Poisson extension of Delzant construction

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Certificate

This is to certify that this dissertation entitled *'Poisson extension of Delzant construction'* towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune, represents study/work carried out by Vansh Aggarwal at Indian Institute of Science Education and Research Pune under the supervision of Mainak Poddar, Professor, Department of Mathematics , during the academic year 2025-2026.

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This thesis is dedicated to my family and friends

Declaration

I hereby declare that the matter embodied in the report entitled *'Poisson extension of Delzant construction'* are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Mainak Poddar and the same has not been submitted elsewhere for any other degree.

Vansh Aggarwal

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Abstract

This thesis investigates the existence and construction of torus equivariant Poisson structures on a class of manifolds with torus action that includes all quasitoric manifolds. This is achieved through the study of reducible Poisson structures on $\mathbb{C}^m$ with respect to suitable torus actions using the Poisson reduction theory of Marsden and Ratiu. We also study the symplectic foliation and Poisson cohomology associated to these Poisson structures.

List of Notations

Throughout this thesis, we adhere to standard mathematical conventions. The following is a list of frequently used symbols and notations for easy reference.

$\mathbb{N}$	The set of natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{Z}$	The ring of integers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{R}^n$	Real n-dimensional Euclidean space
$\mathbb{C}^m$	Complex m-dimensional space
$\mathbb{R}^m_+$	The non-negative orthant in $\mathbb{R}^m$
S^1	The unit circle in the complex plane, $\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$
$\mathbb{T}^m$	The compact n-dimensional torus, $(S^1)^m$
P_Δ	An n-dimensional simple convex polytope
F_i	A facet (Face of codimension -1) of P
i_A	The affine embedding map $i_A(x) = Ax + \mathbf{b}$
$\mathcal{Z}_{P_\Delta}$	The moment-angle manifold associated to the polytope P
μ	The moment map $\mu(z_1, \dots, z_m) = (\lvert z_1\rvert^2, \dots, \lvert z_m\rvert^2)$
Ω_A	A symplectic 2-form
$X^\#$	The fundamental vector field generated by $X \in \mathfrak{g}$
$\mathfrak{g}$	The Lie algebra of a Lie group G

$\mathfrak{g}^*$	The dual vector space to the Lie algebra $\mathfrak{g}$

Note: Specific notations introduced within individual chapters will be defined locally where they first appear.

Introduction and Motivation

There exists a canonical symplectic structure on $\mathbb{C}^m$ and a natural action of $\mathbb{T}^m$ on it which is Hamiltonian with respect to the canonical symplectic structure. Delzant ([Del]) uses symplectic reduction theory on $\mathbb{C}^m$ with $\mathbb{T}^m$ action and constructed a special class of symplectic manifolds with Hamiltonian torus action called symplectic toric manifolds. He goes on to show a one-to-one correspondence between symplectic toric manifolds and a special class of polytopes called Delzant polytopes.

All symplectic structures have a non-degeneracy condition, but to study the dynamics in some systems, we do not need such a strong condition. For that purpose, Poisson structures are developed that are allowed to be degenerate, and symplectic structures are simply a special case of them. One could think of a Poisson structure as a bivector (called the Poisson bivector) on a Poisson manifold satisfying a condition called the Jacobi condition. So a Symplectic structure is a non-degenerate Poisson bivector satisfying the Jacobi condition. Seminal foundational work was done by Weinsten [Wei] on Poisson manifolds that imply the existence of a (possibly singular) symplectic foliation on a Poisson manifold. Using the relaxation of non-degeneracy, Marsden and Ratiu ([Mar]) developed Poisson reduction theory. It states under what conditions we could induce a Poisson structure (from an existing Poisson structure on a Poisson manifold) on a smooth orbit space that was obtained from a free group action on a submanifold of the given Poisson manifold. So one could think of degenerate Poisson structures on $\mathbb{C}^m$ that, under some constraints, can be reduced using Poisson reduction theory.

In Delzant's paper, he constructed a submanifold of $\mathbb{C}^m$ and a subtorus of $\mathbb{T}^m$ corresponding to a Delzant polytope such that the action of the subtorus on the submanifold is free. He concluded that the orbit space of the subtori action on the submanifold adopts a smooth structure induced from the submanifold and a symplectic structure from $\mathbb{C}^m$. But what if we ignore the need for an induced symplectic structure and focus on studying the submanifolds of $\mathbb{C}^m$ with free torus action and its smooth orbit space (called quasitoric manifold) corresponding to a polytope? Quasitoric manifolds (under the name toric manifolds) were first introduced by Davis and Januskiewicz [Dav] and Buchstaber and Panov elaborated on their work. Buchstaber and Panov explained the theory in their book Toric topology ([Buc]). They defined such submanifolds of $\mathbb{C}^m$k as moment angle manifold $\mathcal{Z}_{P_\Delta}$ (corresponding to a simple polytope P_Δ) with $\mathbb{T}^m$ action on it. But they only studied the family of subtori of $\mathbb{T}^m$ with a certain dimension that act freely on $\mathcal{Z}_{P_\Delta}$, and the dimension depends entirely on the polytope. They defined the orbit space corresponding to the free action of these subtori on $\mathcal{Z}_{P_\Delta}$ as Quasitoric manifolds. What if we relax the dimensionality and explore exactly when any subtorus of $\mathbb{T}^m$ acts freely on $\mathcal{Z}_{P_\Delta}$? As a result, we get a much larger family of orbit spaces that have a smooth structure and contain the Quasitoric manifold as a subfamily.

Cain and Arlo, in their paper [Can1], examine certain invariant real Poisson structures on smooth complete toric varieties. They constructed quadratic Poisson structures on $\mathbb{C}^m$ that are algebraic torus invariant and induce a Poisson structure on toric varieties. They also studied its symplectic leaf and Poisson cohomology structure. They published another paper [Can2], in which they examined real $\mathbb{T}_{\mathbb{C}}$- invariant Poisson structures on complex toric manifolds and studied their Poisson cohomology. Hong and Wei [Hon] also studied Poisson geometry of toric varieties, especially the Poisson cohomology of holomorphic toric varieties. Poisson manifolds. A holomorphic toric Poisson manifold is a smooth toric variety $\mathbb{X}$, equipped with a holomorphic Poisson structure, which is invariant under the torus action, and they computed the Poisson cohomology groups for all holomorphic toric Poisson structures on $\mathbb{X} = \mathbb{C}P^n$. Here we don't explore such cases and only focus on constructing and studying reducible Poisson structures.

In this thesis, we answer the above posed questions by first constructing new and degenerate Poisson structures. The canonical Poisson structure is a non-degenerate bivector that pairs radial directions with their corresponding angular directions. An obvious question is whether we could pair angular with other angular directions and radial with other radial directions. In Section 2.4, we constructed 4 such families of degenerate Poisson structures on $\mathbb{C}^m$ and computed the constraints induced by the Jacobi condition on the coefficient function of the terms in the bivector using simple computational methods (Python).

In general, the newly constructed structures are not reducible under Marsden and Ratiu's reduction theory. We needed additional constraints on the way the terms of the Poisson bivectors look like. In Section 3.4, we first studied the tangent space of a general moment angle manifold $\mathcal{Z}_{P_\Delta}$ corresponding to a simple polytope P_Δ. Theorem 3.4.2 states the basis of the tangent space of any point p in $\mathcal{Z}_{P_\Delta}$.

Whether the action of any subtori $\mathbb{T}^d_\Delta$ of $\mathbb{T}^m$ on $\mathcal{Z}_{P_\Delta}$ entirely depends on the combinatorial data of the polytope P_Δ. Section 3.3 explores why and how the freeness depends only on the combinatorial structure of the polytope. Theorem 3.3.1 and Proposition 3.3.2 state the exact condition under which the action of $\mathbb{T}^d_\Delta$ on $\mathcal{Z}_{P_\Delta}$ is free, and the subtorus that gives us quasitoric manifolds is just a special case.

Finally, in Chapter 4, we list reducible Poisson structures and study their symplectic leaf structure and Poisson cohomology. We start by listing 4 families of Poisson bivectors on $\mathbb{C}^m$ that have the same structure as the Poisson bivectors in section 2.4 and are reducible under Poisson reduction theory. Theorem 4.0.1 states and proves that these bivectors are invariant under the natural action of $\mathbb{T}^m$ and satisfy the condition needed for reduction. In section 4.1, we analyse the symplectic leaf structure of these Poisson bivectors on moment angle manifold which can be extended to $\mathbb{C}^m$, and in section 4.2, we analyse Poisson cohomology of $\mathbb{C}^m$ paired with the Poisson bivectors in section 2.4.

Contents

Chapter 1

Symplectic Manifold

In Hamiltonian mechanics, given $\mathbb{R}^{2m}$ with position and momentum coordinates $(q_1, .., q_m, p_1, \ldots, p_m)$ and Hamiltonian function $H : \mathbb{R}^{2m} \to \mathbb{R}$, the trajectory of the particle governed by the given Hamiltonian is $(q_1(t), .., q_m(t), p_1(t), \ldots, p_m(t)$ such that

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Let $z = (q_1, \ldots, p_m)$ and $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, then

$$\dot{z} = -J_0 \nabla H(z)$$

Basically, if there is a particle with starting point z_0 and Hamiltonian function H, then it will follow along the trajectory $z(t)$ such that,

$$z(0) = z_0 \quad \dot{z}(t) = -J_0 \nabla H(z(t))$$

Definition 1.0.1. - *Given $H : \mathbb{R}^m \to \mathbb{R}$, Hamiltonian vector field X_H is a vector field generated by H such that*

$$X_H = -J_0 \nabla H$$

As we know, the gradient of a scalar field gives a vector field such that at each point the vector is perpendicular to the level set of H to which the point belongs. The action of $-J_0$

rotates it. It gives a vector field that is along the level set of H (The trajectory of the particle is such that its total energy or Hamiltonian remains constant along the motion).

Now, this vector field generates a flow, which in turn gives rise to a one-parameter subgroup of diffeomorphisms. These diffeomorphisms are special and belong to a class called symplectomorphisms. Basically, if a path is a solution to a Hamiltonian, then the pushforward of the path by that symplectomorphism will be a solution to the pushforward of the Hamiltonian function. So a symplectomorphism maps trajectory to trajectory. To define it rigorously,

Definition 1.0.2. - *A diffeomorphism $\psi : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is a symplectomorphism if*

$$d\psi(z)^T J_0 d\psi(z) = J_0 \quad \forall z \in \mathbb{R}^{2m}$$

We can describe the above Hamiltonian mechanics using a special 2-form, or bivector.

Definition 1.0.3. - $\omega_{\mathbb{R}^{2m}} = \sum_i dq_i \wedge dp_i$ *and* $\Pi_{\mathbb{R}^{2m}} = \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$

So the Hamiltonian vector field X_H is such that

$$\Pi_{\mathbb{R}^{2m}}(dH, .) = X_H \quad \text{or} \quad i_{X_H}\omega_{\mathbb{R}^{2m}} = dH$$

1.1 Symplectic vector space

Definition 1.1.1. - *A vector space A is a symplectic vector space if it comes with a bilinear map ω such that it is skew-symmetric and nondegenerate.*

A symplectic vector space is always even-dimensional, since if it is odd, the kernel of the bilinear form will be non-trivial.

Definition 1.1.2. - *Given a linear subspace $B \subset A$, its symplectic compliment*

$$B^\omega = \{a \in A | \omega(a, b) = 0 \quad \forall b \in B\}$$

The subspace is
1) Symplectic if $B \cap B^\omega = \{0\}$
2) Coisotropic if $B^\omega \subset B$
3) Isotropic if $B \subset B^\omega$
4) Lagrangian if $B = B^\omega$

Lemma 1.1.3 ([Mc][Ana]). - *Given any subspace $B \subset A$,*

$$dim(B^{\omega}) + dim(B) = dim(A) \quad and \quad (B^{\omega})^{\omega} = B$$

With the help of this lemma, we can show that there exists a special kind of bases of the symplectic vector space A such that w.r.t. to those bases,

$$(\omega) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Or in other words,

Theorem 1.1.4 ([Mc][Ana]). - *Given (A, Ω) with dimension 2m, there exist basis $c_1, ..c_m, d_1, \ldots, d_m$ such that*

$$\Omega(c_i, d_j) = \delta_{ij} \quad \Omega(c_i, c_j) = \Omega(d_i, d_j) = 0$$

and they are called a symplectic basis of A.

In the next section, I will mention the Darboux theorem, which states that given any point in a symplectic manifold, there exists a neighbourhood of that point with coordinate functions such that the basis tangent vectors of the tangent space at each point in the neighbourhood induced by the coordinate functions are a symplectic basis of the symplectic form in a smooth manner.

1.1.1 Linear Symplectic Reduction

Given a Symplectic vector space (A, Ω), we can construct new symplectic vector spaces from it. More specifically, if we have a coisotropic subspace of A, its quotient with its symplectic complement has a canonical symplectic form on it derived from Ω.

Lemma 1.1.5 ([Mc][Ana]). - *Given (A, Ω) and a coisotropic subspace $B \subset A$, $\overline{B} = B/B^{\Omega}$ has a unique symplectic structure $\overline{\Omega}$ such that its pull back with projection map $P : B \to \overline{B}$ equals to $\Omega|_B$.*

We use this concept and generalise it to manifolds. What we do is, given a symplectic manifold with a coisotropic submanifold, if there exists an isotropic regular involutive distribution on it, then the quotient manifold will have a unique symplectic structure induced from the symplectic structure on the given manifold

1.2 Symplectic manifold

We can define a symplectic manifold in two equivalent ways:-

Definition 1.2.1. - *A Symplectic manifold is a manifold A with a nondegenerate 2-form Ω_A such that,*

$$d\Omega_A = 0$$

Ω_A is called a symplectic structure.

Definition 1.2.2. - *A Symplectic manifold is a manifold M with a nondegenerate bivector field Π such that.*

$$\sum_{cyclic} \Pi(f_1, \Pi(f_2, f_3)) = \frac{1}{2} J(f_1, f_2, f_3) = 0 \quad \forall f_1, f_2, f_3 \in C^\infty(M)$$

Π is called a nondegenerate Poisson structure.

These two structures are equivalent, i.e., given a manifold with a symplectic structure, there exists a nondegenerate Poisson structure on it such that they are inverse of each other, and vice versa. Inverse of each other means,

$$\Omega_A(\Pi(\alpha, -), -) = \alpha \quad \forall \alpha \in \Omega^1(M)$$

$$\Pi(\Omega_A(\chi, -), -) = \chi \quad \forall \chi \in \mathfrak{X}^1(M)$$

There is a one-to-one correspondence between Ω_A and Π such that they satisfy the above conditions.

1.2.1 Symplectomorphism

Suppose we have 2 symplectic manifolds. The idea of symplectomorphisms is that they are functions that map trajectories to trajectories.

More specifically, given 2 symplectic manifolds (A, Ω_A) and (C, Ω_C) and a diffeomorphism $\phi : A \to C$,

Definition 1.2.3. - *ϕ is a symplectomorphism if for any $H \in C^\infty(A)$, if $z(t)$ is a trajectory (solution) of X_H, then $\phi(z(t))$ is a trajectory (solution) of $X_{H\circ\phi^{-1}}$.*

A more straightforward way to determine whether a diffeomorphism is a symplectomorphism is to check whether it preserves the symplectic structure.

Definition 1.2.4. - *ϕ is a symplectomorphism if $\phi^*\Omega_C = \Omega_A$*

Theorem 1.2.5 (Darboux's Theorem). *[[Ana]] Let (A, Ω_A) be a $2n$-dimensional symplectic manifold, and let $p \in A$ be any point. Then there exists a coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on the neighborhood U, the symplectic form Ω_A takes the standard form:*

$$\Omega_A|_U = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

So, unlike a Riemannian structure, there is no local invariant of a symplectic structure.

1.3 Symplectic and Hamiltonian Vector Field

Given what we have discussed so far, it's evident that symplectic vector fields and symplectic group action provide us with maps that preserve symplectic structure. One interesting aspect is that on $\mathbb{R}^{2m}$ or any general vector space, the concepts of symplectic and Hamiltonian are the same, but on a general symplectic manifold, they can differ.

Symplectic vector fields on (A, Ω_A) are vector fields such that the one-parameter diffeomorphisms produced by these vector fields are symplectomorphisms. Suppose χ is vector field, then let ϕ^t_χ be the one parameter diffeomorhism group with open set $O_{\phi^t_\chi} \subset A$ as its domain such that,

$$\chi \circ \phi^t_\chi = d\phi^t_\chi \left(\frac{\partial}{\partial t} \right) \quad \forall p \in O_{\phi^t_\chi}$$

Then the vector field is a symplectic vector field if and only if the one-parameter subgroup of diffeomorphism preserves the symplectic structure

$$(\phi^t_\chi)^*(\Omega) = \Omega \quad \forall t$$

equivalently χ is symplectic if the Lie derivative of Ω along χ is zero,

$$L_\chi(\Omega) = 0$$

And by Cartan's formula,(Since $d\Omega = 0$) this is true if and only if,

$$d \circ i_\chi(\Omega) = 0$$

i.e the one form $i_\chi(\Omega)$ we get by interior multiplication of χ on Ω is closed.

Hamiltonian vector field are the symplectic vector field χ_f that we get from a smooth function $f \in C^\infty(A)$ such that it satisfies,

$$i_{\chi_f}(\Omega) = df$$

In other words, a vector field χ is Hamiltonian if and only if $i_\chi(\Omega)$ is exact.

As one can see, if the first De Rham cohomology class ($H^1(A)$) is zero, which is true for vector spaces, then the concept of symplectic and Hamiltonian vector field is the same, but it can differ when $H^1(A)$ is non-zero.

1.4 Symplectic and Hamiltonian Group Actions

Suppose we have a symplectic manifold with a Lie group action. What if the fundamental vector fields produced by the Lie algebra of the Lie group are symplectic or Hamiltonian?

Let (A, Ω_A) be a symplectic manifold, and G be a Lie group acting smoothly on A. We denote the action of a group element $g \in G$ on a point $p \in A$ by $g \cdot p$, and the corresponding diffeomorphism on A by $\Psi_g : A \to A$, where $\Psi_g(p) = g \cdot p$.

