Probing Post-Newtonian Formalism of Binary Black Holes in General Relativity and Beyond

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Certificate

This is to certify that this dissertation entitled "Probing Post-Newtonian Formalism of Binary Black Holes in General Relativity and Beyond" towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune, represents study/work carried out by Pushparaj Chakravarti at Inter-University Centre for Astronomy and Astrophysics (IUCAA) under the supervision of Dr Apratim Ganguly, Scientific Officer - E (R & D), Department of Physics, during the academic year 2023-2024.

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This thesis is dedicated to my father and everyone who believed in me.

Declaration

I hereby declare that the matter embodied in the report entitled "**Probing Post-Newtonian Formalism of Binary Black Holes in General Relativity and Beyond**" are the results of the work carried out by me at the Department of Physics, **Indian Institute of Science Education and Research, Pune** and **Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune** from the period 15-05-2023 to 30-04-2024, under the supervision of Dr Apratim Ganguly and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

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Abstract

This project studies the analytical methods used to model the non-spinning compact binaries in general relativity (GR), scalar-tensor theories, and $f(R)$ theories of gravity. We have studied the post-Newtonian (PN) approximation methods and effective-one-body (EOB) formalism to understand the inspiral phase of compact binary coalescence. We have successfully reproduced the post-Newtonian dynamics in GR, scalar-tensor theory, and $f(R)$ gravity. Additionally, utilizing second-order PN results from GR, we have recovered the 2PN conservative dynamics in the EOB framework, which were initially developed by Alessandra Buannano and Thibault Damour. Using the Hamiltonian in the effective framework, we have described the Innermost stable circular orbit. We have also derived the post-Newtonian dynamics at second-order for the general $f(R)$ gravity. These 2PN results are helpful in modelling compact binary dynamics in $f(R)$ gravity. Additionally, we have performed the conformal mapping between scalar-tensor and $f(R)$ gravity.

Contents

Chapter 1

Introduction

In 1916, Albert Einstein proposed a geometrical treatment of spacetime and its connection with matter embedded in it, the theory of general relativity. Along with quantum mechanics, general relativity (GR) is one of the fundamental pillars of modern physics. Like all physical theories, it should be tested against observations and experiments with the highest accuracy possible. Over time, it has undergone several precision tests. GR has been probed accurately in the *weak-field* limit of relativistic gravity. This includes solar system tests, such as the mercury perihelion precession test [1], gravitational lensing due to the Sun, and gravitational redshift [2]. Along with that, the systems sourced by a compact object involving a strong gravitational field were first tested for the binary pulsar [3]. Observations from such astrophysical tests constrain possible deviations from GR. However, both experimental and theoretical evidence shows that modification of GR is inevitable at small and large energy scales [4].

From a theoretical point of view, GR is purely a classical theory. GR is not perturbatively renormalizable and hence cannot be conventionally quantized, and this limitation can be overcome by accounting for higher-order curvature correction to Einstein-Hilbert action, making it renormalizable [4, 5]. The higher-order curvature corrections provide correction to GR at high energies. On the other hand, from an observational perspective, cosmological measurements provide evidence for the existence of dark energy (nonzero cosmological constant). Dark energy is used to describe the unknown form of energy that permeates through all spacetime and results in the accelerated expansion of the Universe. A form of dark energy, $p/\rho - 1$ (where p is the dark energy pressure and ρ is the energy density) is the cosmological constant. The value of the cosmological constant, Λ, that can be attributed to vacuum energy density, which is a theoretical background energy permeating all of the spacetime curvature. The theoretically calculated value for vacuum energy density comes out to be larger (by 120 orders) than the observed value of the cosmological constant. This discrepancy needs to be solved, and it is called "the cosmological constant problem" [6].

It is worth noting that modifying the aforementioned aspects at both low and high energy levels can introduce an extra degree of freedom while still being constrained by the solar system observations and local tests, such that it rules out many candidate theories of gravity proposed [7]. The work presented in this thesis focuses on the high-energy correction in GR that can be modelled and constrained by studying strong-field gravity. In order to detect gravitational waves (GWs) sourced from compact binary coalescence (CBC), which is one the most important sources of GWs, it is beneficial to know the gravitational waveform that is emitted by inspiraling and merging compact binaries beforehand. Therefore, we will workout to establish an analytical model of the GWs generated by the dynamics of CBCs, with a particular focus on the relativistic two-body problem and its conservative dynamics.

GWs are the distortion in the fabric of spacetime caused by accelerating masses. At present the detectable GWs are sourced from massive astrophysical objects, providing an excellent opportunity to probe prediction by GR in a strong gravity regime. The first direct detection of GWs, known as event GW150914, was confirmed by the Laser Interferometer Gravitational-Wave Observatory (LIGO) and Virgo collaboration [8] from colliding black holes and ushered a new era of gravitational wave astronomy. The detection opened up opportunities for previously impossible astrophysical observations and allowed for the testing of strong-field gravity. GWs can originate from a variety of astrophysical sources, including compact binary sources, unmodelled transients from core-collapse supernovae, continuous waves from pulsars, and stochastic background emitted in the early universe.

The aim is to establish a theoretical framework of CBC dynamics as a test of GR in alternative theories of gravity. In CBC, compact objects, such as black holes on quasi-circular orbits, radiate energy due to the emission of GWs, causing their orbital separation to shrink until they plunge, merge and form a Kerr black hole. In general, the dynamics of CBC can be broadly divided into three phases:

- *Inspiral:* When the individual black holes are relatively far apart with respect to their length scales, their orbit gradually shrinks adiabatically. This process takes a very long time to lose energy and come close by emitting GW in the weak field regime until the effects due to structure become important.

- *Merger:* It is the collision phase of two black holes into one. Before the collapse, the characteristic velocities can reach a significant fraction of the speed of light.

- *Ringdown:* After the merger, the final deformed BH rapidly emits gravitational waves to lose the excitations. These oscillation modes of black holes are called "quasinormal modes". It encodes the information about the mass, spin, and electric charge of the final black hole.

Conventionally, the post-Newtonian method is used to model the inspiral phase. It is an approximation method used in the weak-field and slow-motion region. It expands the motion of matter or the metric field configuration about a dimensionless parameter ϵ ($\epsilon \sim v/c$ where c is the speed of light, and v is the characteristic velocity). However, this method starts to lose accuracy significantly towards the late inspiral at strong gravity due to *highly non-linear* relativistic effects. Numerical relativity (NR) simulation techniques, which employ Einstein field equations, are used to model the merger phase. For modelling the ringdown signals, black hole perturbation theory is used, which is a framework used to analyze and understand the behaviour of small disturbances or perturbations around black holes. Combining all these techniques makes it possible to produce a single waveform for the signal sourced by compact binaries [9, 10].

