y[t], {t, t^2, t^3, t^4, t^5, t^6}]} .
  {{1}, {t + s}, {t^2 + t*s + s^2}, {t^3 + t^2*s + t*s^2 + s^3},
  {t^4 + t^3*s + t^2*s^2 + t*s^3 + s^4},
  {t^5 + t^4*s + t^3*s^2 + t^2*s^3 + t*s^4 + s^5}};
{res} = Resultant[{X}, {Y}, s];

```

```

ParametricPlot[{x[t], y[t]}, {t, -31.1, 32.4}, PlotStyle -> {Black, Thick}]
NRroots[res == 0, t]

```



```

t == -41.818 - 19.1454 i || t == -41.818 + 19.1454 i || t == -30.7288 || t == -30. ||
t == -26.8837 - 2.51139 i || t == -26.8837 + 2.51139 i || t == -22.6519 ||
t == -18.1224 - 20.0177 i || t == -18.1224 + 20.0177 i || t == -10.0715 || t == -7.25308 ||
t == -0.240482 || t == 8.41888 || t == 18.0206 || t == 20. || t == 22.6092 ||
t == 28.7848 - 0.569203 i || t == 28.7848 + 0.569203 i || t == 31.739 || t == 32.2368

```

```

{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12} =
  {-30.72877916248007, -29.99999999974946, -22.651937287862193,
   -10.071488974838084, -7.253082041836597, -0.24048226174477072,
   8.418879029241214, 18.0205707137901, 20.00000000000009,
   22.60921542713383, 31.739005302303298, 32.23681523039284};
{a1, a2, a3, a4, a5, a6} = {c1, c2, c3, c5, c6, c7};
{b1, b2, b3, b4, b5, b6} = {c4, c9, c10, c8, c11, c12};

q1 = {a1^7 - b1^7, a1^6 - b1^6,
  a1^5 - b1^5, a1^4 - b1^4, a1^3 - b1^3, a1^2 - b1^2, a1 - b1};

```

```

q2 = {a2^7 - b2^7, a2^6 - b2^6, a2^5 - b2^5, a2^4 - b2^4,
      a2^3 - b2^3, a2^2 - b2^2, a2 - b2};
q3 = {a3^7 - b3^7, a3^6 - b3^6, a3^5 - b3^5, a3^4 - b3^4,
      a3^3 - b3^3, a3^2 - b3^2, a3 - b3};
q4 = {a4^7 - b4^7, a4^6 - b4^6, a4^5 - b4^5, a4^4 - b4^4,
      a4^3 - b4^3, a4^2 - b4^2, a4 - b4};
q5 = {a5^7 - b5^7, a5^6 - b5^6, a5^5 - b5^5, a5^4 - b5^4,
      a5^3 - b5^3, a5^2 - b5^2, a5 - b5};
q6 = {a6^7 - b6^7, a6^6 - b6^6, a6^5 - b6^5, a6^4 - b6^4,
      a6^3 - b6^3, a6^2 - b6^2, a6 - b6};

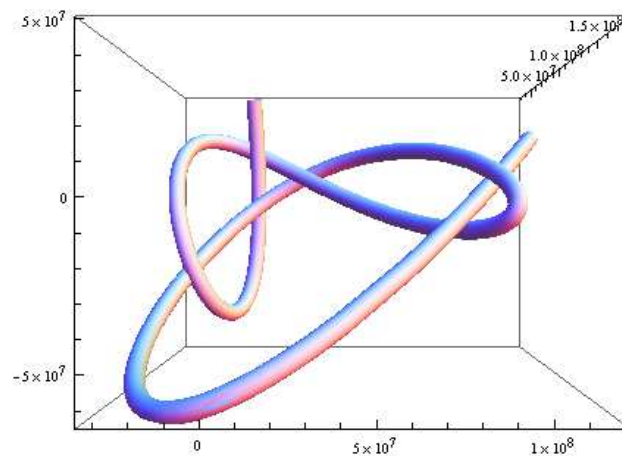
m = {q1, q2, q3, q5, q6}; w = LinearSolve[Transpose[m], Transpose[{q4}]];
{{5 * 10^6, -5 * 10^6, 10^7, -2.25 * 10^7, 2.25 * 10^7}}.w