Associated with the action of G on A, there is a natural Lie algebra homomorphism from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra of vector fields on A. For each element $X \in \mathfrak{g}$, we define the fundamental vector field $X^\#$ on A pointwise by taking the derivative of the action along the one-parameter subgroup $\exp(tX) \subset G$:

$$X_p^\# = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot p \quad \text{for } p \in A.$$

1.4.1 Symplectic Actions

Definition 1.4.1 (Symplectic Action)**.** *An action of a Lie group G on a symplectic manifold (A, Ω_A) is called a symplectic action (or an action by symplectomorphisms) if the action preserves the symplectic form:*

$$\Psi_g^* \Omega_A = \Omega_A \quad \textit{for all } g \in G.$$

Each of the fundamental vector fields $X^\#$ produced by the symplectic group action will be symplectic in nature, since for $X^{\#}$, $\Psi_{exp(tX)}$ will be the one-parameter subgroup of symplectomorphisms.

1.4.2 Hamiltonian Actions and the Moment Map

A stronger condition than the action being symplectic is that the action is Hamiltonian. More precisely, each $X^{\#}$ is a Hamiltonian vector field, and there Hamiltonian function varies smoothly with $X^{\#}$ in some sense. This requires that the closed 1-forms $\iota_{X^{\#}}\Omega_A$ are exact, and that the chosen Hamiltonian functions piece together to form an equivariant map to the dual of the Lie algebra, $\mathfrak{g}^*$.

Definition 1.4.2 (Hamiltonian Action and Moment Map)**.** *A symplectic action of a Lie group G on a symplectic manifold (A, Ω_A) is called a Hamiltonian action if there exists a smooth map $\mu : A \to \mathfrak{g}^*$, called a* ***moment map*** *(or momentum map), satisfying the following two conditions:*

1. ***Hamiltonian Condition:*** *For each $X \in \mathfrak{g}$, let $\mu^X : A \to \mathbb{R}$ be the component of μ along X, defined by $\mu^X(p) = \langle \mu(p), X \rangle$. Then μ^X is a Hamiltonian function for the fundamental vector field $X^{\#}$. That is,*

$$d\mu^X = \iota_{X^{\#}}\Omega_A.$$

2. ***Equivariance Condition:*** *The map μ is equivariant with respect to the given action of G on A and the coadjoint action of G on $\mathfrak{g}^*$. That is, for all $g \in G$ and $p \in A$,*

$$\mu(g \cdot p) = \mathrm{Ad}_g^*(\mu(p)),$$

where Ad_g^* is the coadjoint representation of G.

The existence of a moment map is deeply tied to the topology of the manifold A and the algebraic structure of the Lie group G. For example, if the Lie algebra $\mathfrak{g}$ is semisimple, or if the first de Rham cohomology group $H^1(A;\mathbb{R}) = 0$, then every symplectic action on A admits a moment map. When G is a torus $\mathbb{T}^n$, the coadjoint action is trivial, and the equivariance condition simplifies to the moment map μ being strictly invariant under the $\mathbb{T}^n$-action: $\mu(t \cdot p) = \mu(p)$ for all $t \in \mathbb{T}^n$.

1.5 Symplectic Reduction

As we have seen in symplectic reduction in case of symplectic vector spaces, we will generalise the concept to manifolds. But the problem is that the quotient of a given coisotropic submanifold of a symplectic manifold with any isotropic distribution won't necessarily result in a symplectic manifold.

Let (A, Ω_A) be the symplectic manifold and C be a coisotropic submanifold of it. Let I be the isotropic distribution on C,

$$I_p = T_pC^{\Omega_A} \subset T_PC$$

For the quotient space to be a manifold, we first need the distribution to be integrable to have leaf submanifolds from which we define our equivalence classes as points in the quotient space. It turns out that any such isotropic distribution, as described above, is integrable on its coisotropic submanifold C.

Lemma 1.5.1 ([Mc]). *For every coisotropic submanifold $C \subset A$, the isotropic distribution I is integrable.*

Now we will define an equivalence class on C where p and q belong to the same class, denoted by $[p]$, if and only if there exists a smooth path joining them such that the tangents to the path lie in the distribution, or in other words, they lie on the same leaf.

$$\gamma : [0, 1] \to C \quad \gamma(0) = p \quad \gamma(1) = q \quad \dot{\gamma}(t) \in I_{\gamma(t)} = T_{\gamma(t)}C^{\Omega_A}$$

So we have a quotient map from C to the set of equivalence classes $\overline{C}$.

$$Q : C \to \overline{C}$$

Next, we want our Q to be a submersion and $\overline{C}$ to be a smooth manifold. A sufficient condition is that the isotropic distribution be regular on C. By regularity, we mean that around every point $p \in C$ there exists a slice submanifold S which cuts the leaf submanifold at most once and transversely ($T_pC = I_p \oplus T_pS$) and the quotient space is Hausdorff.

This is a sufficient condition for $\overline{C}$ to have a unique smooth structure such that Q is a submersion. Under this smooth structure, $\overline{S} = \{[p] | p \in S\}$ is the open set of $\overline{C}$ where S is a slice as described above. If one notices, $\overline{S}$ is the set of all leaves that pass through the local slice S. Also, the tangent space at each point of $\overline{S}$ and thus $\overline{C}$ can be canonically identified with the quotient of the tangent space of C with the isotropic subspace we get from the isotropic distribution.

$$T_{[p]}\overline{C} \cong T_pC/I_p$$

Now comes the induced symplectic structure $\overline{\Omega}_C$ on $\overline{C}$. We describe it pointwise by Lemma 2.1.7, Mc Duff[Mc], and the collection of these pointwise skew symmetric bilinear map defines a unique symplectic structure such that its pull back by Q equals $\Omega_A|_C$

$$Q^*\overline{\Omega} = \Omega_A|_C$$

The Marsden reduction theorem is more concerned with Hamiltonian group action on a symplectic manifold. The gist is that every Hamiltonian group action provides us with a coisotropic submanifold and a very nice isotropic distribution that satisfies all the above conditions.

1.6 Delzant construction

Theorem 1.6.1 (Marsden-Weinstein-Meyer). *[[Ana]] Let (A, Ω_A) be a symplectic manifold equipped with a Hamiltonian action of a Lie group G, and let $\mu : A \to \mathfrak{g}^*$ be the corresponding moment map. Let $0 \in \mathfrak{g}^*$ be a regular value of μ. If the action of G restricted to the level set $\mu^{-1}(0)$ is free and proper, then the orbit space (the symplectic quotient):*

$$A_{red} = \mu^{-1}(0)/G$$

is a smooth manifold. Moreover, A_{red} inherits a natural symplectic form $\Omega_{A_{red}}$ satisfying:

$$i^*\Omega_A = \pi^*\Omega_{A_{red}}$$

where $i : \mu^{-1}(0) \hookrightarrow A$ is the inclusion map, and $\pi : \mu^{-1}(0) \to A_{red}$ is the quotient projection.

The theorem guarantees that if we restrict to a regular level set of the moment map, the resulant submanifold is coisotropic and the directions generated by the group action provide us with an isotropic distribution.

A special class of symplectic manifolds with a Hamiltonian action is when the Lie group acting is a torus.

Definition 1.6.2 (Symplectic Toric Manifold)**.** *A **symplectic toric manifold** is a compact connected symplectic manifold (A, Ω_A) of dimension $2n$, equipped with an Hamiltonian action of an n-dimensional torus $\mathbb{T}^n$.*

When the Lie group is torus, the dual of its Lie algebra is canonically isomorphic to euclidian space.

Theorem 1.6.3 (Atiyah-Guillemin-Sternberg Convexity Theorem)**.** *[[Ana]] Let (A, Ω_A) be a compact, connected symplectic manifold equipped with a Hamiltonian action of a compact torus $\mathbb{T}^n$. Let $\mu : A \to \mathbb{R}^n \cong (\mathfrak{t}^n)^*$ be the moment map where $(\mathfrak{t}^n)^*$ is dual of the Lie algebra of the Lie group $\mathbb{T}^n$. Then:*

1. *The level sets of μ are connected.*

2. *The image of the moment map, $\Delta = \mu(A)$, is a **convex polytope** in $\mathbb{R}^n$.*

3. *Furthermore, Δ is precisely the convex hull of the images of the fixed points of the torus action: $\Delta = \mathrm{conv}(\mu(A^{\mathbb{T}^n}))$.*

Delzant classified compact symplectic toric manifolds using a special class of polytopes called delzant polytopes.

Definition 1.6.4 (Delzant Polytope)**.** *A convex polytope $\Delta \subset \mathbb{R}^n$ is called a **Delzant polytope** if it satisfies the following conditions:*

1. ***Simple:** There are exactly n edges meeting at every vertex.*

2. ***Rational:** The edges meeting at the vertex v are of the form $v + tu_i$, where $t \geq 0$ and $u_i \in \mathbb{Z}^n$.*

3. ***Smooth:** The primitive integer edge directions $u_1, \ldots, u_n$ at any vertex form a basis for the lattice $\mathbb{Z}^n$.*

Delzant established a one-to-one correspondence between symplectic toric manifolds and Delzant polytope.

Theorem 1.6.5 (Delzant Classifiaction)**.** *[[Ana]] Symplectic toric manifolds are classified up to equivariant symplectomorphism by their moment polytopes. Two symplectic toric manifolds are equivariantly symplectomorphic if and only if their corresponding Delzant polytopes* *differ by a translation. Furthermore, every Delzant polytope arises in this way.*

Chapter 2

Poisson manifold

In symplectic geometry, we have a non-degenerate bivector satisfying the Jacobian condition, but what if we relax the condition of non-degeneracy?

Definition 2.0.1. - *A Poisson manifold P is a manifold with a bivector called the Poisson bivector.*

$$\Pi = \{,\} : C^{\infty}(P) \times C^{\infty}(P) \to C^{\infty}(P)$$

that satisfies the Jacobi condition

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

Now this bivector provides us with a bundle map and encodes an infinitesimal action of the Lie group $(C^{\infty}(P), \{,\})$ on P.

$$\Pi : T^{*}P \to TP$$

$$\{,\} : C^{\infty}(P) \to (P) \quad f \to X_f$$

And at each point, the tangent vectors belonging to the image of Π_p are called Hamiltonian directions. For each smooth function $f \in C^{\infty}(P)$, X_f is called a Hamiltonian vector field. The whole concept is similar as we get in symplectic geometry but more general.

The Jacobi identity makes the map $\{,\}$ a Lie algebra anti-homomorphism.

$$\{\{f, g\},\} = X_{\{f,g\}} = -[X_f, X_g]$$

and it makes the set of Hamiltonian vector fields ($Ham(\mathfrak{X}(M))$) a Lie sub-algebra of the algebra of vector fields.

Now, similar to symplectic vector fields that preserve the symplectic structure, we have Poisson vector fields that preserve the Poisson bivector.

Definition 2.0.2. - $X \in \mathfrak{X}(P)$ *is a called a Poisson vector field if*

$$L_X(\Pi) = 0$$

Equivalently

$$X\{f, g\} = \{Xf, g\} + \{f, Xg\} \quad \forall f, g \in C^\infty(P)$$

By simple calculation, one can show that the set of Poisson vector fields ($Pois(\mathfrak{X}(M))$) is a Lie sub-algebra of $\mathfrak{X}(M)$ and $Ham(\mathfrak{X}(M))$ lie inside it as a Lie sub-algebra.

$$Ham(\mathfrak{X}(M)) \subset Pois(\mathfrak{X}(M)) \subset \mathfrak{X}(M)$$

Since the bivector Π can be degenerate, there may exist non-constant functions whose differentials lie entirely in the kernel of $\Pi^\sharp : T^*P \to TP$. These functions are called Casimir functions.

Definition 2.0.3 (Casimir Function)**.** *A smooth function $C \in C^\infty(P)$ on a Poisson manifold (P, Π) is called a* ***Casimir function*** *if its Poisson bracket with every other smooth function on P vanishes:*

$$\{C, f\} = \Pi(dC, df) = 0 \quad \textit{for all } f \in C^\infty(P).$$

equivalently $X_C = 0$

If f and g are two Casimir functions, then $f.g$ and obviously $\Pi(f, g)$ are also Casimir functions. Thus, the set of Casimir functions is a sub-algebra of the algebra of smooth functions on P. Let's denote this subalgebra as $C(P, \Pi)$.

2.1 Symplectic Foliation

Theorem 2.1.1 ([Cran]). *-(Weinstein splitting theorem)-Let (P, Π) be a Poisson manifold of dimension (2n+r). For each point $p \in P$ there exist an open chart $(U_p, x_1, .., x_n, y_1, .., y_n, q_1, .., q_r)$ such that $q(p) = (q_1(p), \dots, q_r(p)) = 0$ and*

$$\Pi|_{U_p} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{1 \leq j < k \leq r} \pi_{jk}(q) \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial q_j}$$

where π_{jk} are smooth functions of q_i's. such that $\pi_{jk}(0) = \pi_{jk}(q(p)) = 0$. This is the generalisation of Darboux charts of a symplectic manifold.
So if needed, we could shrink U_p such that it is diffeomorphic to $V \times W$ where V and W are open neighbourhoods around the origin in $\mathbb{R}^{2n}$ and $\mathbb{R}^r$, respectively.

$$D(U_p) = V \times W$$

Let $S_p \subset U_p = \{s \in U_p | q(s) = 0\}$. Thus $S_p = D^{-1}(V \times \{0\})$ and on this submanifold,

$$\Pi|_{S_p} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Thus S_p is a regular submanifold of P such that the Poisson bivector restricted to S_p is non-degenerate and induces a symplectic form on it. So for each $p \in P$ there exists a submanifold with an induced symplectic structure containing p. Now to get the maximal immersed submanifold such that the Poisson bivector restricted to it is non-degenerate and p lies inside of it, we have two use equivalent methods

First, we can define an equivalence class of points in P. $x \sim y$ if and only if there exists a piece-wise smooth path between them such that tangent vectors to the path are made up of Hamiltonian directions.

$$x \sim y \iff \exists g_1, .., g_k \in C^\infty(M) \quad \text{such that} \quad \phi_{X_{g_k}} \circ \dots \circ \phi_{X_{g_1}}(x) = y$$

where the $\phi_{X_{g_i}}$ is the flow produced by the Hamiltonian vector field X_{g_i}. The equivalence class, often called orbits, provides us with the maximal immersed connected submanifold with induced symplectic structure containing x.

Second is by foliation. The distribution produced by the Hamiltonian directions is involutive and thus integrable. The integrability allows us to foliate the entire manifold into maximal connected immersed submanifolds, and the Poisson structure on P induces a symplectic structure on these immersed submanifolds. Let's call it symplectic foliation.

Definition 2.1.2. - *A symplectic leaf is an orbit S of M together with an induced symplectic structure ω_S. The foliation with symplectic leaves as its foliated submanifold is called a symplectic foliation.*

$$F_\Pi = \{(S, \omega_S) | S \textit{ is a sympletic leaf}\}$$

2.2 Schouten Bracket

Some concepts in Poisson geometry can be studied using poly-vector fields and a Lie bracket defined on the algebra of poly-vector fields. This bracket is called the Schouten bracket, which is when restricted on the Lie algebra of vector fields is same as the Lie bracket on $\mathfrak{X}(P)$

Firstly, the way we define a poly-vector field or a multivector field is very similar to the way we define a differential form. A k degree form $\alpha \in \Omega^k(P) = \Gamma(\wedge^k(T^*P))$ is a $C^\infty(P)$ multilinear alternating map on $\mathfrak{X}(P)$.

$$\alpha : \mathfrak{X}(P) \times \cdots \times \mathfrak{X}(P) \to C^\infty(P)$$

Similarly a k degree multi-vector field $X^k \in \mathfrak{X}^k(P) = \Gamma(\wedge^k(TP))$ is a $C^\infty(P)$ multilinear alternating map on $\Omega^1(P)$ the space of one form

$$X^k : \Omega^1(P) \times \cdots \times \Omega^1(P) \to C^\infty(P)$$

It can also be identified as an alternating multiderivation on $C^\infty(M)$

$$L_{X^k} : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M)$$

$$L_{X^k}(f_1,..f_k) = X^k(df_1,..df_k)$$

$$L_{X^k}(f_1,..,f_rg_r,..,f_k) = f_rL_{X^k}(f_1,..,g_r,..,f_k) + g_rL_{X^k}(f_1,..,f_r,..,f_k)$$

Basically, there is a one-to-one correspondence between a multivector fields and alternating derivatives.

From this, we can define the Schouten bracket over the graded commutative algebra of multivector fields.

$$\mathfrak{X}^\circ(M) = \oplus_{i=1}^m \mathfrak{X}^i(M)$$

Definition 2.2.1. - *Given* $\alpha \in \mathfrak{X}^{k+1}(M)$ *and* $\beta \in \mathfrak{X}^{l+1}(M)$, *then* $[\alpha, \beta] \in \mathfrak{X}^{k+l+1}(M)$ *is* $(k+l+1)$ *order multivector field such that*

$$L_{[\alpha,\beta]} = L_\alpha \circ L_\beta - (-1)^{kl} L_\beta \circ L_\alpha$$

where

$$L_\alpha \circ L_\beta(f_1, \ldots, f_{k+l+1}) = \sum_{\sigma \in S_{k,l+1}} (-1)^\sigma L_\alpha(f_{\sigma(1)},..,f_{\sigma(k)}, L_\beta(f_{\sigma(k+1)}, \ldots, f_{\sigma(k+l+1)}))$$

The definition is inspired by the ways we define the Lie bracket over vector fields

$$L_{[X,Y]} = L_XL_Y - L_YL_X$$

where X and Y are vector fields

So we can redefine a few things.
Define Jacobi map $J : C^\infty(\mathbb{C}^m) \times C^\infty(\mathbb{C}^m) \times C^\infty(\mathbb{C}^m) \to C^\infty(\mathbb{C}^m)$

$$J(f, g, h) = \{f, \{g, h\}\} + \{h\{f, g\}\} + \{g, \{h, f\}\}$$

By simple calculations, $\frac{1}{2}J(f, g, h) = [\Pi, \Pi](f, g, h)$. So the condition of Jacobi identity is to $[\Pi, \Pi] = 0$.

A Vector field X is Poisson if and only if $[X, \Pi] = 0$.

2.2.1 Poisson Cohomology

Let (P, Π) be a Poisson manifold, where $\Pi \in \mathfrak{X}^2(P)$ is the Poisson bivector field satisfying $[\Pi, \Pi] = 0$ with respect to the Schouten bracket.

The Poisson structure Π induces a differential on the graded algebra of multivector fields $\mathfrak{X}^\bullet(P) = \bigoplus_k \Gamma(\bigwedge^k TP)$. This differential, called the Poisson coboundary operator, is denoted by $d_\Pi \colon \mathfrak{X}^k(P) \to \mathfrak{X}^{k+1}(P)$ and is defined by taking the Schouten bracket with the Poisson bivector:

$$d_\Pi(P) = [\Pi, P] \tag{2.1}$$

for any $X \in \mathfrak{X}^k(P)$.