In order to accurately describe the dynamics and radiation of various binaries from inspiral to a merger for extreme mass ratio and comparable mass ratio binaries, we will use effective one-body (EOB) formalism. EOB can synergise with NR, PN and Gravitational self-force theory, providing an effective method to compute accurate waveform using a single formalism. The reason is that one could try to combine PN and NR waveforms and evolve into a merger. However, this requires too much computation time and still requires stitching/mapping of waveforms [9] [10]. EOB requires essential inputs from higher-order PN expanded results. Therefore, this work also focuses on reproducing the existing PN results till the second order in GR, the first order in scalar-tensor theory, and using these tools generating second-order PN expanded results in $f(R)$ gravity, which will be essential to develop EOB dynamics in $f(R)$ gravity (leading to a possible test of GR).

There are several proposals to test GR using GW for CBCs. Theory-dependent tests look for signatures of alternative theories, such as scalar-tensor theories, $f(R)$ gravity, etc. However, gravitational waveform modelling within these alternative theories of gravity are relatively less developed than in GR. Theory-agnostic tests have been proposed, including parametrized tests of post-Newtonian theory, parametrized post-Einsteinian framework, and inspiral-merger-ringdown consistency tests. Nonetheless, at present, the bounds derived from such tests often cannot be mapped onto physically meaningful quantities that describe the compact binary dynamics. Hence, a possible development in $f(R)$ gravity can be a good test of GR in strong gravity. Thus, a new extension is proposed, which explicitly extends the post-Newtonian modelling to the second-order and a conformal correspondence between $f(R)$ gravity and scalar-tensor theory. The other possible PN and EOB extensions of GR, such as massive gravity, Horndeski's theories, etc, are not covered in this thesis.

We will focus on the analytical aspects of CBC in the inspiral phase and late inspiral phase, accounting for the post-Newtonian approximation methods and effective-one-body formalism in GR, scalar-tensor theory and $f(R)$ gravity. Chapter 1 provides an introduction to the research presented in the rest of the thesis. In Chapter 2, we briefly review what gravitational waves (GWs) are and how they arise in general relativity (GR). Chapter 3 is an introduction to modified theories of gravity and an introduction to scalar-tensor theory and $f(R)$ gravity. We also show the conformal mapping from between scalar-tensor and $f(R)$ gravity. Chapter 4 explains the analytical methods we have used to explain the inspiral. For post-Newtonian methods, we have reproduced the Lagrangian till 2PN order in GR and 1PN in Scalar-tensor theory. We have derived the second-order post-Newtonian dynamics for the generic $f(R)$ gravity for the first time in this thesis. In Chapter 5, we have explained EOB formalism and re-derived the 2PN conservative dynamics. Finally, chapter 6 concludes the results and the future aspects related to the work done in this thesis.

Chapter 2

Gravitational waves in general relativity

General Relativity (GR) asserts how the curvature of spacetime acts on matter to manifest itself as "gravity" and how energy and momentum influence spacetime to create curvature in spacetime. GR has successfully passed numerous observational tests with flying colours so far. One of its remarkable predictions is the existence of gravitational waves (GWs), geometrically which are ripples in spacetime caused by the accelerated motion of massive objects. The dominant mode comes from the quadrupolar moment, which arises due to the change in the mass distribution of the massive source.

GR formulation tells us that gravity is the property of spacetime, which is described by the Einstein field equations (EFEs). It relates the relation between the curvature, $R_{\mu\nu\rho\gamma}$ and the energy-momentum of the matter distribution given by [11],

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1}$$

in natural units $G = c = 1$ where G is the gravitational constant and c is the speed of light. Here, $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu} \equiv g^{\rho\gamma} R_{\mu\nu\rho\gamma}$ is the Ricci tensor, $R \equiv g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, Λ is the cosmological constant that can be taken as zero for the Universe without dark energy contribution, and $T_{\mu\nu}$ is the stress-energy-momentum tensor containing information regarding matter distribution, which follows the energy conservation relation $\nabla_\mu T^{\mu\nu} = 0$.

The EFEs naturally arise from the action principle. Action is intrinsic to the metric itself rather than depending on our choice of coordinates. The action is given by Einstein-Hilbert action [11]

$$S_G = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R + L_M\right], \tag{2.2}$$

where $\kappa = \sqrt{8\pi G}$, L_M represents the matter Lagrangian. We have used the signature of metric as $(-,+,+,+)$. Extremizing the action should recover the physical laws

$$\delta S/\delta g_{\mu\nu} = \int d^4x\left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\left[\frac{1}{2\kappa^2}R + L_M\right] + \sqrt{-g}\left[\frac{\delta R}{2\kappa^2\delta g^{\mu\nu}} + \frac{\delta L_M}{\delta g^{\mu\nu}}\right]\right] \tag{2.3a}$$

$$= \int d^4x\left[\frac{1}{2\kappa^2}\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}R + \sqrt{-g}\frac{\delta R}{\delta g^{\mu\nu}}\right) + \left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}L_m + \sqrt{-g}\frac{\delta L_M}{\delta g^{\mu\nu}}\right)\right], \tag{2.3b}$$

where the Greek indices $\mu, \nu, \rho, ... = 0, 1, 2, 3$ denotes the spacetime indices. By imposing $\delta S/\delta g_{\mu\nu} = 0$ we can get the field equations. Now by using $\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}$ we can get

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}R + \sqrt{-g}\frac{\delta R}{\delta g^{\mu\nu}} = \kappa^2\sqrt{-g}\left(g_{\mu\nu}L_M - 2\frac{\delta L_M}{\delta g^{\mu\nu}}\right), \tag{2.4}$$

variation of Ricci scalar is given as

$$\delta R = \delta(R_{\mu\nu}g_{\mu\nu}) = g^{\mu\nu}(\nabla_\rho\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho_{\rho\mu}) + R_{\mu\nu}\delta g^{\mu\nu}, \tag{2.5}$$

using the gauge condition, the variation of the Ricci scalar can be given as $\delta R = R_{\mu\nu}\delta g_{\mu\nu}$. Now we can recover EFEs if we can set the stress-energy momentum tensor as

$$T_{\mu\nu} := \left(g_{\mu\nu}L_M - 2\frac{\delta L_M}{\delta g^{\mu\nu}}\right). \tag{2.6}$$

This definition turns out naturally using this treatment. On the other hand, energy-momentum conservation arises as the consequence of Lagrangian symmetry under spacetime translation. Noether's theorem states that given a physical system, for every infinitesimal symmetry, there is a corresponding law of symmetry (conserved quantity). Implies, in four dimension spacetime, the invariance under translation lead to $T^{\mu\nu}$ which obeys $\partial_\nu T^{\mu\nu} = 0$. Therefore inserting 2.4 and 2.6 into 2.3b we arrived at the EFEs [11]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}. \tag{2.7}$$