{{5.06934 * 10^6}}

d = Transpose[
  {{5 * 10^6, -5 * 10^6, 10^7, 5.069338.326090865, -2.25 * 10^7, 2.25 * 10^7}}];
n = {q1, q2, q3, q4, q5, q6}; u = LinearSolve[n, d];
v = {{t^7, t^6, t^5, t^4, t^3, t^2, t}}; {{z}} = v.u;
Factor[z + x[t]]
-0.060283 (-33.329 + 1. t) (376.738 - 37.8892 t + 1. t^2)
(144.276 + 21.4044 t + 1. t^2) (955.986 + 61.5665 t + 1. t^2)

f[t_] := 15 (-29 + t) (-20 + t) (10 + t) (30 + t)^2
g[t_] := (-32 + t) (-6 + t) (4 + t) (30 + t) (-400 + t^2)
h[t_] := -0.06 (-33.329044815 + t) (376.737563885 - 37.8892469397 t + t^2)
(144.275534095 + 21.404400212 t + t^2) (955.985733648 + 61.56649851 t + t^2)
ParametricPlot3D[{f[t], g[t], h[t]}, {t, -31.3, 32.7}, PlotStyle -> {Tube[3 * 10^6]}]

```



A.5 A representation of the knot $3_1\#3_1$

```

x[t_] := 5 t (77.3 - 17.5 t + t^2) (77.3 + 17.5 t + t^2)
y[t_] := (-10.1^2 + t^2) (-7.3^2 + t^2) (-2.2^2 + t^2)

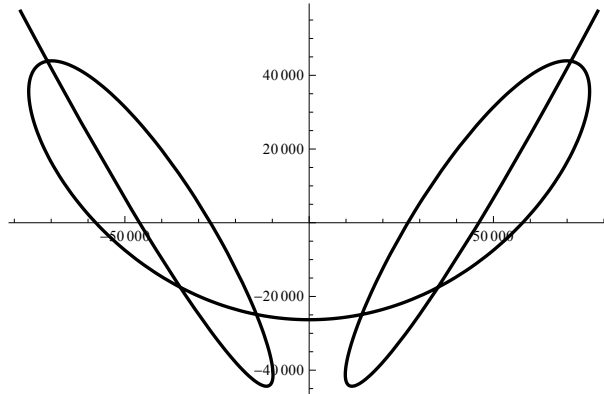
{{X}} = {Coefficient[x[t], {t, t^2, t^3, t^4, t^5}]} . {{1}, {t+s}, {t^2+t*s+s^2},
  {t^3+t^2*s+t*s^2+s^3}, {t^4+t^3*s+t^2*s^2+t*s^3+s^4}};
{{Y}} = {Coefficient[y[t], {t, t^2, t^3, t^4, t^5, t^6}]} .
  {{1}, {t+s}, {t^2+t*s+s^2}, {t^3+t^2*s+t*s^2+s^3},
  {t^4+t^3*s+t^2*s^2+t*s^3+s^4},
  {t^5+t^4*s+t^3*s^2+t^2*s^3+t*s^4+s^5}};
{res} = Resultant[{X}, {Y}, s];

```

```

ParametricPlot[{x[t], y[t]}, {t, -10.55, 10.55}, PlotStyle -> {Black, Thick}]
NRroots[res == 0, t]

```



```

t == -10.4627 || t == -9.88822 || t == -9.1468 - 0.871953 i || t == -9.1468 + 0.871953 i ||
t == -8.75 - 0.858778 i || t == -8.75 + 0.858778 i || t == -8.02489 ||
t == -4.92721 || t == -1.21645 || t == -0.481499 || t == 0.481499 || t == 1.21645 ||
t == 4.92721 || t == 8.02489 || t == 8.75 - 0.858778 i || t == 8.75 + 0.858778 i ||
t == 9.1468 - 0.871953 i || t == 9.1468 + 0.871953 i || t == 9.88822 || t == 10.4627

```

```

{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12} =
  {-10.462689786449586, -9.888216508786305, -8.024893362737378, -4.927208785673205,
  -1.2164458079580254, -0.4814986323142994, 0.4814986323142994, 1.2164458079580254,
  4.9272087856732085, 8.024893362738313, 9.888216508786323, 10.462689786449582};
{a1, a2, a3, a4, a5, a6} = {c1, c2, c3, c7, c8, c9};
{b1, b2, b3, b4, b5, b6} = {c4, c5, c6, c10, c11, c12};

r1 = {a1^7 - b1^7, a1^6 - b1^6,
  a1^5 - b1^5, a1^4 - b1^4, a1^3 - b1^3, a1^2 - b1^2, a1 - b1};

```