By the graded Jacobi identity of the Schouten bracket and the integrability condition $[\Pi, \Pi] = 0$, taking the differential twice yields:

$$d_\Pi(d_\Pi(X)) = [\Pi, [\Pi, X]] = \frac{1}{2}[[\Pi, \Pi], X] = 0. \tag{2.2}$$

Thus, $d_\Pi^2 = 0$, making $(\mathfrak{X}^\bullet(P), d_\Pi)$ a cochain complex, which is known as the Lichnerowicz-Poisson complex.

The cohomology of this complex is the *Poisson cohomology* of (P, Π), denoted by $H^\bullet_\Pi(P)$. The k-th Poisson cohomology group is formally defined as:

$$H^k_\Pi(P) = \frac{\ker(d_\Pi \colon \mathfrak{X}^k(P) \to \mathfrak{X}^{k+1}(P))}{\operatorname{im}(d_\Pi \colon \mathfrak{X}^{k-1}(P) \to \mathfrak{X}^k(P))}. \tag{2.3}$$

The lower degree Poisson cohomology groups have important geometric interpretations:

- $H^0_\Pi(P)$ consists of Casimir functions, which are functions that commute with all smooth functions under the Poisson bracket.
- $H^1_\Pi(P)$ is the quotient of the space of Poisson vector fields by the space of Hamiltonian vector fields, measuring the non-trivial symmetries of the Poisson structure.

2.3 Reduction Theory of Poisson Structure

Similar to Symplectic reduction, we have a more generalised theorem for Poisson reduction. Let $(P, \{\}_P)$ be a Poisson manifold, $M \subset P$ a submanifold and $i : M \to P$ the inclusion. Let G be a lie group acting on P and $\mathfrak{g}$ be the lie algebra of G. As explained in Section 1.4, each element $X \in \mathfrak{g}$ generates a fundamental vector field $X^{\#}$ pointwise. Thus at each point $p \in P$, we get $X_p^{\#} \in T_pP$ corresponding to X. The set

$$V_p = \{X_p^{\#} | X \in \mathfrak{g}\}$$

is a subspace of T_pP. One can also thing of V_p as the subspace of tangent vectors at p that are tangent to the orbits produced by the action of G on P. Let E be the distribution constructed by such tangent vectors. More precisely, E is a distribution on P such that $E_p = V_p$. We want the following assumptions to be true to state the condition to reduce the Poisson structure:

- (A1) $E \cap TM$ is an integrable distribution on TM, and thus it defines a foliation Φ on M.
- (A2) The space of orbits ($O = M/\Phi$) is a manifold with projection map $\pi : M \to O = M/\Phi$ a submersion, or in other words, the foliation Φ is regular.
- (A3)For $K, L \in C^\infty(P)$, if dK and dL vanishes on E, then $d\{K, L\}_P$ also vanishes on E.

Theorem 2.3.1 ([Mar]). - *Assuming (A1)-(A3), M/Φ assumes a unique Poisson structure induced by $\{\}_P$ if and only if*

$$B_x(E_x^0) \subset T_xM + E_x \quad \forall x \in M \tag{2.4}$$

where $E_x^0 = \{\alpha_x \in T_x^*P | \alpha_x(E_x) = 0\}$, $B_x : T_x^*P \to T_xP$ is the linear map induced by the bundle map $\{\}_P : T^*(P) \to T(P)$.

Now the Poisson structure induced on $(O, \{\}_O)$ is defined function-wise. For $f, g \in C^\infty(O)$ (or locally defined), we have a smooth extension (or locally defined extension) F, G of $f \circ \pi = \pi^*(f)$ and $g \circ \pi = \pi^*(g)$ with differential vanishing on E. In other words, we will have smooth extensions F and G that are G-invariant. So $\{f, g\}_O$ is smooth function uniquely defined as,

$$\pi^*\{f,g\}_O = i^*\{F,G\}_P$$

Intuitively speaking, the condition mentioned in the theorem makes the way we define $\{\}_O$ independent of what smooth extension we do, as long as the differential of the extended smooth function vanishes on E or it is G-invariant. Now let's see what constraints assumptions (A1)-(A3) put on the group action.

Since E is produced by a Lie group action, E is integrable, and since M is a submanifold, TM as a distribution on M is also integrable. Therefore, $E \cap TM$ is integrable and (A1) is true for any distribution produced by a group action. The foliation produced by an integrable distribution is regular if the group is free, so (A2) is true if the action is free.

Now what about (A3)? Let $f \in C^\infty(P)$ be a smooth function such that df vanishes on E, i.e, $df|_E = 0$. This is true if $df_p(v) = 0 \quad \forall v \in V_p \quad p \in P$. Thus $df(Y) = 0$ for each section Y of the distribution(a vector field with tangent vectors lying in the distribution). Since each fundamental vector field $X^\#$ is a section of E, $df(X^\#) = X(f) = 0 \quad \forall X \in \mathfrak{g}$. Now this becomes an if and only if statement since V_p as a subspace is generated by $X_p^\#$.

$$df_p(v) = 0 \quad \forall v \in V_p, p \in P \iff df(X^\#) = X^\#(f) = 0 \quad \forall X \in \mathfrak{g}$$

Thus (A3) can be stated as, if $K, L \in C^\infty(P)$ and $X^\#(K) = X^\#(L) = 0$ for all $X \in \mathfrak{g}$ then $X^\#(\{K,L\}_P) = 0$ for all $X \in \mathfrak{g}$. Let Π_P be the bivector field of the Poisson structure $\{\}_P$, then (A3) is true if,

$$L_{X^\#}\Pi = 0 \quad \forall X \in \mathfrak{g} \tag{2.5}$$

Since $(L_{X^\#}\Pi)(e,h) = [X^\#, \Pi](e,h) = X^\#(\Pi(e,h)) - \Pi(X^\# e, h) - \Pi(e, X^\# h)$(Schouten braket of $X^\#$ and Π) for all $e, h \in C^\infty(P)$. (2.5) is true if and only if $X^\#$ is a Poisson vector field on P for each $X \in \mathfrak{g}$. If for each $g \in G$, Ψ_g is a Poisson diffeomorphism

$$d\Psi_g(\Pi) = \Pi \tag{2.6}$$

then $exp(tX)$ are Poisson diffeomorphism for all t exponential map is defined for and for all $X \in \mathfrak{g}$.

Lemma 2.3.2. - Ψ_g *being a Poisson diffeomorphism for all* $g \in G$ *is a sufficient condition for (A3) to hold true.*

If any Poisson bivector on a Poisson manifold satisfies (2.6) under a group action on the Poisson manifold, then the bivector is said to be invariant under the group action.

2.4 Poisson Structures on $\mathbb{C}^m$

We have a standard Poisson structure on $\mathbb{C}^m$ which is non-degenerate.

$$\Pi_{st} = \sum_{q=1}^{m} \frac{\partial}{\partial x_q} \wedge \frac{\partial}{\partial y_q} = \frac{i}{2} \sum_{q=1}^{m} \frac{\partial}{\partial z_q} \wedge \frac{\partial}{\partial \overline{z}_q} \tag{2.7}$$

Take $\mathbb{C}_0^m = \{p \in \mathbb{C}^m | \exists i \in \{1,..,m\} \text{ such that } r_i(p) = 0\}$, where r_i is the i'th radial coordinate. Basically, $\mathbb{C}_0^m$ contains all those points where at least one $z_i = 0$ or equivalently at least one radial coordinate $r_i = 0$. Take $\mathbb{C}_1^m = \mathbb{C}^m - \mathbb{C}_0^m$.

$$\Pi_{st}|_{\mathbb{C}_1^m} = \sum_{q=1}^{m} 2 \frac{\partial}{\partial p_q} \wedge \frac{\partial}{\partial \theta_q}$$

Here x_i and y_i are standard euclidean coordinates on $\mathbb{C}^m$ and $p_i = r_i^2 = x_i^2 + y_i^2$ and θ_i are radial and angular coordinates on $\mathbb{C}_1^m$. We also have complex coordinates on $\mathbb{C}^m$ given as $z_i = x_i + \iota y_i$ and $\overline{z_i} = x_i - \iota y_i$.

Now we will construct several Poisson structures that will be useful in the following sections. Below is a list of bivector fields and the constraints on them by the Schouten bracket condition.

$$\Pi_1 = \sum_{1 \leq i < j \leq m} a_{ij} \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) \wedge \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right) \tag{2.8}$$

$$\Pi_1|_{\mathbb{C}_1^m} = \sum_{1 \leq i < j \leq m} a_{ij} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \theta_j}$$

where each $a_{ij} \in C^\infty(\mathbb{C}^m)$ and $(y_k \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial y_k})(a_{ij}) = 0 \quad \forall 1 \leq k \leq m$. In other words each a_{ij} is a function of only p_k where $k \in \{1,..,m\}$.

$$\Pi_2 = \Pi_{st} + \Pi_1$$

Let's first change the symbol for coefficient functions.

$$\Pi_2 = \sum_{q=1}^{m} \frac{\partial}{\partial x_q} \wedge \frac{\partial}{\partial y_q} + \sum_{1 \leq i < j \leq m} b_{ij} \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) \wedge \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right) \tag{2.9}$$

$$\Pi_2|_{\mathbb{C}^m_1} = \sum_{q=1}^{m} 2\frac{\partial}{\partial p_q} \wedge \frac{\partial}{\partial \theta_q} + \sum_{1\le i<j\le m} b_{ij}\frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \theta_j}$$

here $b_{ij} \in C^\infty(\mathbb{C}^m)$ are smooth functions such that $(y_k\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial y_k})(b_{ij}) = 0$ for all $k \in \{1, .., m\}$ and there are some additional condition. For simplicity, we will take a smaller subset of the solution space satisfying the Schouten bracket condition. So here we will only take those b_{ij}'s such that for each ij pair, b_{ij} is a C^∞ function that depends only on $p_i = r_i^2$ and $p_j = r_j^2$ or only on $x_i^2 + y_i^2$ and $x_j^2 + y_j^2$.

$$b_{ij} = b_{ij}(x_i^2 + y_i^2, x_j^2 + y_j^2)$$

$$\Pi_3 = \sum_{1\le i<j\le m} \frac{c_{ij}}{2}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right) \wedge \left(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right) \tag{2.10}$$

$$\Pi_3|_{\mathbb{C}^m_1} = \sum_{1\le i<j\le m} c_{ij}p_ip_j\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}$$

Again, we will have a differential equation from the Schouten bracket condition, but we will only take a subset of the solution space. So here c_{ij} are constant functions.

$$\Pi_4 = \Pi_3 + \Pi_1$$

$$\begin{aligned}\Pi_4 = &\sum_{1\le i<j\le m} \frac{e_{ij}}{2}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right) \wedge \left(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right)\\ &+ \sum_{1\le r<s\le m} d_{rs}\left(y_r\frac{\partial}{\partial x_r} - x_r\frac{\partial}{\partial y_r}\right) \wedge \left(y_s\frac{\partial}{\partial x_s} - x_s\frac{\partial}{\partial y_s}\right)\end{aligned} \tag{2.11}$$

$$\Pi_4|_{\mathbb{C}^m_1} = \sum_{1\le i<j\le m} e_{ij}p_ip_j\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} + \sum_{1\le r<s\le m} d_{rs}\frac{\partial}{\partial \theta_r} \wedge \frac{\partial}{\partial \theta_s}$$

where e_{ij} and d_{rs} are constant functions

Π_3 Π_4 contains some terms with only radial directions, but as per the theory explained in toric geometry, there exist no additional constraints on c_{ij}'s and e_{ij}'s such that (2.8) becomes reducible under Marsden-Ratiu reduction theory[Mar] with $P = \mathbb{C}^m$ and $M = \mathcal{Z}_{P_\Delta}$(moment angle manifold assiociated to P_Δ).
There exists another such pair of bivector field containing radial terms that could become reducible and induce a Poisson structure on the quotient space containing a smooth structure.

Start by choosing atmost 3 distinct numbers $i_1, i_2, i_3 \in \{1, \ldots, m\}$. Let $R = \{i_1, i_2, i_3\}$ be the set containing the chosen numbers. Define 2 bivector fields Π_3^R and Π_4^R on $\mathbb{C}^m$,

$$\Pi_3^R = \sum_{1 \le k < j \le 3} \frac{c^R_{i_j i_k}}{2} \left(x_{i_j} \frac{\partial}{\partial x_{i_j}} + y_{i_j} \frac{\partial}{\partial y_{i_j}} \right) \wedge \left(x_{i_k} \frac{\partial}{\partial x_{i_k}} + y_{i_k} \frac{\partial}{\partial y_{i_k}} \right) \tag{2.12}$$

$$\Pi_3^R|_{\mathbb{C}^m_1} = \sum_{1 \le k < j \le 3} c^R_{i_j i_k} p_{i_j} p_{i_k} \frac{\partial}{\partial p_{i_j}} \wedge \frac{\partial}{\partial p_{i_k}}$$

In this case, the sufficient condition on the coefficient functions to satisfy the Schouten bracket condition is,

$$c^R_{i_j i_k} = c^R_{i_j i_k}(x_l^2 + y_l^2) \in C^\infty(\mathbb{C}^m)$$

where $l \in \{1, \ldots, m\}$ and $l \neq i_j, i_k$. In other words, $\left(x_{i_j} \frac{\partial}{\partial x_{i_j}} + y_{i_j} \frac{\partial}{\partial y_{i_j}} \right) c^R_{i_j i_k} = \left(x_{i_k} \frac{\partial}{\partial x_{i_k}} + y_{i_k} \frac{\partial}{\partial y_{i_k}} \right) c^R_{i_j i_k} =$ 0 for all pairs $(k, j) \in \{(1,2), (2,3), (1,3)\}$.

$$\Pi_4^R = \Pi_3^R + \Pi_1$$

$$\begin{aligned} \Pi_4^R = & \sum_{1 \le k < j \le 3} \frac{e^R_{i_j i_k}}{2} \left(x_{i_j} \frac{\partial}{\partial x_{i_j}} + y_{i_j} \frac{\partial}{\partial y_{i_j}} \right) \wedge \left(x_{i_k} \frac{\partial}{\partial x_{i_k}} + y_{i_k} \frac{\partial}{\partial y_{i_k}} \right) \\ & + \sum_{1 \le r < s \le m} d_{rs} \left(y_r \frac{\partial}{\partial x_r} - x_r \frac{\partial}{\partial y_r} \right) \wedge \left(y_s \frac{\partial}{\partial x_s} - x_s \frac{\partial}{\partial y_s} \right) \end{aligned} \tag{2.13}$$

$$\Pi_4^R|_{\mathbb{C}^m_1} = \sum_{1 \le k < j \le 3} e^R_{i_j i_k} p_{i_j} p_{i_k} \frac{\partial}{\partial p_{i_j}} \wedge \frac{\partial}{\partial p_{i_k}} + \sum_{1 \le r < s \le m} d_{rs} \frac{\partial}{\partial \theta_r} \wedge \frac{\partial}{\partial \theta_s}$$

In this case, the sufficient condition on the coefficient functions to satisfy the Schouten bracket condition is,

$$e^R_{i_j i_k} = e^R_{i_j i_k}(x_l^2 + y_l^2) \in C^\infty(\mathbb{C}^m)$$

where $l \in \{1, \ldots, m\}$ and $l \neq i_j, i_k$. In other words, $\left(x_{i_j} \frac{\partial}{\partial x_{i_j}} + y_{i_j} \frac{\partial}{\partial y_{i_j}} \right) e^R_{i_j i_k} = \left(x_{i_k} \frac{\partial}{\partial x_{i_k}} + y_{i_k} \frac{\partial}{\partial y_{i_k}} \right) e^R_{i_j i_k} =$ 0 for all pairs $(k, j) \in \{(1,2), (2,3), (1,3)\}$ and d_{rs} are constant functions.

Proof. The proof is done computationally using Python. The code computed the conditions on coefficient functions when the trivector $([\Pi_i', \Pi_i'])$ is restricted on $\mathbb{C}^3_1$. We do it to use radial and angular coordinates. By smoothness, under the same conditions on coefficient functions, the trivector field $([\Pi_i', \Pi_i'])$ is zero on the entire $\mathbb{C}^3$. We can easily generalise the

entire result for $\mathbb{C}^m$. Similar computation was also done for $\mathbb{C}^4$ and $\mathbb{C}^5$. I have attached the Github link of the code below.
https://github.com/Lexion-god/Diffrential-Geometry/tree/main

$\square$

Chapter 3

Toric Geometry

3.1 Convex Polytopes

Let $P_\Delta \subset \mathbb{R}^n$ be an n-dimensional simple polytope with m facets $F_1, \ldots, F_m$. We present P_Δ as the intersection of m closed half-spaces in $\mathbb{R}^n$:

$$P_\Delta = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle + b_i \geq 0 \quad \text{for } i = 1, \ldots, m\}$$

where $a_i \in \mathbb{R}^n$ is the inward-pointing normal vector to the facet F_i, and $b_i \in \mathbb{R}$. We can also write this as a matrix inequality:

$$P_\Delta = \{x \in \mathbb{R}^n \mid Ax + \mathbf{b} \geq \mathbf{0}\}$$

where A is the $m \times n$ matrix whose rows are $a_1, \ldots, a_m$, and $\mathbf{b} \in \mathbb{R}^m$.

$$A = \begin{pmatrix} a_1 \\ .. \\ a_m \end{pmatrix} \quad a_i = (a_{i1}, .., a_{in})$$

Consider the affine map $i_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by:

$$i_A(x) = Ax + \mathbf{b}.$$

Since P_Δ has non-empty interior and normal vectors to its facets span $\mathbb{R}^n$, the matrix A has full column rank n. Thus, i_A is an affine embedding. The image $i_A(\mathbb{R}^n)$ is an n-dimensional affine plane traversing $\mathbb{R}^m$.

The condition $Ax + \mathbf{b} \geq \mathbf{0}$ interprets that the intersection of non-negative orthant $\mathbb{R}^m_+$ and the n dimensional plane $i_A(\mathbb{R}^n)$ is the embedded P_Δ in $\mathbb{R}^m$.

$$P_\Delta \cong i_A(\mathbb{R}^n) \cap \mathbb{R}^m_+.$$

Intuitively, instead of looking at P_Δ as bounded by flat walls in $\mathbb{R}^n$, we lift P_Δ into a higher-dimensional space $\mathbb{R}^m$, where it is simply a flat, n-dimensional diagonal "slice" through the corner of a room (the positive orthant).