2.1 Linearized gravity

The first step to understanding GWs arising in GR is to study the expansion of EFEs 2.1 around the flat Minkowski metric (background spacetime) [12]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \tag{2.8}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small perturbation. Expansion up to the leading order in $h_{\mu\nu}$ is called *linearized gravity*. Using the ansatz 2.8 and solving for the EFEs 2.1 for the perturbative radiation field to the first order in $h_{\mu\nu}$ gives the Christoffel as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) \tag{2.9}$$

that gives the $R_{\mu\nu}$ up to leading order in $h_{\mu\nu}$

$$\begin{aligned} R_{\mu\nu} &= \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu} + O(h^2_{\mu\nu}) \\ &= \frac{1}{2}\eta^{\alpha\lambda}(\partial_\mu\partial_\alpha h_{\nu\lambda} + \partial_\nu\partial_\alpha h_{\mu\lambda} - \partial_\alpha\partial_\lambda h_{\mu\nu}) - \frac{1}{2}\eta^{\alpha\rho}(\partial_\mu\partial_\mu h_{\alpha\rho} + \partial_\mu\partial_\alpha h_{\rho\mu} - \partial_\mu\partial_\rho h_{\alpha\mu}) \\ &= \frac{1}{2}(\partial_\rho\partial_\mu h^{\rho}{}_{\nu} + \partial_\rho\partial_\nu h^{\rho}{}_{\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \end{aligned} \tag{2.10}$$

where $h = h^{\mu}_{\mu} = \eta^{\mu\nu}h_{\mu\nu}$, by using 2.10 in to the 2.1 we arrived at

$$\left(\partial_\rho\partial_\mu h^{\rho}{}_{\nu} + \partial_\rho\partial_\nu h^{\rho}{}_{\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \eta_{\mu\nu}\Box h\right) = \frac{16\pi G}{c^4}T_{\mu\nu} \tag{2.11}$$

as it is more convenient to use trace-reversed metric perturbation $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. Using 2.11, we can arrive at

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^\alpha\partial_\nu\bar{h}_{\mu\alpha} - \partial^\alpha\partial_\mu\bar{h}_{\nu\alpha} = -\frac{16\pi G}{c^4}T_{\mu\nu}. \tag{2.12}$$

The last three terms on the left-hand side of the equation 2.12 serve to keep the expression gauge-invariant. For a vector field ξ^α, the infinitesimal coordinate transformation $x^\mu \rightarrow x^{\mu\prime} = x^\mu + \xi^\mu$, the perturbation $h'_{\mu\nu}$ to $h_{\mu\nu}$ can be given as

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\alpha\xi^\alpha), \tag{2.13}$$

without loosing generality we can find ξ^α such that the gauge condition is verified

$$\partial^\mu \bar{h}_{\mu\nu} = 0. \tag{2.14}$$

The chosen gauge condition is called as *Lorentz gauge* (also called the harmonic gauge or the De Donder gauge). Using the 2.14 and 2.12 we can derive the Lineraized Einstein's field equation [13] in natural units as $\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$, in SI units we get

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{2.15}$$

2.2 Gravitational waves and its propagation in space-time

In vacuum, the energy-momentum tensor is identically zero ($T^{\mu\nu} = 0$). Therefore 2.15 takes the form of a wave equation, which tells the metric perturbations to propagate at the speed of light c. It can be given as [13]

$$\Box \bar{h}_{\mu\nu} = \left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2} \right) \bar{h}_{\mu\nu} = 0, \tag{2.16}$$

where the simplest solution for plane waves can be given as

$$\bar{h}_{\mu\nu} = \mathrm{Re}(A_{\mu\nu} \exp i k_\alpha x^\alpha) = A_{\mu\nu} \cos(k_\alpha x^\alpha), \tag{2.17}$$

here 'Re' represents the real part, $A_{\mu\nu}$ denotes the constant components and k_α wave vector. Plugging 2.17 into 2.16 we give

$$0 = \Box \bar{h}_{\mu\nu} = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \bar{h}_{\mu\nu} = \eta^{\alpha\beta} \partial_\alpha (i k_\beta \bar{h}_{\mu\nu}) = -\eta^{\alpha\beta} k_\alpha k_\beta \bar{h}_{\mu\nu}, \tag{2.18}$$

because all the components of $\bar{h}_{\mu\nu}$ will not be zero everywhere. so we must have

$$k_\alpha k^\alpha = 0, \tag{2.19}$$

i.e. the wave vector is a null vector, which gives the frequency of the plane wave as $\omega = \sqrt{k_1^2 + k_2^2 + k_3^2} = k_0$, where k_0 is the time component and the k_i's are spatial component

$(i = 1, 2, 3)$. Imposing the harmonic gauge condition to the plane wave $\partial_\mu \bar{h}^{\mu\nu}$ will give $k_\mu A^{\mu\nu} = 0$. These four equations will reduce the number of independent components of tensor $A_{\mu\nu}$ from 10 to 6. It can be further reduced by using the Lorentz gauge condition that will modify the metric perturbation for any transformed component ξ^μ as

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu, \tag{2.20}$$

here $\bar{h}'_{\mu\nu}$ represents the new metric perturbation, such that the new coefficients satisfy $A^\mu_\mu = 0$ and $A_{0\nu} = 0$. So we have a total of four additional constraints that reduce the number of independent components of tensor $A_{\mu\nu}$ from 6 to 2. These two degrees of freedom will correspond to the physical information characterizing of the plane wave in this gauge. Considering the wave propagation along the z-axis, $k_3 = \omega$ and $k_x = k_y = 0$. Therefore, the rank $(0, 2)$ tensor $A_{\mu\nu}$ will become

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.21}$$

from the harmonic gauge condition, we have gone to sub-gauge, which is known as *transverse traceless gauge* $h^{\mathrm{TT}}_{\mu\nu}$ [13]. It is called transverse traceless because the metric perturbation is traceless and perpendicular to the wave vector. Therefore for the rest frame observer $u^\alpha = (1, 0, 0, 0)$, the gravitational wave propagation along z-direction, and with ω frequency the metric 2.21 is simply written as [13]

$$h^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{\mathrm{TT}}_{11} & h^{\mathrm{TT}}_{12} & 0 \\ 0 & h^{\mathrm{TT}}_{12} & -h^{\mathrm{TT}}_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.22}$$

conventionally h^{TT}_{11} corresponds to the h_+ polarization, and h^{TT}_{12} corresponds to the $h_\times$ polarization. Here h_+ and $h_\times$ are the free functions of retarded time $t - z$. It has two degrees of freedom of the metric. So gravitational waves possess two linearly independent polarization states.

Chapter 3

Modified theories of gravity

General relativity can be modified in several ways. However, these modifications are constrained by the experiments and cosmological observations. Inherently in GR, spacetime can have curvature singularities, which could be resolved by a quantum theory of gravity (as discussed in [4]). Therefore, it is clear that GR needs to be modified at high energy scales.

