Investigating Unitarity and Symmetry Constraints on Cosmological Observables

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Certificate

This is to certify that this dissertation entitled *"Investigating Unitarity and Symmetry Constraints on Cosmological Observables"* towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Naman Kapoor at Indian Institute of Science Education and Research under the supervision of Diptimoy Ghosh, Associate Professor, Department of Physics, during the academic year 2024-2025.

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This thesis is dedicated to Physics and my loving & supportive family

Declaration

I hereby declare that the work presented in this thesis entitled *"Investigating Unitarity and Symmetry Constraints on Cosmological Observables"* is my own work and has not been submitted elsewhere for any degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions

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Abstract

This thesis explores the application of the principles of unitarity and symmetries in possible theories governing early universe in deriving corresponding identities that must be satisfied by the cosmological observables that we measure. In the first part of the thesis, an attempt at re-deriving cosmological cutting rules is made using a recently-proposed definition of Bunch-Davies S-matrix. In the second part, we make an attempt in offering a resolution to some puzzle in the literature about soft theorems involving pure-graviton correlators.

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Chapter 1

Introduction

The classical inflationary paradigm has been really successful in resolving the standard puzzles posed by the Hot Big Bang model. Furthermore, the small inhomogeneities in temperature that we observe in CMB find their natural origin in the dynamics of some quantum fluctuations in the inflationary era [1, 2, 3]. The existence of some superhorizon mode that stays constant regardless of reheating details has been shown [4, 5] and is crucial to reliably make such a contact between CMB observations and our theoretical attempt to explain them by inflation.

Some of these theoretical ideas can be partially tested in a *model-independent* manner by focusing on the fundamental principles underlying them and deriving observational consequences that follow directly. In doing so, we can potentially falsify broad claims such as: "quantum fluctuations seeded the CMB correlations," "Einstein gravity was a valid description during the inflationary era," or "only a single scalar degree of freedom existed at that time." If our measurements are inconsistent with the conditions required by these assumptions, then the foundational ideas themselves may be called into question.

A partial test for determining if the statistical correlations we observe in CMB are indeed seeded by *quantum* fluctuations is to first establish the general signatures of ***unitarity*** that quantum correlations must carry and then check if we do find those signatures in our observations. In QFT, the optical theorem is such a statement, but in terms of some theoretical quantity, S-matrix [6], which encodes the scattering amplitudes of particles. In cosmology, however, the concept of particle is not that well-defined and the natural observables we actually measure are correlations of relevant fields evaluated at different spatial points but at the *same time*.

In a recent work [7], a definition of some suitable S-matrix for cosmology, and its relation with the natural cosmological observables was established. The authors aimed to demonstrate that many well-established tools and results from quantum field theory and particle physics can be directly applied to cosmological settings, potentially yielding novel insights [8]. Motivated by this perspective, we explore the use of this cosmological S-matrix to provide an alternative, more transparent, and non-perturbative derivation of the unitarity constraints that wavefunction coefficients must obey—commonly referred to as the cosmological cutting rules [9, 10, 11]

In the second part of the thesis, we first study and review the inflationary soft theorems, with the aim of addressing an apparent puzzle in the literature concerning their applicability [12]. Broadly speaking, these theorems are model-independent statements about in-in correlation functions that arise from the presence of non-linearly realized asymptotic symmetries in the theory. They impose important constraints on observable quantities within a given theoretical framework. From a phenomenological perspective, their significance is even greater: by providing testable predictions tied to symmetry principles, they can be used to rule out entire classes of inflationary models, thereby helping to narrow down the viable theories of the early universe. The early results were limited, restricted to few simplest symmetries and were understood via ”background-wave” arguments [13, 14]. However, it was later recognized that the true origin of these soft theorems lies in the non-linear realization of symmetries and the satisfaction of the so-called physical mode condition[15, 16]

Unlike many of its competing theories, the theory of inflation naturally predicts tensorial fluctuations (primordial gravitational waves). Cosmologists have thus been and will continue to look for graviton correlations in CMB in coming decades. It is thus important to have theoretical calculations of observables ready, to test our theories by comparing their predictions with the actual measurements. The complicated nature of calculations involving gravitons is a well-known fact. In a recent work by Pajer et al. [12], such a calculation was performed - in an exact de Sitter background, working in $\delta\phi$-gauge, they computed the 4-point pure-graviton in-in correlator. In addition to being of phenomenological interest, soft theorems (*consistency* relations) are practically really powerful tools in a theoretician's toolkit; they provide a useful consistency check on the final result of such long, tedious calculations. Pajer et al. [12] found it challenging to find an appropriate consistency relation to check their own result. The soft theorems involve taking derivatives of correlators of fields, and taking derivatives of polarization tensors of gravitons is ill-defined. This makes it challenging to explicitly check soft-theorems involving pure-graviton correlators alone. Despite this, [12] tried to confirm their own calculation of 4-point pure-graviton correlator by some naive means and found it getting violated, to their surprise. This is puzzling since they have confirmed their tedious calculation of the 4-point correlators by several independent ways.

In this thesis, we show that there is no fundamental obstacle to the explicit verifiability of soft theorems involving gravitons. Focusing on the de Sitter $\delta\phi$ gauge calculation presented in [12], we explicitly verify the 3–2 soft theorem and provide reasoning for why its validity is to be expected. The preceding analysis can be extended in a straightforward manner to the 4–3 case, which is required to resolve the associated puzzle. However, the Mathematica implementation of this explicit check is currently underway.

The thesis is structured as follows: Chapter 1 introduces the study, sets it in context of ongoing research, states the questions we attempt to address, and discusses the background motivation for why all this is important. Chapter 2, along with the Appendix A, reviews some of the basic, but relevant aspects of the theoretical machinery of inflationary cosmology. The main content of this chapter, however, is the outlining of our theoretical attempt at obtaining unitarity constraints on wavefunction coefficients of cosmology. Chapter 3 reviews the well-studied idea of symmetries and their effect on cosmological correlators that we measure. We review the derivation of ward identities and consistency relations that follow from these symmetries. We also collect and emphasize important results, for their later use in the thesis. Chapter 4, after outlining a puzzle in the literature, that concerns soft theorems involving just pure-graviton-correlators, discusses our attempt at resolving it. 5 summarizes our findings & conclusions, and discusses limitations of our study and future directions.

Chapter 2

Part 1: Investigating Unitarity Constraints on Cosmological Observables

2.1 Theoretical Background and Basic Definitions

The setting here is that of QFT in a (rigid) curved spacetime, where we ignore the dynamical nature of gravity, and consider quantum matter fields on a fixed de Sitter background. Thus the (free) action of a single scalar field is given by (this is the appropriate analogue of inflaton, the particle/field hypothesized to be responsible for inflation in simplest inflationary models):

$$S_{\text{free}} = -\int d\tau\, d^d x \sqrt{-g}\, \frac{Z^2}{2} \left(g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + m^2 \varphi^2 \right)$$

In terms of the canonically normalized field, $\varphi(\tau,k) \equiv (-\tau)^{-d/2} \phi(\tau,k)$, convenient for FLRW backgrounds, the free equation of motion in fourier space then reads:

$$\mathscr{E}[k\tau]\, \varphi(\tau,k) \equiv \left((\tau\partial_\tau)^2 + k^2\tau^2 + \mu^2 \right) \varphi(\tau,k) = 0$$

To fix the mode function (for mode expansion of quantized field operator), we make a natural choice for cosmological applications - Bunch-Davies vacuum:

$$\hat{\varphi}(\tau,k) = f_-(k\tau)\hat{a}_{-k} + f_+(k\tau)\hat{a}_k^\dagger$$

$$f_+(z) \equiv \frac{\sqrt{\pi}}{2iZ} e^{+\frac{\pi}{2}\mu} H_{i\mu}^{(2)}(-z) = f_-(z^*)^* \tag{2.1}$$

where $H^{(2)}$ is the Hankel function of 2nd kind. This choice is ”natural” in the sense that they behave like modes in Minkowski, in the far past, when all modes are deep inside the horizon.

To study the cosmic effects due to self-interaction of the inflaton (perturbatively), we introduce 3 key relevant observables

S-matrix

Note that due to time dependence of FLRW scale-factor, the free hamiltonian is already time-dependent. It means, for example, that defining interacting in and out states in the scattering theory need specification of time for the free asymptotic eigenstates.

The S-matrix encodes the transition probabilities between initial and final states. The definition of the Bunch-Davies S-matrix, as first defined in the recent paper [7], $S_{n'\to n}$, is given by:

$$S_{n'\to n} \equiv {}_{\text{out}}\langle n, -\infty | n', -\infty\rangle_{\text{in}} \tag{2.2}$$

where the label of $-\infty$ indicates that these states asymptote to eigenstates of free hamiltonian evaluated at $-\infty$, in the far past (for in states)/ far future (for out states).

The Green's function $G_{n'\to n}$ for this process is expressed as:

$$G_{n'\to n} \equiv {}_{\text{out}}\langle 0| T \prod_{b=1}^{n} \hat{\varphi}(\tau_b, -\mathbf{k}_b) \prod_{b'=1}^{n'} \hat{\varphi}(\tau'_{b'}, \mathbf{k}'_{b'})|0\rangle_{\text{in}} \tag{2.3}$$

The amputated green's function and the on-shell S-matrix elements can then be shown to be given by the usual LSZ procedure for scattering theory in QFT (see [7]):

$$\mathscr{G}_{n'\to n} \equiv \left[\prod_{b=1}^{n} iZ^2 \hat{\mathscr{E}}[k_b \tau_b]\right] \left[\prod_{b'=1}^{n'} iZ^2 \hat{\mathscr{E}}[k'_{b'} \tau'_{b'}]\right] G_{n\to n'} \tag{2.4}$$

$$S_{n'\to n} = \left[\prod_{b=1}^{n} \int_{-\infty}^{0} \frac{d\tau_b}{-\tau_b} f^{+}(\mathbf{k}_b \tau_b)\right] \left[\prod_{b'=1}^{n'} \int_{-\infty}^{0} \frac{d\tau'_{b'}}{-\tau'_{b'}} f^{-}(\mathbf{k}'_{b'} \tau'_{b'})\right] \mathscr{G}_{n'\to n} \tag{2.5}$$

Clearly therefore, we obtain the Feynman rules for S-matrix elements by the procedure completely analogous to that in flat space.

$$\langle 0| T \varphi(\mathbf{k}, \tau) \varphi(\mathbf{k'}, \tau')|0\rangle = f^{-}(k\tau_{>}) f^{+}(k\tau_{<}) (2\pi)^d \delta^d(\mathbf{k} + \mathbf{k'}) \tag{2.6}$$

is the propagator (sometimes conveniently denoted by $G_2(k\tau, k'\tau')$) and the incoming/outgoing particles in the external legs contribute a factor of $f^{-}(k\tau)/f^{+}(k\tau)$.

Wavefunction Coefficients

The wavefunction coefficients $\psi_n(\tau_0)$ capture the time evolution of the vacuum state in cosmology. Starting with the "in" vacuum some time in the far past, the information in the wavefunction of the universe at late times, τ_0, can be broken down into infinite set of wavefunction coefficients as follows:

$$\Psi_{\tau_0}[\phi] \equiv \langle\phi(\tau_0)|0,-\infty\rangle_{\text{in}} \equiv \exp\left(\sum_n \frac{1}{n!}\prod_{b=1}^{n}\int \frac{d^d\mathbf{k}_b}{(2\pi)^d}\phi(\mathbf{k}_b)\psi_n(\tau_0)\right) \tag{2.7}$$

The coefficients $\psi_n(\tau)$ are then given by the connected part of RHS below:

$$\psi_n(\tau_0) = \left.\frac{\delta^n\Psi_{\tau_0}[\phi]}{\Psi_{\tau_0}[0]\,\delta\phi_{k_1}\cdots\delta\phi_{k_n}}\right|_{\phi=0} = \langle\phi(\tau_0)=0|\prod_{b=1}^{n}\frac{i\hat{\Pi}(\tau_0,\mathbf{k}_b)}{Z^2}|0,-\infty\rangle \tag{2.8}$$

Their evaluation in perturbation theory is analogously simplified by "Witten Diagrams" and associated rules for writing them. Everything works essentially the same way as the usual Feynman diagrams/rules, except here, there are 2 main building blocks - bulk-bulk and bulk-boundary propagators:

$$\begin{aligned}
\text{Bulk-to-boundary:}\quad & \langle 0_{\tau_0}|\hat{\Pi}_k(\tau_0)\hat{\phi}_k(\tau)|\Omega\rangle = K_k(\tau,\tau_0) = \frac{(-\tau)^{\frac{d}{2}}f^+(k\tau)}{(-\tau_0)^{\frac{d}{2}}f^+(k\tau_0)}\\
\text{Bulk-to-bulk:}\quad & \langle 0_{\tau_0}|T\hat{\phi}_k(\tau)\hat{\phi}_{-k}(\tau')|\Omega\rangle = G_k(\tau,\tau',\tau_0) = i\Big[G_2(k\tau,k\tau')\\
& -\frac{f^-(k\tau_0)}{f^+(k\tau_0)}\cdot(-\tau)^{d/2}f^+(k\tau)\cdot(-\tau')^{d/2}f^+(k\tau')\Big]
\end{aligned} \tag{2.9}$$

In-In correlators

The temperature fluctuations we observe in the CMB, with the use of appropriate transfer functions, can be most directly related to these theoretical objects - primordial correlation functions at constant time slice in the far past of our universe. They are called in-in correlators and are simply the expectation values of product of field operators close to the beginning of Hot Big-Bang (late times for inflationary/de Sitter phase, evolving from Bunch-Davies vacuum in the far past).

$$_{\text{in}}\langle 0|\phi_H(\mathbf{k}_1,\tau_0)..\phi_H(\mathbf{k}_n,\tau_0)|0\rangle_{in} = \langle 0|U_I^\dagger(\tau_0,-\infty^+)\phi_I(\mathbf{k}_1,\tau_0)..\phi_I(\mathbf{k}_n,\tau_0)U_I(\tau_0,-\infty^-)|0\rangle \tag{2.10}$$

Again, this interaction picture representation sets the stage for doing perturbation theory and we can again derive the feynman rules for the witten diagrams. They are basically the same, again, except for the difference in propagators and the fact that there are more witten diagrams contributing here - interaction vertices can be taken both above and below the final time slice. There are again 2 propagators here: Wightman propagator for bulk-boundary and boundary-crossing contractions; the usual Feynman propagator/its complex conjugate for bulk-bulk contractions below/above the time slice.

Note: At tree-level, we prefer to work with wavefunction coefficients since they are directly obtained from on-shell action, which is easy to work with, and then we can use the standard relations between ψ and $\langle\phi...\phi\rangle'$ etc to obtain the latter..

2.2 Goal, Approach and Challenges

The utility and definitions of cosmological observables are well known to any cosmologist but the newly proposed definition of S-matrix in [7], as just described above, was intended to strengthen the connection between the tools/language used in high energy physics and cosmology. The hope was that it could lead to new insights in cosmology since S-matrix is a far more well-understood object than the natural observables of cosmology. The paper derived the relation betwen $\psi_n(\tau_0)$ and $S_{0\to n}$, for example ($\tau_0 = 0$ in de Sitter):

$$\lim_{\tau\to 0}\psi_{k_1\ldots k_n}(\tau) = \sum_{j=0}^{\infty}\left[\prod_{\ell=1}^{j}\int_{q_\ell q'_\ell} P_{q_\ell q'_\ell}(\tau)\right]\frac{(-1)^j S_{0\to k_1\ldots k_n q_1 q'_1\ldots q_j q'_j}}{\left[\prod_{b=1}^{n}(-\tau)^{d/2}f^+(k_b\tau)\right]\left[\prod_{c=1}^{j}(-\tau)^d f^+(q_c\tau)f^+(q'_c\tau)\right]}, \tag{2.11}$$

where the integral is a double integral over internal momenta q_l, q'_l, for each l

This horrible looking equation can be schematically written as follows for clarity, for $\psi_{k_1k_2k_3}(\tau_0)$ for example,:

$$\mathscr{F}_3^+\psi_3 = S_3 - \int\int P_{qq'}S_{3qq'} + \int\int P_{q_1q'_1}\int\int P_{q_2q'_2}S_{3q_1q'_1q_2q'_2}\cdots\infty$$

where $\mathscr{F}_3^+$ is just the product $\prod_{b=1}^{3}(-\tau)^{d/2}f^+(k_b\tau)$

We (Elaf and I) read the paper mainly with the goal of using unitarity of S and finding its consequences on ψ. The idea was to use the general non-perturbative statement of optical theorem of S and possibly derive the cosmological cutting rules of wavefunction coefficients in full generality. This would have clearly been a more illuminating approach than found in the literature [11].

For this we had to invert this general relation and express some general S element in terms of ψ. We found a simple relation:

$$S_{0\to n} = \prod_{b=1}^{n}(-\tau)^{d/2} f^{+}(k_b\tau)\Big[\psi_n - \int\int P_{qq'}\psi_{nqq'}\Big] \tag{2.12}$$

Having found this relation, we went ahead and used the generalized optical theorem of S:

$$S_{i\to f} + S^{*}_{f\to i} = \sum_n S_{i\to n}S^{*}_{f\to n} \tag{2.13}$$

and substituting S from (2.12) into (2.13), we derived the corresponding set of constraints on ψ's.

We explicitly computed some examples (for some ψ_n in ϕ^3 theory etc) to check the constraints, and except for few simple cases, we were unable to reproduce the known cutting rules. There were several technical gaps for us here and so it took a long time for us to figure out what was really going on. We eventually realized that the simple inverted relation (2.12) between S and ψ was indeed wrong and the fault had to do with our technical misreading/misunderstanding of the original relation (2.11) of [7].