Definition 3.1.1. *A convex polytope P_Δ of dimension n is **simple** if every vertex is the intersection of exactly n facets.*

Basically, at each vertex, n facets meet and from now on, we assume the polytope we deal with to be simple.

3.1.1 Gale Dual

How do we describe an n-dimensional affine plane $i_A(\mathbb{R}^n)$ inside $\mathbb{R}^m$? Any subspace of $\mathbb{R}^m$ can be described by specifying its orthogonal complement. An n-dimensional plane in $\mathbb{R}^m$ is defined by exactly $k = m - n$ linearly independent normal directions.

These normal directions are precisely what the Gale dual captures. Let Γ be a $k \times m$ matrix whose rows form a basis for the left null space (cokernel) of A.

$$\Gamma = \begin{pmatrix} \gamma_1 & .. & .. & \gamma_m \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ .. \\ \lambda_{m-n} \end{pmatrix}$$

That is, Γ is a matrix of full rank $k = m - n$ satisfying:

$$\Gamma A = 0 = A^T \Gamma^T.$$

Geometrically, the rows λ_i of Γ are orthogonal to the column space of A. Since the column space of A gives the directions of our affine plane $i_A(\mathbb{R}^n)$, the rows of Γ are exactly the normal vectors perpendicular to the embedded plane in $\mathbb{R}^m$.

Let $y \in \mathbb{R}^m$. When does a point y lie in our affine plane? It belongs to the plane if and only if its projection onto this orthogonal normal space is a specific constant. Specifically, if $y = Ax + \mathbf{b}$, multiplying by Γ yields:

$$\Gamma y = \Gamma A x + \Gamma \mathbf{b} = 0 + \Gamma \mathbf{b} = \mathbf{c}.$$

where $\mathbf{c} = \Gamma \mathbf{b} \in \mathbb{R}^{m-n}$.

So we have redefined our polytope while it is embedded in $\mathbb{R}^m$.

$$i_A(P_\Delta) = \{y \in \mathbb{R}^m_+ \mid \Gamma y = c\}. \tag{3.1}$$

The columns of Γ, denoted $\gamma_1, \ldots, \gamma_m \in \mathbb{R}^{m-n}$, forms the Gale dual A.

3.2 Moment-Angle Manifolds: Spinning in Complex Space

We have situated our polytope as a slice $i_A(P_\Delta) = \{y \in \mathbb{R}^m_{\geq 0} \mid \Gamma y = \mathbf{c}\}$ of the positive orthant.

Now, consider the space $\mathbb{C}^m$. The map $\pi : \mathbb{C}^m \to \mathbb{R}^m_+$ defined by taking the squared magnitude of each coordinate:

$$\pi(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$$

"folds" complex space down onto the positive orthant. To visualize this: over every point y strictly interior to the orthant ($y_i > 0$), the preimage $\pi^{-1}(y)$ is an m-dimensional torus $(S^1)^m$, where the radius of the i-th circle is $\sqrt{y_i}$. If a coordinate $y_i = 0$ (meaning we hit a wall of the orthant), the corresponding circle collapses to a point.

The moment-angle manifold $\mathcal{Z}_{P_\Delta}$ is simply the preimage of our polytope slice $i_A(P_\Delta)$ under this map:

$$\mathcal{Z}_{P_\Delta} = \pi^{-1}(i_A(P_\Delta)).$$

$$
\begin{tikzcd}
\mathcal{Z}_{P_\Delta} \arrow[r, "i"] \arrow[d, "\pi_{\mathcal{Z}_{P_\Delta}}"] & \mathbb{C}^m \arrow[d, "\pi"] \\
P_\Delta \arrow[r, "i_A"] & \mathbb{R}^m_+
\end{tikzcd}
$$

Intuitively, we take the flat n-dimensional polytope slice $P_\Delta \subset \mathbb{R}^m_+$ and "spin" it in complex space by restoring the m independent phase angles over every point. Where the polytope hits a facet F_i ($y_i = 0$), the corresponding circle pinches off, creating a smooth, closed topology.

By combining the Gale presentation of P_Δ with the map μ, we substitute $y_i = |z_i|^2$ directly into the linear equations $\Gamma y = \mathbf{c}$.

Theorem 3.2.1 (Buchstaber–Panov)**.** *[[Buc]] The moment-angle manifold $\mathcal{Z}_{P_\Delta}$ is exactly the intersection of $m-n$ real homogeneous quadrics (Hermitian quadrics) in $\mathbb{C}^m$, determined entirely by the Gale transform Γ and translation vector $\mathbf{c}$:*

$$\mathcal{Z}_{P_\Delta} = \left\{ \mathbf{z} \in \mathbb{C}^m \;\middle|\; \sum_{i=1}^{m} \Gamma_i |z_i|^2 = \mathbf{c} \right\}.$$

Because P_Δ is a simple polytope, these quadrics intersect transversally, guaranteeing that $\mathcal{Z}_{P_\Delta}$ is a smooth closed manifold of dimension $m+n$. Furthermore, the m-dimensional compact torus T^m acts naturally on $\mathcal{Z}_{P_\Delta}$ by rotating the phases of the coordinates z_i, and the orbit space modulo this action is exactly the original polytope P_Δ.

3.2.1 Quasitoric Manifolds and Symplectic Reduction

Having constructed the moment-angle manifold $\mathcal{Z}_{P_\Delta}$ with its natural T^m-action, we can construct a class of manifolds known as quasitoric manifolds. This is achieved by taking the quotient of $\mathcal{Z}_{P_\Delta}$ by a freely acting $(m-n)$ subtorus.

Let $P_\Delta \subset \mathbb{R}^n$ be a simple n polytope with m facets $F_1, \ldots, F_m$. A quasitoric manifold Q over P_Δ is a $2n$-dimensional compact smooth manifold with a locally standard action of the n-dimensional torus $\mathbb{T}^n$, such that the orbit space modulo this action is diffeomorphic, as a manifold with corners, to P_Δ.

The combinatorial data required to build Q is captured by a **characteristic map**. We assign to each facet F_i a primitive vector $\eta_i \in \mathbb{Z}^n$. This assignment extends to a surjective homomorphism of tori:

$$\ell : \mathbb{T}^m \to \mathbb{T}^n$$

where the standard basis vectors $\mathbf{e}_i$ of the Lie algebra of $\mathbb{T}^m$ are mapped to η_i. For the resulting torus action on the manifold to be locally standard, the characteristic vectors must satisfy the *smoothness condition*: if n facets $F_{i_1}, \dots, F_{i_n}$ intersect at a vertex of P_Δ, their corresponding vectors $\eta_{i_1}, \dots, \eta_{i_n}$ must form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. The $n \times m$ integer matrix η formed by these column vectors is called the **characteristic matrix**.

The kernel of the characteristic map ℓ defines a closed subtorus $H \subset \mathbb{T}^m$ of dimension $k = m - n$:

$$H = \ker(\ell) \cong \mathbb{T}^{m-n}.$$

The smoothness condition guarantees that the $m - n$ dimensional subtorus H acts **freely** on the moment-angle manifold $\mathcal{Z}_{P_\Delta}$.

Because the action of H on $\mathcal{Z}_{P_\Delta}$ is free and proper, the orbit space is a smooth manifold of dimension $(m + n) - (m - n) = 2n$. We define the quasitoric manifold directly as this quotient:

$$Q = \mathcal{Z}_{P_\Delta}/H.$$

The residual torus $T^m/H \cong \mathbb{T}^n$ naturally acts on Q, and the orbit space is simply $(\mathcal{Z}_{P_\Delta}/H)/(T^m/H) \cong \mathcal{Z}_{P_\Delta}/\mathbb{T}^m \cong P_\Delta$.

Marsden-Weinstein-Meyer theorem (symplectic reduction) allows us to induce symplectic structure on many quasitoric manifolds (with a valid characteristic map).

3.3 Free Torus Action

In toric geometry, we construct action of $m - n$ torus $\mathbb{T}^{m-n}$ on the moment-angle manifold $\mathcal{Z}_{P_\Delta}$ such that it is free. The freeness leads the quotient map

$$\pi : \mathcal{Z}_{P_\Delta} \to O = \mathcal{Z}_{P_\Delta}/\mathbb{T}^{m-n}$$

to be a smooth submersion. We will explore the family of subtorus of $\mathbb{T}^m$ (with a natural action on $\mathcal{Z}_{P_\Delta}$) that are free on $\mathcal{Z}_{P_\Delta}$. Now, any d-dimensional subgroup of $\mathbb{T}^m$ looks like

$$\mathbb{T}^d_\Delta = \{(e^{2\pi i<\delta_1,\phi>}, \dots, e^{2\pi i<\delta_m,\phi>}) \in \mathbb{T}^m\}$$

where $\phi = (\phi_1, \dots, \phi_d) \in \mathbb{R}^d$ is an d dimensional parameter and $\Delta = (\delta_1, .., \delta_m)$ is a set of m vectors in $\mathbb{R}^d$. Denote any element of $\mathbb{T}^d_\Delta$ as $g(\phi) = (e^{2\pi i<\delta_1,\phi>}, \dots, e^{2\pi i<\delta_m,\phi>})$

Now $\mathbb{T}^d_\Delta$ is a subtorus if the set of m vectors satisfies the following equivalent condition : Define the linear map,

$$L : \mathbb{R}^d \to \mathbb{R}^m$$

$$[L] = \begin{bmatrix} \delta_1 \\ .. \\ .. \\ \delta_m \end{bmatrix} = \begin{bmatrix} \mu_1 & \dots & \dots & \mu_d \end{bmatrix}$$

1) There exist $\mathbb{R}$-linearly independent $z_1, .., z_d \in \mathbb{Z}^m$ such that each $z_i \in \mathrm{Im}(L)$
2) There exist linearly independent vectors $v_1, ..., v_d \in \mathbb{R}^m$ such that they form the basis of $L(\mathbb{R}^d)$ and each vector lies in $\mathbb{Q}^m$.

Now comes the freeness part. It turns out that the freeness of the group action of $\mathbb{T}^d_\Delta$ on $\mathcal{Z}_{P_\Delta}$ does not depend on the geometric realisation of the polytope, or in other words, the normal vectors to the facets of our moment polytope. It only depends on the combinatorics of the polytope, or in other words, which edge meets where.

So the idea is suppose we have $z \in \mathcal{Z}_{P_\Delta}$ such that $z = (0, .., 0, z_{r+1}, .., z_m)$ with $z_i \neq 0 \quad \forall r + 1 \leq i \leq m$ and $\mathbb{T}^d_\Delta$ is acting on it. Now, given any $g \in \mathbb{T}^d_\Delta$, no matter what g looks like, the first r components of g will act trivially on the first r components of z. So how g will act on z depends only on its last $m - r$ components, and thus it will act trivially on z if and only if the latter $m - r$ components are identity, regardless of what the first r components are. For $\mathbb{T}^d_\Delta$ to act freely on z, if there is an element g such that its later $m - r$ components are zero, then necessarily its first r component needs to be zero. So that only identity will act trivially on z.

Suppose, there exist a ϕ such that $< \delta_j, \phi > \in \mathbb{Z} \quad \forall m \geq j \geq r+1$. So for freeness whenever there exist such ϕ, then $< \delta_i, \phi >$ must be an integer for all $1 \leq i \leq r$.

Now we will expand the idea for each $m \in \mathcal{Z}_{P_\Delta}$.

We define a stratification of $\mathcal{Z}_{P_\Delta}$ with the help of the moment polytope:

$$\begin{tikzcd} \mathcal{Z}_{P_\Delta} \arrow[r, "i"] \arrow[d, "\pi_{\mathcal{Z}_{P_\Delta}}"] & \mathbb{C}^m \arrow[d, "\pi"] \\ P_\Delta \arrow[r, "i_A"] & \mathbb{R}^m_+ \end{tikzcd}$$

Let P_Δ be an n-polytope in $\mathbb{R}^n$ described my m equations,

$$P_\Delta : \{x \in \mathbb{R}^n | < x, a_i > + b_i \geq 0, 1 \leq i \leq m\}$$

Define a stratification of P_Δ by its faces. Let F be a face of P_Δ with $\dim F = n - r$. Then F is characterized (as a subset of P_Δ) by r equations

$$\langle x, a_i \rangle + b_i = 0, \quad i = i_1, \ldots, i_r.$$

We write $F = F_I$ where $I = (i_1, \ldots, i_r)$ and $1 \leq i_1 < i_2 \ldots < i_r \leq m$. One can also think of $i_A(F_I)$ as the face we get from the intersection of $i_A(P_\Delta)$ and the plane $Pl_I = \{y \in \mathbb{R}^m_+ | y_i = 0 \quad i \in I\}$. Let $z = (z_1, \ldots, z_m) \in M$ such that $z \in \pi^{-1}(i_A(interior(F_I)) = N_I$. Since for all such F_I, N_I are closed under T^m action, and so the stratification of $\mathcal{Z}_{P_\Delta}$ is just the stratification of $\mathcal{Z}_{P_\Delta}$ into $\mathbb{T}^d_\Delta$ orbit types. More specifically, the stabilizer of N_I is

$$Stabilizer(z) = \mathbb{T}_I = \{(e^{2\pi i t_1}, \ldots, e^{2\pi i t_m}) \mid e^{2\pi i t_s} = 1, \forall s \notin I\}$$

since if $z \in N_I$ then $z_i = 0 \iff i \in I$

So $\mathbb{T}^d_\Delta$ will act freely on $\mathcal{Z}_{P_\Delta} \iff \mathbb{T}^d_\Delta \cap \mathbb{T}_I = (1, \ldots, 1) = e \; \forall I$ such that F_I is a face of P_Δ. For $I = (i_1, \ldots, i_r)$, define J as its compliment or more precisely let $J = (j_1, \ldots, j_{m-r})$ and

$1 \leq j_1 < .. < j_{m-r} \leq m$ such that no j_k lies in I. Define two matrices with certain δ_i's as column vectors.

$$\Delta_I^c = \begin{pmatrix} \delta_{j_1} & \dots & \delta_{j_{m-r}} \end{pmatrix} = \begin{pmatrix} w_1 \\ .. \\ w_d \end{pmatrix}$$

$$\Delta_I = \begin{pmatrix} \delta_{i_1} & \dots & \delta_{i_r} \end{pmatrix} = \begin{pmatrix} u_1 \\ .. \\ u_d \end{pmatrix}$$

Theorem 3.3.1. - $\mathbb{T}^d_\Delta \cap \mathbb{T}_I = (1, \dots, 1) = e$ *if and only if , whenever there exist* $(\phi_1, \dots, \phi_d) \in \mathbb{R}^d$ *such that* $\sum_i \phi_i w_i \in \mathbb{Z}^{m-r}$ *then* $\sum_i \phi_i u_i \in \mathbb{Z}^r$.

Proof. - Assume $\mathbb{T}^d_\Delta \cap \mathbb{T}_I = (1, \dots, 1) = e$ and suppose there exist ϕ such that $\sum_i \phi_i w_i \in \mathbb{Z}^{m-r}$. Now this is equivalent $< \phi, \delta_{j_k} > \in \mathbb{Z}$ for all $j_k \notin I$. That means the corresponding $g(\phi) \in \mathbb{T}^d_\Delta$ also belongs in $\mathbb{T}_I$. By the above assumption, $g(\phi) = e$ and thus $< \phi, \delta_{i_k} > \in \mathbb{Z}$ for all $i_k \in I$ which is equivalent to $\sum_i \phi_i u_i \in \mathbb{Z}^r$.
Now assume the latter and let $g(\phi) \in \mathbb{T}^d_\Delta \cap \mathbb{T}_I$. Since $g(\phi) \in \mathbb{T}_I$, the $j_k \notin I$ component of $g(\phi)$ will be identity and thus $< \phi, \delta_{j_k} > \in \mathbb{Z}$ for all $j_k \notin I$. This implies $\sum_i \phi_i w_i \in \mathbb{Z}^{m-r}$ and by assumption $\sum_i \phi_i u_i \in \mathbb{Z}^r$. At last we get $< \phi, \delta_{i_k} > \in \mathbb{Z}$ for all $i_k \in I$ and so $g(\phi) = e$. □

$\mathbb{T}_I$ is largest when F_I is a vertex because $\mathbb{T}_J \subset \mathbb{T}_I$ is true if and only if $F_I \subset F_J$. Let v be the vertex face and $\mathbb{T}_v$ be the stabilizer subgroup of N_I corresponding to the vertex face v.

Proposition 3.3.2. $\mathbb{T}^d_\Delta$ *is free if and only if* $\mathbb{T}^d_\Delta \cap \mathbb{T}^d_v = (1, \dots, 1) = e \quad \forall$ *vertices* v *in* P_Δ.

Proof. If the action of $\mathbb{T}^d_\Delta$ is free then its intersection with each stabilizer subgroup $\mathbb{T}_I$ must be trivial. If the latter is true then since each stabilizer subgroup $\mathbb{T}_J$ lies within a stabilizer subgroup of a vertex, the intersection of $\mathbb{T}^d_\Delta$ with each stabilizer subgroup will be trivial and thus the action of $\mathbb{T}^d_\Delta$ will be free. □

3.4 Tangent Space of $\mathcal{Z}_{P_\Delta}$

Given the polytope P_Δ, let $\gamma_1, \ldots, \gamma_m \in \mathbb{R}^{m-n}$ be the gale dual of the normal vectors $a_1, \ldots, a_m \in \mathbb{R}^n$ and polytope constant $b = (b_1, .., b_m) \in \mathbb{R}^m$. Here

$$a_i^T = (a_{i1}, .., a_{in}) \qquad \gamma_j^T = (\gamma_{1j}, \ldots, \gamma_{(m-n)j})$$

Let's define two matrices,

$$A^T = \begin{pmatrix} a_1 & .. & .. & a_m \end{pmatrix} \qquad \Gamma = \begin{pmatrix} \gamma_1 & .. & .. & \gamma_m \end{pmatrix} \quad A^T \Gamma^T = 0$$

So $A^T : \mathbb{R}^m \to \mathbb{R}^n$ and $\Gamma : \mathbb{R}^m \to \mathbb{R}^{m-n}$ are two linear map and the rows of matrix A^T spans the $ker(\Gamma) = \Gamma_{ke}$ and vice versa. Then

$$\mathcal{Z}_{P_\Delta} = \{z \in \mathbb{C}^m | \quad \Gamma|z|^2 = \Gamma b = c\}$$

where $(|z|^2)^T = (|z_1|^2, .., |z_m|^2)$. Let

$$\mu : \mathbb{C}^m \to \mathbb{R}^{m-n} \qquad \mu(z) = \Gamma|z|^2$$

So $\mathcal{Z}_{P_\Delta} = \mu^{-1}(c)$ and $T_p\mathcal{Z}_{P_\Delta} = ker(d\mu_p) \subset T_p\mathbb{C}^m$ for $p \in \mathcal{Z}_{P_\Delta}$.