These possible constrained modifications can be classified based on Lovelock theorem [14]. The theorem states that GR is the only theory of gravity that emerges under certain assumptions, which can be precisely expressed as

"*In four spacetime dimensions, the only divergence-free symmetric rank-2 tensor constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, and preserving diffeomorphism invariance, is the Einstein tensor plus a cosmological term.*"

The modification of Einstein's gravity can be broadly categorized into four classes, as shown in Fig 3.1. The most natural extension to GR is by adding an extra field. It can be dynamical or nondynamical. Nondynamical fields consider the metric and the affine connections as independent field variables, implying that the background metric is a fixed structure against which the dynamical field evolves, an example is the Palatini formulation. Whereas in the case of dynamical field, the metric and the affine connections are dependent field variables and allow $g_{\mu\nu}$ to couple with the additional (scalar, vector, tensor) fields. This additional coupling violates the strong equivalence principle, as it can vary the energy flux in strong

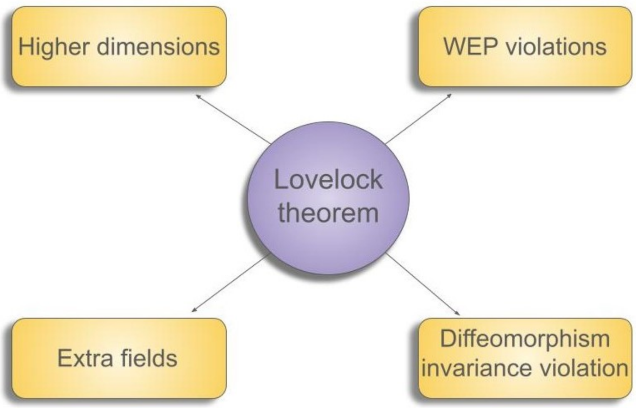

Figure 3.1: An illustration of obtaining modified gravity theories by violating Lovelock theorem. Adapted from Fig 2.1 of [4]

gravity. Hence, deviation from the geodesic motion. The examples are scalar-tensor, metric $f(R)$, Einstein-Gauss-Bonnet, Einstein-Aether, TeVeS gravity, etc. In this work, we will focus on scalar-tensor and $f(R)$ gravity theories.

3.1 Scalar-tensor theory

Scalar-tensor gravity is one of the most natural extensions of GR, in which the scalar degree of freedom nonminimally couples with the metric, i.e., the Ricci scalar in the Einstein-Hilbert action is multiplied by a function of the scalar field. The scalar-tensor theory can serve to explain the accelerated expansion of the Universe [15]. Scalar-Tensor theories are often regarded as potential candidates for unified theories of gravity, which aim to unite gravity with other fundamental forces of nature, such as electromagnetism and the strong and weak nuclear forces [4]. By comprehending these theories, we may eventually achieve a more comprehensive understanding of the fundamental laws of physics.

In the Einstein-Hilbert action, we add a single massless scalar field, which nonminimally couples with the curvature. The action is called the Jordan-Brans-Dicke action [16]. It is given by

$$S = \frac{1}{16\pi\,\tilde{G}} \int d^4x \sqrt{-\tilde{g}} \left[\varphi R - \frac{\omega(\varphi)}{\varphi} \tilde{g}^{\mu\nu} \left(\partial_\mu \varphi\right) \left(\partial_\nu \varphi\right) \right] + S_M[\Psi, \tilde{g}_{\mu\nu}]. \tag{3.1}$$

here $\omega(\varphi) = \omega_{\rm BD}$ is a constant, and terms with '$\sim$' represent the Jordan frame. The most general action of scalar-tensor gravity was studied by Bergmann and Wagoner, 3.1 for a massive scalar field which can introduce the additional potential term in action [17]. The variational principle, with standard topological and surface term assumptions which satisfy the weak equivalence principle, results in

$$\delta_m \int dx^4 \sqrt{-\tilde{g}} S_M = 0. \tag{3.2}$$

The equation 3.2 can be used to obtain the geodesic equation of a test particle. However, it is not applicable for an extended particle. When matter interacts with the scalar field through second-order coupling with the metric, it can lead to deviations from the weak equivalence principle. We have used the same treatment as GR, see section 2.1, to obtain the dynamics in Jordan-Brans-Dicke theory. The variation of the action with respect to the metric field will give the equation for the gravitational field and matter source

$$R_{\alpha\beta}\varphi - \frac{1}{2}g_{\alpha\beta}R\varphi = T^{\rm M}_{\alpha\beta} + \nabla_\alpha\nabla_\beta\varphi - g_{\alpha\beta}\Box\varphi + \frac{\omega(\varphi)}{\varphi}\left(\nabla_\alpha\varphi\nabla_\beta\varphi - \frac{1}{2}g_{\alpha\beta}\nabla_\rho\varphi\nabla^\rho\varphi\right), \tag{3.3}$$

whereas variation with respect to the scalar field will give

$$\omega(\varphi)\left(\frac{2\Box\varphi}{\varphi} - \frac{\nabla_\rho\varphi\nabla^\rho\varphi}{\varphi^2}\right) = -R. \tag{3.4}$$

We can make Eq. 3.1 a minimally coupled action by making a conformal transformation from the Jordan to the Einstein frame. The conformal transformation of the metric is given by

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \tag{3.5}$$

where Ω is the conformal factor. Let us study the energy-momentum conservation equation, the curvature tensor and its conservation equation under the conformal transformation 3.5. It is the rescaling of the coordinate system such that the angle between vectors at any point remains unchanged so that it preserves the causal structure of the manifold. The Ricci tensor and Ricci scalar after the conformal rescaling is given by

$$\begin{aligned}\tilde{R}_{\alpha\beta} =& R_{\alpha\beta} - 2\nabla_\alpha\nabla_\beta\ln\Omega - g_{\alpha\beta}g^{\rho\sigma}\nabla_\rho\nabla_\sigma\ln\Omega + 2(\nabla_\alpha\ln\Omega)\nabla_\beta\ln\Omega \\ &- 2g_{\alpha\beta}g^{\rho\sigma}(\nabla_\rho\ln\Omega)\nabla_\sigma\ln\Omega, \\ \tilde{R} =& \Omega^{-2}[R - 6g^{\alpha\beta}\nabla_\alpha\nabla_\beta\ln\Omega - 6g^{\alpha\beta}(\nabla_\alpha\ln\Omega)\nabla_\beta ln\Omega],\end{aligned} \tag{3.6}$$

also given in [18]. Let us make the conformal transformation using these conformal rescaled variables 3.6 into the equation 3.1 from Jordan to Einstein frame. Therefore for $\omega(\varphi) = 1/2$ we obtain 3.1 in the Einstein frame as

$$S = \frac{1}{16\pi G} \int d^4x\sqrt{-g}\left[R - 2g^{\mu\nu}\left(\partial_\mu\varphi\right)\left(\partial_\nu\varphi\right)\right] + S_M[\Psi, \Omega^2 g_{\mu\nu}], \tag{3.7}$$

where S_M represents the matter action over ψ as the matter field and the conformally transformed metric tensor $\Omega^2 g_{\mu\nu}$ that means the global rescaling of the coordinate is given by $\tilde{x}^\mu = A_0 x^\mu$, where the "0" subscript indicates the quantity evaluates at $\varphi(\phi_0) \equiv \varphi_0$. Because at infinity, the system will get imposed by cosmology, and we get the asymptotically constant value of the scalar field (φ_0) that is at a large distance from the source, the scalar field will approach a constant value. The field equation in the Einstein frame can be obtained by varying the action 3.7 with respect to the metric tensor $g^{\mu\nu}$ and the scalar field φ. We have obtained it as [19]

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi T_{\mu\nu}, \\ \Box_g\varphi + g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi &= -4\pi\alpha(\varphi)T. \end{aligned} \tag{3.8}$$

where $G = c = 1$, $\Box_g$ is the d'Alembartian operator with respect to spacetime metric, and $\alpha(\varphi) = d\ln\Omega/d\varphi$ is the coupling factor.