Clearly there are 2 kinds of momenta arguments for S in (2.11), external (k's) and internal (q's), with latter being integrated over. We realized that the equation (2.11) is not to be read literally. The k's are considered distinct while q's here are to be considered equivalent. This becomes crucial in determining the combinatorial symmetry factor of diagrams contributing (in S_n). For example, consider the expression for ψ_4 in ϕ^3 theory at 2nd order in perturbation theory (see [7]) :

$$\mathscr{F}_4^{+}\psi_{k_1k_2k_3k_4} = S_{0\to k_1k_2k_3k_4} - \int P_{qq'}S_{k_1k_2k_3k_4qq'} = S_{0\to k_1k_2k_3k_4} - \Big(\int P_{qq'}S_{k_1k_2q}S_{0\to k_3k_4q'} + \underset{k_1,k_2,k_3,k_4}{\text{perm}}\Big) \tag{2.14}$$

First off, it is clear that this equation is essentially just the relation between the Feynman propagator, $G_F(k\tau_1, k\tau_2)$ and bulk-boundary propagator G_ψ, which schematically looks like $G_\psi = G - K^2$ (see equation 2.9). This is so because clearly, omitting the expressions ($\sim KKKK$ (i.e. K^4)) for 4 external legs (common to all the terms involved here); and also omitting combinatorial factors for now, we then have $\psi_4 \sim G_\psi \sim G - K^2$, $S_{0\to 4} \sim G$ and $S_{0\to 6} = S_{0\to 3}S_{0\to 3} \sim K^2$

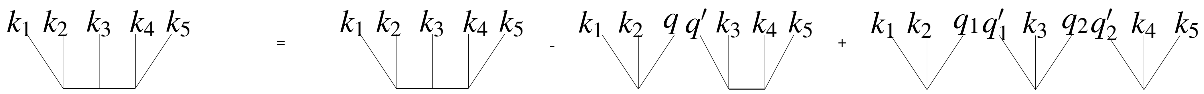
Coming to the combinatorial factors, since we are working at a fixed, 2nd order in perturbation theory, all the terms involved have a common factor of $x = \frac{\lambda^2}{2!3!^2}$. Furthermore, the permutation of vertices for any term here (2!) and the internal permutation of contractions on each vertex ($3!^2$) are common for each term as well. What differs is the different groupings, which is where it becomes essential to know how to treat q,q'.

For the equation above to be correct, we can now see that q and q' shouldn't be considered distinct for the grouping procedure counting. This grouping combinatorial factor for each term is: $\psi_4 : \frac{4!}{2!^2 2!}$, $S_{0\to4} : \frac{4!}{2!^2 2!}$. Assuming q, q' to be "distinct" (just as external k's are), we must have $S_{0\to3}S_{0\to3} : \frac{4!}{2!^2}$ because (12q)(34q') and (34q)(12q') are to be considered distinct (whereas in case of just external momenta, (12)(34) and (34)(12) were to be considered same (and thus not counted twice in the group counting) because we had already accounted for their difference earlier in permuting vertices (2!) (the first step!)). We thus immediately see that this "prescription" of taking them distinct gives us the wrong formula $G_\psi = G - \frac{K^2}{2}$ instead. To fix it, we proposed the prescription of keeping q, q' on equal footing, since that clearly makes $S_{0\to3}S_{0\to3} : \frac{4!}{2!^2 2!}$ back again.

We checked the validity of this prescription for higher-point diagrams and realized that the prescription more generally is to consider *all* internal momenta on equal footing; in other words, "just permute the external momenta and don't worry about which set of k's goes with which q..". But we further realized that even this was incorrect. The "general" prescription above worked only at tree level apparently. I now present below the expressions of ψ_5 at tree-level in ϕ^3 theory, and ψ_4 at loop-level in ϕ^4 theory, to illustrate the points just made (omitting/cancelling again the bulk-boundary propagator/f^+ term for outgoing particles in S-matrix, for external legs, i.e. $\sim K^4$ terms common to LHS and RHS):

Tree Level:

$$3!^4 x \frac{5!}{2!^3}(G-K^2)(G-K^2) = 3!^4 x \frac{5!}{2!^3}(G^2) \; - \; 3!^4 x \frac{5!}{2!^2}(GK^2) \; + \; 3!^4 x \frac{5!}{2!^3}(K^4)$$



$$\psi_5 \quad = \quad S_5 \quad - \quad \int P_{qq'} S_{5,qq'} \quad + \quad \int P_{q_1 q_1'} \int P_{q_2 q_2'} S_{5,q_1 q_1',q_2 q_2'}$$

Figure 2.1: Graphical representation of ψ_5 at tree-level in terms of S's

Here, $x = \frac{\lambda^3}{3!3!^3}$. We thus see that the prescription indeed works for this higher, 5-point case too. We checked 6,7 cases too, taking cases with and without symmetry in the diagrams to ensure we are not dealing with special cases. We thus concluded that at tree level, the prescription definitely seems to work.

Loop Level:

$$2!4!^2 y \frac{4!}{2!^3}(G-K^2)(G-K^2) \; = \; 2!4!^2 y \frac{4!}{2!^3}(G^2) \quad - \quad 2!4!^2 y \frac{4!}{2!^3}(GK^2) \quad + \quad 2!4!^2 y \frac{4!}{2!^3}(K^4)$$

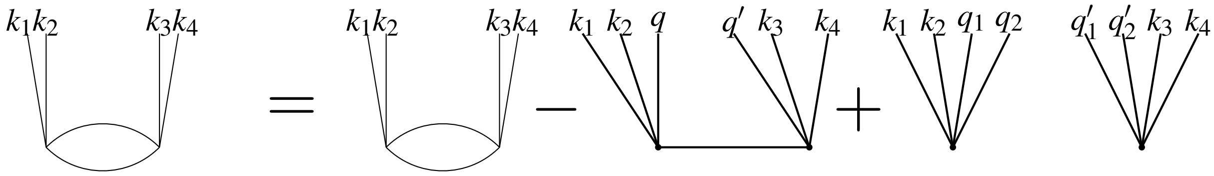


Figure 2.2: Graphical representation of ψ_4 at 1-loop level in terms of S's

Here, y = $\frac{\lambda^2}{2!4!^2}$. The prescription seems to be failing at loop level. In the example above, there is a factor of 2 missing in the middle term in the RHS.

All-in-all, we realized that we had failed to understand the general relation between ψ and S properly. We got stuck at this point and couldn't find a resolution and appropriate prescription. In any case, the inversion formula we had derived relied on a specific prescription, which was actually incorrect. Therefore that derivation doesn't go through anymore and thus despite pushing on still, trying to invert the relation 2.11, we struggled and couldn't make much progress.

Chapter 3

Part 2: Investigating Symmetry Constraints on Cosmological Observables

3.1 Symmetries: General Discussion

General FLRW Metric: The metric

$$ds^2 = a^2(\tau)(d\tau^2 - dx^2) = dt^2 - a^2(t)\,dx^2$$

possesses three translational and three rotational symmetries, reflecting the homogeneity and isotropy of the universe. These symmetries manifest in the behavior of perturbations around this background. For a perturbed FLRW metric of the form

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 + 2B_i\,dx^i d\eta + (\delta_{ij} + 2E_{ij})\,dx^i dx^j\right],$$

transformations associated with the background symmetries lead to *linear transformations* of the perturbations. This linearity arises from the fact that the perturbations themselves are scalar, vector, and tensor modes. For example, under an infinitesimal spatial translation $x'^i = x^i + a^i$, the scalar perturbation transforms as $A'(x) = A(x) - a^i \partial_i A$.

On the other hand, the spatial slices of FLRW are also conformally invariant—admitting one dilation and three special conformal transformations (SCTs). Under such transformations, the perturbations are affected by a local conformal factor $\Omega(x)$, resulting in a *non-linear transformation* (a field-independent term—effectively a "shift" in field space; more precisely, such transformations should be referred to as sub-linear). These transformations are, therefore, naturally realized nonlinearly on the fields. More generally, most transformations—whether symmetries of the background or not—are nonlinearly realized on the perturbations of the metric.

Soft theorems are concerned with precisely such transformations: non-linearly realized symmetries of the gauge-fixed and constraint-imposed physical action. They constrain the behavior of correlators, especially in the soft (low-momentum) limit.

de Sitter Metric:

$$ds^2 = \frac{1}{\tau^2}(d\tau^2 - dx^2)$$

This is a special case within the FLRW class, notable for its full invariance under the conformal group. In this case, dilations and SCTs are realized *linearly* on perturbations, and as a result, their associated soft theorems reduce to standard Ward identities for de Sitter fluctuations.

Linearly realized symmetries constrain the *form* of correlation functions. The greater the symmetry, the more constrained the form becomes. For instance, in conformal field theories (CFTs), one often leverages the additional dilation and special conformal symmetries to constrain correlators. These symmetries are linearly realized on primary operators and allow one to completely fix the two- and three-point functions, while higher-point functions are determined up to functions of conformally invariant cross-ratios. This ability to constrain correlators using linearly realized symmetries forms one of the central pillars of the ***conformal bootstrap*** program.

Non-linearly realized symmetries, on the other hand, provide partial information in the form of *recursion relations* between $(N+1)$-point and N-point correlation functions. In principle, if these relations were fully known, one could, in theory, compute all higher-point correlators recursively starting from the most basic two-point function. Schematically it looks like:

$$\frac{\langle N, q\rangle}{P(q)} = D^0\langle N\rangle + q^i D^1_i\langle N\rangle + \frac{q^i q^j}{2} D^2_{ij}\langle N\rangle + \cdots$$

where D^n corresponds to n-symmetry, in the hierarchy of different symmetries labelled by n.. D^0 and D^1 are nothing but the linear generators of dilations and SCTs acting on N-pt functions on RHS. For higher n, they are suitable combinations of D's with different indices.

3.1.1 Linearly Realized Symmetries and Ward Identities

Assuming that vacuum is invariant under relevant transformations, which are linearly realized, we get the following relation for translations and rotations (finite and infinitesimal):

For translations, we have the following relations:

In x-space,

$$T^\dagger_{\vec{a}}\phi(\vec{x})T_{\vec{a}} = \phi(\vec{x}+\vec{a}) \implies \langle\phi(\vec{x}_1+\vec{a}) - \phi(\vec{x}_n+\vec{a})\rangle = \langle\phi(\vec{x}_1) - \phi(\vec{x}_n)\rangle \tag{3.1}$$

$$[\phi(\vec{x}), P_i] = -i\partial_i\phi(\vec{x}) \implies \sum_{A=1}^{m} \frac{\partial}{\partial x^i_A}\langle\phi(\vec{x}_1) - \phi(\vec{x}_n)\rangle = 0 \tag{3.2}$$

And in k-space,

$$T_{\vec{a}}^{\dagger}\phi(\vec{k})T_{\vec{a}} = e^{-i\vec{k}\cdot\vec{a}}\phi(\vec{k}) \implies e^{-i\vec{k}_T\cdot\vec{a}}\langle\phi(\vec{k}_1) - \phi(\vec{k}_n)\rangle = \langle\phi(\vec{k}_1) - \phi(\vec{k}_n)\rangle \tag{3.3}$$

$$[\phi(\vec{k}), P_i] = -k_i\phi(\vec{k}) \implies \sum_{A=1}^{m} k_A^i\langle\phi(\vec{k}_1) - \phi(\vec{k}_n)\rangle = 0 \tag{3.4}$$

Similarly for rotations, we have:

$$R_O^{\dagger}\phi(\vec{x})R_O = \phi(R\vec{x}) \implies \langle\phi(R\vec{x}_1) - \phi(\vec{R}x_n)\rangle = \langle\phi(\vec{x}_1) - \phi(\vec{x}_n)\rangle \tag{3.5}$$

$$[\phi(\vec{x}), J_i] = -i\varepsilon_{ijk}x^j\partial^k\phi(\vec{x}) \implies \sum_{A=1}^{m}\left(x_A^i\partial^j - x_A^j\partial^i\right)\langle\phi(\vec{x}_i) - \phi(\vec{x}_n)\rangle = 0 \tag{3.6}$$

In k-space:

$$R_O^{\dagger}\phi(\vec{k})R_O = \phi(R\vec{k}) \implies \langle\phi(R\vec{k}_1) - \phi(R\vec{k}_n)\rangle = \langle\phi(\vec{k}_1) - \phi(\vec{k}_n)\rangle \tag{3.7}$$

$$[\phi(\vec{k}), J_i] = -i\varepsilon_{ijk}k^j\frac{\partial}{\partial k^k}\phi(\vec{k}) \implies \sum_{A=1}^{m}\left(k_A^i\frac{\partial}{\partial k^j} - k_A^j\frac{\partial}{\partial k^i}\right)\langle\phi(\vec{k}_i) - \phi(\vec{k}_n)\rangle = 0 \tag{3.8}$$

For inflaton ϕ, dilations are also linearly realized.

Action

$$\begin{aligned} S[\Phi] &= \int d^dx d\tau\sqrt{-g(\tau)}g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi \\ &= \int d^dx d\tau\, a^{d+1}(\tau)\left[\frac{1}{a^2(\tau)}\left(\dot{\phi}^2 - (\nabla\phi)^2\right)\right] \\ &= \int d^dx d\tau\, a^{d-1}(\tau)\left(\dot{\phi}^2 - \nabla\phi^2\right). \end{aligned} \tag{3.9}$$

Dilated Action

$$\begin{aligned} S[\Phi'] &= \int d^dx d\tau\sqrt{-g(\tau)}g^{\mu\nu}(x)\partial_\mu\phi'\partial_\nu\phi' \\ &= \int d^dx' d\tau'\sqrt{-g(\tau')}g^{\mu\nu}(x')\partial'_\mu\phi'\partial'_\nu\phi' \\ &= \int d^dx d\tau\lambda^{d-1}a^{d-1}(\lambda\tau)\left[(\phi'(x'))'^2 - (\nabla\phi(x))^2\right] \end{aligned} \tag{3.10}$$

Dilation Symmetry For dilation to be symmetry of FLRW-ϕ action, let $\phi'(x') = \lambda^{-\Delta}\phi(x)$ be the transformation rule for ϕ. Then we get $\lambda^{d-1-2\Delta}a^{d-1}(\lambda\tau) = a^{d-1}(\tau)$ for symmetry. Thus we get $a(\lambda\tau) = a(\tau)\lambda^{\frac{2\Delta}{d-1}-1}$. For $a \propto \tau^n$, we see thus $\Delta = \frac{(n+1)(d-1)}{2}$
For example, for dS, d = 3 & n = -1, thus $\Delta = 0$..

Under the transformation $x'^{\mu} = \lambda x^{\mu}$ (and use: $\lambda = 1+\varepsilon$ for infinitesimal transformations), the quantum scalar field ϕ_Δ of scaling dimension Δ transforms as:

$$\phi'_\Delta(x) \equiv U_\lambda^\dagger \phi_\Delta(x) U_\lambda = \lambda^{-\Delta}\phi(x/\lambda) = \lambda^{-\Delta}\phi(\vec{x}/\lambda, \tau/\lambda)$$

Restricting to infinitesimal transformations, it is easy to see that the generator of dilations, $D_{\phi_\Delta(x)}$, for the scalar field representation of dilations, is given by $D_{\phi_\Delta(x)} = -(\Delta + \tau\partial_\tau + \vec{x}\cdot\nabla_{\vec{x}})$ where $\phi'_\Delta(x) = \phi_\Delta(x) + \varepsilon D_{\phi_\Delta(x)}$

In k-space, we correspondingly get:

$$\phi'_\Delta(\vec{k},\tau) \equiv U_\lambda^\dagger \phi_\Delta(\vec{k},\tau) U_\lambda = \int d^d x d\tau e^{-i\vec{k}\cdot\vec{x}} U_\lambda^\dagger \phi_\Delta(\vec{x},\tau) U_\lambda = \lambda^{d-\Delta}\phi(\lambda\vec{k}, \tau/\lambda) \tag{3.11}$$

Note the similarity/contrast between the transformation rule in $\vec{x}$-space and $\vec{k}$-space.

Again restricting to infinitesimal transformations, we obtain the representation of dilation generator on k-space fields:

$$D_{\phi_\Delta(\vec{k},\tau)} = d - (\Delta + \tau\partial_\tau - \vec{k}\cdot\nabla_{\vec{k}}) \quad (\text{w/ } \phi'_\Delta(\vec{k},\tau) = \phi_\Delta(\vec{k},\tau) + \varepsilon D_{\phi_\Delta(\vec{k},\tau)}) \tag{3.12}$$

Ward Identities for Dilation

$\boxed{D_{\langle\phi_1\ldots\phi_n\rangle} = \sum_{a=1}^{n} D_{\phi_a} (\text{for both } \mathbf{x} \text{ and } \mathbf{k})}$ & $\boxed{D_{\langle\ldots\rangle'_k} = D_{\langle\ldots\rangle_k} - d}$ (prove using chain rule & $\vec{k_T}\cdot\nabla_{\vec{k_T}}\delta^d(\vec{k_T})$ =

And Ward Identities are then simply derived: $\boxed{D_{\langle\ldots\rangle}(\langle\ldots\rangle) = 0}$

In the context of Maldacena action, we know that ϕ is in fact a scalar field (by principle of general covariance), i.e. $\phi'(x') = \phi(x)$ under arbitrary coordinate transformations and thus dilations too, and thus scaling dimension is 0 anyway. In any case, dilations are clearly linearly realized here due to simple, tensorial transformations.

Therefore all-in-all, we see that linearly realized symmetries indeed constrain the form of correlators $\langle\phi_1\ldots\phi_n\rangle$ for fixed n. For example, in 4d Euclidean CFTs, which has 15 linearly realized symmetries (SO(4,2)), one can completely fix the functional form of 2 and 3-point functions (upto specific coupling strengths of course).

In this thesis however, we are investigating soft theorems which do not constrain the form, but instead provide relations between N and N-1 correlators. They too follow from symmetries but non-linearly realized ones. The realization of a transformation is more about the quantity being transformed, than the transformation itself. For example, dilation in $\delta\phi$-gauge is a linearly realized symmetry on the inflaton $\delta\phi$ giving us respective ward identities, as derived above, for inflaton correlators. For curvature perturbations in ζ-gauge however, they are non-linearly realized and give us the leading order (in q) relation between N and N-1 correlators (see 3.18). Therefore, we see that *same* symmetry can have *different consequences* for different observables due to different realizations of the symmetries. We now turn to studying non-linearly realized symmetries and their consequences on cosmological correlators.

3.2 Soft Theorems: Collecting Important Definitions and Equations

I mainly used [15], [17] to learn this material. Before reviewing the logic behind the derivation of soft theorems, I will first state the key equations and results that will be useful later in this thesis.