I will explain the tangent space by stratifying the manifold, and I will use a similar method as I have explained in Appendix B

Define the face E_I as interior of F_I and $N_I = \pi^{-1}(i_A(E_I))$ where $I = (i_1, \ldots, i_r)$ and $1 \le i_1 < i_2 \ldots < i_r \le m$. When I is the null set, then $r = 0$ and $E_I = E_0$ is the interior of the polytope P_Δ. Let $J = (j_1, .., j_{m-r})$ such that $1 \le j_1 < j_2 \ldots < j_{m-r} \le m$ and $j_i \not\in I$. By construction

$$N_I = \{z \in i(\mathcal{Z}_{P_\Delta}) | z_k = 0, z_l \neq 0 \quad k \in I, l \in J\}$$

From now on, we will assume $\mathcal{Z}_{P_\Delta}$ is canonically embedded in $\mathbb{C}^m$, and we will use the natural global chart of $\mathbb{C}^m$. So at $p = (x_1, y_1, \ldots, x_m, y_m) \in \mathcal{Z}_{P_\Delta}$,

$$(d\mu_p) = \begin{pmatrix} 2\gamma_{11}x_1 & 2\gamma_{11}y_1 & \ldots & 2\gamma_{1j}x_j & 2\gamma_{1j}y_j & \ldots & 2\gamma_{1m}x_m & 2\gamma_{1m}y_m \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 2\gamma_{(m-n)1}x_1 & 2\gamma_{(m-n)1}y_1 & \ldots & 2\gamma_{(m-n)j}x_j & 2\gamma_{(m-n)j}y_j & \ldots & 2\gamma_{(m-n)m}x_m & 2\gamma_{(m-n)m}y_m \end{pmatrix}$$

Define a family of linear maps indexed by I,

$$L_I : \mathbb{R}^{m-r} \to \mathbb{R}^{m-n}$$

$$(L_I) = \begin{pmatrix} \gamma_{j_1} & .. & \gamma_{j_i} & .. & \gamma_{j_{m-r}} \end{pmatrix}$$

where the column vectors are the specified Gale dual vectors.
By choice of the polytope P_Δ, if F_I is a face of P_Δ then the set of vectors $\{a_{i_1}, .., a_{i_r}\}$ are linearly independent. So by theorem 1.2.4 of [Buc], the set of vectors $\{\gamma_{j_1}, .., \gamma_{j_{m-r}}\}$ spans $\mathbb{R}^{m-n}$. By rank-nullity theorem, dimension of $K_I = ker(L_I)$ is $(n-r)$. Let $\{\nu_1^I, \ldots, \nu_{n-r}^I\}$ be the basis of K_I where $(\nu_j^I)^T = ({}_1\nu_j^I, .., {}_{(m-r)}\,\nu_j^I)$. Construct vectors $\mu_1^I, \ldots, \mu_{n-r}^I \in \mathbb{R}^m$,

$$(\mu_k^I)^T = ({}_1\mu_k^I, .., {}_m\,\mu_k^I)$$

where ${}_{i_\alpha}\mu_k^I = 0$, ${}_{j_\beta}\mu_k^I = {}_\beta\,\nu_k^I$ and $1 \leq \alpha \leq r, 1 \leq \beta \leq m-r$. Basically, we are adding zeros in the coordinate places with index number belonging to I, and the rest is from ν_j^I arranged in the same order.

Proposition 3.4.1. $\Gamma\mu_k^I = 0 \quad \forall k$. *Basically span*$\{\mu_1^I, \ldots, \mu_{n-r}^I\} \subset \Gamma_{ke}$.

Proof. - The proof is simple.

$$\Gamma\mu_k^I = \sum_{i=1}^{m}({}_i\mu_k^I\gamma_i) = \sum_{\alpha=1}^{r}({}_{i_\alpha}\mu_k^I\gamma_{i_\alpha}) + \sum_{\beta=1}^{m-r}({}_{j_\beta}\mu_k^I\gamma_{j_\beta}) = \sum_{\beta=1}^{m-r}({}_\beta\nu_k^I\gamma_{j_\beta}) = 0$$

□

Construct vectors $\{\chi_1^I, .., \chi_r^I, \rho_1^I, \ldots, \rho_r^I, \zeta_1^I, \ldots, \zeta_{n-r}^I, \phi_1^I, \ldots, \phi_{m-r}^I\} \in T_p\mathbb{C}^m$ where $p \in N_I$ (we will assume the derivatives below are at p unless specified otherwise),

$$\chi_k^I = \frac{\partial}{\partial x_{i_k}} \quad \rho_k^I = \frac{\partial}{\partial y_{i_k}} \quad \zeta_k^I = \sum_{l=1}^{m-r}\left({}_l\nu_k^I\left(\frac{x_{j_l}}{p_{j_l}}\frac{\partial}{\partial x_{j_l}} + \frac{y_{j_l}}{p_{j_l}}\frac{\partial}{\partial y_{j_l}}\right)\right) \quad \phi_k^I = \left(y_{j_k}\frac{\partial}{\partial x_{j_k}} - x_{j_k}\frac{\partial}{\partial y_{j_k}}\right)$$

Theorem 3.4.2. -$\{\chi_1^I, .., \chi_r^I, \rho_1^I, \ldots, \rho_r^I, \zeta_1^I, \ldots, \zeta_{n-r}^I, \phi_1^I, \ldots, \phi_{m-r}^I\}$ *forms the basis of* $T_p\mathcal{Z}_{P_\Delta}$.

Proof. - First, I will show that each of them belongs to the kernel of $d\mu_p$.

$$d\mu_p(\chi_k^I) = \begin{pmatrix} 0+0+..+2\gamma_{1i_k}x_{i_k}+0+..+0+0 \\ .... \\ 0+0+..+2\gamma_{(m-n)i_k}x_{i_k}+0+..+0+0 \end{pmatrix} = \begin{pmatrix} 0 \\ .. \\ 0 \end{pmatrix}$$

Since $i_k \in I \implies x_{i_k} = 0$ Similarly $d\mu_p(\rho_k^I) = 0$. For the next type,

$$d\mu_p(\zeta_k^I) = \begin{pmatrix} \sum_l ({}_l\nu_k^I \gamma_{1j_l}(x_{j_l}^2 + y_{j_l}^2)/p_{j_l}) \\ .... \\ \sum_l ({}_l\nu_k^I \gamma_{(m-n)j_l}(x_{j_l}^2 + y_{j_l}^2)/p_{j_l}) \end{pmatrix} = \begin{pmatrix} \sum_l ({}_l\nu_k^I \gamma_{1j_l}) \\ .... \\ \sum_l ({}_l\nu_k^I \gamma_{(m-n)j_l}) \end{pmatrix} = \sum_l ({}_l\nu_k^I \gamma_{j_l}) = \begin{pmatrix} 0 \\ .. \\ 0 \end{pmatrix}$$

for last

$$d\mu_p(\phi_k^I) = \begin{pmatrix} 0+0+..+2\gamma_{1j_k}x_{j_k}y_{j_k} - 2\gamma_{1j_k}y_{j_k}x_{j_k}+..+0+0 \\ .... \\ 0+0+..+2\gamma_{(m-n)j_k}x_{j_k}y_{j_k} - 2\gamma_{(m-n)j_k}y_{j_k}x_{j_k}+..+0+0 \end{pmatrix} = \begin{pmatrix} 0 \\ .. \\ 0 \end{pmatrix}$$

It's easy to show that the tangent vectors are linearly independent, and since the dimension of $\mathcal{Z}_{P_\Delta}$ is $m+n$, the set of tangent vectors spans the tangent space of $\mathcal{Z}_{P_\Delta}$. □

In the construction of our moment angle manifold, one can see that we only take points with modular vector ($|z| \in \mathbb{R}^m$) lying in the polytope. Hence, in the inverse image, the entire angular part that corresponds to that modular/radial part belongs to it.

Proposition 3.4.3. *Let K be a tuple like I such that $K \subset I$ and F_K is a Face. Then $E_I \subset F_K$ and span$(\zeta_1^I, .., \zeta_{n-r}^I) \subset T_p\mathcal{Z}_{P_\Delta}$ for all $p \in N_K$.*

Proof. - Well, the proof is similar to the above theorem, and one needs to just plug it in. □

Proposition 3.4.4. *Span$(\phi_1, \dots, \phi_m\}$ is a subspace of $T_p\mathcal{Z}_{P_\Delta} \quad \forall p \in \mathcal{Z}_{P_\Delta}$ where $\phi_i = y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}$.*

Proof. - If I show that $\phi_i \in ker((d\mu_p)) \quad \forall p \in \mathcal{Z}_{P_\Delta}$ then its enough. So for $p = (x_1, y_1, \dots, x_m, y_m) \in$

$\mathcal{Z}_{P_\Delta}$

$$[d\mu_p](\phi_i) = \begin{pmatrix} 0+0+..+2\gamma_{1i}x_iy_i - 2\gamma_{1i}y_ix_i + .. + 0 + 0 \\ .... \\ 0+0+..+2\gamma_{(m-n)i}x_iy_i - 2\gamma_{(m-n)i}y_ix_i + .. + 0 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ .. \\ 0 \end{pmatrix}$$

$\square$

If one notices,

$$\zeta_k^I = \sum_{l=1}^{m-r} 2\left({}_l\nu_k^I \frac{\partial}{\partial p_{j_l}} \right)$$

So if one labels $(p_1, \ldots p_m)$ as your standard coordinate in $\mathbb{R}^m$, then this construction is the same as constructing the tangent space of points in HC_I. Since E_I is a submanifold of HC_I in $\mathbb{R}^m$, the common points will have the same tangent space.

Chapter 4

Reducible Poisson Structure on $\mathbb{C}^m$

Finally, we will construct Poisson bivectors on $\mathbb{C}^m$ that are reducible under Marsden-Ratiu reduction theory[Mar] with $P = \mathbb{C}^m$ and $M = \mathcal{Z}_{P_\Delta}$(moment angle manifold assiociated to P_Δ). So to that they are reducible, as mentioned in section 2.3, we need to prove condition (A3) and condition (2.4) in theorem 2.3.1. Let, P_Δ be the moment polytope with normal vectors $a_1, \ldots, a_m \in \mathbb{R}^n$, polytope constant $b = (b_1, .., b_m) \in \mathbb{R}^m$ and $\gamma_1, \ldots, \gamma_m \in \mathbb{R}^{m-n}$ be the gale dual. Let $\mathcal{Z}_{P_\Delta}$ be the corresponding moment angle manifold. Firstly we will construct 4 families of bivector fields($\Pi_1', \Pi_2',_K \Pi_3^R,_K \Pi_4^R$) corresponding to P_Δ that are Poisson by theorem 2.4.1. Let

$$\mathbb{T}^d_\Delta = \{(e^{2\pi i<\delta_1,\phi>}, \ldots, e^{2\pi i<\delta_m,\phi>}) \in \mathbb{T}^m\}$$

be a d-dimensional subtorus of $\mathbb{T}^m$ acting freely on $\mathcal{Z}_{P_\Delta}$ and $E = E^d_\Delta$ be the distribution produced by the group action of $\mathbb{T}^d_\Delta$ on $\mathbb{C}^m$ as mentioned in section 2.3. As we have already seen in section 2.3 that this distribution satisfies conditions (A1) and (A2). By lemma 2.3.2, to satisfy condition (A3), its enough to show that for each $g' \in \mathbb{T}^d_\Delta$

$$d\Psi_g'(\Pi_1') = \Pi_1' \quad d\Psi_g'(\Pi_2') = \Pi_2' \quad d\Psi_g'({}_K\Pi_3^R) =_K \Pi_3^R \quad d\Psi_g'({}_K\Pi_4^R) =_K \Pi_4^R \tag{4.1}$$

In theroem 4.0.1 stated below, the first part of the proof will prove (4.1).
In the second part of the proof, I will show under what additional constraints on the polytope P_Δ and the subtorus $\mathbb{T}^d_\Delta$ will these bivector fields becomes reducible.

$$\Pi_1' = \sum_{1\le i<j\le m} \alpha_{ij}\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right)\wedge\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right) \tag{4.2}$$

where each $\alpha_{ij}\in C^\infty(\mathbb{C}^m)$ and $(y_k\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial y_k})(\alpha_{ij}) = 0 \quad \forall 1\le k\le m$. So the conditions on α_{ij}'s are same as a_{ij}'s. In other words each α_{ij} is a function of only p_k where $k\in\{1,..,m\}$.

$$\Pi_2' = \sum_{q=1}^{m}\frac{\partial}{\partial x_q}\wedge\frac{\partial}{\partial y_q} + \sum_{1\le i<j\le m}\beta_{ij}\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right)\wedge\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right) \tag{4.3}$$

here $\beta_{ij}\in C^\infty(\mathbb{C}^m)$ are smooth functions such that $(y_k\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial y_k})(\beta_{ij}) = 0$ for all $k\in\{1,..,m\}$ and there are some additional condition. For simplicity, we will take a smaller subset of the solution space satisfying the Schouten bracket condition. So here we will only take those β_{ij}'s such that for each ij pair, β_{ij} is a C^∞ function that depends only on $p_i = r_i^2$ and $p_j = r_j^2$ or only on $x_i^2+y_i^2$ and $x_j^2+y_j^2$.

$$\beta_{ij} = \beta_{ij}(x_i^2+y_i^2, x_j^2+y_j^2)$$

. So the conditions on β_{ij}'s are same as b_{ij}'s.

To describe the other 2 families, we will only focus on those $I = (i_1,\dots,i_r)(1\le i_1 < i_2 \dots < i_r\le m)$ such that F_I is a r codimension face of P_Δ and $m-r\le 3$. Let the set of such I's for P_Δ be $I^3_{P_\Delta}$. For each $K=(k_1,\dots,k_r)\in I^3_{P_\Delta}$, let $L=(l_1,\dots,l_{m-r})$ be the set of tupels such that $1\le l_1<\dots<l_{m-r}\le m$ and $L\cap K=\emptyset$. So for each such K, define $P^K = \prod_{q=1}^{m-r} p_{l_q}$ and two Poisson bivector fields

$$_K\Pi_3^R = \sum_{1\le i<j<(n-r)} P^K\sigma_{ij}^K\zeta_i^K\wedge\zeta_j^K \tag{4.4}$$

where σ_{ij}^K are constant functions.

$$_K\Pi_4^R = \sum_{1\le i<j<(n-r)} P^K\upsilon_{ij}^K\zeta_i^K\wedge\zeta_j^K + \sum_{1\le i<j\le m}\tau_{ij}\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right)\wedge\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right) \tag{4.5}$$

where υ_{ij}^K and τ_{ij} are constant functions.

Theorem 4.0.1. - *The above construction defines a family of Poisson structures that can be reduced as per the reduction theory of Poisson structures with $P = \mathbb{C}^m$ and $M = \mathcal{Z}_{P_\Delta}$ (moment angle manifold assiociated to P_Δ) and $E = E^d_\Delta$ corresponding to free torus action $\mathbb{T}^d_\Delta$*

Proof. - **Part – 1**
To show that (A3) is satisfied, it is sufficient to prove (4.1) for each element $g' \in \mathbb{T}^d_\Delta$. I will prove (4.1) for each element in $\mathbb{T}^m$ with its canonical action on $\mathbb{C}^m$, and since $\mathbb{T}^d_\Delta$ is a subtorus of $\mathbb{T}^m$ with action induced from the canonical action of $\mathbb{T}^m$, it easily follows that (4.1) is true for each $g' \in \mathbb{T}^d_\Delta$.