3.2 $f(R)$ gravity

Over the years, there has been a growing interest in understanding the physical causes behind the observed cosmic acceleration. The accelerated cosmic expansion has postulated the existence of dark energy, which comprises about 70 per cent of the critical energy density of the Universe. This has led to the exploration of f(R)-gravity, which is considered a promising approach to explain cosmic acceleration. Infrared corrections to GR are also often explored through the prototypical $f(R)$ function, a curvature term in the Einstein-Hilbert action dependent on the Ricci scalar $f(R)$.

In principle, any alternative theory of gravity should recover the positive (well-tested) results of GR. Therefore, a systematic analysis of such theories is needed on a short scale and within a low energy limit. In this thesis, section 4.3, we have discussed the short-scale behaviour of higher-order (fourth-order) gravity. Meanwhile, any theory of gravity is constrained by observations in order to be a physical theory. For $f(R)$ gravity, the action is given by [20]

$$S_{f_R} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M\left[\Psi, g_{\mu\nu}\right] \tag{3.9}$$

the series expansion of the $f(R)$ is given as

$$f(R) = \ldots + \frac{\rho_2}{R^2} + \frac{\rho_1}{R} - 2\Lambda + R + \frac{R^2}{\eta_2} + \frac{R^3}{\eta_3} + \ldots \tag{3.10}$$

where the coefficients have appropriate dimensions [20]. Extremizing the action by variation with respect to the metric by using the variations in GR 2.5 will give the field equation. Therefore the variation with respect to the metric given as

$$\delta S_{f_R}/\delta g^{\mu\nu} = \frac{1}{16\pi G}\int \mathrm{d}^4x\left(\frac{\delta f(R)}{\delta g^{\mu\nu}}\sqrt{-g} + f(R)\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right) \tag{3.11a}$$

$$= \frac{1}{16\pi G}\int \mathrm{d}^4x\left(F(R)\frac{\delta R}{\delta g^{\mu\nu}}\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\right) \tag{3.11b}$$

$$= \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(F(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]F(R)\right) \tag{3.11c}$$

where $F(R)$ represents $F(R) = df(R)/dR$. Demanding $\delta S_{f_R}/\delta g^{\mu\nu} = 0$, we get the field equation as

$$H_{\mu\nu} := f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right]f'(R) = \kappa^2 T_{\mu\nu} \tag{3.12}$$

where κ is $\sqrt{8\pi G}$ as shown in [20]. When $f(R) = R$, the theory reduces to GR.

3.3 Conformal mapping between scalar-tensor and $f(R)$ gravity

In scalar-tensor theory, the additional scalar field appears to be coupled in a non-minimal way to gravity in the Jordan frame. This can be recast in the Einstein frame by using the conformal transformation such that the scalar field gets minimally coupled with gravity, as we have shown in section 3.1. In this section, we show the conformal mapping from scalar-tensor to $f(R)$ gravity. The conformal transformation is given as $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where is the conformal factor, the tilde represents the Einstein frame, and the non-tilde represents the Jordan frame. We considered Jordan's frame to be a "physical" frame where the curvature is non-minimally coupled with the curvature term. Though working in Einstein's frame is convenient as the additional scalar fields get coupled with the matter. Here we have shown the conformal transformation 3.5 from $f(R)$ action 3.9 to the scalar-tensor action in the Einstein frame 3.7. We can start from the action for $f(R)$ in the Jordan frame 3.9 by adding and subtraction a first-order Taylor expansion as expanded in 4.56 is given by [7]

$$S_{f_R} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(f(R) + f'(R)R - f'(R)R) + S_M\left[\Psi, g_{\mu\nu}\right] \tag{3.13a}$$

$$= \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}f'(R)R - U\right) + \int d^4x L_M\left[\Psi, g_{\mu\nu}\right] \tag{3.13b}$$

where $U = f'(R)R - f(R)/2\kappa^2$. Introducing the new scalar field $\varphi(x)$ such that for the choice of conformal factor $\Omega^2 = f'(R) = e^{-\kappa Q\varphi(x)}$. A simple relation can be established

$$\varphi(x) = -\frac{\ln f'(R)}{2Q\kappa} \tag{3.14}$$

Now using the relation 3.6 into the 3.13 we obtain

$$S_{f_R} = \int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{2\kappa^2}f'(R)\Omega^{-2}(\tilde{R} + 6\Box\ln f'(R) - 6\tilde{g}^{\mu\nu}\partial_\mu\ln f'(R)\partial_\nu\ln f'(R)) - \Omega^{-4}U\right) + \int d^4x L_M\left[\Psi, \Omega^{-2}\tilde{g}_{\mu\nu}\right] \tag{3.15}$$

replace $f'(R)$ with Ω^2 so that we can write S_{f_R} as S_E which represents the action in Einstein frame

$$S_E = \int d^4x\left[\frac{1}{2\kappa}\tilde{R} - 6\frac{\kappa^2 Q^2}{2\kappa^2}(\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi) - \Omega^{-4}U\right] + \int d^4x L_M(\tilde{g}_{\mu\nu}/f'(R), \Psi_M) \tag{3.16}$$

For the only choice of $Q = -1/\sqrt{6}$ we can recover equations of the same form as 3.7. Therefore the action we obtain as

$$S_E = \int d^4x\left\{\frac{1}{2\kappa}\tilde{R} - \frac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V(\varphi)\right\} + \int d^4x L_M(\Omega^2\tilde{g}_{\mu\nu}, \Psi_M) \tag{3.17}$$

where $V(\varphi)$ is the Lagrangian density for the scalar-field φ [7], $L_\phi = -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$. In the previous section, we have considered a massless scalar field such that no $V(\varphi)$ term appeared. For scalar-tensor theory to be physical, it should have a massive scalar field. Otherwise, the field possesses to be long range, which contradicts the solar system tests. We use the conformal transformation to go to the Einstein frame so that curvature in the field equations takes a similar form to Einstein's curvature. Whereas the additional terms appear to be in the form of coupling with matter and Lagrangian density. However we transform back to Jordan to discuss the physical quantities.