Gauge transformations of ζ, γ

We work in ζ-gauge which is fully fixed with respect to small gauge transformations ($\zeta(\vec{x} \to \infty) = 0$). With respect to large gauge transformations however, it is not a fully fixed gauge. There are residual gauge transformations remaining: $x'^i = x^i + \xi^i$ where ξ is given by:

$$\xi_i \equiv \sum_{n=0}^{\infty} \xi_i^{(n)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} M_{i\ell_0 \dots \ell_n} x^{\ell_0} \dots x^{\ell_n} \tag{3.13}$$

In momentum space, we have the following infinitesimal gauge transformations of scalar and tensor perturbations in comoving gauge, at a fixed order in tensor perturbations (scalar being exact)

$$\partial\zeta^{(n)} = \frac{(-i)^n}{n!} \left[\frac{M_{ii[n]}}{3} \partial^{[n]} \tilde{\delta} - M_{ij[n]} \left(\delta^{ij} \partial^{[n]} + \frac{k^i \nabla^j}{n+1} \partial^{[n]} \right) \zeta + M_{ij[n]} Y^{ijkl} \partial^{[n]} \gamma_{kl} + O(\gamma^2) \right] \tag{3.14}$$

$$\partial\gamma_{ij}^{(n)} = \frac{(-i)^n}{n!} \left[M_{ij[n]}^{Conf} \partial^{[n]} \tilde{\delta} - M_{ab[n]} \left(\delta^{ab} \partial^{[n]} + \frac{k^a \nabla^b}{n+1} \partial^{[n]} \right) \gamma_{ij} + M_{ab[n]} \Gamma^{ab}{}_{ij}{}^{cd} \partial^{[n]} \gamma_{cd} + O(\gamma^2) \right] \tag{3.15}$$

where $M_{ij[n]}^{Conf} = M_{ij[n]} + M_{ji[n]} - \frac{2}{3} \delta_{ij} M_{aa[n]}$;

$Y^{ijkl}(\hat{k}) = \frac{1}{4}(\delta^{ij}\hat{k}^k\hat{k}^l - \frac{1}{2}\delta^{ik}\hat{k}^j\hat{k}^l - \frac{1}{2}\delta^{il}\hat{k}^j\hat{k}^k)$

$$\Gamma_{abijkl}(\hat{k}) = -\frac{1}{2}\left(\delta_{ij}+\hat{k}_i\hat{k}_j\right)\left(\delta_{ab}\hat{k}_k\hat{k}_\ell - \frac{1}{2}\delta_{ak}\hat{k}_\ell\hat{k}_b - \frac{1}{2}\delta_{a\ell}\hat{k}_k\hat{k}_b\right) + \delta_{b(i}\delta_{j)(k}\delta_{l)_a} - \delta_{a(i}\delta_{j)(k}\delta_{l)b}$$
$$-\delta_{b(i}\hat{k}_{j)}\delta_{a(k}\hat{k}_{l)} + \delta_{a(i}\hat{k}_{j)}\delta_{b(k}\hat{k}_{l)} - \delta_{a(k}\delta_{l)(i}\hat{k}_{j)}\hat{k}_b - \delta_{b(k}\delta_{l)(i}\hat{k}_{j)}\hat{k}_a + 2\delta_{ab}\hat{k}_{(i}\delta_{j)(k}\hat{k}_{l)}$$

Here, the first term in the equations above is the "non-linear/sub-linear transformation" part, while the second term is the linear transformation part.

$\boxed{\textbf{\textit{Soft Theorems}}}$:

They basically are resulting by looking at the full transformation above, and giving non-linear part to LHS and linear and higher order part to RHS of the Ward identity ($\langle\Omega|[Q,\mathscr{O}]|\Omega\rangle = -i\langle\Omega|\delta\mathscr{O}|\Omega\rangle$). This will become clear in the derivation. I write the identities for connected (unprimed) correlation functions below:

$$\begin{aligned}\lim_{q\to 0} M_{ij[n]}(\hat{q})\partial^{[n]}\left(\frac{\delta^{ij}}{3}\frac{\langle\zeta(q)O(k_1\ldots k_N)\rangle_c}{P_\zeta(q)} + \frac{\langle\gamma^{ij}O(k_1\ldots k_N)\rangle_c}{P_\zeta(q)}\right)&\\
= M_{ij[n]}(\hat{q})\Bigg[-\sum_{a=1}^{N}\partial_a^{[n]}\left(\delta^{ij}+k_a^i\nabla_a^j\partial_a^{[n]}\right)\langle O(k_1\ldots k_N)\rangle_c&\\
+\sum_{b=1}^{M}Y^{ijkl}(\hat{k}_b)\partial_b^{[n]}\langle O^{\zeta}_{(1\ldots(b-1)}\gamma_{kl}(k_b)_{(b+1)\ldots M)}O^{\gamma}_{(M+1)\ldots N}\rangle_c&\\
+\sum_{c=M+1}^{N}\Gamma^{ij}{}_{p_{i_c}q_{j_c}}{}^{kl}(\hat{k}_c)\partial_c^{[n]}\langle O^{\zeta}_{(1\ldots M)}O^{\gamma}_{(M+1\ldots(c-1)}\gamma_{kl}(\hat{k}_c)_{(c+1)..N)}\rangle_c\Bigg]&\\
+O(\gamma^2)&\end{aligned} \tag{3.16}$$

where $\partial^{[n]}$ means $\frac{\partial^n}{\partial k_1\partial k_2\ldots\partial k_n}$

$\boxed{\textbf{\textit{Soft Theorems for n = 0}}}$:

$$
\begin{aligned}
\lim_{q\to 0} M_{ij}(\hat{q})\left(\frac{\delta^{ij}}{3}\frac{\langle\zeta(q)O(k_1\dots k_N)\rangle_c}{P_\zeta(q)}+\frac{\langle\gamma^{ij}O(k_1\dots k_N)\rangle_c}{P_\zeta(q)}\right)& \\
= M_{ij}(\hat{q})\Bigg[-\sum_{a=1}^{N}\left(\delta^{ij}+k_a^i\nabla_a^j\right)\langle O(k_1\dots k_N)\rangle_c& \\
+\sum_{b=1}^{M} Y^{ijcd}(\hat{k}_b)\langle O^{\zeta}_{(1\dots(b-1)}\gamma_{cd}(k_b)_{(b+1)\dots M)}O^{\gamma}_{(M+1)\dots N}\rangle_c& \\
+\sum_{c=M+1}^{N}\Gamma^{ij}{}_{p_{ic}q_{jc}}{}^{kl}(\hat{k}_c)\langle O^{\zeta}_{(1\dots M)}O^{\gamma}_{(M+1\dots(c-1)}\gamma_{kl}(\hat{k}_c)_{(c+1)\dots N)}\rangle_c\Bigg]& \\
+O(\gamma^2)&
\end{aligned}
\tag{3.17}
$$

For ex, for dilations, $M_{ij}=\lambda\,\delta_{ij}$, using traceless and transverse conditions for γ, we quickly obtain the usual consistency relations:

$$
\lim_{q\to 0}\langle\frac{\langle\zeta(q)O(k_1\dots k_N)\rangle_c}{P_\zeta(q)}=-\sum_{a=1}^{N}\left(3+k_a\cdot\nabla_{k_a}\right)\langle O(k_1\dots k_N)\rangle_c \tag{3.18}
$$

The soft theorems that we will explicitly check later in 4.5, also correspond to n = 0, but not to dilations. Instead, they are associated with a different symmetry—one that is non-linearly realized by γ, in contrast to dilations—namely, the *anisotropic rescaling symmetry*. For this, $M_{ij}(\hat{q})=S_{ij}(\hat{q})$ where $S_{ii}=0$, $q^iS_{ij}=0$. i.e. M_{ij} for this symmetry transformation is basically like TT polarization tensor/graviton. The 3-2 soft theorem for this symmetry then looks like:

$$
\begin{aligned}
\lim_{\vec{q}\to 0} S_{ij}(\hat{q})\frac{\langle\gamma^{ij}(\vec{q})\gamma_{ab}(\vec{k}_1)\gamma_{cd}(\vec{k}_2)\rangle'_c}{P_\gamma(q)} &= S_{ij}(\hat{q})\Bigg[-\left(k_1^i\nabla_1^j\right)\langle\gamma_{ab}(\vec{k}_1)\gamma_{cd}(-\vec{k}_1)\rangle'_c \\
&+\Gamma^{ij}{}_{ab}{}^{pq}(\hat{k}_1)\langle\gamma_{pq}(\vec{k}_1)\gamma_{cd}(-\vec{k}_1)\rangle'_c+\Gamma^{ij}{}_{cd}{}^{pq}(\hat{k}_1)\langle\gamma_{ab}(\vec{k}_1)\gamma_{pq}(-\vec{k}_1)\rangle'_c\Bigg]
\end{aligned}
\tag{3.19}
$$

where I stripped the delta functions in writing this identity, and used the even-ness of Γ in the 2nd line.

3.3 Reviewing the Derivation of Inflationary Soft Theorems

In essence, the usual soft theorems are really just the consequences of some special non-linearly realized symmetries (on propagating fields) of the action on correlation functions. They provide consistency relations between N and N-1 point functions of those fields, providing a partial recursion relation of sorts, thus recursively giving us information in higher correlators that we can already infer by symmetries. Of course, it is not possible to fully fix the recursion relation, since otherwise interactions won't have any role left in determining the correlators.

We first present the argument for how residual gauge transformations in a partially-fixed gauge can be lifted to become genuine symmetries of the action in that partially-fixed gauge. In the context of inflationary action, the "partially-fixed" gauge is actually fully-fixed gauge, but with respect to *small gauge transformations* only ($\xi(t,\vec{x}\to\infty)=0$); and the residual gauge transformations will thus be *large gauge transformations*, which will be non-linearly realized on the propagating fields. To lift these large gauge transformations to genuine symmetries requires imposing the usual *adiabatic mode condition* (see [4]). In the context of inflation, we will be mainly working with the ζ, $\delta\phi$ gauges (fixed with respect to small gauge transformations). Having obtained genuine (non-linearly realized) symmetries of the action then, we now expect to get non-trivial statements about observables in the theory. Following [17], we then present their argument to deduce the early-late time consistency relations from such symmetries in the simplest setting of just a single scalar field. We thus realize that the purely late-time consistency relations *do not* follow from non-linearly realized symmetries alone. For some special kind of symmetries however, satisfying so-called *physical mode condition*, we can turn the early-late relations into purely late-time ones.

3.3.1 Large RGT → Symmetries: Adiabatic Mode Condition

The revered principle of general coordinate covariance in General Relativity says mathematically, restricting ourselves here to infinitesimal coordinate transformations (gauge transformations):

$$S=\int d^nx\sqrt{h}N\mathscr{L}(h_{ij},N,N^i,\Phi)\overset{x'^\mu=x^\mu+\xi^\mu}{=}\int d^nx\sqrt{h}N'\mathscr{L}(h'_{ij},N',N'^i,\Phi') \tag{3.20}$$

Let the small-gauge fixing conditions be $f_\mu(h_{ij},N,N^i,\Phi)=0$.

We are then interested in Residual GT, i.e. restricting ourselves to only those transformations $\tilde{x}^\mu=x^\mu+\xi^\mu_{RGT}$ such that, we further have $f_\mu(\tilde{h}_{ij},\tilde{N},\tilde{N}^i,\tilde{\Phi})=0$

So we start now with gauge-fixed (small) action:

$$S=\int d^nx\sqrt{h}N\mathscr{L}(h_{ij},N,N^i,\Phi)\quad\text{with } f_\mu(h_{ij},N,N^i,\Phi)=0. \tag{3.21}$$

Perform a residual GT, then since it's a GT:

$$S=\int d^nx\sqrt{\tilde{h}}\tilde{N}\mathscr{L}(\tilde{h}_{ij},\tilde{N},\tilde{N}^i,\tilde{\Phi})\quad\text{trivially} \tag{3.22}$$

But importantly, being ***residual***, it follows that:

$$f_\mu(\tilde{h}_{ij}, \tilde{N}, \tilde{N}^i, \tilde{\Phi}) = 0 \quad \text{too.} \tag{3.23}$$

Therefore, under a residual gauge transformation, we have the following:

$$S = \int d^n x \sqrt{\tilde{h}} \tilde{N} \mathscr{L}(\tilde{h}_{ij}, \tilde{N}, \tilde{N}^i, \tilde{\Phi}) \quad \text{with } f_\mu(\tilde{h}_{ij}, \tilde{N}, \tilde{N}^i, \tilde{\Phi}) = 0. \tag{3.24}$$

It is immediately clear then that the partially-gauge fixed action is invariant under Res GT (i.e. ***GT : Action :: Res GT : Partially Gauge-fixed Action***) as expected, of course.

So far, we are just talking of residual gauge transformation. Now, we want to move towards genuine symmetry transformations of the action of purely physical fields (propagating dof, for ex, ζ, γ in comoving gauge etc). To obtain that action, we solve the constraint equation (again, gauge invariant, and thus Res GT invariant too) and put the solution of constrained variables in terms of dynamical ones (for ex, in comoving gauge, $N(\zeta), N^i(\zeta)$) back into the partially-gauge fixed action.

That is, we now start with the condition (which basically says, in some partially-chosen gauge, we start with the action expressed purely in terms of dynamical fields):

$$S = \int d^n x \sqrt{h} N \mathscr{L}(h_{ij}, N, N^i, \Phi) \quad \text{with} \quad \begin{aligned} f_\mu(h_{ij}, N, N^i, \Phi) &= 0, \\ N &= N(\zeta), \\ N^i &= N^i(\zeta) \end{aligned} \tag{3.25}$$

Therefore for some Residual GT to ***continue to be symmetry*** of this condition, we have to impose the *further condition* that $\tilde{N} = N(\tilde{\zeta}), \tilde{N}^i = N^i(\tilde{\zeta})$, namely that these Residual GT must be symmetries, not only of constraint equations, but also of the constraint solutions. This is referred to as the *adiabatic mode condition* in the literature.

Note:
Symmetries of Equations: Map solutions to solutions.
Symmetries of (Particular) Solution: Map solution to itself

In specific contexts, such as single-field inflation (maldacena action), one needs to check whether such transformations exist at all; but one is assured that if they do, then they are genuine symmetries and thus must have observable consequences. For single-field inflationary action in ζ-gauge, following the procedure outlined above explicitly, it was shown in [15] that there are indeed infinitely many such symmetries of maldacena action, $x'^i = x^i + \xi^i(t, \vec{x})$ ($\xi^0 = 0$ since inflaton has to be set to zero as part of the definition of the small-gauge-fixing condition), at cubic order in interactions (to impose the adiabatic mode condition above, one has to solve the hamiltonian and momentum constraint equations above; they solved them at 1st order in perturbations, which implies they are working at the level of cubic action):

$$\xi^i = \left(1 + \int^t \frac{\mathrm{d}t'}{H(t')}\vec{\nabla}^2\right)\bar{\xi}^i \quad \text{where} \quad \bar{\xi}_i \equiv \sum_{n=0}^{\infty} \bar{\xi}_i^{(n)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} M_{i\ell_0\ldots\ell_n} x^{\ell_0} \ldots x^{\ell_n} \tag{3.26}$$

The time dependence of ξ^i above was obtained by demanding the adiabatic mode condition. M's are also constrained since ξ^i is supposed to be residual gauge transformation in the first place (i.e. in ζ-gauge, $\delta \partial_i \gamma_{ij} = 0$):

$$M_{i\ell\ell\ell_2\ldots\ell_n} = -\frac{1}{3} M_{\ell i\ell_2\ldots\ell_n} \quad (n \geq 1) \tag{3.27}$$

The paper then goes on and argues that there is actually one more condition required for these transformations to be genuine symmetries:

$$\tilde{q}^i \left(M_{i\ell_0\ell_1\ldots\ell_n}(\hat{q}) + M_{\ell_0 i\ell_1\ldots\ell_n}(\hat{q}) - \frac{2}{3}\delta_{i\ell_0} M_{\ell\ell\ell_1\ldots\ell_n}(\hat{q}) \right) = 0 \tag{3.28}$$

I could not understand the derivation of this equation, though.

3.3.2 Symmetries → Early-Late time Consistency Relations

In the previous subsection, we derived the form of transformations that are genuine symmetries of single-field inflationary action (at cubic order in perturbations) in ζ-gauge. Now we wish to find the consequences of these symmetries on in-in correlators. Here we briefly review the logic behind the derivation of early-late-time consistency relation as was first proved in [17]. We refer the reader to the paper for more details. Note that the argument here involves a single scalar field for simplicity, but it is generalizable.

Note: We work in the Heisenberg picture throughout and present the key steps of the derivation using path integral formalism below.