```

r2 = {a2^7 - b2^7, a2^6 - b2^6, a2^5 - b2^5, a2^4 - b2^4,
      a2^3 - b2^3, a2^2 - b2^2, a2 - b2};
r3 = {a3^7 - b3^7, a3^6 - b3^6, a3^5 - b3^5, a3^4 - b3^4,
      a3^3 - b3^3, a3^2 - b3^2, a3 - b3};
r4 = {a4^7 - b4^7, a4^6 - b4^6, a4^5 - b4^5, a4^4 - b4^4,
      a4^3 - b4^3, a4^2 - b4^2, a4 - b4};
r5 = {a5^7 - b5^7, a5^6 - b5^6, a5^5 - b5^5, a5^4 - b5^4,
      a5^3 - b5^3, a5^2 - b5^2, a5 - b5};
r6 = {a6^7 - b6^7, a6^6 - b6^6, a6^5 - b6^5, a6^4 - b6^4,
      a6^3 - b6^3, a6^2 - b6^2, a6 - b6};

n = {r1, r2, r3, r4, r5}; w = LinearSolve[Transpose[n], Transpose[{r6}]];
{{10 000, -10 000, 10 000, 10 000, -10 000}}.w

{{10 000.}}

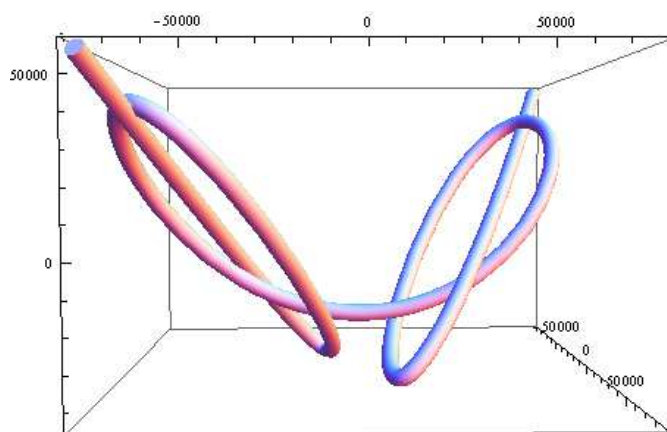
d = Transpose[{{10 000, -10 000, 10 000, 10 000, -10 000, 9999.999999941221}}];
m = {r1, r2, r3, r4, r5, r6}; u = LinearSolve[m, d];
v = {{t^7, t^6, t^5, t^4, t^3, t^2, t}}; {{z}} = v.u;
Factor[z]

-0.149218 t (-9.98476 + 1. t) (-8.25331 + 1. t)
(8.25331 + 1. t) (9.98476 + 1. t) (0.0253677 - 8.75601 × 10-14 t + 1. t2)

f[t_] := 5 t (77.3 - 17.5 t + t2) (77.3 + 17.5 t + t2)
g[t_] := (-102.01 + t2) (-53.29 + t2) (-4.84 + t2)
h[t_] := -0.15 t (-99.695462027 + t2) (-68.11720396 + t2) (0.025367747 + t2)

ParametricPlot3D[{f[t], g[t], h[t]}, {t, -10.55, 10.55}, PlotStyle -> {Tube[3000]}]

```



A.6 A representation of the knot $3_1\#3_1^*$

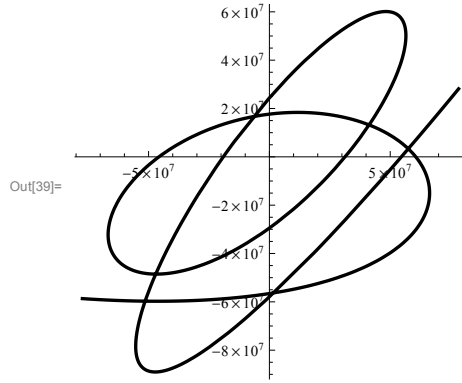
```

In[34]:= x[t_] := 30 (-32.5 + t) (-21.3 + t) (-3.3 + t) (16.2 + t) (28 + t)
y[t_] := (-34 + t) (-23 + t) (-6.8 + t) (12 + t) (21.7 + t) (33.1 + t)

{{X}} = {Coefficient[x[t], {t, t^2, t^3, t^4, t^5}]} . {{1}, {t + s}, {t^2 + t*s + s^2},
  {t^3 + t^2*s + t*s^2 + s^3}, {t^4 + t^3*s + t^2*s^2 + t*s^3 + s^4}};
{{Y}} = {Coefficient[y[t], {t, t^2, t^3, t^4, t^5, t^6}]} .
  {{1}, {t + s}, {t^2 + t*s + s^2}, {t^3 + t^2*s + t*s^2 + s^3},
  {t^4 + t^3*s + t^2*s^2 + t*s^3 + s^4},
  {t^5 + t^4*s + t^3*s^2 + t^2*s^3 + t*s^4 + s^5}};
{res} = Resultant[{X}, {Y}, s];

ParametricPlot[{x[t], y[t]}, {t, -29.8, 34.5}, PlotStyle -> {Black, Thick}]
NRroots[N[res] == 0, t]

```