Now let's make the conformal transformation for the field equations 3.12 using the conformal relation **??** such that the conformal relation for the curvature become 3.6. So we can get the relation

$$R = e^{\sqrt{\frac{2}{3}}\kappa\varphi}\left\{\tilde{R} + \kappa\sqrt{6}\tilde{g}^{\alpha\gamma}\nabla_\alpha\nabla_\gamma\varphi - \kappa^2\tilde{g}^{\alpha\gamma}\nabla_\alpha\varphi\nabla_\gamma\varphi\right\} \tag{3.18}$$

in the 3.12 $f(R)$ can be replaced by using 4.54 as

$$f(R) = \frac{3}{2}\Box\Omega^2 + \frac{1}{2}\Omega^2 R - \frac{\kappa T}{2}. \tag{3.19}$$

Therefore by introducing the relation 3.18 and 3.19 to the $f(R)$ field equations 3.12 we obtain the field equation in Einstein frame as

$$\begin{aligned} \tilde{R}_{\mu\nu} - 2\nabla_\mu\varphi\nabla_\nu\varphi &= \kappa^2\left(T_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}T\right), \\ \Box_{\tilde{g}}\varphi &= \tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\kappa^2}{2}\frac{\partial\ln\Omega}{\partial\varphi}T \end{aligned} \tag{3.20}$$

hence we have recovered the field equations using the conformal transformation 3.5 as we have derived previously i section 3.8. This conformal relation can now further be used to construct the first-order conformal mapping between scalar-tensor and $f(R)$ gravity.

Chapter 4

Analylitical approximation methods for relativistic two-body system

In the case of the two-body problem, the dynamics (considering the masses to be the point sources) is characterized solely by their masses and, possibly, their spins, moving under their mutual, purely gravitational interaction. Numerical solution of non-linear Einstein field equations in the strong field is obtained through numerical relativity (NR) simulations. These simulations offer the most precise waveforms for comparable mass binaries within the LIGO band. These simulations are rooted in the 3+1 decomposition of spacetime within the Arnowitt-Deser-Misner (ADM) formalism. However, their computational demands are substantial, requiring weeks to months on supercomputers. This limits their practicality for fully exploring the parameter space or generating extended waveforms. Hence, efficient analytical waveform models are needed to complement NR simulations. These models provide fast computational capabilities and contribute to enhancing our understanding of two-body dynamics and the emission of gravitational waves. Moreover, they support the exploration of modified gravity theories, assisting the tests of GR.

The work in this thesis is focused on the analytical methods to model the inspiral phase of binary black hole dynamics, one is post-Newtonian methods, and the other is effective-one-body formalism. The dynamics corresponding to gravitational self-force (GSF) theory is not part of this thesis. The possible case when $v/c \sim 1$ but $GM/rc^2 << 1$ called post-Minkowskian approximation is also not part of this thesis. In this section, we have introduced the analytical methods in GR, scalar-tensor theory and $f(R)$ gravity. We have followed the work by Nathalie Deruelle and Felix-Louis Julie [21]. Our approach was to learn and derive the first-order post-Newtonian Lagrangian in General Relativity. Then, by adding an additional scalar degree of freedom, the Einstein-Hilbert action gets modified, and the 1PN Lagrangian can be obtained.

4.1 Post-Newtonian approximation method in general relativity

The description of coalescing of compact binary systems, like binary black-hole systems, can be achieved through the post-Newtonian approximation of general relativity. This approximation holds true particularly during the early inspiral phase, given the conditions of a weak gravitational field. In PN, the Einstein field equations are successively approximated by a series on the expansion about the parameter $\epsilon \sim v/c \sim (Gm/rc^2)$, where v the magnitude of their relative velocity and r is the separation between the two point masses. Einstein studied the post-Newtonian calculations for the first time to explain the perihelion precession of Mercury's orbit. [22]. Later, it has been developed for higher orders. The Einstein field equation for the effective energy-momentum tensor, $\mathbb{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}$ is given by

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}T^{\sigma}{}_{\sigma}g_{\mu\nu}\right) = \frac{8\pi G}{c^4}\mathbb{T}_{\mu\nu} \tag{4.1}$$

where $T^{\sigma}{}_{\sigma} = g^{\sigma\kappa}T_{\kappa\sigma}$ is the trace. From this section, we will use the SI units unless it is specified.

To develop the post-Newtonian limit, we consider the individual masses in the system to be point masses, that is, the system is weakly internally stressed ($T^{i,j} << T^{00}$), slow-moving ($T^{0i} << T^{00}$), and weakly self-gravitating ($U/c^2 << 1$, where U is the gravitational potential). It is worth noting that in a classical system, the conservative dynamics remain unchanged under time reversal. Under time reversal, the components g_{00} and g_{ij} exhibit even symmetry, while g_{0i} displays odd symmetry. Thus, g_{00} and g_{ij} can only contain even powers of velocity, while g_{0i} can only contain odd powers as long as the invariance under time reversal is preserved. If we expand g_{00} up to the order of ϵ^n, then g_{0i} must be expanded up to ϵ^{n-1}, and g_{ij} up to ϵ^{n-2}. The perturbation expansion of in metric tensor can be expressed as

$$\begin{aligned} g_{00} &= -1 + \epsilon^2 g_{00}^{(2)} + \epsilon^4 g_{00}^{(4)} + \epsilon^6 g_{00}^{(6)} + \mathcal{O}(c^{-8}), \\ g_{0i} &= \epsilon^3 g_{0i}^{(3)} + \epsilon^5 g_{0i}^{(5)} + \mathcal{O}(c^{-7}), \\ g_{ij} &= \delta_{ij} + \epsilon^2 g_{ij}^{(2)} + \epsilon^4 g_{ij}^{(4)} + \mathcal{O}(c^{-6}) \end{aligned} \tag{4.2}$$

where the Latin indices $i, j, k, l... = 1, 2, 3$ denotes the spatial indices. For simplicity, in the rest of the thesis, we omitted the power of ϵ's, and the superscripts represent the order of ϵ the individual terms contain unless it is not specified. We have invoked an order of c when it is necessary to maintain the units. Similar to metric expansion, the perturbation expansion of the energy-momentum tensor can be expressed as

$$\begin{aligned} T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(4)}T^{00} + {}^{(6)}T^{00} + \ldots, \\ T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + {}^{(5)}T^{0i} + \ldots, \\ T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + {}^{(6)}T^{ij} + \ldots. \end{aligned} \tag{4.3}$$