Consider the vacuum expectation value of a general operator at late times, τ_f: $O_{\varphi_f}(\vec{k}_1, \ldots, \vec{k}_N)$, built from the fundamental ϕ field (e.g., a product of ϕ's at different momenta):

$$_{in}\langle 0|O_{\varphi_f}(\vec{k}_1, \ldots, \vec{k}_N)|0\rangle_{in} = \int \mathscr{D}\varphi_f \, O_{\varphi_f}(\vec{k}_1, \ldots, \vec{k}_N)|\Psi[\varphi_f]|^2. \tag{3.29}$$

Next, consider an abstract symmetry which acts on the fields ϕ, and which may consist of both a nonlinear and a linear piece. We can write such a symmetry schematically as

$$\delta\varphi_f = \delta_{\mathrm{NL}}\varphi_f + \delta_{\mathrm{L}}\varphi_f. \tag{3.30}$$

We now perform the change of variable $\varphi_f \to \varphi_f + \delta\varphi_f$ in the integral. This does not affect the late-time correlation function, because φ_f is merely a dummy variable of integration. This implies that

$$\langle O_{\varphi_f}(\vec{k}_1,\dots,\vec{k}_N)\rangle = \int \mathscr{D}(\varphi_f+\delta\varphi_f)\left[O_{\varphi_f}(\vec{k}_1,\dots,\vec{k}_N)+\delta O_{\varphi_f}(\vec{k}_1,\dots,\vec{k}_N)\right]|\Psi[\varphi_f+\delta\varphi_f]|^2. \tag{3.31}$$

We assume that the path integral measure is invariant under this change of variable. Expanding in $\delta\varphi_f$ (assumed infinitesimal), the zeroth-order term gives the original expectation value, so the first-order terms must sum to zero:

$$0 = \int \mathscr{D}\varphi_f\,\delta O_{\varphi_f}(\vec{k}_1,\dots,\vec{k}_N)|\Psi[\varphi_f]|^2 + \int \mathscr{D}\varphi_f\,O_{\varphi_f}(\vec{k}_1,\dots,\vec{k}_N)(\Psi^*[\varphi_f]\delta\Psi[\varphi_f]+\text{c.c.}). \tag{3.32}$$

To proceed, we need an expression for $\delta\Psi$. Rather than assuming anything about the late-time wavefunctional, we infer its transformation from that of the initial Gaussian vacuum. Inserting identity operator in terms of initial time field eigenstates, we get:

$$\Psi[\varphi_f+\delta\varphi_f] = \langle\varphi_f+\delta\varphi_f|0\rangle_{in} = \int \mathscr{D}\varphi_i \int_{\phi(\tau_i)=\varphi_i}^{\phi(\tau_f)=\varphi_f+\delta\varphi_f} \mathscr{D}\phi\, e^{iS[\phi]}\Psi_0[\varphi_i]. \tag{3.33}$$

We now shift $\varphi_i \to \varphi_i+\delta\varphi_i$, giving

$$\Psi[\varphi_f+\delta\varphi_f] = \int \mathscr{D}\varphi_i \int_{\phi(\tau_i)=\varphi_i+\delta\varphi_i}^{\phi(\tau_f)=\varphi_f+\delta\varphi_f} \mathscr{D}\phi\, e^{iS[\phi+\delta\phi]}\Psi_0[\varphi_i+\delta\varphi_i]. \tag{3.34}$$

Because the transformation is a symmetry, the path integral is invariant under the change $\phi\to\phi+\delta\phi$, so:

$$\int_{\phi(\tau_i)=\varphi_i+\delta\varphi_i}^{\phi(\tau_f)=\varphi_f+\delta\varphi_f} \mathscr{D}\phi\, e^{iS[\phi]} = \int_{\phi(\tau_i)=\varphi_i}^{\phi(\tau_f)=\varphi_f} \mathscr{D}\phi\, e^{iS[\phi]}. \tag{3.35}$$

Therefore,

$$\delta\Psi[\varphi_f] = \int \mathscr{D}\varphi_i\langle\varphi_f|\varphi_i\rangle\delta\Psi_0[\varphi_i]. \tag{3.36}$$

Using the explicit form of the Gaussian wavefunctional,

$$\delta\Psi_0[\varphi_i] = -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_i(p)[\varphi_i(\vec{p})\delta\varphi_i(-\vec{p})+\delta\varphi_i(\vec{p})\varphi_i(-\vec{p})]\Psi_0[\varphi_i]. \tag{3.37}$$

We write $\delta\varphi_i = \delta_{\rm NL}\varphi_i+\delta_{\rm L}\varphi_i$. The nonlinear part is assumed to be nonzero at spatial infinity and can be written as

$$\delta_{\rm NL}\varphi_i(\vec{q}) = (2\pi)^3\delta^3(\vec{q})D_{-\vec{q}}(\tau_i), \tag{3.38}$$

with $D_{-\vec{q}}(\tau_i)$ being some operator. Consider dilations and SCTs as examples:
For the dilation symmetry:

$$D_{\vec{q}}(\tau_i) = \lambda, \quad \delta_L = \sum_{a=1}^{N-1} -\lambda\left(3+\vec{k}_a\cdot\vec{\nabla}_{k_a}\right). \tag{3.39}$$

For SCTs:

$$D_{\vec{q}}(\tau_i) = -2i\vec{b}\cdot\vec{\nabla}_q, \quad \delta_L = \sum_{a=1}^{N-1} i\left[6\vec{b}\cdot\vec{\nabla}_{k_a} - (\vec{b}\cdot\vec{k}_a)\nabla^2_{k_a} + 2(\vec{k}_a\cdot\vec{\nabla}_{k_a})(\vec{b}\cdot\vec{\nabla}_{k_a})\right]. \tag{3.40}$$

Substituting into the above gives:

$$\delta\Psi_0[\varphi_i] = -D_{\vec{q}}(\tau_i)[E_i(q)\varphi_i(\vec{q})]_{\vec{q}=0}\Psi_0[\varphi_i] - \int\frac{d^3p}{(2\pi)^3}E_i(p)\varphi_i(\vec{p})\delta_{\mathrm{L}}\varphi_i(-\vec{p})\Psi_0[\varphi_i]. \tag{3.41}$$

Substitute into the variation of the wavefunctional and then into the earlier identity, leading to:

$$\begin{aligned}
-\int\mathscr{D}\varphi_f O_{\varphi_f}\Psi^*[\varphi_f]\int\mathscr{D}\varphi_i\langle\varphi_f|\varphi_i\rangle\int\frac{d^3p}{(2\pi)^3}E_i(p)\varphi_i(\vec{p})\delta_{\mathrm{L}}\varphi_i(-\vec{p})\Psi_0[\varphi_i]\\
-D_{\vec{q}}(\tau_i)\left[E_i(q)\int\mathscr{D}\varphi_f O_{\varphi_f}\Psi^*[\varphi_f]\int\mathscr{D}\varphi_i\langle\varphi_f|\varphi_i\rangle\varphi_i(\vec{q})\Psi_0[\varphi_i]\right]_{\vec{q}=0} + \mathrm{c.c.}\\
+\langle\delta O_{\varphi_f}\rangle = 0.
\end{aligned} \tag{3.42}$$

Assuming parity-invariant interactions and in-vacuum (i.e. symmetry not broken spontaneously), and further, $D_{\vec{q}} = D^*_{-\vec{q}}$, this expression can be simplified by returning to Heisenberg-picture operators:

$$\mathscr{V} + D_{\vec{q}}(\tau_i)\left[E_i(q)\langle 0_{\mathrm{in}}|O_{\phi_f}(\vec{k}_1,\ldots,\vec{k}_N)\phi_i(\vec{q})|0_{\mathrm{in}}\rangle + \mathrm{c.c.}\right]_{\vec{q}=0} = \langle 0_{\mathrm{in}}|\delta O_{\phi_f}(\vec{k}_1,\ldots,\vec{k}_N)|0_{\mathrm{in}}\rangle, \tag{3.43}$$

where the field insertions are $\phi_a(\vec{k}) \equiv \phi(\tau_a,\vec{k})$ and the correction term is

$$\begin{aligned}
\mathscr{V} = \int\frac{d^3p}{(2\pi)^3}E_i(p)\Big[\langle 0_{\mathrm{in}}|O_{\phi_f}(\vec{k}_1,\ldots,\vec{k}_N)\phi_i(\vec{p})\delta_L\phi_i(-\vec{p})|0_{\mathrm{in}}\rangle\\
-\langle 0_{\mathrm{in}}|O_{\phi_f}(\vec{k}_1,\ldots,\vec{k}_N)|0_{\mathrm{in}}\rangle\langle 0_{\mathrm{in}}|\phi_i(\vec{p})\delta_L\phi_i(-\vec{p})|0_{\mathrm{in}}\rangle + \mathrm{c.c.}\Big].
\end{aligned} \tag{3.44}$$

In the far past limit $\tau_i \to -\infty$, the term $\mathscr{V}$ vanishes being an integral of rapidly oscillating exponential (appearing in interaction picture, in perturbation theory analysis) Thus,

$$D_{\vec{q}}(\tau_i)\left[E_i(q)\langle 0_{\mathrm{in}}|O_{\phi_f}\phi_i(\vec{q})|0_{\mathrm{in}}\rangle_c + \mathrm{c.c.}\right]_{\vec{q}=0} = \langle 0_{\mathrm{in}}|\delta_L O_{\phi_f}|0_{\mathrm{in}}\rangle_c. \tag{3.45}$$

Removing delta functions, we write:

$$D_{\vec{q}}(\tau_i)\left[E_i(q)\langle 0_{\mathrm{in}}|O_{\phi_f}\phi_i(\vec{q})|0_{\mathrm{in}}\rangle'_c + \mathrm{c.c.}\right]_{\vec{q}=0} = \langle 0_{\mathrm{in}}|\delta_L O_{\phi_f}|0_{\mathrm{in}}\rangle'_c. \tag{3.46}$$

This is the early-late time consistency relation that encodes the constraints on in-in correlators which follow from the requirement of symmetry alone.

Note: If the symmetry is linearly-realized, LHS vanishes, and we retrieve the usual Ward-identities for such symmetries discussed earlier, as expected.

3.3.3 Physical Mode Condition on Symm. $\rightarrow$ Late time Consistency Relations

The consistency relation derived in the previous subsection looks almost like the usual late-time consistency relation except there are some quantities being evaluated at early times. The "physical mode condition" proposed in [17] provides sufficient conditions to connect the two quantities evaluated at these different times and thus express everything at late times alone, resulting in the late-time consistency relations, as desired.

The conditions are as follows:

- $$\lim_{\vec{q}\to 0} \phi_i(\vec{q})|0_{\text{in}}\rangle = \phi_f(\vec{q})|0_{\text{in}}\rangle \frac{g(\tau_i)}{g(\tau_f)} \tag{3.47}$$

- $$D_{\vec{q}}(\tau_i) = D_{\vec{q}}(\tau_f)\frac{g(\tau_i)}{g(\tau_f)} \tag{3.48}$$

- $$E_i(q)\langle 0_{\text{in}}|\mathscr{O}_{\phi_f}(\vec{k}_1\ldots\vec{k}_N)\phi_i(\vec{q})|0_{\text{in}}\rangle'_c = (\text{Re}\,E_i(q))\left(\text{Re}\langle 0_{\text{in}}|\mathscr{O}_{\phi_f}(\vec{k}_1\ldots\vec{k}_N)\phi_i(\vec{q})|0_{\text{in}}\rangle'_c\right) + \bar{\Delta}(q) \tag{3.49}$$

where, in the soft limit, we are assuming that the small corrections (like $\bar{\Delta}(q)$) can be ignored in the consistency relation, as part of the condition. See [17] for details.

First condition highlights the specific solution/mode for which we are computing the correlation functions, called dominant/growing mode in the context of cosmology (the other solution is usually decaying and is thus observationally irrelevant). This condition turns both - initial power spectrum $P_{\phi_i}(q)$ to $P_{\phi_f}(q)$ ($P_{\phi_i}(q) = P_{\phi_f}(q)\left(\frac{g(\tau_i)}{g(\tau_f)}\right)^2 + \tilde{\Delta}(q)$), and the initial-time field insertion in correlator to the final-time one. The second condition in itself has been termed as the "physical mode condition"; it basically says that the non-linear part of the transformation has the same time dependence as the zero momentum dominant (growing) mode of the field. The third condition is somewhat of a technical assumption imposed to bring the relation into standard form, which finally looks (after using all these conditions directly in the early-late time relation just derived):

$$D_{\vec{q}}(\tau_f)\left(\frac{1}{P_{\phi_f}(q)}\langle 0_{\text{in}}|\mathscr{O}_{\phi_f}(\vec{k}_1\ldots\vec{k}_N)\phi_f(\vec{q})|0_{\text{in}}\rangle'_c\right)\Bigg|_{\vec{q}=0} = \langle 0_{\text{in}}|\delta_L\mathscr{O}_{\phi_f}(\vec{k}_1\ldots\vec{k}_N)|0_{\text{in}}\rangle'_c \tag{3.50}$$

Chapter 4

A Puzzle and Steps to its Resolution

I first introduce the basic concepts related to spin-2 gravitons, as it will help us understand the ambiguity problem in taking derivatives of polarization tensors. I found [18] a useful reference for studying gravitational waves in cosmology context.

4.1 Polarization Tensors - Basics

Gravitational waves are tensor perturbations to the spatial metric,

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]. \tag{4.1}$$

Since $h_{ij}(\vec{k},\tau)$ has 6 degrees of freedom, we can write:

$$h_{ij}(\vec{k},\tau) = \sum_{a=1}^{6} h^a(\vec{k},\tau)\varepsilon^a_{ij} \tag{4.2}$$

We define the standard basis matrices for a symmetric 3×3 matrix as follows:

$$\varepsilon^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

The remaining basis matrices, which have 1 on the off-diagonal elements, are defined as follows:

$$\varepsilon^4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varepsilon^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.4}$$

h_{ij} being symmetric matrix, can then be expressed as a linear combination of these basis matrices, abbreviating $h^a(\vec{k},\tau)$ as simply h^a:

$$h_{ij}(\vec{k},\tau) = \sum_{a=1}^{6} h^a(\vec{k},\tau)\,\varepsilon^a_{ij} = \begin{pmatrix} h^1 & h^4 & h^5 \\ h^4 & h^2 & h^6 \\ h^5 & h^6 & h^3 \end{pmatrix} \tag{4.5}$$

Now we fix the gauge - TT gauge: $\partial^i h_{ij} = 0, \quad h_{ii} = 0$. In Fourier space, the Transverse condition simply becomes, $\forall \vec{k}$:

$$\hat{k}^i h_{ij} = 0 \Rightarrow \sum_{a=0}^{6} h^a(\vec{k},\tau)\hat{k}^i \varepsilon^a_{ij} = 0 \tag{4.6}$$

After giving it a little thought, it becomes clear that this means that, for example, h^4, h^5, h^6 can be linearly eliminated *in a $\hat{k}$ dependent way* in favour of h^1, h^2, h^3, using these 3 (j runs from 1 to 3) $\hat{k}$-dependent constraint equations linearly relating the 6 dof.

Therefore we can define new polarization tensors that absorb all this $\hat{k}$-dependent h^4 etc in terms of h^1 etc, and express everything in terms of just the independent components:

$$h_{ij}(\vec{k},\tau) = \sum_{a=1}^{3} h^a(\vec{k},\tau)\tilde{\varepsilon}^a_{ij}(\hat{k}) \tag{4.7}$$

This is where the polarization tensors primarily get their k-dependence from.. We are almost done now; traceless condition simply eliminates, for example, h^3, in terms of h^1, h^2 now and we finally get the expected 2-polarization expression for the tensor perturbations:

$$\boxed{h_{ij}(\vec{k},\tau) = \sum_{a=1}^{2} h^a(\vec{k},\tau)\varepsilon^a_{ij}(\hat{k})} \tag{4.8}$$

Since these h^1, h^2 are finally independent fields now (no constraints/relations..); due to linear expression of h_{ij} above, it's clear that all the constraints on h_{ij} are supposed to be encoded in the structure of polarization tensor itself (as should be clear from the argument also). Therefore, in particular, The TT-gauge condition implies directly then $\boxed{\hat{k}^i \varepsilon^a_{ij}(\hat{k}) = 0, \quad \varepsilon^a_{ii} = 0}$

- ***Getting Exact form of ε^a_{ij}***: Any Symmetric tensor h_{ij} can be written as following:

$$\begin{aligned} h_{ij}(k,\tau) = {} & a(k,\tau)m_i m_j + b(k,\tau)n_i n_j + c(k,\tau)k_i k_j + d(k,\tau)(m_i n_j + n_i m_j) + e(k,\tau)(m_i k_j + k_i m_j) \\ & + f(k,\tau)(n_i k_j + k_i n_j) \end{aligned} \tag{4.9}$$

where m, n are orthogonal unit vectors, perpendicular to direction of propagation, k (suppressing ˆ notation here..). Imposing $k^i h_{ij} = 0$ and $h_{ii} = 0$ gives directly:

$$h_{ij}^{TT}(k,\tau) = a(k,\tau)(m_i m_j - n_i n_j) + d(k,\tau)(m_i n_j + n_i m_j)$$
$$\doteq \quad h^{+}(k,\tau)e_{ij}^{+} + h^{\times}(k,\tau)e_{ij}^{\times} \tag{4.10}$$

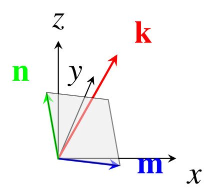


Figure 4.1: 3D space with unit vectors **k**, **m**, and **n**, all orthogonal to each other.

Here we just arrived at the definitions of $+/\times$ basis. In this thesis, we mainly use helicity basis though. I have summarized the relevant definitions, conventions and useful identities in the Appendix B

4.2 Understanding the Ambiguity in Derivatives of Polarization Tensors

Referring to the image above, consider the following equations:

$$\begin{aligned}
\hat{n}\cdot\hat{n} = 1 &\Rightarrow \frac{\partial \hat{n}_a}{\partial k_b} = \alpha \hat{m}_a + \beta \hat{k}_a, \\
\hat{m}\cdot\hat{m} = 1 &\Rightarrow \frac{\partial \hat{m}_a}{\partial k_b} = \gamma \hat{n}_a + \delta \hat{k}_a, \\
\hat{n}\cdot k = 0 &\Rightarrow k^a \frac{\partial \hat{n}_a}{\partial k^b} = -\hat{n}_b, \\
\hat{m}\cdot k = 0 &\Rightarrow k^a \frac{\partial \hat{m}_a}{\partial k^b} = -\hat{m}_b, \\
\hat{n}\cdot\hat{m} = 0 &\Rightarrow \hat{n}^a \frac{\partial \hat{m}_a}{\partial k^b} + \hat{m}^a \frac{\partial \hat{n}_a}{\partial k^b} = 0.
\end{aligned} \tag{4.11}$$

$$\left.\begin{aligned}
(3) + (1) &\Rightarrow \beta = -\frac{\hat{n}_b}{k}, \\
(4) + (2) &\Rightarrow \delta = -\frac{\hat{m}_b}{k}
\end{aligned}\right\} \xrightarrow{(5)} \alpha = -\gamma \tag{4.12}$$

$$\begin{aligned}
\frac{\partial \hat{n}_a}{\partial k^b} &= \alpha \hat{m}_a - \frac{\hat{k}_a \hat{n}_b}{k} \\
\frac{\partial \hat{m}_a}{\partial k^b} &= -\alpha \hat{n}_a - \frac{\hat{k}_a \hat{m}_b}{k}
\end{aligned} \tag{4.13}$$

The free parameter α can obviously not be fixed by any condition, after all it makes perfect geometric sense that just the orthonormality of basis vectors doesn't fix the $\hat{m}$'s absolute direction in the plane orthogonal to $\vec{k}$. Therefore, when we change k^a, and ask how $\hat{m}$ changes, it can always make a change along $\hat{n}$ direction since that was never fixed by any constraint (the constraints here are that all basis vectors are orthogonal to each other - TT gauge just demands that $\hat{m},\hat{n}$ are orthogonal to k, the other conditions are just our convention that we wish to work with orthonormal basis; that's how we *define* polarization tensors.

Using the definition 4.10, and inserting the derivative expressions above 4.13, we immediately get the expressions for derivatives of polarization tensors:

$$\boxed{\begin{aligned} \frac{\partial \varepsilon^{+}_{ij}}{\partial k^a} &= -2\alpha\varepsilon^{\times}_{ij} - \frac{k_i\varepsilon^{+}_{aj} + k_j\varepsilon^{+}_{ia}}{k^2}, \\ \frac{\partial \varepsilon^{\times}_{ij}}{\partial k^a} &= 2\alpha\varepsilon^{+}_{ij} - \frac{k_i\varepsilon^{\times}_{aj} + k_j\varepsilon^{\times}_{ia}}{k^2}. \end{aligned}} \tag{4.14}$$

4.2.1 The Puzzle

Thus we finally see how derivatives of polarization tensors are not well defined. This in turn means that derivatives of pure graviton correlators $\langle\gamma\gamma\gamma\rangle$ etc, are also ill-defined naively. This is the challenge [12] referred to, namely they couldn't verify the results of $\langle\gamma\gamma\gamma\gamma\rangle$ they obtained in de Sitter by an appropriate soft theorem due to derivatives acting on correlators in the soft theorems, which are naively ill-defined as just discussed. However, the authors also stated that by some other, naive means that they tried to check, the trispectrum soft theorem was *not* satisfied, thus resulting in a puzzle (since they have confirmed their main result (calculation of $\langle\gamma\gamma\gamma\gamma\rangle$ in $\delta\phi$-gauge in exact de Sitter) through multiple independent methods, this strongly suggests that their result for $\langle\gamma\gamma\gamma\gamma\rangle$ is reliable.)