```

Out[40]= t == -29.4334 - 0.642336 i || t == -29.4334 + 0.642336 i || t == -29.2862 || t == -27.9464 ||
  t == -21.2574 || t == -19.549 || t == -15.682 || t == -2.00257 - 4.09262 i ||
  t == -2.00257 + 4.09262 i || t == -1.98026 || t == 6.34472 - 6.68474 i ||
  t == 6.34472 + 6.68474 i || t == 8.18692 || t == 21.8758 || t == 25.9478 || t == 26.6674 ||
  t == 32.5602 || t == 34.064 || t == 34.0909 - 1.45431 i || t == 34.0909 + 1.45431 i

```

```

{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12} =
  {-29.286231462142837, -27.946357909173386, -21.25744461940997,
  -19.549037204110213, -15.682027791559056, -1.9802583986743842,
  8.186921456677068, 21.87581287082495, 25.947783314268374,
  26.667395058578226, 32.56023844415642, 34.06397022774709};
{a1, a2, a3, a4, a5, a6} = {c1, c2, c3, c4, c5, c6};
{b1, b2, b3, b4, b5, b6} = {c10, c11, c12, c7, c8, c9};

r1 = {a1^7 - b1^7, a1^6 - b1^6,
  a1^5 - b1^5, a1^4 - b1^4, a1^3 - b1^3, a1^2 - b1^2, a1 - b1};

```

```

r2 = {a2^7 - b2^7, a2^6 - b2^6, a2^5 - b2^5, a2^4 - b2^4,
      a2^3 - b2^3, a2^2 - b2^2, a2 - b2};
r3 = {a3^7 - b3^7, a3^6 - b3^6, a3^5 - b3^5, a3^4 - b3^4,
      a3^3 - b3^3, a3^2 - b3^2, a3 - b3};
r4 = {a4^7 - b4^7, a4^6 - b4^6, a4^5 - b4^5, a4^4 - b4^4,
      a4^3 - b4^3, a4^2 - b4^2, a4 - b4};
r5 = {a5^7 - b5^7, a5^6 - b5^6, a5^5 - b5^5, a5^4 - b5^4,
      a5^3 - b5^3, a5^2 - b5^2, a5 - b5};
r6 = {a6^7 - b6^7, a6^6 - b6^6, a6^5 - b6^5, a6^4 - b6^4,
      a6^3 - b6^3, a6^2 - b6^2, a6 - b6};

n = {r1, r2, r3, r4, r5}; w = LinearSolve[Transpose[n], Transpose[{r6}]];
{{-2 * 10^7, 2 * 10^7, -2 * 10^7, 1.8 * 10^7, -1.8 * 10^7}}.w

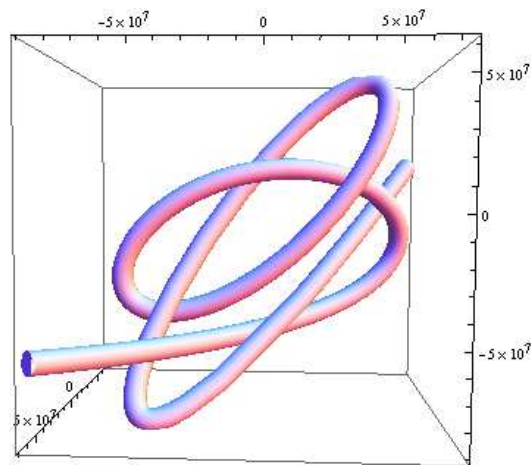
{{1.82154 * 10^7}}

d = Transpose[
  {{-2 * 10^7, 2 * 10^7, -2 * 10^7, 1.8 * 10^7, -1.8 * 10^7, 18215443.132480267}}];
m = {r1, r2, r3, r4, r5, r6}; u = LinearSolve[m, d];
v = {{t^7, t^6, t^5, t^4, t^3, t^2, t}}; {{z}} = v.u;
Factor[z]
0.0393543 t (-32.8074 + 1. t) (-24.2097 + 1. t)
(15.2573 + 1. t) (28.2892 + 1. t) (0.00437177 - 0.00820683 t + 1. t^2)

f[t_] := 30 (-32.5 + t) (-21.3 + t) (-3.3 + t) (16.2 + t) (28 + t)
g[t_] := (-34 + t) (-23 + t) (-6.8 + t) (12 + t) (21.7 + t) (33.1 + t)
h[t_] := -0.03 t (t + 28.289226) (t + 15.257278)
(t - 24.209735) (t - 32.807367) (0.0043718 - 0.0082068 t + t^2)

ParametricPlot3D[{f[t], g[t], h[t]}, {t, -30, 34.6}, PlotStyle -> {Tube[5 * 10^6]}]

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List of Publications

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