4.1.1 1PN expansion for non-spinning compact binaries

To obtain the first-order 1PN Lagrangian of a binary system from the Einstein-Hilbert action, which is also called the first-order post-Keplerian (1PK) approximation, we need to consider up to the order of $1/c^2$ in the expansion. We can obtain the metric for first-order correction by using an order-by-order comparison between the Ricci tensor $R_{\mu\nu}$ and the effective energy-momentum tensor ($\mathbb{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T^{\sigma}{}_{\sigma} g_{\mu\nu}$) in EFEs. The metric expansion is considered until the first order, and it is given as

$$\begin{aligned} g_{00} &= -1 + g_{00}^{(2)} + g_{00}^{(4)} + \mathcal{O}(c^{-6}) \\ g_{0j} &= g_{0i}^{(3)} + \mathcal{O}(c^{-5}) \\ g_{ij} &= \delta_{ij} + g_{ij}^{(2)} + \mathcal{O}(c^{-4}) \end{aligned} \tag{4.4}$$

using 4.4 to solve for the Christoffel so that we can determine the Ricci tensors. The Christoffels are given in the appendix A.2. Using these relations of Christoffels we can obtain the Ricci tensors till the first order. On the other hand, these relations will help us to obtain the higher-order metric terms using the iteration method. Using the relation $g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu$ we can obtain the relations $g^{(2)00} = -g^{(2)00}$, $g_{ij}^{(2)} = -g^{(2)ij}$, $g^{(4)00} = \left(g_{00}^{(2)}\right)^2 - g_{00}^{(4)}$, and $g_{0i}^{(3)} = g^{(3)0i}$ we can impose the gauge condition ($g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$) to simplify the expressions for the Ricci tensor. The Ricci tensors are given in appendix A.2.

Using the gauge condition $g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0$, it follows that $g^{\mu\nu}\Gamma^{0}_{\mu\nu}$ and $g^{\mu\nu}\Gamma^{i}_{\mu\nu}$ up to order of $\mathcal{O}(c^{-3})$ becomes

$$\frac{1}{2c}\frac{\partial g^{(2)}_{00}}{\partial t} + \frac{1}{2c}\frac{\partial g^{(2)}_{kk}}{\partial t} - \frac{\partial g^{(3)}_{0k}}{\partial x^k} = 0, \qquad \frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial x^i} + \frac{\partial g^{(2)}_{ik}}{\partial x^k} - \frac{1}{2}\frac{\partial g^{(2)}_{kk}}{\partial x^i} = 0 \tag{4.5}$$

where k is the summed up index, taking the covariant derivative of gauge, $\nabla_\rho(g^{\mu\nu}\Gamma^{\rho}_{\mu\nu}) = 0$ will give the orders of $\mathcal{O}(c^{-3})$ and $\mathcal{O}(c^{-4})$ to impose on $R^{(4)}_{00}$ and $R^{(3)}_{0i}$. Therefore we obtain the component of the Ricci tensor as

$$R^{(2)}_{00} = -\frac{1}{2}\nabla^2 g^{(2)}_{00} \tag{4.6a}$$

$$R^{(4)}_{00} = -\frac{1}{2}\nabla^2 g^{(4)}_{00} - \frac{1}{2c^2}\frac{\partial^2 g^{(2)}_{00}}{\partial t^2} - \frac{g^{(2)ij}}{2}\frac{\partial^2 g^{(2)}{}_{00}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial x^i}\frac{\partial g^{(2)}_{00}}{\partial x^i} \tag{4.6b}$$

$$R^{(3)}_{0i} = -\frac{1}{2}\nabla^2 g^{(3)}_{0i} \tag{4.6c}$$

$$R^{(2)}_{ij} = -\frac{1}{2}\nabla^2 g^{(2)}_{ij} \tag{4.6d}$$

Now to equate the $R_{\mu\nu}$'s with the $\mathbb{T}_{\mu\nu}$ let's solve for the expansions in the effective energy-momentum tensor, where the trace of the energy-momentum tensor is given as

$$T = T^{\mu}_{\mu} = g_{\mu\nu}T^{\mu\nu} = g_{00}T^{00} + 2g_{0i}T^{0i} + g_{ij}T^{ij}. \tag{4.7}$$

Here we consider the components energy-momentum up to the order $\mathcal{O}(c^{-4})$

$$g_{00}T^{00} = (-1 + g^{(2)}_{00} + g^{(4)}_{00})({}^{(0)}T^{00} + {}^{(2)}T^{00}) \tag{4.8a}$$

$$= {}^{(0)}T^{00} - {}^{(2)}T^{00} + g^{(2)}_{00}{}^{(0)}T^{00} + g^{(2)}_{00}{}^{(2)}T^{00} + g^{(4)}_{00}{}^{(0)}T^{00} + g^{(4)}_{00}{}^{(2)}T^{00}, \tag{4.8b}$$

$$g_{0i}T^{0i} \equiv 2g_{0i}T^{0i} = 2g^{(3)}_{0i}{}^{(1)}T^{0i}, \tag{4.8c}$$

$$g_{ij}T^{ij} = (\delta_{ij} + g^{(2)}_{ij})({}^{(2)}T^{ij}) = {}^{(2)}T^{i}_{i} + g^{(2)}_{ij}{}^{(2)}T^{ij}. \tag{4.8d}$$

As we have defined the effective energy-momentum tensor $\mathbb{T}$ is given as $\mathbb{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = g_{\mu\sigma}g_{\nu\tau}\left(T^{\sigma\tau} - \frac{1}{2}T^{\rho}_{\rho}g^{\sigma\tau}\right)$. Using the expansion of energy-momentum tensor 4.3 till order of c^{-4} we can write the expansions for the effective energy-momentum tensor. The expansion is given as follows

$$\begin{aligned}
\mathbb{T}_{00} &= \mathbb{T}^{(0)}_{00} + \mathbb{T}^{(2)}{}_{00} + \mathcal{O}(c^{-6}),\\
\mathbb{T}_{0i} &= \mathbb{T}^{(1)}_{0i} + \mathbb{T}^{(3)}{}_{0i} + \mathcal{O}(c^{-7}),\\
\mathbb{T}_{ij} &= \mathbb{T}^{(0)}_{ij} + \mathbb{T}^{(2)}{}_{ij} + \mathcal{O}(c^{-6}).
\end{aligned} \tag{4.9}$$

The individual components of effective energy-momentum tensors are given in appendix A.2. Whereas for the two body dynamics, the energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_a m_a \frac{d\tau_a}{dt} \frac{dx^\mu_a}{d\tau_a} \frac{dx^\nu_a}{d\tau_a} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)) \tag{4.10}$$

where $d\tau_a/dt = 1/\gamma_a$, γ_a is the lorentz factor for a^{th} body. In the N-body system, the metric felt by a particle, labelled as b, is obtained by taking as a source the energy-momentum tensor of all the other particles, i.e. replacing $\sum_a$ with $\sum_{a\neq b}$. Imposing $d\tau_a/dt$ upto order $\mathcal{O}(v^2/c^2)$ that is $d\tau_a/dt = 1 + \phi - v_a^2/(2c^2)$ (where ϕ is the Newtonian gravitational potential defined by the Poisson's equation $\nabla^2\phi = -4\pi G \sum_a m_a c^2 \delta^{(3)}(\mathbf{r} - \mathbf{r}_a(t))$) in the expression 4.10. We get 4.10 as

$${}^{(0)}T^{00}(t, \boldsymbol{x}) = \sum_{a\neq b} m_a c^2 \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)), \tag{4.11a}$$

$${}^{(2)}T^{00}(t, \boldsymbol{x}) = \sum_{a\neq b} m_a \left(\frac{1}{2} v_a^2 + \phi c^2\right) \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)), \tag{4.11b}$$

$${}^{(1)}T^{0i}(t, \boldsymbol{x}) = c \sum_{a\neq b} m_a v_a^i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)), \tag{4.11c}$$

$${}^{(2)}T^{ij}(t, \boldsymbol{x}) = \sum_{a\neq b} m_a v_a^i v_a^j \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)). \tag{4.11d}$$