4.3 Clues to solving the puzzle

After going through the paper [15] carefully, where the relevant soft theorems are derived, it did not appear that odd to us that [12]'s results are not satisfying the soft theorem.

The reason was that the soft theorems have been derived specifically in the ζ-gauge, for variables ζ and $\tilde{\gamma}$, say. On the other hand, the calculation of [12] is in the context of de sitter (where ζ-gauge is not even well-defined!), working in $\delta\phi$-gauge and involves, not $(\zeta,\tilde{\gamma})$, but $(\delta\phi,\gamma)$ (i.e. they are concerned with correlators $\langle\gamma\gamma\gamma\rangle$, $\langle\gamma\gamma\gamma\gamma\rangle$ etc, and not $\langle\tilde{\gamma}\tilde{\gamma}\tilde{\gamma}\rangle$ etc)

But it turns out that this logic doesn't in general rule out the possibility of satisfaction of soft theorems by γ correlators. Indeed, as we will show below in detail, at late times, in slow-roll limit (going from quasi-dS to exact dS), we indeed have 3-2 consistency relations being satisfied in dS in $\delta\phi$-gauge! The reason this happens is simply because $\langle\tilde{\gamma}\tilde{\gamma}\tilde{\gamma}\rangle = \langle\gamma\gamma\gamma\rangle$, and

$\langle\tilde{\gamma}\tilde{\gamma}\rangle = \langle\gamma\gamma\rangle$.

From this, it is clear that one should first establish when the appropriate correlators are equal in the 2 gauges. As for the $\langle\tilde{\gamma}\tilde{\gamma}\tilde{\gamma}\tilde{\gamma}\rangle$, $\langle\gamma\gamma\gamma\gamma\rangle$ relation, some terms remain to be evaluated to make any definitive statement. The explicit checking of soft theorems is also a work in progress in mathematica.

Note: We are working at *tree level* here. One only needs to perform the field redefinition at $(n-1)^{th}$ order (in fluctuations) for relating the n-point correlators $\langle\tilde{\gamma}_1\tilde{\gamma}_2...\tilde{\gamma}_n\rangle$ and $\langle\gamma_1\gamma_2...\gamma_n\rangle$. The reason is that the $(n-1)^{th}$ order term from field redefinition is the highest term (in the complete field redefinition), that can affect the interaction terms up until nth order in action (which is the highest power term that can contribute to n-point correlator at tree level) coming from quadratic part of action. Conversely, n^{th} and higher order terms (in the field redefinition) always generate $(n+1)^{th}$ order type or higher interaction terms, which do not contribute to lower, tree-level n-point correlators.

Plan:

1. Work out the field redefinitions between $(\zeta, \tilde{\gamma})$ and $(\delta\phi, \gamma)$ at cubic order in fluctuations. And thus check if $\langle\tilde{\gamma}\tilde{\gamma}\tilde{\gamma}\tilde{\gamma}\rangle = \langle\gamma\gamma\gamma\gamma\rangle$ in $\tau, \varepsilon \to 0$ limit

2. Check soft theorems (3-2, 4-3) explicitly to be sure of the conclusions made.

Note: The problem of ambiguity in taking derivatives of polarization tensors is not relevant for explicit verifiability of soft theorems. The soft theorems derived in [15] have graviton field in terms of components, with helicities summed over. It's easy to show that such correlators have definite k-dependence since polarization tensors, when summed over helicities, turn into projection tensors (see B).

4.4 Working out Field Redefinitions

First, we review the simpler calculation, with final results that were first shown in the appendix of [19], working out the quadratic relation between $\zeta, \tilde{\gamma}$ and $\delta\phi, \gamma$

4.4.1 Working out Field Redefinitions at 2nd order

$$\begin{aligned} x: &\quad N = 1 + N_1, \quad N^i = \delta^{ij}\partial_j\psi, \quad h_{ij} = e^{2\rho(t)}(e^{\gamma(t,x)})_{ij}, \quad \phi(t,x) = \bar{\phi}(t) + \delta\phi(t,x) \\ \tilde{x}: &\quad \tilde{N} = 1 + \tilde{N}_1, \quad \tilde{N}^i = \delta^{ij}\partial_j\tilde{\psi}, \quad \tilde{h}_{ij} = e^{2\rho(\tilde{t})+2\zeta(\tilde{t},\tilde{x})}(e^{\tilde{\gamma}(\tilde{t},\tilde{x})})_{ij}, \quad \tilde{\phi}(\tilde{t},\tilde{x}) = \bar{\phi}(\tilde{t}) \end{aligned} \tag{4.15}$$

Note: I use γ' and $\tilde{\gamma}$ interchangeably below for ease of writing; they mean the same thing.

The Overall Idea: There are really only 1 + 2 = 3 (1 from scalar, 2 from tensor) degrees of freedom in the system ($\zeta, \tilde{\gamma}$ in ζ gauge, or equivalently $\delta\phi, \gamma$ in $\delta\phi$-gauge etc). The idea is that, we have computed everything in one gauge (for example in Maldacena computation of cubic action, it is easy to compute everything in the $\delta\phi$-gauge), and all quantities are thus known functions of $\delta\phi$, say, and we now wish to rewrite the action in terms of another physical, scalar quantity (like ζ, for reasons of phenomenological interest due to constancy at superhorizon scales etc). Clearly then it should be the case that $\delta\phi = \delta\phi[\zeta, \tilde{\gamma}]$, $\gamma = \gamma[\zeta, \tilde{\gamma}]$, a "field redefinition", since there is really only one scalar; they all are functions of each other. Not just the matter and metric components, but also the gauge transformation parameters, T, L^i should be $T[\zeta, \tilde{\gamma}], L^i[\zeta, \tilde{\gamma}]$ too. So we will calculate everything, expanding every quantity $A[\zeta, \tilde{\gamma}]$ in power series in $\zeta, \tilde{\gamma}$: $A_1 + A_2 + \cdots$

Since the 2 gauges are really defined by $\phi(t,x)$ and h_{ij} (and N, N^i just work out themselves by the constraint equtions..), we will need to focus only on their transformation properties to find the field redefinitions we are interested in. Let's start with the simplest object: Scalar Inflaton, ϕ. Again, we have, in the 2 gauges:

$$\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t) \qquad , \qquad \phi'(t',x') = \bar{\phi}(t') \tag{4.16}$$

Imposing the *scalar* transformation property: $\phi(x,t) = \phi'(t',x')$, we immediately get (dropping the overhead-bar from $\bar{\phi}$ from now on):

$$\delta\phi(x,t) = T_1\dot{\phi} + T_2\dot{\phi} + T_1^2\ddot{\phi}/2 + O(3) \tag{4.17}$$

Our goal now is to find the exact functions $T_1 \sim \zeta$ and $T_2 \sim \zeta^2$ so that we can find $\delta\phi[\zeta] = \delta\phi_1 + \delta\phi_2$, i.e. the expression of inflaton perturbation in terms of curvature perturbation at second order (in power series). For this purpose we have to use the other and only remaining piece of information we know - how h_{ij} look in both gauges. We will be able to use the gauge conditions of h_{ij} to find what T_1, T_2 should be and thus plug that in the relation above to find $\delta\phi[\zeta]$ as desired.

We will use the transformation rule $g_{\alpha\beta}(x) = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g'_{\mu\nu}(x')$. It might appear awkward to use this "inverted" formula but it is useful to prefer this over the "direct" one since it's calculationally a bit simpler. Since we know the coordinate transformation are

$$t' = t + T(t,x) \quad , x'^{i} = x^{i} + L^{i}(t,x) \tag{4.18}$$

we can immediately and exactly calculate $\frac{\partial x'^{\mu}}{\partial x^{\alpha}}$, while to compute the inverse of this, we will have to do it order by order. As we'll see, we will anyways have to calculate $\frac{\partial x^{\mu}}{\partial x'^{\alpha}}$ also but the point of doing things this way is that if we want to calculate the field redefinition at Nth order, then calculating the latter needs to be done only at (N-1)th order if we use the inverted formula. In contrast, if we choose to calculate with "direct" formula, we can immediately see, we need to find $\frac{\partial x^{\mu}}{\partial x'^{\alpha}}$ at Nth order itself. So we can save ourselves from doing unnecessary and tedious algebra if we do the calculation this way.

We thus obtain, setting α, β to spatial indices i, j:

$$\begin{aligned} h_{ij}(x,t) = {} & h'_{ij}(t',x') + h'_{ia}\partial_j L^a + h'_{ja}\partial_i L^a + \partial_i L^a \partial_j L^b h'_{ab} \\ & + \partial_i T \partial_j T(-N'^2 + N' \cdot N') + \partial_i T N'_j + \partial_j T N'_i + \partial_i T \partial_j L^k N'_k + \partial_j T \partial_i L^k N'_k \end{aligned} \tag{4.19}$$

Note that RHS quantities are expressed in terms of (t',x') which are implicitly themselves functions of ζ and since we are solving the equations perturbatively, we need to write everything in (t,x) coordinates so as to write down equations at any given fixed order in ζ consistently.

The transformation formula relating h_{ij} and h'_{ij}, N'_i, N' expanded to second order, including both scalar and tensor parts, is the following:

$$\begin{aligned} a^2(\delta_{ij} + \gamma^{(1)}_{ij} + (\gamma^{(2)}_{ij} + \gamma^{2(1)}_{ij}/2) + O(3) = {} & a^2\Big[\delta_{ij}[1 + 2(T_1\dot{\rho} + \zeta) + 2(T_2\dot{\rho} + T_1^2\ddot{\rho}/2 + T_1^2\dot{\rho}^2 + T_1\dot{\zeta} + \zeta^2 \\ & + 2T_1\zeta\dot{\rho} + L_1^i\partial_i\zeta)] \\ & + \gamma'_{ij}[1 + 2(T_1\dot{\rho} + \zeta)] + \gamma'^2_{ij}/2 + T_1\dot{\gamma}'_{ij} + L_1^a\partial_a\gamma'_{ij}\Big] \\ & + a^2\Big[2[\ \delta_{(ia}(1 + 2(T_1\dot{\rho} + \zeta)) + \gamma'_{(ia}\ ]\ \partial_{j)}L_1^a] + a^2\partial_i L_1^a\partial_j L_1^a \\ & + a^2[2(\partial_{(i}T\partial_{j)}\psi^{(1)} + \partial_{(i}(L_2)_{j)}] + \partial_i T_1\partial_j T_2 + O(3) \end{aligned} \tag{4.20}$$

$$\boxed{0^{th}\text{ Order:}}\quad e^{2\rho(t)}\delta_{ij} = e^{2\rho(t)}\delta_{ij} \quad \text{(trivially satisfied)}$$
$$\boxed{1^{st}\text{ Order:}}\quad e^{2\rho(t)}\gamma^{(1)}_{ij} = e^{2\rho(t)}[\gamma'_{ij} + 2\delta_{ij}(T_1\dot\rho + \zeta) + \partial_i(L_1)_j + \partial_j(L_1)_i] \tag{4.21}$$

The only (gauge) conditions remaining to impose now are tensorial, and thus we solve the 1^{st} order equation using them:

$$\begin{aligned} Tr(\gamma) = Tr(\gamma') = 0 &\implies 6(T_1\dot\rho + \zeta) + 2\partial\cdot L_1 = 0 \\ \partial\cdot\gamma = \partial\cdot\gamma' = 0 \quad &\implies 2\partial_j(T_1\dot\rho + \zeta) + \partial^2(L_1)_j + \partial_j(\partial\cdot L_1) = 0 \end{aligned} \tag{4.22}$$

Substituting 1st eqn into second, we find the divergence of conformal equation for L_1, and since all variables here are assumed to be "small" (vanishing at spatial infinity), it is easy to see immediately that $L_1 = 0$ and thus $T_1\dot\rho + \zeta = 0 \implies T_1 = -\zeta/\dot\rho$ and thus $\gamma^{(1)}_{ij} = \gamma'_{ij}$

Putting in T_1 and L_1 solutions, we write the 2^{nd} Order equation directly in terms of ζ, γ' and 2^{nd} order quantities to find:

$$\begin{aligned} \boxed{2^{nd}\text{ Order:}}\quad e^{2\rho(t)}(\gamma^{(2)}_{ij} + \cancel{\gamma'^2_{ij}/2}) = e^{2\rho(t)}\Big[&2\delta_{ij}(T_2\dot\rho - \frac{\varepsilon\zeta^2}{2} - \zeta\dot\zeta/\dot\rho) + \cancel{\gamma'^2_{ij}/2} - \zeta\dot\gamma'_{ij}/\dot\rho \\ &+ 2\Big[\partial_{(i}(L_2)_{j)} - \frac{1}{\dot\rho}\partial_{(i}\zeta\partial_{j)}\psi^{(1)}\Big]\Big] + \frac{1}{\dot\rho^2}\partial_i\zeta\partial_j\zeta \end{aligned} \tag{4.23}$$

Again, we use the TT-gauge conditions for γ, γ' by taking δ^{ij} and $\partial^{-2}\partial^i\partial^j$ on both sides and obtain thus:

$$\begin{aligned} \delta^{ij}: \quad & 0 = 6\dot\rho T_2 + 3\varepsilon\zeta^2 - 6\zeta\dot\zeta/\dot\rho + 2\partial\cdot L_2 - \frac{2}{\dot\rho}\partial_i\zeta\partial_i\psi^{(1)} + \frac{e^{-2\rho}}{\dot\rho^2}(\partial\zeta)^2 \\ \partial^{-2}\partial^i\partial^j: \quad & 0 = 2\dot\rho T_2 - \varepsilon\zeta^2 - 2\zeta\dot\zeta/\dot\rho + 2\partial\cdot L_2 - \frac{\partial^{-2}\partial^i\partial^j}{\dot\rho}\Big[2\partial_i\zeta\partial_j\psi^{(1)} - \frac{e^{-2\rho}}{\dot\rho}\partial_i\zeta\partial_j\zeta\Big] - \frac{\partial^{-2}[\partial^i\partial^j\zeta\dot\gamma'_{ij}]}{\dot\rho} \end{aligned} \tag{4.24}$$

Subtracting the 2nd equation from 1st, we obtain T_2 and thus finally the 2nd order relation between scalar perturbations ($\delta\phi = T_1\dot\phi + T_1^2\ddot\phi/2 + T_2\dot\phi + O(3)$:

$$\begin{aligned} T_2 \quad = \quad & \frac{\varepsilon}{2}\zeta^2 + \frac{\zeta\dot\zeta}{\dot\rho} + \frac{1}{2\dot\rho^2}\partial_i\zeta\partial_i\psi^{(1)} - \frac{1}{4\dot\rho}\Big[\frac{e^{-2\rho}}{\dot\rho^2}(\partial\zeta)^2 + \frac{\partial^{-2}\partial^i\partial^j}{\dot\rho}\Big[2\partial_i\zeta\partial_j\psi^{(1)} - \frac{e^{-2\rho}}{\dot\rho}\partial_i\zeta\partial_j\zeta\Big] \\ & + \frac{\partial^{-2}[\partial^i\partial^j\zeta\dot\gamma'_{ij}]}{\dot\rho}\Big] \end{aligned} \tag{4.25}$$

and thus $\delta\phi = -M_{pl}\sqrt{2\varepsilon}(\zeta - f(\zeta,\gamma'))$ where $f(\zeta)$ is given by:

$$
\begin{aligned}
f(\zeta,\gamma') = \frac{1}{2}\left[\frac{\ddot{\phi}}{\dot{\phi}\dot{\rho}} + \varepsilon\right]\zeta^2 + \frac{1}{\dot{\rho}}\zeta\dot{\zeta} - \frac{1}{4}\frac{e^{-2\rho}}{\dot{\rho}^2}\left[\partial_i\zeta\partial^i\zeta - \partial^{-2}\partial_i\partial_j(\partial^i\zeta\partial^j\zeta)\right] \\
+ \frac{1}{4\dot{\rho}}\left[\partial_i\zeta\partial_i\psi^{(1)} - \partial^{-2}\partial_i\partial_j(\partial^i\zeta\partial^j\psi^{(1)} - \partial^{-2}[\partial^i\partial^j\zeta\dot{\gamma}'_{ij}]\right]
\end{aligned} \tag{4.26}
$$

Note that this matches almost exactly with Maldacena's 2nd order result (see appendix of [19]) except the extra factor of ∂^{-2} in the last term here. In fact, [19] has a typo apparently (easily seen using dimensional analysis), and the expression we just derived seems correct .

Moving further, we also obtain $\partial \cdot L_2$

$$
\partial \cdot L_2 = \frac{1}{4\dot{\rho}^2}\left[\delta^{ij} - 3\partial^{-2}\partial^i\partial^j\right]\left[e^{-2\rho}\partial_i\zeta\partial_j\zeta - 2\dot{\rho}\partial_i\zeta\partial_j\psi^{(1)}\right] + \frac{3}{4\dot{\rho}}\partial^{-2}[\partial^i\partial^j\zeta\dot{\gamma}'_{ij}] \tag{4.27}
$$

Next, focusing on the TT tensor part at 2nd order,

$$
\gamma^{(2)}_{ij} = -\frac{\dot{\gamma}'_{ij}\zeta}{\dot{\rho}} + \partial^{Conf}_i(L_2)_j - \frac{2}{\dot{\rho}}\left[\partial_{(i}\zeta\partial_{j)}\psi^{(1)} - \frac{\delta_{ij}}{3}\partial_a\zeta\partial_a\psi^{(1)}\right] + \frac{e^{-2\rho}}{\dot{\rho}^2}\left[\partial_i\zeta\partial_j\zeta - \frac{\delta_{ij}}{3}(\partial\zeta)^2\right] \tag{4.28}
$$

I tried solving for L_2 and did get an expression, but I am not able to reproduce the Maldacena's highly simple expression for $\gamma^{(2)}$yet.

$$
\begin{aligned}
(L_2)_j = \partial^{-2}\Bigg[\dot{\gamma}'_{ij}\partial_i\zeta/\dot{\rho} - \frac{\partial_j}{12\dot{\rho}^2}\left[\delta^{ab} - 3\partial^{-2}\partial^a\partial^b\right]\left[e^{-2\rho}\partial_a\zeta\partial_b\zeta - 2\dot{\rho}\partial_a\zeta\partial_b\psi^{(1)}\right] \\
- \frac{\partial_j}{4\dot{\rho}}\partial^{-2}[\partial^a\partial^b\zeta\dot{\gamma}'_{ab}] - \frac{e^{-2\rho}}{\dot{\rho}^2}\left[\partial^2\zeta\partial_j\zeta + \frac{1}{3}\partial_i\zeta\partial_i\partial_j\zeta\right] \\
+ \frac{2\partial_i}{\dot{\rho}}\left[\partial_{(i}\zeta\partial_{j)}\psi^{(1)} - \frac{\delta_{ij}}{3}\partial_a\zeta\partial_a\psi^{(1)}\right]\Bigg]
\end{aligned} \tag{4.29}
$$

Somehow Maldacena (see [19]) doesn't seem to get any contribution to $\gamma^{(2)}_{ij}$ from L_2. I couldn't derive this result and moved on.