Suppose we have a point $p \in \mathbb{C}^m$ such that $r_i = 0 \iff i \in I$. Without loss of generality, let's shuffle the coordinates for our own convenience and assume that only the first s radial parts are zero.

$$p = (0, .., 0, r_{s+1}e^{i2\pi\theta_{s+1}}, \dots, r_m e^{i2\pi\theta_m})$$

so for any element $g = (e^{i2\pi t_1}, .., e^{i2\pi t_m}) \in \mathbb{T}^m$ $g.p = (0, .., 0, r_{s+1}e^{i2\pi(\theta_{s+1}+t_{s+1})}, \dots, r_m e^{i2\pi(\theta_m+t_m)}$
So the group action doesn't change the radial data of p ($r_i(p) = r_i(g.p) \quad \forall 1 \le i \le m \quad g \in G$). Since these radial and angular coordinates are not defined everywhere, I will show some results in standard coordinates $(x_1, y_1, ..., x_m, y_m)$ of $\mathbb{C}^m$,

$$g.(x_1, y_1, .., x_m, y_m) = \Psi_g(x_1, y_1, .., x_m, y_m) = (x_1\cos(t_1) - y_1\sin(t_1), x_1\sin(t_1) + y_1\cos(t_1), \dots, x_m\cos(t_m) - y_m$$

By simple computation, $d\Psi_g$ is a block-diagonal matrix,

$$R(t_i) = \begin{pmatrix} \cos(t_i) & -\sin(t_i) \\ \sin(t_i) & \cos(t_i) \end{pmatrix}$$

$$d\Psi_g = \begin{pmatrix} R(t_1) & & & & \\ & . & & & \\ & & . & & \\ & & & . & \\ & & & & R(t_m) \end{pmatrix}$$

let's first see how it acts on $\frac{\partial}{\partial \theta_i}|_p$ and $p_i \frac{\partial}{\partial p_i}|_p$.

$$
\begin{aligned}
d\Psi_g\left(\frac{\partial}{\partial \theta_i}|_p\right) &= d\Psi_g\left(-y_i(p)\frac{\partial}{\partial x_i}|_p + x_i(p)\frac{\partial}{\partial y_i}|_p\right) \\
&= \left((-y_i(p)\cos(t_i) - x_i(p)\sin(t_i))\frac{\partial}{\partial x_i}|_{g.p} + (-y_i(p)\sin(t_i) + x_i(p)\cos(t_i))\frac{\partial}{\partial y_i}|_{g.p}\right) \\
&= -y_i(g.p)\frac{\partial}{\partial x_i}|_{g.p} + x_i(g.p)\frac{\partial}{\partial y_i}|_{g.p} \\
&= \frac{\partial}{\partial \theta_i}|_{g.p}
\end{aligned}
$$

With similar calculations, one can show that.

$$
d\Psi_g\left(p_i\frac{\partial}{\partial p_i}|_p\right) = p_i\frac{\partial}{\partial p_i}|_{g.p}
$$

So for a bivector,

$$
\begin{aligned}
d\Psi_g\left(\alpha_{ij}(p)\frac{\partial}{\partial \theta_i}|_p \wedge \frac{\partial}{\partial \theta_j}|_p\right) &= \alpha_{ij}(p)\left(d\Psi_g\frac{\partial}{\partial \theta_i}|_p\right) \wedge \left(d\Psi_g\frac{\partial}{\partial \theta_j}|_p\right) \\
&= \alpha_{ij}(p)\left(\frac{\partial}{\partial \theta_i}|_{g.p}\right) \wedge \left(\frac{\partial}{\partial \theta_j}|_{g.p}\right) \\
&= \alpha_{ij}(g.p)\left(\frac{\partial}{\partial \theta_i}|_{g.p}\right) \wedge \left(\frac{\partial}{\partial \theta_j}|_{g.p}\right)
\end{aligned}
$$

Since the group action preserves the radial data and α_{ij} 's are functions of p_i's only, so $\alpha_{ij}(p) = \alpha_{ij}(g.p) \quad \forall 1 \leq i, j \leq m, g \in \mathbb{T}^m$. Since $d\Psi_g$ are linear on bivectors,

$$
d\Psi_g(\Pi'_1|_p) = \sum_{1\leq i<j\leq m} d\Psi_g\left(\alpha_{ij}(p)\left(y_i(p)\frac{\partial}{\partial x_i}|_p - x_i(p)\frac{\partial}{\partial y_i}|_p\right) \wedge \left(y_j(p)\frac{\partial}{\partial x_j}|_p - x_j(p)\frac{\partial}{\partial y_j}|_p\right)\right) = \Pi'_1|_{g.p}
$$

Now if we were to simplify ${}_K\Pi_3^R$,

$$
{}_K\Pi_3^R = \sum_{1\leq i<j\leq m} \eta_{ij}\left(p_i\frac{\partial}{\partial p_i} \wedge p_j\frac{\partial}{\partial p_j}\right)
$$

where η_{ij} are polynomial functions of p_k's where $k \neq i, j$. So if we do similar calculations as above, under this Poisson structure, the $\mathbb{T}^m$ action is also Poisson in nature. And since ${}_K\Pi_4^R = \Pi_1' +_K \Pi_3^R$ so by linearity of $d\Psi_g$, this will also make $\mathbb{T}^m$ action a Poisson group action.

To prove $(A3)$ for Π_2', we could use a similar method as above.

$$d\Psi_g\left(\frac{\partial}{\partial x_i}|_p\right) = \cos(t_i)\frac{\partial}{\partial x_i}|_{g.p} + \sin(t_i)\frac{\partial}{\partial y_i}|_{g.p}$$

$$d\Psi_g\left(\frac{\partial}{\partial y_i}|_p\right) = \cos(t_i)\frac{\partial}{\partial y_i}|_{g.p} - \sin(t_i)\frac{\partial}{\partial x_i}|_{g.p}$$

By simple computation,

$$d\Psi_g\left(\frac{\partial}{\partial x_i}|_p \wedge \frac{\partial}{\partial y_i}|_p\right) = d\Psi_g\left(\frac{\partial}{\partial x_i}|_p\right) \wedge d\Psi_g\left(\frac{\partial}{\partial y_i}|_p\right) = \frac{\partial}{\partial x_i}|_{g.p} \wedge \frac{\partial}{\partial y_i}|_{g.p}$$

So by linearity and the fact that coefficient functions of Π_2' are either constants or functions of p_i's, it will also make the action Poisson. So (4.1) holds true for each $g \in \mathbb{T}^m$ and thus for each $g' \in \mathbb{T}^d_\Delta$.

Remark: A very similar proof can be done to show that in general, the Poisson bivectors constructed in section 2.4 are invariant under the standard $\mathbb{T}^m$ action.

Part − 2
Now comes the fourth condition (2.4) we need for reduction

$$B(E^0) \subset T\mathcal{Z}_{P_\Delta} + E$$

In case of Π_1',${}_K\Pi_3^R$ and ${}_K\Pi_4^R$, for all $p \in \mathcal{Z}_{P_\Delta}$, $\Pi_1'|_p$,${}_K\Pi_3^R|_p$ and ${}_K\Pi_4^R|_p$ are made up of tangent vectors that lie in $T_p\mathcal{Z}_{P_\Delta}$. So no matter what E_p^0 is, $B_p(E_p^0) \subset T_p\mathcal{Z}_{P_\Delta}$ (I have explained this properly in Lemma 4.1.3). Thus, no matter what $\mathbb{T}^m_\Delta$is, the fourth condition is always satisfied for these Poisson structures.
In case of Π_2', additional constraints need to be put on P_Δ and $\mathbb{T}^d_\Delta$. The conditions that Delzant[Del] imposed on the polytope and the subtorus is sufficient enough for Π_2' to reduce.
□

4.1 Symplectic Leaves of Reducible Poisson Structure

To study the symplectic leaf structure of our quotient manifold with the induced Poisson structure, we first study the symplectic leaf structure of the induced family of Poisson bivectors on the Moment angle manifold $\mathcal{Z}_{P_\Delta}$. We will use the methods mentioned in Section 2.1 that uses Weinstein splitting theorem 2.1.1 to show that we get a symplectic leaf by integrating along the hamiltonian directions.

Lemma 4.1.1. - *$\mathcal{Z}_{P_\Delta}$ admits Poisson structures induced from $\Pi'_{1},_K \Pi_3^R$ and ${}_K\Pi_4^R$*

Proof. We just need to take $\mathbb{T}^d_\Delta$ as the zero dimension torus ($d = 0$), or in other words, $\mathbb{T}^d_\Delta$ is the trivial group containing only the identity element. In that case the orbit space will be $\mathcal{Z}_{P_\Delta}$. □

We will study each of the $\Pi'_{1},_K \Pi_3^R$ and ${}_K\Pi_4^R$ Poisson structures individually, but before that, let's discuss some properties of an arbitrary bivector field χ^2 on a manifold N of dimension n. Assume the distribution produced by it (as the image of χ^2 acting as bundle map from cotangent bundle to tangent bundle) is integrable, and at any point $q \in N$, χ^2_q looks like

$$\chi^2_q = \sum_{1\leq i<j\leq n} w_i \wedge w_j$$

where w_i are tangent vectors in $T_q(N)$.

Definition 4.1.2. - *$v \in T_qN$ is called field vector at p if $v \in \mathrm{Im}(\chi^2_q)$. $T_q(\chi^2)$ is the subspace of T_pN spanned by the field vector at p and is called the field space at q. The distribution $\Delta(\chi^2)$ produced by it is called the field distribution.*

So, in our case the bivector field is Poisson and field vectors are called Hamiltonian vectors.

Lemma 4.1.3. - *If v is a field vector at q, then v is a linear combination of w_i's*

$$v = \sum_i a_i w_i$$

where a_i are constants.

Proof. Assume any dual basis vectors of $T^*_q(N)$. Since for each dual basis vector, its image under the linear map χ^2_q is a linear combination of w_i's, therefore if z is any linear combination of dual basis vectors, then $\chi^2_q(z)$ is a linear combination of w_i's. □

Let F be the foliation produced the field distribution of χ^2 and S be a submanifold of N such that,

$$T_p(\chi^2) = \Delta(\chi^2)_p \subset T_pS \quad \forall p \in S$$

So the distribution $\Delta(\chi^2)|_{T(S)}$ induced by $\Delta(\chi^2)$ is equal to

$$\Delta(\chi^2)|_{T(S)}(p) = \Delta(\chi^2)(p) \quad \forall p \in S$$

Since $[\Delta(\chi^2), \Delta(\chi^2)] \subset \Delta(\chi^2)$ is true $\implies [\Delta(\chi^2)|_{T(S)}, \Delta(\chi^2)|_{T(S)}] \subset \Delta(\chi^2)|_{T(S)}$. We get a foliation F_S of S such that $F_S \subset F$. In other words, one can think of S as a union of a collection of leaves that is a sub-collection of F.

Moving forward, for our purpose, I will give an idea on how to think of the complex plane $\mathbb{C}^m$ and what kind of submanifolds we get in the foliation of an integrable distribution made of either angular directions or radial directions.

Let $W = (w_1^2, .., w_m^2)$ be a tuple of m real numbers such that each $w_i \geq 0$, i.e $W \in \mathbb{R}_+^m$. Denote,

$$\mathbb{T}_W^m = \{(w_1 e^{2\pi i t_1}, \ldots\ldots, w_m e^{2\pi i t_m}) | \quad t_1, \ldots, t_m \in \mathbb{R}\}$$

So $\mathbb{T}_W^m$ is a subset of $\mathbb{C}^m$ and it contains all those points p whose radial coordinates equals to that of W ($r_i(p) = w_i \quad p_i(p) = w_i^2 \quad \forall 1 \leq i \leq m$). Think of $\mathbb{C}^m$ as a disjoint union of different radius tori $\mathbb{T}_w^m$.

$$\mathbb{C}^m = \{(w_1 e^{2\pi i t_1}, \ldots\ldots, w_m e^{2\pi i t_m}) | w_1, .., w_m \in \mathbb{R}_+ \quad \text{and} \quad t_1, .., t_m \in \mathbb{R}\}$$

So I associate to each point $q \in \mathbb{C}^m$, the tori $\mathbb{T}_W^m$ (with $w_i = r_i(q) \quad 1 \leq i \leq m$) it lies in and for each point there exist only one such torus.

Given an integrable distribution I_Θ on $\mathbb{C}^m$ that is made up of only angular directions, i.e let d be a tangent vector at $q \in \mathbb{C}^m$ then if it lies in the subspace induced by the distribution I_Θ at q then,

$$d = \sum_i \alpha_i \frac{\partial}{\partial \theta_i}$$

Then one can see that if we integrate along these directions, then each integral submanifold lying in the foliation will lie inside a single torus and will not spill or be part of any other tori. To be more precise, given any point, the integral submanifold it lies in will lie inside the tori $\mathbb{T}^m_W$ associated to the point.

Remark - Depending on α_i's, the integral submanifold need not be a subtorus of $\mathbb{T}^m_W$ and might wraps around $\mathbb{T}^m_W$ in a wild manner. It might just be a non compact subgroup of $\mathbb{T}^m_W$.

Similarly if the integrable distribution I_r is made up of radial directions, i.e let d be a tangent vector at $q \in \mathbb{C}^m$ then if it lies in the subspace induced by the distributionI_r at q then,

$$d = \sum_i \beta_i p_i \frac{\partial}{\partial p_i} \quad \forall p \in \mathbb{C}^m$$

It's like the trajectory of the point is moving from a point of one torus to the point of another torus with the same angular coordinates but of a different radial coordinates. In other words, a submanifold belonging to the foliation will intersect each torus $\mathbb{T}^m_R$ at most one, and the points in the submanifold will have the same angular coordinates wherever it is defined. It will also preserve the zeros of the points, i.e if $z_i(p) = 0$ where $p \in \mathbb{C}^m$,then for all the points lying in the integral submanifold in which p belongs to will have the ith z coordinate equal to zero

4.1.1 Leaf structure of Π'_1 on $\mathcal{Z}_{P_\Delta}$

Based on this, let's first analyse Π'_1

$$\Pi'_1 = \sum_{1 \le i < j \le m} \alpha_{ij} \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) \wedge \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right)$$

where each $\alpha_{ij} \in C^\infty(\mathbb{C}^m)$ and $(y_k \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial y_k})(\alpha_{ij}) = 0 \quad \forall 1 \le k \le m$. So the conditions on α_{ij}'s are same as a_{ij}'s. In other words each α_{ij} is a function of only p_k where $k \in \{1, .., m\}$. The first thing to notice is that our bivector is made up of angular directions only, and the second is that α_{ij} are smooth functions that only depend on the radial coordinates.

$$\alpha_{ij} = \alpha_{ij}(p_1, \ldots, p_m)$$

$$\left(y_k \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial y_k} \right) (\alpha_{ij}) = 0 \quad \forall 1 \le k \le m$$

So, within a torus $\mathbb{T}^m_W$, α_{ij} will remain constant on it, so let's call these constant values α^W_{ij}. Since at each point of $\mathbb{T}^m_W$, the Hamiltonian directions lie in the tangent space of $\mathbb{T}^m_W$, the symplectic leaves we get in foliation will lie inside these tori. In addition, the leaves will be subgroups that are immersed(not necessarily embedded) in $\mathbb{T}^m_W$.

So if we took $N = \mathbb{C}^m$, $S = \mathbb{T}^m_w$ and $\chi^2 = \Pi'_1$ then we get a foliation of the torus, let's call it $F_{\mathbb{T}^m_W}$ and the foliation produced is just the disjoint union of $F_{\mathbb{T}^m_W}$ for all $R \in \mathbb{R}^m_+$

$$F_{\mathbb{C}^m} = \bigcup_{W \in \mathbb{R}^m_+} F_{\mathbb{T}^m_W}$$

Since $\mathcal{Z}_{P_\Delta} = \bigcup_{W \in i_A(P)} \mathbb{T}^m_W$,

$$F_{\mathcal{Z}_{P_\Delta}} = \bigcup_{R \in i_A(P)} F_{\mathbb{T}^m_W}$$

4.1.2 Leaf structure of ${}_K\Pi^R_3$ on $\mathcal{Z}_{P_\Delta}$

Now let's analyse ${}_K\Pi^R_3$

$${}_K\Pi^R_3 = \sum_{1 \le i < j < (n-r)} P^K \sigma^K_{ij} \zeta^K_i \wedge \zeta^K_j$$

where σ^K_{ij} are constant functions.

Here we will analyse it face by face. The reason is,

Theorem 4.1.4. *If $p \in \pi^{-1}(E_I)$ and L_p is the symplectic leaf in which p lies in, then $\pi(L_p) \subset E_I$ (where $\pi : \mathbb{C}^m \to \mathbb{R}^m_+$ is quotient map we defined in the commutative square in section 3.2).*

Proof. Suppose for some $p \in \pi^{-1}(E_I)$ there exist I and K such that $I \subset K$, E_I and E_K are faces and, $\pi(L_p) \cap E_I$ and $\pi(L_p) \cap E_K$ are not null set. Thus as mentioned in section 2.1, there exist a point $q \in \pi^{-1}(E_K)$ and $g_1, .., g_k \in C^\infty(\mathbb{C}^m)$ such that $\phi_{X_{g_k}} \circ \cdots \circ \phi_{X_{g_1}}(p) = q$. Thus $p \sim q$ and rank of ${}_K\Pi^R_3|_p$ = rank of ${}_K\Pi^R_3|_q$

Let $i, j \in K - I$, then the radial coordinate functions p_i and p_j take non zero value on each point in $\pi^{-1}(E_I)$. Thus on each point in $\pi^{-1}(E_I)$, θ_i and θ_j as coordinate functions are defined and, $\frac{\partial}{\partial\theta_i}$ and $\frac{\partial}{\partial\theta_j}$ are non zero tangent vectors. The opposite is true for each point in $\pi^{-1}(E_K)$. Construct a bivector field Π^{ij} on $\mathbb{C}^m$,

$$\Pi^{ij} =_K \Pi^R_3 + \left(\frac{\partial}{\partial\theta_i} \wedge \frac{\partial}{\partial\theta_j} \right)$$

This is Poisson and reducible since it is of the form ${}_K\Pi_4^R$. Let's denote ${}_K\Pi_3^R$ by Ra_{ij} and $\left(\frac{\partial}{\partial\theta_i} \wedge \frac{\partial}{\partial\theta_j}\right)$ by An_{ij} So for any function $f \in C^\infty(\mathbb{C}^m)$, $\Pi^{ij}(f)$ has radial terms at one side due to $Ra_{ij}(f)$ and angular terms at other side due to $An_{ij}(f)$. Tangent directions in $Ra_{ij}(f)$ coincides with the tangent directions we get in ${}_K\Pi_3^R(f)$.
Now for each g_k, we get $\Pi^{ij}(g_k)$, and by the choice of g_k's to show $p \sim q$, when we intergrate along $Ra_{ij}(g_k)$ and $An_{ij}(g_k)$, we get to a point q' in $N_K = \pi^{-1}(E_K)$. Basically what is happening is the Ra part integrates similarly to ${}_K\Pi_3^R$ and the points in the integrated paths we get will have the same radial coordinates. This is true Since the An part is only changing the angular coordinate of the points along the path. Thus under the Poisson structure Π_{ij}, $p \sim q'$ and the rank of the structure must be same. Also the radial coordinates of q' equals to that of q ($p_l(q) = p_l(q') \quad \forall 1 \leq l \leq m$).
Now since the image of Ra_{ij} and An_{ij} are linearly independent (properly explained in 4.1.3) and disjoint, at any point $p \in \mathcal{Z}_{P_\Delta}$, the rank of Π^{ij} equal to rank of Ra_{ij} plus rank of An_{ij}. Now since the coefficient function of $Ra_{ij} =_K \Pi_3^R$ only depend on the radial coordinate functions and using the main assumption, the rank of $Ra_{ij}|_{q'}=$ rank of $Ra_{ij}|_q = Ra_{ij}|_p$. By construction, rank of $An_{ij}|_{q'}$ equal to zero and that of p is non zero. Here we get a contradiction that rank of $\Pi^{ij}|_p =$ rank of $\Pi^{ij}|_{q'}$. Thus our original assumption cannot hold true. $\square$

It means the set of r_i's that are zero remains the same throughout a symplectic leaf. That means each symplectic leaf lies within the submanifold that is inverse of a face ($\pi^{-1}(E_I)$). To get an intuitive idea, let's take the interior of the polytope, i.e E_0. Since for each point in E_0 all p_i are non zero, each angular coordinate θ_i is well defined and $\pi^{-1}(E_0) \subset \mathbb{C}^m$ is diffeomorphic to $E_0 \times \mathbb{T}^m$. So it's like in the moment angle manifold, we will get a product group bundle with base E_0, where each slice of the bundle is diffeomorphic to E_0 and the points in that slice have fixed angular coordinates $\theta \in \mathbb{T}^m$. Now this will happen with each $\pi^{-1}(E_I)$, which is diffeomorphic to $E_I \times \mathbb{T}^{m-r}$, and each slice in this bundle is diffeomorphic to E_I.