4.4.2 Finding Late-time limit of 2nd and 3rd order Relations

The expressions are already quite messy at 2nd order itself, as we just saw in the preceding calculation. We actually just need to focus on the late time limit after all, since what really concerns us here is whether γ and γ' are equal at late times, $\tau \to 0$ (and also in $\varepsilon \to 0$ limit, but we will come to that later, since ζ diverges in this limit, therefore we will first need to turn ζ to $\delta\phi$ first, and then take that limit, which we will do later).

We directly go on to doing the 3rd order computation now, with the useful observation that at late times, at O(2) level, we have the following relations:

$$0 = \dot{\zeta} = \dot{\gamma}'_{ij} = \partial_i \psi^{(1)} = 1/a = L_1 = \partial \cdot L_2 = L_2 = \gamma^{(2)}_{ij} \quad \text{and} \quad T_1 = -\zeta/\dot{\rho}, \quad T_2 = \frac{\varepsilon \zeta^2}{2\dot{\rho}} \tag{4.30}$$

where, although we didn't derive the expression of $\gamma^{(2)}$ explicitly, we can take the late-time limit of the 2nd order equation to deduce this anyway.

Moving on to peforming our calculation of interest here - field redefinition at 3rd order - let us start with **LHS**:

$$a^2\left[\delta_{ij} + \gamma_{ij} + \gamma^2_{ij}/2 + \gamma^3_{ij}/6\right] = a^2\left[\left(\delta_{ij}\right) + \left(\gamma^{(1)}_{ij}\right) + \left(\gamma^{(2)}_{ij} + \gamma^{2(1)}_{ij}/2\right) + \left(\gamma^{(3)}_{ij} + \gamma^{(1)}_{ik}\gamma^{(2)}_{kj} + \gamma^{3(1)}_{ij}/6\right)\right] \tag{4.31}$$

Here, we have clearly separated the exression by the order of quantities. We have already dealt with pieces upto O(2) in the last subsection, and we'll now focus only on O(3) piece (the last bracket).

Let us now calculate the ***RHS***. Since it is quite messy, we will focus on just the O(3) piece of RHS, instead of the full expression. The basic expression to begin with is (***in (t',x') coordinates***:

$$\begin{aligned} & h'^{(3)}_{ij} + 2h'^{(0)}_{(ia}\partial_{j)}L^{a(3)} + 2h'^{(1)}_{(ia}\partial_{j)}L^{a(2)} + \cancel{2h'^{(2)}_{(ia}\partial_j L^{a(1)}} + \cancel{\partial_i L^{a(1)}\partial_j L^{b(1)}h'^{(1)}_{ab}} + \cancel{2\partial_{(i}L^{a(2)}\partial_{j)}L^{b(1)}h'^{(0)}_{ab}} \\ & -2\partial_{(i}T^{(1)}\partial_{j)}T^{(2)} - 2\partial_i T^{(1)}\partial_j T^{(1)}N'^{(1)} + 2h'^{(1)}_{(ia}\partial_{j)}T^{(1)}N'^{a(1)} + 2h'^{(0)}_{(ia}\partial_{j)}T^{(1)}N'^{a(2)} + 2h'^{(0)}_{(ia}\partial_{j)}T^{(2)}N'^{a(1)} \\ & + \cancel{2h'^{(0)}_{ab}\partial_{(i}T^{(1)}\partial_{j)}L^{a(1)}N'^{b(1)}} \end{aligned} \tag{4.32}$$

We first calculate $h'_{ij}(t',x')$ till 3rd order as required here, and obviously we need to turn from coordinates (t',x') to (t,x) for doing the perturbative analysis properly:

$$h'_{ij}(t',x') = e^{2\rho(t) + 2\left[T\dot{\rho} + \frac{T^2\ddot{\rho}}{2} + \frac{T^3\dddot{\rho}}{6}\right] + 2\left[\zeta + (T\dot{\zeta} + L^i\partial_i\zeta) + \left(\frac{T^2\dddot{\zeta}}{2} + TL^i\partial_i\dot{\zeta} + \frac{L^iL^j\partial_i\partial_j\zeta}{2}\right)\right]}\left[\delta_{ij} + \gamma'_{ij} + \gamma'^2_{ij}/2 + \gamma'^3_{ij}/6\right] \tag{4.33}$$

We also have to turn γ' terms to that in (t,x) coordinates, and then open the exponential of scalar part in front to 3rd order too. To see what is happening in the tedious algebra here, just realize the following structure:

$$
\begin{aligned}
e^{a_0+a_1+a_2+a_3}(b_0+b_1+b_2+b_3) = &\left[\left(e^{a_0}\right)\left(b_0\right)\right] \\
&+\left[\left(e^{a_0}\right)\left(b_1\right)+\left(e^{a_0}(a_1)\right)\left(b_0\right)\right] \\
&+\left[\left(e^{a_0}\right)\left(b_2\right)+\left(e^{a_0}(a_1)\right)\left(b_1\right)+\left(e^{a_0}(a_2+a_1^2/2\right)\left(b_0\right)\right] \\
&+\left[\left(e^{a_0}\right)\left(b_3\right)+\left(e^{a_0}(a_1)\right)\left(b_2\right)+\left(e^{a_0}(a_2+a_1^2/2)\right)\left(b_1\right)\right. \\
&\left.+\left(e^{a_0}(a_3+a_1a_2+a_1^3/6)\right)\left(b_0\right)\right]
\end{aligned}
\tag{4.34}
$$

All said and done, after canceling several terms using the first order relations, $T^{(1)}\dot{\rho}+\zeta = L_1^i = 0$, we eventually get the following "simple" expression for $h'^{(3)}_{ij}$:

$$
\begin{aligned}
h'^{(3)}_{ij} = e^{2\rho(t)}\Bigg[&T^{2(1)}\ddot{\gamma}'_{ij}/2+T^{(2)}\dot{\gamma}'_{ij}+L^{(2)a}\partial_a\gamma'_{ij}+T^{(1)}\dot{\gamma}'_{(ik}\gamma'_{kj)}+\gamma'^3_{ij}/6 \\
&+2\gamma'_{ij}\left[T^{(2)\dot{\rho}}+T^{2(1)}\ddot{\rho}/2+T^{(1)}\dot{\zeta}\right] \\
&+2\delta_{ij}\left[T^{(3)}\dot{\rho}+T^{(1)}T^{(2)}\ddot{\rho}+\frac{T^{3(1)}\dddot{\rho}}{6}+T^{(2)}\dot{\zeta}+L^{(2)i}\partial_i\zeta+T^{2(1)}\ddot{\zeta}/2\right]\Bigg]
\end{aligned}
\tag{4.35}
$$

Cancel $e^{2\rho(t)}$ on both LHS and RHS. Now, we take $\tau\to 0$ limit, using the relations summarized in this limit earlier. Therefore we finally get the cubic relation between γ and γ' (LHS = RHS, at O(3), in $\tau\to 0$ limit):

$$
\gamma^{(3)}_{ij} = \frac{1}{2M^2}\gamma'_{ij}\delta\phi^2+2\delta_{ij}\left[T^{(3)}\dot{\rho}+(\frac{\varepsilon}{2})^2\zeta^3-\frac{\dddot{\rho}}{\dot{\rho}^3}\zeta^3\right]+2\partial_{(i}L^{(3)}_{j)}+2\partial_{(i}T^{(1)}N'^{j)(2)}
\tag{4.36}
$$

I couldn't simplify the expression further much since solving for 2nd order constraint equation, in presence of tensorial perturbation, was too tedious to compute. And actually we don't need to calculate anything from this point either.

We are getting a $\gamma'_{ij}\delta\phi^2$ type term (which *does not vanish* in the $\tau,\varepsilon\to 0$ limit on its own), and clearly, it cannot be canceled from any other term in the RHS. The $T^{(3)}$ term comes with a δ_{ij}, which doesn't contribute to traceless γ. $\partial_i L^{(3)}_j$ has a spatial derivative, whereas $\gamma'_{ij}\delta\phi^2$ term has no such spatial derivative. And finally, $\partial_{(i}T^{(1)}N'^{j)(2)}$ also has the structure of $\partial\zeta$ times something, again, of a completely different structure than the term of our interest, thus unable to cancel it no matter what $N^{(2)}$ looks explicitly.

Therefore, we have found at least one term in the cubic O(3) relation between γ and γ' that does not cancel in the $\tau,\varepsilon\to 0$ limit. This means that the quartic actions in terms of γ and γ' do not look the same; furthermore, since free theory wightman propagator $\langle\gamma\gamma\rangle = \langle\gamma'\gamma'\rangle$, we can conclude therefore that the tree-level $\langle\gamma\gamma\gamma\gamma\rangle \neq \langle\gamma'\gamma'\gamma'\gamma'\rangle$. But note also that at O(2), in $\tau \to 0$ limit, $\gamma_{ij}^{(2)}$ vanishes and thus $\gamma_{ij} = \gamma'_{ij}$ at O(2) in this limit. Therefore, we have then $\langle\gamma\gamma\gamma\rangle = \langle\gamma'\gamma'\gamma'\rangle$

Note: There is one important caveat in making the conclusion above that I have not understood well yet. The $\gamma'\delta\phi^2$ term in field redefinition will lead to $\gamma'^2\delta\phi^2$ type interaction term in the action. This will contribute to $\langle\gamma\gamma\gamma\gamma\rangle$ by 1-loop exchange diagram (just the $\delta\phi$ propagators in the loop). Since we are supposedly working at tree level, this contribution should not appear in the soft theorem and thus despite the analysis above, we should naively expect the 4-3 consistency relation getting satisfied (at tree-level). But as we will see in the next section, the 4-3 consistency relation is indeed *not* satisfied. Strictly speaking, we are not working at tree-level, but in some order of M_{pl} in perturbation theory. Both the 4-point contact and graviton-exchange diagrams will be taken in $\langle\gamma\gamma\gamma\gamma\rangle$ (in section 4.5) because they both are at $O(\frac{1}{M_{pl}^6})$; while this 1-loop diagram contributes at $O(\frac{1}{M_{pl}^4})$ (which is M_{pl}^2-suppressed and hence can be ignored).

4.5 Checking Soft Theorems Explicitly

4.5.1 Confirming Validity of 3-2 Consistency Relation

As per the logic discussed earlier, we now expect that the 3-2 consistency relations are expected to hold, while the 4-3 consistency relations are not expected to hold (for γ correlators). We now check this explicitly for the simplest soft theorems valid in this context, corresponding to n = 0, anisotropic rescaling symmetry (see discussion above 3.19):

$$\lim_{\vec{q}\to 0} S_{ij}(\hat{q})\frac{\langle\gamma^{ij}(\vec{q})\gamma_{ab}(\vec{k}_1)\gamma_{cd}(\vec{k}_2)\rangle'_c}{P_\gamma(q)} = S_{ij}(\hat{q})\Big[-\left(k_1^i\nabla_1^j\right)\langle\gamma_{ab}(\vec{k}_1)\gamma_{cd}(-\vec{k}_1)\rangle'_c \\ +\Gamma^{ij}{}_{ab}{}^{pq}(\hat{k}_1)\langle\gamma_{pq}(\vec{k}_1)\gamma_{cd}(-\vec{k}_1)\rangle'_c+\Gamma^{ij}{}_{cd}{}^{pq}(\hat{k}_1)\langle\gamma_{ab}(\vec{k}_1)\gamma_{pq}(-\vec{k}_1)\rangle'_c\Big] \tag{4.37}$$

As argued earlier, and can be easily checked, the expression $\langle\gamma'\gamma'\gamma'\rangle$ that Maldacena first derived in [19] and $\langle\gamma\gamma\gamma\rangle$ that is obtained in de Sitter in $\delta\phi$-gauge in [12], are really one and the same. Forgetting about γ vs γ' then, I now use the Maldacena's expression (for convenience):

$$\langle\gamma^{h_1}_{k_1}\gamma^{h_2}_{k_2}\gamma^{h_3}_{k_3}\rangle = -\frac{H^4 I_{123}}{2M^4k_1^3k_2^3k_3^3}\varepsilon^{*h_1}_{ii'}\varepsilon^{*h_2}_{jj'}\varepsilon^{*h_3}_{ll'}(k_2^i\delta_{jl}+k_3^j\delta_{il}+k_1^l\delta_{ij})(k_2^{i'}\delta_{j'l'}+k_3^{j'}\delta_{i'l'}+k_1^{l'}\delta_{i'j'})\tilde{\delta} \tag{4.38}$$

where $I_{123} = -k_T + \sum_{i>j}\frac{k_ik_j}{k_T}+\frac{k_1k_2k_3}{k_T^2}$

Further, we will also need the 2-point expressions:

$$\langle\gamma_{ab}(\vec{k}_1)\gamma_{cd}(-\vec{k}_1)\rangle'_c = \frac{2H^2}{M^2k_1^3}\Pi^{k_1}_{abcd} \quad , \quad P_\gamma(k) = \frac{H^2}{M^2k^3} \tag{4.39}$$

Since S_{ij} is known to be TT, let $S_{ij} = \varepsilon^{*h}_{ij}(\hat{q})$ for some helicity h. Let's first evaluate the LHS of the soft theorem 4.37, but before stripping the delta functions first:

$$LHS = \lim_{\vec{q}\to 0}\sum_{ij}\varepsilon^{*h}_{ij}(\hat{q})\frac{\sum_{h_q,h_1,h_2}\varepsilon^{h_q}_{ij}(\hat{q})\varepsilon^{h_1}_{ab}(\hat{k}_1)\varepsilon^{h_2}_{cd}(\hat{k}_2)}{P_\gamma(q)}\langle\gamma^{h_q}(\vec{q})\gamma^{h_1}(\vec{k}_1)\gamma^{h_2}(\vec{k}_2)\rangle_c \tag{4.40}$$

Using the normalization property for helicity basis polarization tensors (see B), we simplify:

$$LHS = \lim_{\vec{q}\to 0}\frac{-H^2I}{M^2k_1^3k_2^3}\tilde{\delta}(\vec{k}_1+\vec{k}_2+\vec{q})\sum_{h_1h_2}\varepsilon^{h_1}_{ab}(\hat{k}_1)\varepsilon^{h_2}_{cd}(\hat{k}_2)\varepsilon^{*h}_{ii'}\varepsilon^{*h_1}_{jj'}\varepsilon^{*h_2}_{ll'}(k_1^i\delta_{jl}+k_2^j\delta_{il}+q^l\delta_{ij})(k_1^{i'}\delta_{j'l'}+k_2^{j'}\delta_{i'l'}+q^{l'}\delta_{i'j'}) \tag{4.41}$$

In the $\lim_{\vec{q}\to 0}$, we have the following:

1. $\tilde{\delta}(\vec{k}_1+\vec{k}_2+\vec{q}) \to \tilde{\delta}(\vec{k}_1+\vec{k}_2)$. In other words, $\vec{k}_2 = -\vec{k}_1$, $k_2 = k_1$
2. $\mathrm{I} = -3k_1/2$
3. $\varepsilon^*\varepsilon^*\varepsilon^*()() \to 2\delta^{h_1h_2}\varepsilon^h_{ab}(\hat{q})k_1^a k_1^b$

Substituting all in, we get

$$LHS = \lim_{\vec{q}\to 0} \varepsilon^{*h}_{ij}(\hat{q})k_1^i k_1^j (2\Pi^{k_1}_{abcd})\frac{3H^2}{M^2k_1^5} \tag{4.42}$$

where I again dropped the $\tilde{\delta}$, since we are interested in soft theorems involving primed correlators.

Let us now turn to evaluating RHS. The first term involves taking derivatives. Let's first ignore the tensor structure $\Pi^{k_1}_{abcd}$ and take derivative of its coefficient, $\frac{2H^2}{M^2k_1^3}$, only. Therefore doing just this, we get on the RHS:

$$RHS \supset \lim_{\vec{q}\to 0} \varepsilon^{*h}_{ij}(\hat{q})[-k_1^i(-\frac{6H^2k_1^j}{M^2k_1^5})\Pi^{k_1}_{abcd}] = LHS \tag{4.43}$$

i.e., taking the derivative of non-tensorial part of 2-point function produces the LHS all by itself. Thus, if this 3-2 consistency relation is to hold, the derivative of tensorial part, $\Pi^{k_1}_{abcd}$, should cancel the other terms on the RHS, the Γ terms. This is what we check next.

Taking derivative of Π_{ijkl}, we obtain:

$$\begin{aligned}
k_i\partial_j\Pi_{abcd} = k_i\Big[&-2\frac{k_jk_ak_bk_ck_d}{k^6} + \frac{1}{2k^4}\Big[3(\delta_{ja}k_bk_ck_d + \delta_{jc}k_ak_bk_d) \\
&+2(\delta_{bd}k_jk_ak_c + \delta_{ac}k_jk_bk_d + \delta_{bc}k_jk_ak_d + \delta_{ad}k_jk_bk_c) \\
&-2(\delta_{cd}k_jk_ak_b + \delta_{ab}k_jk_ck_d)\Big] \\
&-\frac{1}{2k^2}\Big[\delta_{ja}\delta_{bd}k_c + \delta_{jc}\delta_{bd}k_a + \delta_{ja}\delta_{bc}k_d + \delta_{jb}\delta_{ac}k_d + \delta_{jd}\delta_{ac}k_b \\
&+\delta_{jc}\delta_{bd}k_a + \delta_{jb}\delta_{ad}k_c + \delta_{jd}\delta_{ac}k_b \\
&-(\delta_{ja}\delta_{cd}k_b + \delta_{jb}\delta_{cd}k_a + \delta_{jc}\delta_{ab}k_d + \delta_{jd}\delta_{ab}k_c)\Big]\Big]
\end{aligned} \tag{4.44}$$

Now we will show that the derivative term of tensor structure Π indeed cancels the $\Gamma\Pi$ terms:

$$-k_1^{(i}\partial^{j)}\Pi_{abcd}(k_1) + \Gamma^{ij}{}_{ab}{}^{pq}(k_1)\Pi_{pqcd}(k_1) + \Gamma^{ij}{}_{cd}{}^{pq}(k_1)\Pi_{abpq}(k_1) = 0 \tag{4.45}$$

so that LHS = RHS (symmetrization on k and ∂ appears because everything is contracted with symmetric polarization tensor ε at the end and so to get independent "matrix equation" in these indices, we need to symmetrize first).

Recall that (see below eqn 3.3)
$\Gamma_{abijkl}(\hat{k}) = -\frac{1}{2}\left(\delta_{ij} + \hat{k}_i\hat{k}_j\right)\left(\delta_{ab}\hat{k}_k\hat{k}_\ell - \frac{1}{2}\delta_{ak}\hat{k}_\ell\hat{k}_b - \frac{1}{2}\delta_{a\ell}\hat{k}_k\hat{k}_b\right) + \delta_{b(i}\delta_{j)(k}\delta_{l)_a} - \delta_{a(i}\delta_{j)(k}\delta_{l)b}$
$- \delta_{b(i}\hat{k}_{j)}\delta_{a(k}\hat{k}_{l)} + \delta_{a(i}\hat{k}_{j)}\delta_{b(k}\hat{k}_{l)} - \delta_{a(k}\delta_{l)(i}\hat{k}_{j)}\hat{k}_b - \delta_{b(k}\delta_{l)(i}\hat{k}_{j)}\hat{k}_a + 2\delta_{ab}\hat{k}_{(i}\delta_{j)(k}\hat{k}_{l)}$

Not all the terms in this long expression are relevant for the calculation here. In the expression on RHS, Γ is contracted with ε and Π and so using the symmetry in indices and the property $k^a\Pi_{abcd}(k) = 0$, it's easy to see that the only terms in Γ that contribute in the $\Gamma\Pi$ type terms are $\Gamma_{ijabcd}(\hat{k}) \supset -\delta_{i(c}\delta_{d)(a}\hat{k}_{b)}\hat{k}_j - \delta_{j(c}\delta_{d)(a}\hat{k}_{b)}\hat{k}_i$

And further, the expression of Π is:

$$\Pi_{abcd}(k) = \frac{1}{2}\left[\left(\delta_{ac} - \frac{k_ak_c}{k^2}\right)\left(\delta_{bd} - \frac{k_bk_d}{k^2}\right) + \left(\delta_{ad} - \frac{k_ak_d}{k^2}\right)\left(\delta_{bc} - \frac{k_bk_c}{k^2}\right)\right.$$
$$\left. - \left(\delta_{ab} - \frac{k_ak_b}{k^2}\right)\left(\delta_{cd} - \frac{k_ck_d}{k^2}\right)\right]$$

It is easy to see that, in the $\Gamma\Pi$ terms, there are no terms of power k^0, as expected. We just expect to get $\frac{1}{k^2}, \frac{1}{k^4}, \frac{1}{k^6}$ type terms, and thus we can match the corresponding terms we got above in 4.44.

Focusing on the $\frac{1}{k^6}$ part in $\Gamma\Pi$ term , we get:

$$\begin{aligned} = \frac{-1}{2k^6}\Big[&(\delta_{i(p}\delta_{q)(a}\hat{k}_{b)}\hat{k}_j + \delta_{j(p}\delta_{q)(a}\hat{k}_{b)}\hat{k}_i)(k_pk_qk_ck_d) \\ &+\delta_{i(p}\delta_{q)(c}\hat{k}_{d)}\hat{k}_j + \delta_{j(p}\delta_{q)(c}\hat{k}_{d)}\hat{k}_i)(k_pk_qk_ak_b)\Big] \end{aligned} \tag{4.46}$$

We can clearly drop symmetrization on p,q indices and thus implement δ directly. The rest is straightforward algebra and finally we get $= -\frac{2k_ik_jk_ak_bk_ck_d}{k^6}$, which indeed cancels the corresponding term in 4.44 as expected.

Next, the $\frac{1}{k^4}$ term looks:

$$= \frac{1}{2k^4}(\delta_{i(p}\delta_{q)(a}\hat{k}_{b)}\hat{k}_j + \delta_{j(p}\delta_{q)(a}\hat{k}_{b)}\hat{k}_i)\Big[\delta_{pc}k_qk_d + \delta_{qd}k_pk_c + \delta_{pd}k_qk_c + \delta_{qc}k_pk_d - \delta_{pq}k_ck_d - \delta_{cd}k_pk_q\Big] \tag{4.47}$$

Again, the term in square brackets is symmetric in p,q and thus we can directly implement δ with p or q indices to sum over them. After doing some algebra, we again get the expected expression that cancels with the corresponding term in 4.44. Similar steps can be done for $\frac{1}{k^6}$ term and we again get the expected term.

4.5.2 4-3 Consistency Relation

The relevant 4-3 consistency relation to check is the following (momentum-conserving δ's stripped):

$$\lim_{q\to 0} S_{ij}(\vec{q})\frac{\langle\gamma_{ij}(\vec{q})\gamma_{ab}(\vec{k}_1)\gamma_{cd}(\vec{k}_2)\gamma_{ef}(\vec{k}_3)\rangle_c'}{P_\gamma(q)} = S_{ij}(\vec{q})\Big[-(k_1^i\nabla_1^j + k_2^j\nabla_2^j)\langle\gamma_{ab}(\vec{k}_1)\gamma_{cd}(\vec{k}_2)\gamma_{ef}(\vec{k}_3)\rangle_c' \\ +\Gamma^{ij}_{ab\,pq}(k)\langle\gamma_{pq}(\vec{k}_1)\gamma_{cd}(\vec{k}_2)\gamma_{ef}(\vec{k}_3)\rangle_c' \\ +\Gamma^{ij}_{cd\,pq}(k)\langle\gamma_{ab}(\vec{k}_1)\gamma_{pq}(\vec{k}_2)\gamma_{ef}(\vec{k}_3)\rangle_c' \\ +\Gamma^{ij}_{ef\,pq}(k)\langle\gamma_{ab}(\vec{k}_1)\gamma_{cd}(\vec{k}_2)\gamma_{pq}(\vec{k}_3)\rangle_c'\Big] \tag{4.48}$$

Focusing on LHS initially, we have:

$$\begin{aligned}\text{LHS} &= \lim_{q\to 0}\varepsilon^{*h_q}_{ij}(\vec{q})\sum_{h,h_1,h_2,h_3}\varepsilon^{h}_{ij}(\vec{q})\varepsilon^{h_1}_{ab}(\vec{k}_1)\varepsilon^{h_2}_{cd}(\vec{k}_2)\varepsilon^{h_3}_{ef}(\vec{k}_3)\frac{\langle\gamma^h_q\gamma^{h_1}_{k_1}\gamma^{h_2}_{k_2}\gamma^{h_3}_{k_3}\rangle_c'}{P_\gamma(q)} \\ &= \lim_{q\to 0}\frac{2\varepsilon^{h_1}_{ab}(\vec{k}_1)\varepsilon^{h_2}_{cd}(\vec{k}_2)\varepsilon^{h_3}_{ef}(\vec{k}_3)\langle\gamma^{h_q}_q\gamma^{h_1}_{k_1}\gamma^{h_2}_{k_2}\gamma^{h_3}_{k_3}\rangle_c'}{P_\gamma(q)}\end{aligned} \tag{4.49}$$

The paper [12] evaluates wavefunction coefficients first and then uses the relation between ψ_3, ψ_4 and $\langle\gamma\gamma\gamma\gamma\rangle$ to find the desired in-in correlator:

$$\lim_{q\to 0}\langle\gamma^{h_q}_q\gamma^{h_1}_{k_1}\gamma^{h_2}_{k_2}\gamma^{h_3}_{k_3}\rangle_c' = \lim_{q\to 0}\Big[\underbrace{4\sum_h\sum_{\text{cyclic}(k_1,k_2,k_3)}P_\gamma(k_1)\,\mathrm{Re}\,\Psi_{h_qh_1h}(\vec{q},\vec{k}_1,-\vec{q}-\vec{k}_1)\cdot\mathrm{Re}\,\Psi_{h_2h_3h}(k_2,\vec{k}_3,\vec{k}_1+\vec{q})}_{A} \\ -\underbrace{2\,\mathrm{Re}\,\Psi_{h_qh_1h_2h_3}(\vec{q},\vec{k}_1,\vec{k}_2,\vec{k}_3)}_{B}\Big]\frac{H^8}{M^8q^3k_1^3k_2^3k_3^3} \tag{4.50}$$

We have already seen Ψ_3 in the last section. Thus we can immediately evaluate the A-term:

$$\sum_h \mathrm{Re}\,\Psi_3 \cdot \mathrm{Re}\,\Psi_3 = \frac{M^4}{16H^4} I(0,\vec{k}_1,-\vec{k}_1)\, I(\vec{k}_2,\vec{k}_3,\vec{k}_1)$$
$$\sum_h \Big[\varepsilon^{*h_q}_{ii'}(\vec{q})\,\varepsilon^{*h_1}_{jj'}(\vec{k}_1)\,\varepsilon^{*h}_{ll'}(-\vec{k}_1)\,(k_1^i\delta_{jl} - k_1^j\delta_{il})(k_1^{i'}\delta_{j'l'} - k_1^{j'}\delta_{i'l'})$$
$$\times\, \varepsilon^{*h_2}_{pp'}(\vec{k}_2)\,\varepsilon^{*h_3}_{qq'}(\vec{k}_3)\,\varepsilon^{*h}_{rr'}(\vec{k}_1)$$
$$\times\,(k_3^p\delta_{qr} + k_1^q\delta_{pr} + k_2^r\delta_{pq})(k_3^{p'}\delta_{q'r'} + k_1^{q'}\delta_{p'r'} + k_2^{r'}\delta_{p'q'})\Big] \tag{4.51}$$

$$= 2\,k_1^i\,\varepsilon^{*h_q}_{ij}(\vec{q})\,k_1^j\,\underbrace{\varepsilon^{*h_1}_{pp'}(\vec{k}_1)\,\varepsilon^{*h_2}_{qq'}(\vec{k}_2)\varepsilon^{*h_3}_{rr'}(\vec{k}_3)\,t_{pqr}\,t_{p'q'r'}}_{X_3} \tag{4.52}$$

$$\implies A = -\frac{3M^2}{4H^2} X_3 I_{123}\left[\frac{k_1^T\,\varepsilon^*(\vec{q})\,k_1}{k_1^2} + \frac{k_2^T\,\varepsilon^*(\vec{q})\,k_2}{k_2^2} + \frac{k_3^T\,\varepsilon^*(\vec{q})\,k_3}{k_3^2}\right] \tag{4.53}$$

Ψ_4 (the B term) is what was the main result derived recently in [12]. I collect their results, and associated definitions involved below. See appendix (C) for further details:

$$\psi_{\gamma^4}(\{\mathbf{k}\}) = \frac{M_{\mathrm{Pl}}^2}{16H^2}\left[\psi^{(s)}_{\gamma^4}(\{\mathbf{k}\}) + \psi^{(t)}_{\gamma^4}(\{\mathbf{k}\}) + \psi^{(u)}_{\gamma^4}(\{\mathbf{k}\})\right] \tag{4.54}$$

$$\psi^{(s)}_{\gamma^4}(\{\mathbf{k}\}) = M_{\mathrm{Pl}}^2 H^2 \psi^{(s)}_{\phi^4}(\varepsilon_1\cdot\varepsilon_2)^2(\varepsilon_3\cdot\varepsilon_4)^2 + 2f^{(s)}_{(2,2)}\left[2\Pi^{(s)}_{1,1}(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)W_s + 2W_s^2\right]$$
$$- 2f^{(s)}_{(2,0)}\Pi^{(s)}_{1,0}(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)W_s + g^{(s)}_{\mathrm{cont.}}(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4)$$
$$- 2f_{\mathrm{cont.}}W_s W_s^{(c)} \tag{4.55}$$

$$W_s = (\varepsilon_1\cdot\varepsilon_2)\left[(\varepsilon_3\cdot k_1)(\varepsilon_4\cdot k_2) - (\varepsilon_3\cdot k_2)(\varepsilon_4\cdot k_1)\right] + (\varepsilon_3\cdot\varepsilon_4)\left[(\varepsilon_1\cdot k_3)(\varepsilon_2\cdot k_4) - (\varepsilon_1\cdot k_4)(\varepsilon_2\cdot k_3)\right]$$
$$+ \left[(\varepsilon_1\cdot k_2)\varepsilon_2 - (\varepsilon_2\cdot k_1)\varepsilon_1\right]\cdot\left[(\varepsilon_3\cdot k_4)\varepsilon_4 - (\varepsilon_4\cdot k_3)\varepsilon_3\right] \tag{4.56}$$

$$W_s^{(c)} = (\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3) - (\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4) \tag{4.57}$$

$$f_{\mathrm{cont.}} = \prod_{i=1}^{4}(1 - k_i\partial_{k_i})k_T(\log k_T - 1) = \frac{2\mathfrak{e}_4}{k_3^2} + \frac{\mathfrak{e}_3}{k_2^2} + \frac{\mathfrak{e}_2}{k_T} - k_T \tag{4.58}$$

$$
\begin{aligned}
g^{(s)}_{\text{cont}} = &-(k_{12}k_{34}+s^2)\left[\frac{4\mathfrak{e}_4}{k_T^3} - \frac{(k_{12}-k_{34})(k_{12}^2+2k_1k_2-k_{34}^2-2k_3k_4)}{2k_T^2} + \frac{3(k_1k_2+k_3k_4)}{k_T} - \frac{3k_T}{2}\right] \\
&- \frac{4\mathfrak{e}_4}{k_T} + \frac{8k_{12}^2k_{34}^2-4k_{12}^4-4k_{34}^4-2k_1k_2k_{34}^2-2k_3k_4k_{12}^2+8k_1k_2k_{12}^2+8k_3k_4k_{34}^2}{3k_T} \\
&+ \frac{14}{3}k_T(k_1k_2+k_3k_4) - \frac{10k_T^3}{9} + \frac{16}{9}\sum_{a=1}^{4}k_a^3
\end{aligned}
\tag{4.59}
$$

Note on Notation: $\varepsilon_{ij}(\hat{k}) \doteq \varepsilon_i(\hat{k})\varepsilon_j(\hat{k})$; therefore, $\varepsilon_1 \cdot k_2 = \varepsilon_i(\hat{k}_1)k_2^i$ etc

This is algebraically messy to handle and we thus turned to mathematica. The explicit expression of the RHS has been obtained. LHS of the expression remains to be processed by mathematica - it's a long expression. After getting the expression hopefully soon, we will be able to compare LHS and RHS and state the final result.

Chapter 5

Conclusion

5.1 Unitarity and Cutting Rules

In this work, we revisited the problem of non-perturbative unitarity constraints on cosmological observables—namely, the cosmological cutting rules for wavefunction coefficients—by examining the recently introduced Bunch-Davies S-matrix [7]. We identified what appears to be an oversight in the purportedly non-perturbative ψ–S relation presented there: specifically, the relation appears valid only at tree level, once the appropriate prescription is taken into account. This highlights that transferring unitarity constraints from the S-matrix to wavefunction coefficients is less straightforward than one might expect. The underlying complication stems from combinatorial factors in Witten/Feynman diagrams, indicating a subtlety that must be treated carefully when attempting to apply unitarity beyond the perturbative regime.

5.2 Symmetries and Soft Theorems

As noted in the Introduction, graviton correlator calculations are often cumbersome. A recent example is [12], where the authors computed the four-point pure-graviton in-in correlator in an exact de Sitter background, working in the $\delta\phi$-gauge. Soft theorems ordinarily provide powerful consistency checks for such involved computations, but the authors found it challenging to identify a suitable consistency relation for their final result.

Motivated by this, we revisited the problem and observed that there exists a 4–3 consistency relation that bypasses the naive problem of ambiguity in taking derivatives of polarization tensors. We also clarify some misunderstanding in the literature as regards the general explicit verifiability of soft theorems involving gravitons.

The explicit check of 4-3 soft theorem for the calculation of [12] is a work in progress but we have neveretheless provided an outline for the analysis of explicit check and field redefinition based explanation for the results of explicit check, in full detail for the 3-2 case, which can be straightforwardly extended to the 4-3 case. The task has now been reduced to merely the mathematica implementation of this tedious calculation and evaluation of some terms in the third order field redefinition to offer a complete resolution to the puzzle.

Appendix A

Maldacena Action

The setting here is that of QFT in curved spacetime, with quantized metric fluctuations too, against the fixed, homogeneous quasi-de Sitter background.

A.1 Reproduction of Quadratic computation

The study of inflationary perturbations begins with the standard action for a scalar field coupled to gravity. The action can be written as:

$$S = \frac{1}{2}\int \sqrt{-g}\left[R - (\nabla\phi)^2 - 2V(\phi)\right] \tag{A.1}$$

First we describe the classical, homogeneous solutions: We consider a flat FLRW metric (with $H(t) \equiv \dot{\rho}(t)$), given by:

$$ds^2 = -dt^2 + e^{2\rho(t)} dx^i dx_i$$

The Einstein equation (Friedmann equation) and energy-momentum conservation equation for the classical, background fields $\rho(t)$ and $\phi(t)$ are then obtained from this action:

$$\begin{aligned} 3\dot{\rho}^2 &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ \ddot{\rho} &= -\frac{1}{2}\dot{\phi}^2 \\ 0 &= \ddot{\phi} + 3\dot{\rho}\dot{\phi} + V'(\phi) \end{aligned}$$

Having described the homogeneous equations of motion, we now move on to describing main objects of interest - perturbations on top of the homogeneous background. The ADM formalism is naturally suited for it. The ADM metric is written as:

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

We work in comoving gauge as follows:

$$\delta\phi = 0, \quad h_{ij} = e^{2\rho}[(1+2\zeta)\delta_{ij} + \gamma_{ij}], \quad \partial_i \gamma_{ij} = 0, \quad \gamma_{ii} = 0$$

Making use of the Hamiltonian and Momentum constraints, and putting them back into the action (A.1), we simplify everything in the integrand to quadratic order in perturbations (keeping just the free part of the action) and perform integration by parts at several stages to finally obtain the highly simple form of the quadratic action:

$$S = \frac{1}{2}\int dt\, d^3x \frac{\dot{\phi}^2}{\dot{\rho}^2}\left[e^{3\rho}\dot{\zeta}^2 - e^{\rho}(\nabla\zeta)^2\right]$$

After quantizing this theory, we arrive at the well-known 2-point in-in correlation function (at late times, τ_*):

$$\langle \zeta_{\mathbf{k}}(t)\zeta_{\mathbf{k}'}(t)\rangle \sim (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\frac{1}{2k^3}\frac{\dot{\rho}_*^2}{M_{\text{pl}}^2}\frac{\dot{\rho}_*^2}{\dot{\phi}_*^2}$$

where * means that the quantity is to be evaluated at the time of horizon-crossing for k-mode (thus creating a non-trivial implicit dependence of k through τ_*). The logic here is to set $\varepsilon(t)$ to $\varepsilon(t_*)$. Since $\varepsilon(t)$ is essentially an unknown function of time (because it depends on the unknown self-interactions, $V(\phi)$, of inflation), and since it appears in the action, it also appears in the final observables - N-pt functions we calculate, inside the time integrals (that come from vertices of course, for both tree and loop level).