What the Poisson structure on the moment angle will do is foliate each slice of each stack that we get from the interior of faces into symplectic leaves. And the union of these symplectic leaves is the foliation that is produced by the given Poisson structure.

4.1.3 Leaf structure of ${}_K\Pi_4^R$ on $\mathcal{Z}_{P_\Delta}$

Now let's analyse the ${}_K\Pi_4^R$

$$
{}_K\Pi_4^R = \sum_{1\leq i<j<(n-r)} P^K \upsilon_{ij}^K \zeta_i^K \wedge \zeta_j^K + \sum_{1\leq i<j\leq m} \tau_{ij} \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) \wedge \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right)
$$

where υ_{ij}^K and τ_{ij} are constant functions.
Now in this case we have the radial part collectively at one place and the angular part in second place. So lets denote it as

$$
{}_K\Pi_4^R = Ra + An
$$

Now, based on the above analysis, the image of the radial part and the angular part are linearly independent as defined in Appendix A3.

At each point $p \in M$, ${}_K\Pi_4^R|_p$ as a linear map from $T_p^*(M)$ to $T_p(M)$ is the sum of two linearly independent linear maps $\Pi_1'|_p$ and ${}_K\Pi_3^R|_p$. Since the image of $\Pi_1'|_p$ is made up of only angular directions and that of ${}_K\Pi_3^R|_p$ are only made up of radial directions, thus the images are disjoint and so the image of ${}_K\Pi_4^R|_p$ is the direct sum of the images of $\Pi_1'|_p$ and ${}_K\Pi_3^R|_p$.

Intuitively, we can get the leaf submanifold by first integrating along the angular direction and then the radial directions or reverse the order.

Suppose $p \in N_I$ and $\mu(p) = (r_1, \ldots, r_m) = R_p \in F_I$. So if we integrate along the angular directions that are Hamiltonian, by previous analysis of Π_1', we get a subgroup (may not be a Lie sub group of $\mathbb{T}_W^m$) of $\mathbb{T}_W^m$ as submanifold of the symplectic leaf that p belongs to. Denote this subtorus as $\mathbb{T}_{R_p}^m$

Let F_{Ra} be the foliation produced by Ra as an individual Poisson bivector and Z be the leaf in which p lies.

So if we integrate each point in $\mathbb{T}_{R_p}^m$ along the radial Hamiltonian directions, its like every point in this subtorus belongs to a slice diffeomorphic to N_I and when we integrate along radial directions, it is independent of the angular coordiantes of the point will integrate in the similar manner within their own slices. Let $Z \times \mathbb{T}_{R_p}^m$ be the symplectic leaf belonging to foliation produced by ${}_K\Pi_4^R$ such that p lies in it. Then

$$Z \times \mathbb{T}^m_{R_p} = \bigcup_{q \in Z} \mathbb{T}^m_{R_q}$$

Another way to look at it is if we first integrate along the radial directions, we will get Z. Then at each point of Z, when we integrate along the angular directions, its like we attach the subtorus $\mathbb{T}^m_{R_p}$ to each point of Z and thus we get $Z \times \mathbb{T}^m_{R_p}$.

Remark - $Z \times \mathbb{T}^m_{R_p}$ is actually diffeomorphic to the product manifold we get as a product Z and $\mathbb{T}^m_{R_p}$. This is true since when we get Z by integrating along the radial Hamiltonian direction, $\mathbb{T}^m_{R_q}$ looks the same for each point $q \in Z$. This is true because each τ_{ij} (constant functions) does not vary along any radial direction.

4.2 Poisson Cohomology

Before studying the Poisson cohomology of the newly constructed Poisson structure on $\mathbb{C}^m$, let's see the commutative relation between radial and angular vector fields. Define,

$$Y_{p_i} = p_i \frac{\partial}{\partial p_i} = \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) \quad 1 \leq i \leq m$$

$$Y_{\theta_j} = \frac{\partial}{\partial \theta_j} = \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right) \quad 1 \leq j \leq m$$

Lemma 4.2.1. - *Lie bracket of radial and angular vector fields defined above is zero.*

$$[Y_{p_i}, Y_{p_k}] = 0 \quad [Y_{\theta_j}, Y_{\theta_l}] = 0 \quad [Y_{p_i}, Y_{\theta_j}] = 0 \quad \forall 1 \leq i, j, k, l \leq m$$

It means for any smooth function $g \in C^\infty(\mathbb{C}^m)$,

$$Y_{p_i}(Y_{p_k}(g)) = Y_{p_k}(Y_{p_i}(g)) \quad Y_{\theta_j}(Y_{\theta_l}(g)) = Y_{\theta_l}(Y_{\theta_j}(g)) \quad Y_{p_i}(Y_{\theta_j}(g)) = Y_{\theta_j}(Y_{p_i}(g))$$

$\forall 1 \leq i, j, k, l \leq m$

Proof. I will proof the lemma for $[Y_{p_i}, Y_{p_k}] = 0$ and the similar proof can be done for the other two cases. For any function $h \in C^\infty(\mathbb{C}^m)$,

$$\begin{aligned}
Y_{p_i}Y_{p_k}h &= x_i\frac{\partial}{\partial x_i}\left(x_k\frac{\partial h}{\partial x_k}\right) + x_i\frac{\partial}{\partial x_i}\left(y_k\frac{\partial h}{\partial y_k}\right) + y_i\frac{\partial}{\partial y_i}\left(x_k\frac{\partial h}{\partial x_k}\right) + y_i\frac{\partial}{\partial y_i}\left(y_k\frac{\partial h}{\partial y_k}\right) \\
&= x_k\frac{\partial}{\partial x_k}\left(x_i\frac{\partial h}{\partial x_i}\right) + x_k\frac{\partial}{\partial x_k}\left(y_i\frac{\partial h}{\partial y_i}\right) + y_k\frac{\partial}{\partial y_k}\left(x_i\frac{\partial h}{\partial x_i}\right) + y_k\frac{\partial}{\partial y_k}\left(y_i\frac{\partial h}{\partial y_i}\right) \\
&= \left(x_k\frac{\partial}{\partial x_k} + y_k\frac{\partial}{\partial y_k}\right)\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right)(h) \\
&= Y_{p_k}Y_{p_i}h
\end{aligned}$$

Thus $[Y_{p_i}, Y_{p_k}] = \left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right)\left(x_k\frac{\partial}{\partial x_k} + y_k\frac{\partial}{\partial y_k}\right)(h) - \left(x_k\frac{\partial}{\partial x_k} + y_k\frac{\partial}{\partial y_k}\right)\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right)(h) = 0$ □

Now what about the angular vector fields that are preserved along a radial direction? Define 2 vector fields on $\mathbb{C}^m$,

$$f_{p_i}Y_{\theta_j} = f_{p_i}\frac{\partial}{\partial\theta_j} \quad \text{and} \quad f_{p_i}Y_{p_k} = f_{p_i}\frac{\partial}{\partial p_k} \quad 1 \le j, k \le m$$

where f_{p_i} is a smooth function such that $p_i\frac{\partial}{\partial p_i}f_{p_i} = 0$.
Lemma 4.2.2. - *$f_{p_i}Y_{\theta_j}$ and $f_{p_i}Y_{\theta_j}$ commutes with Y_{p_i}.*

$$[f_{p_i}Y_{\theta_j}, Y_{p_i}] = 0 \quad \textit{or} \quad f_{p_i}Y_{\theta_j}(Y_{p_i}(g)) = Y_{p_i}(f_{p_i}Y_{\theta_j}(g))$$

$$[f_{p_i}Y_{p_k}, Y_{p_i}] = 0 \quad \textit{or} \quad f_{p_i}Y_{p_k}(Y_{p_i}(g)) = Y_{p_i}(f_{p_i}Y_{p_k}(g))$$

for all $g \in C^\infty(\mathbb{C}^m)$

Proof. Again the proof will be for 1st case only and 2nd case will follow similar proof. So for any function $g \in C^\infty(\mathbb{C}^m)$,

$$[f_{p_i}Y_{\theta_j}, Y_{p_i}](g) = f_{p_i}[Y_{\theta_j}, Y_{p_i}](g) - (Y_{p_i}(f_{p_i}))(Y_{\theta_j}(g))$$

By lemma 4.2.1, $f_{p_i}[Y_{\theta_j}, Y_{p_i}](g) = 0$ and since $p_i\frac{\partial}{\partial p_i}f_{p_i} = 0 \implies (Y_{p_i}(f_{p_i}))(Y_{\theta_j}(g)) = 0$

□

So if we have two bivector fields defined below,

$$f_{p_i} Y_{\theta_{j,k}} = f_{p_i} Y_{\theta_j} \wedge Y_{\theta_k}$$

$$f_{p_i} Y_{p_{j,k}} = f_{p_i} Y_{p_j} \wedge Y_{p_k}$$

such that f_{p_i} is same as defined previously, then the bivector fields are preserved along $p_i \frac{\partial}{\partial p_i}$ direction,

Lemma 4.2.3. *The Schouten bracket of* $p_i \frac{\partial}{\partial p_i}$ *and* $f_{p_i} Y_{\theta_{j,k}}$ *and* $p_i \frac{\partial}{\partial p_i}$ *and* $f_{p_i} Y_{p_{j,k}}$ *are zero*

$$\left[f_{p_i} Y_{\theta_{j,k}}, p_i \frac{\partial}{\partial p_i} \right] = 0 \quad \textit{and} \quad \left[f_{p_i} Y_{p_{j,k}}, p_i \frac{\partial}{\partial p_i} \right] = 0$$

Proof. Since The Schouten bracket satisfies the graded Leibniz rule:

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]$$

for $P \in \mathfrak{X}^p(P)$, $Q \in \mathfrak{X}^q(P)$, and $R \in \mathfrak{X}^r(P)$ for any manifold P then by using lemma 4.2.2, we can easily proof the above lemma.

□

To further study the Poisson cohomology class, suppose we have a Poisson vector field X on a Poisson manifold (P, Π), then under what conditions on $f \in C^\infty(P)$ will fX will also be a Poisson vector field.

Since X is a Poisson vector field,

$$[X, \Pi](g, h) = X(\Pi(g, h)) - \Pi(Xg, h) - \Pi(g, Xh) = 0 \quad \forall g, h \in C^\infty(P)$$

Lemma 4.2.4. *Under above assumption,* fX *is a Poisson vector field if and only if*

$$X(g)\Pi(f, h) + X(h)\Pi(g, f) = 0 \quad \forall g, h \in C^\infty(P) \tag{4.6}$$

Proof. fX is a Poisson vector field if and only if $[fX, \Pi] = 0$. So for any function $g, h \in C^\infty(P)$,

$$[fX, \Pi](g, h) = f([X, \Pi](g, h)) + X(g)\Pi(f, h) + X(h)\Pi(g, f)$$

Since X is a Poisson vector field, $[fX, \Pi] = 0 \iff X(g)\Pi(f, h) + X(h)\Pi(g, f) = 0$. □

Suppose we have a Casimir function $c_X \in C^\infty(P)$ such that $X(c_X)$ is non-zero almost everywhere. Then if we substitute $g = c$, then the necessary condition for fX to be Poisson is $X(c_X)\Pi(f,h) = 0 \quad \forall h \in C^\infty(P)$. Since $X(c_X)$ is non-zero almost everywhere, thus $\Pi(f,h)$ needs to be zero almost everywhere and by smoothness $\Pi(f,h)$ needs to be zero everywhere. And so f must be a Casimir function. Also by above equation, if f is a Casimir function then fX is a Poisson vector field. Thus if c_X exists

$$fX \text{ is Poisson} \iff f \text{ is a Casimir function}$$

If no such c_X exist, then one side of the implication is not garaunted and f being a Casimir function is sufficient for fX to be Poisson

$$f \text{ is a Casimir function} \implies fX \text{ is Poisson}$$

Suppose there exist another Poisson vector field Y then

$$[eY + fX, \Pi](g,h) = Y(g)\Pi(e,h) + Y(h)\Pi(g,e) + X(g)\Pi(f,h) + X(h)\Pi(g,f) \tag{4.7}$$

So if there exist c_X and c_Y such that it the above mentioned conditions plus

$$X(c_Y) = Y(c_X) = 0$$

then substituting them one by one in (4.7) will prove

$$eY + fX \text{ is Poisson} \iff e, f \text{ are Casimir functions}$$

Theorem 4.2.5. *Let $X_i \quad (1 \leq i \leq n)$ are n Poisson vector fields on (P, Π) and if there exist c_{X_i} satisfying above conditions then*

$$\sum_{i=1}^{n} f_i X_i \textit{ is Poisson} \iff f_i\textit{'s are Casimir functions}$$

Proof. Using a similar expression as (4.7) and substituting c_{X_i} one by one, the proof easily follows from that. $\square$

4.2.1 Poisson cohomology classes on $\mathbb{C}^m$

As proven in theorem 4.0.1, The standard torus action of $\mathbb{T}^m$ on $\mathbb{C}^m$ is Poisson for all 4 Poisson structures. Thus any vector field produced by the action will be Poisson for all 4 Poisson structures. let's study each structure individually

Poisson cohomology of Π_1

$$\Pi_1 = \sum_{1\leq i<j\leq m} a_{ij}\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right)\wedge\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right)$$

$$\Pi_1|_{\mathbb{C}^m_1} = \sum_{1\leq i<j\leq m} a_{ij}\frac{\partial}{\partial\theta_i}\wedge\frac{\partial}{\partial\theta_j}$$

where each $a_{ij}\in C^\infty(\mathbb{C}^m)$ and $(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i})(a_{ij}) = (y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j})(a_{ij}) = 0$. In other words a_{ij} is only a function of p_k where $k\neq i,j$ and $k\in\{1,..,m\}$.

For this Poisson structure, all p_i's as smooth functions on $\mathbb{C}^m$ will belong to the algebra of Casimir functions and thus polynomials with p_i's as variables will also belong to the algebra and each such polynomial will belong to its unique class in 0th Poisson cohomology $H^0_{\Pi_1}(\mathbb{C}^m)$.

To analyze $H^1_{\Pi_1}(\mathbb{C}^m)$, we will use above lemmas.

Theorem 4.2.6. *If there exist $l\in\{1,..,m\}$ such that $p_l\frac{\partial}{\partial p_l}(a_{ij}) = (x_l\frac{\partial}{\partial x_l} + y_l\frac{\partial}{\partial y_l})(a_{ij}) = 0$ for all $1\leq i<j\leq m$, then Y_{p_l} is a Poisson vector field on $(\mathbb{C}^m,\Pi_1)$*

Proof. The proof is easy. It includes lemma 4.2.3 and linearity of Schouten bracket. □

Since p_l is a Casimir function such that $Y_{p_l}(p_l) = p_l$ is non zero almost everywhere, thus p_l satisfies the conditions of c_X for $X = Y_{p_l}$. So fY_{p_l} will be a Poisson if and only if $f\in C(\mathbb{C}^m,\Pi_1)$.

Suppose there exist $l_1,l_2\in\{1,..,m\}$ such that $p_{l_1}\frac{\partial}{\partial p_l}(a_{ij}) = p_{l_2}\frac{\partial}{\partial p_l}(a_{ij}) = 0$ for all $1\leq i<j\leq m$, then since $p_{l1}$and p_{l_2} are the Casimir functions that satisfies the conditions of $c_{Y_{p_{l_1}}}$ and $c_{Y_{p_{l_2}}}$, thus $f_1Y_{p_{l_1}} + f_2Y_{p_{l_2}}$ is a Poisson vector field on $(\mathbb{C}^m,\Pi_1)$ if and only if $f_1,f_2\in C(\mathbb{C}^m,\Pi_1)$.

Proposition 4.2.7. *If each a_{ij} is a constant function, then the vector field $fX_p = \sum_{l=1}^m f_lY_{p_l}$ is a Poisson vector field on $(\mathbb{C}^m,\Pi_1)$ if and only if $f_l\in C(\mathbb{C}^m,\Pi_1)$ for all $1\leq l\leq m$.*

Proof. If each a_{ij} is a constant function, then $p_l \frac{\partial}{\partial p_l}(a_{ij}) = (x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l})(a_{ij}) = 0 \quad \forall 1 \leq l \leq m$. Thus each X_{p_l} is a Poisson vector field. So for any Casimir function $f_l \in C(\mathbb{C}^m, \Pi_1)$, $f_l Y_{p_l}$ is a Poisson vector field and sum of them will also be a Poisson vector field.
The reverse implication easily follows from theorem 4.2.5. $\square$

So each of the radial Poisson vector field of the form $fX_p = \sum_{l=1}^m f_l Y_{p_l} \quad f_l \in C(\mathbb{C}^m, \Pi_1)$ lie in there unique class of $H^1_{\Pi_1}(\mathbb{C}^m)$. This is true since every Hamiltonian vector field only contains angular terms and no two distinct radial Poisson vector fields differ by an angular vector field.

Poisson cohomology of Π_3 and Π_3^R

$$\Pi_3 = \sum_{1\leq i<j\leq m} \frac{c_{ij}}{2}\left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}\right) \wedge \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right)$$

$$\Pi_3|_{\mathbb{C}_1^m} = \sum_{1\leq i<j\leq m} c_{ij} p_i p_j \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}$$

where c_{ij} are constant functions

$$\Pi_3^R = \sum_{1\leq k<j\leq 3} \frac{c^R_{i_j i_k}}{2}\left(x_{i_j} \frac{\partial}{\partial x_{i_j}} + y_{i_j} \frac{\partial}{\partial y_{i_j}}\right) \wedge \left(x_{i_k} \frac{\partial}{\partial x_{i_k}} + y_{i_k} \frac{\partial}{\partial y_{i_k}}\right)$$

$$\Pi_3^R|_{\mathbb{C}_1^m} = \sum_{1\leq k<j\leq 3} c^R_{i_j i_k} p_{i_j} p_{i_k} \frac{\partial}{\partial p_{i_j}} \wedge \frac{\partial}{\partial p_{i_k}}$$

$$c^R_{i_j i_k} = c^R_{i_j i_k}(x_l^2 + y_l^2) \in C^\infty(\mathbb{C}^m)$$

where $l \in \{1, \ldots, m\}$ and $l \neq i_j, i_k$. In other words, $\left(x_{i_j} \frac{\partial}{\partial x_{i_j}} + y_{i_j} \frac{\partial}{\partial y_{i_j}}\right) c^R_{i_j i_k} = \left(x_{i_k} \frac{\partial}{\partial x_{i_k}} + y_{i_k} \frac{\partial}{\partial y_{i_k}}\right) c^R_{i_j i_k} = 0$ for all pairs $(k, j) \in \{(1, 2), (2, 3), (1, 3)\}$.

For general case, no such simple Casimir function exists but under special conditions they can exist.

Lemma 4.2.8. *p_l is a Casimir function of $(\mathbb{C}^m, \Pi_3)$ if and only if $c_{lj} = 0 \quad \forall 1 \leq j \leq m$.*

Proof. $\Pi_3(p_l) = -\sum_{j=1}^{l-1} \frac{c_{il}}{2} p_i \frac{\partial}{\partial p_i} + \sum_{k=l+1}^m \frac{c_{lk}}{2} p_k \frac{\partial}{\partial p_k}$. So for p_l to be Casimir, the vector field needs to be zero. Since each term $p_j \frac{\partial}{\partial p_j}$ of this vector field is linearly independent with each other almost everywhere, each c_{lj} need to be zero. $\square$

Lemma 4.2.9. *p_l where $1 \leq l \leq m$ and $l \neq i_1, i_2, i_3$ is a Casimir function of* $(\mathbb{C}^m, \Pi_3^R)$

Proof. The proof is similar to lemma 4.2.8 □

The same proof can used to show that p_{i_j} where $i_j \in R$ is Casimir function of $(\mathbb{C}^m, \Pi_3^R)$ if and only if $c^R_{i_j i_k} = c^R_{i_j i_l} = 0$ where $i_k, i_l \in R$ and $i_j \neq i_k \neq i_l$.
Now we will study $H^1_{\Pi_3}(\mathbb{C}^m)$ and $H^1_{\Pi_3^R}(\mathbb{C}^m)$. The condition under which the vector field Y_{p_i} will be Poisson or not is same as in case of Π_1.
Theorem 4.2.10. *If there exist $l \in \{1, .., m\}$ such that $p_l \frac{\partial}{\partial p_l}(c_{ij}) = (x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l})(c_{ij}) = 0$ for all $1 \leq i < j \leq m$, then Y_{p_l} is a poisson vector field on* $(\mathbb{C}^m, \Pi_3)$

Proof. The proof is same as theorem 4.2.6 □

Since each c_{ij} is a constant function, each Y_{p_l} is a Poisson vector field on $(\mathbb{C}^m, \Pi_3)$
Theorem 4.2.11. *If there exist $l \in \{1, .., m\}$ such that $p_l \frac{\partial}{\partial p_l}(c^R_{ij}) = (x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l})(c^R_{ij}) = 0$ for all $1 \leq i < j \leq m$, then Y_{p_l} is a poisson vector field on* $(\mathbb{C}^m, \Pi_3^R)$

Proof. The proof is same as theorem 4.2.6 □

Remark - Here we have an example of a function that is not Casimir but still produced Poisson vector field. Take the Poisson bivector field

$$\Pi = p_1 \frac{\partial}{\partial p_1} \wedge p_2 \frac{\partial}{\partial p_2}$$

By theorem 4.2.8, Y_{p_1} is a poisson vector field for this bivector and by lemma 4.2.7, p_2 is not a Casimir function. Since p_2 satisfy the condition in lemma 4.2.4, $p_2 Y_{p_1}$ is a Poisson vector field.

Proposition 4.2.12. *Since each c_{ij} is a constant function, then the vector field $fY_p = \sum_{l=1}^m f_l Y_{p_l}$ is a Poisson vector field on $(\mathbb{C}^m, \Pi_3)$ if $f_l \in C(\mathbb{C}^m, \Pi_3)$ for all $1 \leq l \leq m$.*

Proof. The Proof is similar to that of proposition 4.2.7. □

Similar proposition is for Π_3^R.

Unlike Π_1, in case of Π_3 and Π_3^R, the Hamiltonian vector field contain only radial terms. Thus any two distinct radial Poisson vector field of the form $fX_p = \sum_{l=1}^m f_l Y_{p_l} \quad f_l \in C(\mathbb{C}^m, \Pi_1)$ may lie in the same 1st Poisson cohomology class in case of both Π_3 and Π_3^R.

Poisson cohomology of Π_4 and Π_4^R

$$\Pi_4 = \sum_{1\le i<j\le m} \frac{e_{ij}}{2}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right)\wedge\left(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right) + \sum_{1\le r<s\le m} d_{rs}\left(y_r\frac{\partial}{\partial x_r} - x_r\frac{\partial}{\partial y_r}\right)\wedge\left(y_s\frac{\partial}{\partial x_s} - x_s\frac{\partial}{\partial y_s}\right)$$

$$\Pi_4|_{\mathbb{C}_1^m} = \sum_{1\le i<j\le m} e_{ij}p_ip_j\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial p_j} + \sum_{1\le r<s\le m} d_{rs}\frac{\partial}{\partial\theta_r}\wedge\frac{\partial}{\partial\theta_s}$$

The Schouten bracket condition implies, e_{ij} and d_{rs} are constant functions.

$$\begin{aligned}\Pi_4^R = &\sum_{1\le k<j\le 3} \frac{e^R_{i_ji_k}}{2}\left(x_{i_j}\frac{\partial}{\partial x_{i_j}} + y_{i_j}\frac{\partial}{\partial y_{i_j}}\right)\wedge\left(x_{i_k}\frac{\partial}{\partial x_{i_k}} + y_{i_k}\frac{\partial}{\partial y_{i_k}}\right)\\ &+ \sum_{1\le r<s\le m} d_{rs}\left(y_r\frac{\partial}{\partial x_r} - x_r\frac{\partial}{\partial y_r}\right)\wedge\left(y_s\frac{\partial}{\partial x_s} - x_s\frac{\partial}{\partial y_s}\right)\end{aligned}$$

$$\Pi_4^R|_{\mathbb{C}_1^m} = \sum_{1\le k<j\le 3} e^R_{i_ji_k}p_ip_j\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial p_j} + \sum_{1\le r<s\le m} d_{rs}\frac{\partial}{\partial\theta_r}\wedge\frac{\partial}{\partial\theta_s}$$

$$e^R_{i_ji_k} = e^R_{i_ji_k}(x_l^2 + y_l^2) \in C^\infty(\mathbb{C}^m)$$

where $l \in \{1,\dots,m\}$ and $l \ne i_j, i_k$. In other words, $\left(x_{i_j}\frac{\partial}{\partial x_{i_j}} + y_{i_j}\frac{\partial}{\partial y_{i_j}}\right)e^R_{i_ji_k} = \left(x_{i_k}\frac{\partial}{\partial x_{i_k}} + y_{i_k}\frac{\partial}{\partial y_{i_k}}\right)e^R_{i_ji_k} = 0$ for all pairs $(k,j) \in \{(1,2),(2,3),(1,3)\}$ and d_{rs} are constant functions.

The results are simply the combination of the results we got in Π_1, Π_3 and Π_3^R.

Lemma 4.2.13. *p_l is a Casimir function of $(\mathbb{C}^m, \Pi_4)$ if and only if $e_{lj} = 0 \quad \forall 1 \le j \le m$.*

Proof. The proof is same as Lemma 4.2.8 □

Lemma 4.2.14. *p_l where $1 \le l \le m$ and $l \ne i_1, i_2, i_3$ is a Casimir function of $(\mathbb{C}^m, \Pi_4^R)$*

Proof. The proof is similar to lemma 4.2.8 □

The same proof can used to show that p_{i_j} where $i_j \in R$ is Casimir function of $(\mathbb{C}^m, \Pi_3^R)$ if and only if $e^R_{i_j i_k} = e^R_{i_j i_l} = 0$ where $i_k, i_l \in R$ and $i_j \neq i_k \neq i_l$.
Now since each d_{rs} are constant functions,

Theorem 4.2.15. *If there exist a $l \in \{1, .., m\}$ such that $p_l \frac{\partial}{\partial p_l}(e_{ij}) = (x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l})(e_{ij}) = 0$ for all $1 \leq i < j \leq m$, then Y_{p_l} is a poisson vector field on $(\mathbb{C}^m, \Pi_4)$*

Proof. The proof is same as theorem 4.2.6. □

Since each e_{ij} is a constant function, each Y_{p_l} is a Poisson vector field on $(\mathbb{C}^m, \Pi_4)$

Theorem 4.2.16. *If there exist $l \in \{1, .., m\}$ such that $p_l \frac{\partial}{\partial p_l}(e^R_{ij}) = (x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l})(e^R_{ij}) = 0$ for all $1 \leq i < j \leq m$, then Y_{p_l} is a poisson vector field on $(\mathbb{C}^m, \Pi_4^R)$*

Proof. The proof is same as theorem 4.2.6 □

Proposition 4.2.17. *Since each e_{ij} is a constant function, then the vector field $fX_p = \sum_{l=1}^m f_l Y_{p_l}$ is a Poisson vector field on $(\mathbb{C}^m, \Pi_4)$ if $f_l \in C(\mathbb{C}^m, \Pi_4)$ for all $1 \leq l \leq m$.*

Proof. The Proof is similar to that of proposition 4.2.7. □

Similar to Π_3 and Π_3^R, since there exist Hamiltonian vector field containing only radial terms, any two distinct radial Poisson vector field of the form $fY_p = \sum_{l=1}^m f_l Y_{p_l} \quad f_l \in C(\mathbb{C}^m, \Pi_4)$ may lie in the same 1st Poisson cohomology class of $(\mathbb{C}^m, \Pi_4)$. Same is for $(\mathbb{C}^m, \Pi_4^R)$

Appendix A

Skew Symmetric Maps

A.1 Skew Symmetric Bilinear Form

Suppose we have a vector space V and a bilinear Skew symmetric form B

$$B : V \times V \to \mathbb{R}$$

$$B(v, w) = -B(w, v)$$

Then w.r.t to any basis of (V, β), the matrix of B will be skew symmetric

$$[B]_{\beta}^{T} = -[B]_{\beta}$$

If $\beta = \{\beta_1, \ldots, \beta_m\}$ then the dual basis will be $\beta^* = \{\beta_1^*, \ldots, \beta_m^*\}$ and B will look like,

$$B = \sum_{1 \leq i < j \leq m} a_{ij} \beta_i^* \wedge \beta_j^*$$

where a_{ij} are constants determined by

$$a_{ij} = B(\beta_i, \beta_j)$$

This skew symmetric bilinear map induces a canonical linear map

$$B^* : V \to V^*$$

$$B^*(v) = B(v, -)$$

We have a few convenient results to get an idea of rank of B^* often called rank of B.
Now in case of skew symmetric matrices M, square of its pfaffian is equal to its determinant

$$pf(M) = \sqrt{\det(M)}$$

If the dimension of the matrix is odd then $pf(M)$ is always zero and in case of even, it can be non zero. Also the rank of a skew symmetric matrix is always even.

Suppose we have a bivector field B^2 on a Manifold M. Then it will induce a bilinear skew symmetric map at each point of M on its cotangent space

$$B_p^2 : T_P^*M \times T_P^*M \to \mathbb{R}$$

and a linear map

$$(B^2)_p^* : T_P^*M \to T_PM$$

In other words, it will induce a bundle map,

$$B^2 : T^*M \to TM$$

such that at each point w.r.t any basis, B^2 will give a skew symmetric matrix

In case we have 2 form, it will do the same except cotangent space and tangent space will exchange places

A.2 Image of skew symmetric map

Suppose we have a Injective linear maps between finite dimension vector spaces V and W with basis $\{v_1, ..v_{m-n}\}$ and $\{w_1, .., w_m\}$

$$l : V \to W$$

Then it will induce a dual map on dual space

$$l^* : W^* \to V^*$$

Suppose we have a skew symmetric form Π on V^*, then it will look like

$$\Pi = \sum_{1 \le i < j \le m-n} a_{ij} v_i \wedge v_j$$

We want to look at its image as a linear map,

$$\Pi' : V^* \to V$$

Since l^* is a surjective map, $V^* = l^*(W^*)$, thus

$$\Pi : l^*(W^*) \times l^*(W^*) \to \mathbb{R}$$

and for any $a, b \in W^*$

$$\Pi(l^*(a), l^*(b)) = \sum_{1 \le i < j \le m-n} a_{ij} (l(v_i) \wedge l(v_j))(a, b)$$

thus Π can be realized as a skew symmetric bilinear map $l(\Pi)$on W^* which will induce a map

$$l(\Pi)' : W^* \to l(V)$$

And its easy to that there will be an isomorphism between

$$\mathrm{Im}(\Pi') \tilde{=} \mathrm{Im}(l(\Pi)')$$

and the isomorphism is given by the linear map l

Now we can do it at a vector bundle level. The case we want to study is when we have a submanifold N (with bivector field Π) of a manifold M with an inclusion map,

$$i : N \to M \quad di : TN \to TM \quad i^* : T^*M \to T^*N$$

So if we do the above analysis pointwise, $\Pi|_p(T_p^*N)$ is isomorphic to $(di_p(\Pi|_p))(T_p^*M)$.

Appendix B

Planes in $\mathbb{R}^m$

A plane HR^m of codimension r can be realised as a intersection of r hyperplanes (codimension = 1) given by

$$H_i = \{y \in \mathbb{R}^m | < \gamma_i, y >= c_i\} \quad 1 \leq i \leq r$$

where $\gamma_i \in \mathbb{R}^m$ and $c_i \in \mathbb{R}$ or a solution of r linear equations given as

$$\Gamma^T(y) = C$$

where $\Gamma = \begin{pmatrix} \gamma_1 & \ldots.. & \gamma_r \end{pmatrix}$ and $C^T = \begin{pmatrix} c_1 & \ldots & c_r \end{pmatrix}$ such that the Γ is an injective linear map.

Since γ_i is normal to H_i and $H = \bigcap_i H_i$ so each γ_i is normal to H. So any vector that is perpendicular to each γ_i is parallel to H or in other words if $a = (a_1, .., a_m)$ is perpendicular to each γ_i and $p \in H$ then $p + at \in H \quad \forall t \in \mathbb{R}$.

So if we treat H as a manifold then at point $p \in H$

$$\alpha = \sum_i a_i \frac{\partial}{\partial x_i} \quad \in T_p\mathbb{R}^>$$

is a tangent vector of p in H, $\alpha \in T_pH$. Since the dim of A (subspace of vectors perpendicular to γ_i) and dim of tangent space of H is equal, so the above method gives an isomorphism between them. Also A is the kernel of the matrix transformation Γ with γ_i as its row

$$\Gamma = \begin{pmatrix} .. & \gamma_1 & .. \\ .. & .. & .. \\ .. & \gamma_r & .. \end{pmatrix} = \begin{pmatrix} .. & .. & .. \\ \delta_1 & .. & \delta_m \\ .. & .. & .. \end{pmatrix} : \mathbb{R}^m \to \mathbb{R}^r$$

Now we will only consider planes H such that no coordinated axis is perpendicular to it or in other words no coordinate axis is parallel to it. Let

$$C_i = \{x \in \mathbb{R}^m | x_i = 0\}$$

be the hyperplanes with one coordinate zero (x_i axis is perpendicular to it), called coordinate hyperplane. Then $HC_i = H \cap C_i$ is a plane of co dimension $r+1$. There are two ways to get the subspace A_i of A that is parallel to HC_i.

First is to take intersection of vector subspaces A and D_i of $\mathbb{R}^m$ where

$$D_i = \{v \in \mathbb{R}^m | e_i^*(v) = 0\}$$

Basically vectors with i th coordinate zero. Then

$$A_i = A \cap D_i$$

Second is to construct a matrix transformation Γ_i whose matrix is Γ with ith column removed.

$$\Gamma_i = \begin{pmatrix} .. & .. & .. & .. & .. & .. \\ \delta_1 & .. & \delta_{i-1} & \delta_{i+1} & .. & \delta_m \\ .. & .. & .. & .. & .. & .. \end{pmatrix} : \mathbb{R}^{m-1} \to \mathbb{R}^r$$

Define linear transformation O_i that maps $\mathbb{R}^{m-1}$ to $\mathbb{R}^m$ with ith coordinate zero

$$O_i(b = (b_1, \dots, b_{m-1})) = (b_1, .., b_{i-1}, 0, b_i, .., b_{m-1})$$

So $A_i = O_i(ker(\Gamma_i))$

Its easy to see that if $v \in A_i$ and $q \in HC_i$ then $q + vt \in HC_i$ and A_i and T_qHC_i are isomorphic.

Similarly if we take further intersection of HC_i with another coordinate hyperplane, we will do the similar technique as above to get its tangent space as a vector subspace of $\mathbb{R}^m$. For convinience, if $I = \{i_1, .., i_r\}$ are r tuples in increasing order between 1 and m then define

$$HC_I = \bigcap_{i_j \in I} H \cap C_{i_j}$$

In further sections, we will be taking intersection of H with $\mathbb{R}^m_+$ as to get a polytope P immersed in it and the faces of the immersed polytope will be its intersection with coordinate hyperplanes. I will label $R^m_+ \cap HC_I$ as F_I

Appendix C

Sum of Linear Maps

We dont have much theory on the image of sum of two linear maps with respect to there individual theory but here I will explain a special case.

Let say we have two linear maps

$$L_1 : V \to W \quad L_2 : V \to W$$

between finite dimension vector spaces with basis $\{v_1, ..v_n\}$, and $\{w_1, .., w_m\}$.
Let $I = \{i_1, .., i_r\}$ and $J = \{j_1, \ldots, j_s\}$ be set of tuples in increasing order such that they range between 1 and n and $I \cap J = \phi$ and Let $V_I = span\{v_{i_1}, \ldots, v_{i_r}\}$ and $V_J = span\{v_{j_1}, \ldots, v_{j_s}\}$. let's also assume that $v_k \in \mathbf{ker}(L_1) \iff k \notin I$ and $v_r \in \mathbf{ker}(L_2) \iff r \notin J$. We assume that V. So by assumption, $V_I \bigcap V_J = 0$, $V_I \subset \mathbf{ker}(L_2)$ and $V_J \subset \mathbf{ker}(L_1)$.

Definition C.0.1. *The pair of linear maps satisfying above hypothesis are called linearly independent by domain.*

Define a linear map $L = L_1 + L_2 : V \to W$. Normally, it's hard to analyse the image of such a linear map. But when the linear maps are linearly independent from domain then its easy to prove that the image of the sum of the linear maps is sum of the image of the linear maps.

Lemma C.0.2. $\boldsymbol{Im}(L) = \boldsymbol{Im}(L_1) + \boldsymbol{Im}(L_2)$

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