Therefore, naively it seems impossible to even compute those observables, but we can in fact compute the answer approximately since, in terms of τ, the overall integrands always look like $\tau^n e^{ik\tau}\varepsilon(t)^m$ and by Wick rotating it becomes evident that it's like the integral of $e^{-kx}x^n\varepsilon(x)^m$ and therefore we expect a good approximation to the actual answer, if we set $\varepsilon(x) = \varepsilon(x_*)$ where x_* is the time where $x^n e^{-kx}$ is maximum.

Appendix B

Polarization Tensors - Collecting Useful Identities

B.1 Collecting Useful Identities

$\boxed{+, \times \ \textbf{Basis}}$

$$e^r_{ij}(\hat{\mathbf{k}})\, e^{r'}_{ij}(\hat{\mathbf{k}}) = 2\delta_{rr'}\,, \tag{B.1}$$

$$\Pi_{ij,lm} \equiv \frac{\sum_{r=+,\times} e^r_{ij}(\hat{\mathbf{k}})\, e^r_{lm}(\hat{\mathbf{k}})}{2} = \frac{P_{il}P_{jm} + P_{im}P_{jl} - P_{ij}P_{lm}}{2}\,. \tag{B.2}$$

where $P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j = \hat{m}_i\hat{m}_j + \hat{n}_i\hat{n}_j$ is the projector on the plane perpendicular to $\mathbf{k}$.

$\Pi_{ij,kl}$ is the projector for TT symmetric tensors, i.e.,

$$h^{TT}_{ij}(k,\tau) = \Pi_{ij,kl}(\hat{k})\; h_{kl}(k,\tau). \tag{B.3}$$

$$\begin{aligned}
\varepsilon^h_{ii}(\mathbf{k}) = k^i\varepsilon^h_{ij}(\mathbf{k}) = 0 &\quad \text{(transverse and traceless)},\\
\varepsilon^h_{ij}(\mathbf{k}) = \varepsilon^h_{ji}(\mathbf{k}) &\quad \text{(symmetric)},\\
\varepsilon^h_{ij}(\mathbf{k})\varepsilon^h_{jk}(\mathbf{k}) = P_{ij} &\quad \text{(Projector)},\\
\varepsilon^h_{ij}(\mathbf{k})\varepsilon^{h'}_{ij}(\mathbf{k})^* = 2\delta_{hh'} &\quad \text{(normalization)},\\
\varepsilon^h_{ij}(\mathbf{k})^* = \varepsilon^h_{ij}(\mathbf{k}) &\quad (\gamma_{ij}(x) \text{ is real}),\\
\varepsilon^+_{ij}(-\mathbf{k}) = \varepsilon^+_{ij}(\mathbf{k}), &\quad \text{(+ is Parity Even)}\\
\varepsilon^\times_{ij}(-\mathbf{k}) = -\varepsilon^\times_{ij}(\mathbf{k}) &\quad (\times \text{ is Parity Odd}).
\end{aligned} \tag{B.4}$$

$\boxed{\textbf{Helicity Basis}}$ $\varepsilon^{\pm 2}_{ij} = (\varepsilon^+_{ij} \pm i\varepsilon^\times_{ij})/\sqrt{2}$

Clearly they are complex conjugate of each other. The expression in terms of basis vectors

$\hat{m}, \hat{n}$ is:

$$\varepsilon^2_{ij} = \frac{(\hat{m}_i + i\hat{n}_i)(\hat{m}_j + i\hat{n}_j)}{\sqrt{2}} \equiv \sqrt{2} e^2_i e^2_j \tag{B.5}$$

where $e^2_i \equiv \frac{\hat{m}_i + i\hat{n}_i}{\sqrt{2}}$

$$\begin{aligned}
\varepsilon^h_{ii}(\mathbf{k}) = k^i \varepsilon^h_{ij}(\mathbf{k}) &= 0 && \text{(transverse and traceless)},\\
\varepsilon^h_{ij}(\mathbf{k}) &= \varepsilon^h_{ji}(\mathbf{k}) && \text{(symmetric)},\\
\varepsilon^h_{ij}(\mathbf{k})\varepsilon^h_{jk}(\mathbf{k}) &= 0 && \text{(lightlike)},\\
\varepsilon^h_{ij}(\mathbf{k})\varepsilon^{h'}_{ij}(\mathbf{k})^* &= 2\delta^{hh'} && \text{(normalization)},\\
\varepsilon^{h*}_{ij}(\mathbf{k}) &= \varepsilon^{-h}_{ij}(\mathbf{k}) = \varepsilon^h_{ij}(-\mathbf{k}) && \text{(Helicity flips on flipping } \mathbf{k}\text{)}.
\end{aligned} \tag{B.6}$$

Appendix C

Notation and Definitions

C.1 Notation

I try to stick to the notation of [12] in section 4.
We define the usual primed correlators as ones with 3-momentum conserving delta functions stripped:

$$\langle\mathcal{O}(\mathbf{k}_1)\dots\mathcal{O}(\mathbf{k}_n)\rangle \equiv \langle\mathcal{O}(\mathbf{k}_1)\dots\mathcal{O}(\mathbf{k}_n)\rangle'\delta^{(3)}\left(\sum\mathbf{k}_a\right) \tag{C.1}$$

The three-momenta $\mathbf{k}_a$ have components k_a^i and norms $k_a = |\mathbf{k}_a|$. The corresponding massless four-momenta have components $k_a^\mu = (k_a, \mathbf{k}_a)$. We define $k_{ab} = k_a + k_b$. The elementary symmetric polynomials in three variables are written as

$$e_1 = k_1 + k_2 + k_3, \qquad e_2 = k_1k_2 + k_1k_3 + k_2k_3, \qquad e_3 = k_1k_2k_3, \tag{C.2}$$

while those in four variables are written as

$$\mathfrak{e}_1 = k_1 + k_2 + k_3 + k_4, \qquad \mathfrak{e}_2 = k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4, \tag{C.3}$$

$$\mathfrak{e}_3 = k_1k_2k_3 + k_1k_2k_4 + k_1k_3k_4 + k_2k_3k_4, \qquad \mathfrak{e}_4 = k_1k_2k_3k_4. \tag{C.4}$$

We will often use k_T in place of e_1 and $\mathfrak{e}_1$ where no confusion arises. We also define

$$s = |\mathbf{k}_1 + \mathbf{k}_2|, \qquad t = |\mathbf{k}_1 + \mathbf{k}_3|, \qquad u = |\mathbf{k}_1 + \mathbf{k}_4|. \tag{C.5}$$

C.2 Definition of Quantities in Sec 4.5

In this section, I have mainly collected together the relevant results and definitions from the same paper [12].

The s-channel, 4-point scalar wavefunction coefficient (built from 4-point contact and graviton-exchange diagrams at tree-level) is given by:

$$\psi^{(s)}_{\phi^4}(\{\mathbf{k}\}) = \frac{1}{6H^2M_{\text{Pl}}^2}\left[f^{(s)}_{(2,2)}s^4\Pi^{(s)}_{2,2} + f^{(s)}_{(2,1)}s^2\Pi^{(s)}_{2,1} + f^{(s)}_{(2,0)}(E_LE_R - sk_T)\Pi^{(s)}_{2,0} + f_c\right] \tag{C.6}$$

The full permutation-invariant answer is then given by summing over all channels:

$$\psi_{\phi^4}(\{\mathbf{k}\}) = \psi^{(s)}_{\phi^4}(\{\mathbf{k}\}) + \psi^{(t)}_{\phi^4}(\{\mathbf{k}\}) + \psi^{(u)}_{\phi^4}(\{\mathbf{k}\}) \tag{C.7}$$

where $\{\mathbf{k}\}$ is shorthand for $\{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4\}$ and $\Pi^{(s)}_{m,n}$ are contractions of projectors with external momenta whose explicit form is given below on the next page. The f's are functions of the external momenta, defined by:

$$\begin{aligned} f^{(s)}_{(2,2)} = &\frac{1}{k_T} - \frac{s^2}{k_TE_LE_R} + \frac{sk_1k_2}{k_TE_L^2E_R} + \frac{sk_3k_4}{k_TE_R^2E_L} + \frac{2sk_1k_2k_3k_4}{k_T^2E_L^2E_R^2} - \frac{s(k_1k_2+k_3k_4)}{k_T^2E_LE_R} \\ &+ \frac{k_1k_2}{k_T^2E_L} + \frac{k_3k_4}{k_T^2E_R} + \frac{2k_1k_2k_3k_4}{k_T^3E_LE_R} \end{aligned} \tag{C.8}$$

$$f^{(s)}_{(2,1)} = -\frac{2k_1k_2k_3k_4}{k_T^3} - \frac{k_{12}k_3k_4 + k_{34}k_1k_2}{k_T^2} - \frac{k_{12}k_{34}}{k_T} \tag{C.9}$$

$$E_L = k_1 + k_2 + s, \qquad E_R = k_3 + k_4 + s \tag{C.10}$$

$$f^{(s)}_{(2,0)} = -f^{(s)}_{(2,1)}, \tag{C.11}$$

$$\begin{aligned} f_c = &-\frac{2k_1k_2k_3k_4(k_{12}k_{34}+s^2)}{k_T^3} - \frac{1}{k_T^2}(k_1k_2k_3 + k_1k_2k_4 + k_1k_3k_4 + k_2k_3k_4)(k_{12}k_{34}+s^2) \\ &- \frac{1}{4k_T}\Bigg[24k_1k_2k_3k_4 + 8(k_1k_3+k_2k_3+k_1k_4+k_2k_4)^2 \\ &\quad + (k_1k_2+k_3k_4)\left[9(k_1k_3+k_2k_3+k_1k_4+k_2k_4) - 12(k_1k_2+k_3k_4)\right] \\ &\quad + \frac{1}{2}(k_{12}^2+k_{34}^2+2s^2)\left(3(k_1k_2+k_3k_4) + 4(k_1k_3+k_2k_3+k_1k_4+k_2k_4)\right)\Bigg] \\ &- \frac{3(k_1-k_2)^2(k_3-k_4)^2(k_{12}+k_{34})^2}{4s^2k_T} - 3(k_1k_2k_3 + k_1k_2k_4 + k_3k_4k_1 + k_3k_4k_2) \\ &+ \frac{k_T}{8}\Big[21(k_1k_2+k_3k_4) + 34(k_1k_3+k_2k_3+k_1k_4+k_2k_4) \\ &\quad + 3(k_{12}^2+k_{34}^2+2s^2) - \frac{6}{s^2}(k_1-k_2)^2(k_3-k_4)^2\Big] - \frac{9}{8}k_T^3 \end{aligned} \tag{C.12}$$

The transverse-traceless spin-2 projector is

$$\Pi^{\mathbf{k}}_{ijkl} = \frac{1}{2}\sum_s e^s_{ij}(\mathbf{k})e^s_{kl}(-\mathbf{k}) = \frac{1}{2}\pi^{\mathbf{k}}_{ik}\pi^{\mathbf{k}}_{jl} + \frac{1}{2}\pi^{\mathbf{k}}_{il}\pi^{\mathbf{k}}_{jk} - \frac{1}{2}\pi^{\mathbf{k}}_{ij}\pi^{\mathbf{k}}_{kl} \tag{C.13}$$

where $\pi^{\mathbf{k}}_{ij} = \delta_{ij} - \frac{1}{k^2}k_ik_j$. The following contractions of this appears in wavefunction coefficients:

$$\begin{aligned}
\Pi^{(s)}_{2,2} &= \frac{24}{s^4}\Pi^{\mathbf{s}}_{ijkl}k^i_1k^j_2k^k_3k^l_4 \\
&= -\frac{3}{4} + \frac{3}{2s^2}\sum_{a=1}^{4}k_a^2 - 3\frac{4(k_1^2+k_2^2)(k_3^2+k_4^2) - 2k_1^2k_2^2 - 2k_3^2k_4^2 + \sum_{a=1}^4 k_a^4 - 2(t^2-u^2)^2}{4s^4} \\
&\quad + \frac{3}{2s^6}\left[2(t^2-u^2)(k_1^2-k_2^2)(k_3^2-k_4^2) + (k_3^2+k_4^2)(k_1^2-k_2^2)^2 + (k_1^2+k_2^2)(k_3^2-k_4^2)^2\right] \\
&\quad + \frac{3(k_1-k_2)^2(k_3-k_4)^2k_{12}^2k_{34}^2}{4s^8}
\end{aligned} \tag{C.14}$$

$$\begin{aligned}
\Pi^{(s)}_{2,1} &= \frac{3(k_1-k_2)(k_3-k_4)}{s^4}\pi^s_{ij}(k^i_1-k^i_2)(k^j_3-k^j_4) \\
&= \frac{3(k_1-k_2)(k_3-k_4)}{s^6}\left[s^2(t^2-u^2) + k_{12}k_{34}(k_1-k_2)(k_3-k_4)\right]
\end{aligned} \tag{C.15}$$

$$\Pi^{(s)}_{2,0} = \frac{1}{4}\left(1 - \frac{3(k_1-k_2)^2}{s^2}\right)\left(1 - \frac{3(k_3-k_4)^2}{s^2}\right) \tag{C.16}$$

$$\Pi^{(s)}_{1,1} \equiv \frac{k_{12}k_{34}(k_1-k_2)(k_3-k_4) + s^2(t^2-u^2)}{s^4} \tag{C.17}$$

$$\Pi^{(s)}_{1,0} \equiv \frac{(k_1-k_2)(k_3-k_4)}{s^2} \tag{C.18}$$

Appendix D

FLRW Killing Equation (Deriving SCTs for dS)

An isometry transformation, by definition, is one that preserves the functional form of the metric: $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$. Working at infinitesimal level, expanding this we get the expected equation of Lie Derivative of g vanishing:

$$\partial_\mu \varepsilon^\alpha g_{\alpha\nu} + \partial_\nu \varepsilon^\alpha g_{\alpha\mu} = -\varepsilon^\alpha \partial_\alpha g_{\mu\nu} \tag{D.1}$$

Let's work with only those metrics that are conformally minkowski, $g_{\mu\nu} = a^2\eta_{\mu\nu}$ (I will restrict myself to FLRW-type here, but clearly the main argument holds for any conformally minkowski throughout):

$$\partial_\mu \varepsilon^\alpha \eta_{\alpha\nu} + \partial_\nu \varepsilon^\alpha \eta_{\alpha\mu} = -2\mathscr{H}\varepsilon^0 \eta_{\mu\nu} \tag{D.2}$$

If we *define* ε_μ as $\eta_{\mu\nu}\varepsilon^\nu$, then clearly the equation above looks just like the usual conformal transformation equation, for the minkowski background:

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x)\eta_{\mu\nu} \tag{D.3}$$

except here, $f(x)$ is not just $\frac{2}{d}\partial_\mu \varepsilon^\mu$ but also $= -2\mathscr{H}\varepsilon^0$

Therefore the usual arguments there of CFT work verbatim here and we finally get that ε is a quadratic quantity. $\varepsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\alpha\beta}x^\alpha x^\beta$. Putting this back into the conformal group condition above, we get obviously the 15 independent conformal transformations:

$$\boxed{x'^\mu = x^\mu + a^\mu + \lambda x^\mu + \eta_{\alpha\beta}\omega^{\mu\alpha}x^\beta + b^\mu(\eta_{\alpha\beta}x^\alpha x^\beta) - 2(\eta_{\alpha\beta}b^\alpha x^\beta)x^\mu} \tag{D.4}$$

But obviously we now need to impose the condition that $f(x) = \boxed{\frac{2}{d}(\partial_\mu \varepsilon^\mu) = -2\mathscr{H}\varepsilon^0}$ to cut those 5 extra parameters, giving us the 10 isometries we expect for any background metric (in d = 4 spacetime dimensions)..

Putting equation D.4 in the condition above gives:

$$\boxed{2\lambda - 4\eta_{\mu\nu}b^\mu x^\nu = -2\mathscr{H}(\tau)[a^0 + \lambda\tau + \eta_{ij}w^{0i}x^j + b^0(\tau^2 - \vec{x}^2) - 2b^0\tau^2 + 2(\vec{b}\cdot\vec{x})\tau]} \tag{D.5}$$

The analysis will clearly depend on the form of $\mathscr{H}(\tau)$ and thus we do it specifically for Flat and dS:

- Flat ($\mathscr{H}(\tau) = 0$): RHS of equation above vanishes in this case. We immediately see that $\lambda = b^\mu = 0$ (15-1-4 = 10)
  Further, we obtain the form of infinitesimal isometry transformations for Flat minkowski to be, as expected the Poincare transformations:

$$x'^\mu = x^\mu + a^\mu + \eta_{\alpha\beta}\omega^{\mu\alpha}x^\beta \tag{D.6}$$

- dS ($\mathscr{H}(\tau) = -\frac{1}{\tau}$): Doing the algebra, matching coefficients of each monomial of τ^n or $\vec{x}^n$ on both sides, we get: $a^0 = w^{0i} = b^0 = 0$. Therefore, as expected, we again see 5 parameters (15-1-3-1 = 10) getting eliminated.
  Again, the form of the isometries of dS is then:

$$\tau' = \tau + \lambda\tau + 2\tau(\vec{b}\cdot\vec{x}) \tag{D.7}$$

$$x'^i = x'^i + a^i + \lambda x^i + \eta_{\alpha\beta}\omega^{i\alpha}x^\beta + b^i(\tau^2 - \vec{x}^2) + 2(\vec{b}\cdot\vec{x})x^i \tag{D.8}$$

Thus we obtain 3 translations (a^i's), 3 rotations (ω), 1 dilation (λ) and 3 SCTs (b^i's) as the isometries of 4d dS spacetime (expanding poincare patch of global de Sitter, to be more precise)

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