Using 4.11and 4.9 we can simply get the individual components of the effective energy-momentum tensor. Therefore the individual effective energy-momentum tensors are

$$\mathbb{T}^{(0)}_{00} = \frac{1}{2} {}^{(0)}T^{00} = \frac{1}{2} \sum_a m_a c^2 \delta^{(3)}(\mathbf{r} - \mathbf{r}_a(t)) = \frac{1}{2} m_a c^2, \tag{4.12a}$$

$$\mathbb{T}^{(2)}_{00} = \frac{1}{2}({}^{(2)}T^{00} + 2g^2_{00}\,{}^{(0)}T^{00} + {}^{(0)}T^{ii}) = \frac{1}{2} \sum_a m_a \left(\frac{1}{2} v_a^2 + \phi + 4\phi + v_a^i v_a^i\right) \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)) \tag{4.12b}$$

$$= \frac{1}{2} \sum_a m_a c^2 \left(\frac{3v_a^2}{2c^2} + \frac{5\phi}{c^2}\right) \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)), \tag{4.12c}$$

$$\mathbb{T}^{(1)}_{0i} = -c\sum_a m_a v^i_a \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)) = -m_a c v^i_a, \tag{4.13a}$$

$$\mathbb{T}^{(0)}_{ij} = \sum_a m_a v^i_a v^j_a \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)) = m_a (v^i_a)^2, \tag{4.13b}$$

now we can equate the field equation 4.1 using 4.12 and 4.6. Such that we obtained the first-order metric components as

$$(i) \quad g^{(2)}_{00} = \frac{2\phi}{c^2}, \tag{4.14a}$$

$$(ii) \quad g^{(2)}_{ij} = \delta_{ij}\frac{2\phi}{c^2}, \tag{4.14b}$$

$$(iii) \quad g^{(3)}_{0i} = \frac{G}{2c^3}\sum_a \frac{m_a}{|\mathbf{r} - \mathbf{r}_a|}\left[7v_{ia} + (v_a \cdot n_a)n_{ia})\right], \tag{4.14c}$$

$$(iv) \quad g^{(4)}_{00} = \frac{2\phi^2}{c^4} - \frac{G}{c^4}\sum_a \frac{m_a(3(v_a)^2 + \phi')}{|\boldsymbol{x} - \boldsymbol{x}_a(t)|} \tag{4.14d}$$

where a new potential is defined here as

$$\phi'_a = -G{\sum_b}' \frac{m_b}{|\mathbf{x_a} - \mathbf{x_b}|}. \tag{4.15}$$

Refer to the appendix section (A.3) for detailed calculations to obtain the component values of metric tensor 4.14a, 4.14b, A.24, and A.26c by equating the field equation 4.1 order by order. We have also introduced a new potential ψ given as

$$\nabla^2\psi = \frac{1}{c^4}\frac{\partial^2\phi}{\partial t^2} + \frac{4\pi G}{c^4}\left[{}^{(2)}T^{00} +^{(2)} T^{ii}\right] \tag{4.16a}$$

$$\Rightarrow \psi(t, \boldsymbol{x}) = -\int \frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|}\left\{\frac{1}{4\pi}\partial^2_0\phi + \frac{G}{c^4}\left[{}^{(2)}T^{00}(t, \boldsymbol{x}') +^{(2)} T^{ii}(t, \boldsymbol{x}')\right]\right\} \tag{4.16b}$$

then the defined form of $g^{(4)}_{00}$ is

$$g^{(4)}_{00} = \frac{2}{c^4}\left(\phi^2 + \psi\right). \tag{4.17}$$

Therefore the final metric for the first-order perturbation as

$$g_{00} = -1 + \frac{2\phi}{c^2} + \frac{2}{c^4}\left(\phi^2 + \psi\right) + \mathcal{O}(c^{-6}), \tag{4.18a}$$

$$g_{0i} = -\frac{4V_i}{c^3} + \mathcal{O}(c^{-5}), \tag{4.18b}$$

$$g_{ij} = \left(1 + \frac{2\phi}{c^2}\right)\delta_{ij} + \mathcal{O}(c^{-4}). \tag{4.18c}$$

Let us introduce a quantity $V = c^2(\phi + \psi)$. So 4.18a in the form of potentials, we can write it as

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.19a}$$

$$g_{0\boldsymbol{i}} = -\frac{4}{c^3}V_i + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{4.19b}$$

$$g_{ij} = \delta_{ij} + \frac{2}{c^2}V + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.19c}$$

where V, V_i, ψ are potentials as a function of $T^{\mu\nu}$ for some position vector $\boldsymbol{x}_a$ of particle a given as

$$V(t,\boldsymbol{x}) = -\frac{G}{c^2}\int d^3x' \frac{T^{00}(t,\boldsymbol{x}') + T^{ii}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{4.20a}$$

$$V_i(t,\boldsymbol{x}) = -\frac{G}{c}\int d^3x' \frac{T^{0i}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{4.20b}$$

$$\psi(t,\boldsymbol{x}) = -\int \frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}\left\{\frac{1}{4\pi}\partial_0^2\phi + \frac{G}{c^2}\left[T^{00}(t,\boldsymbol{x}') + T^{ii}(t,\boldsymbol{x}')\right]\right\}. \tag{4.20c}$$

we can expand the potentials V and V_i at large distances from the source, i.e. at $r << d$, using

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{r} + \frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{r^3} + \dots \tag{4.21}$$

The Next step is to estimate all the components of the Ricci tensor, where the Ricci tensor is given in terms of Christoffel symbols and Christoffels in terms of the metric tensor and equate them with the effective energy-momentum tensor.

Following this route, using the above metric tensor 4.18a we obtain the first-order post-

Newtonian Lagrangian as

$$
\begin{aligned}
L =& - mc\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} && \text{(4.22a)}\\
=& - mc^2\sqrt{-g_{00} - 2g_{0j}v^j/c - g_{jk}v^jv^k/c^2} && \text{(4.22b)}\\
=& \frac{1}{2}v_a^2 m_a + \frac{1}{2}v_b^2 m_b + \frac{Gm_bm_a}{r_{ab}} + \frac{1}{8c^2}m_bv_b^4 && \text{(4.22c)}\\
&+ \frac{Gm_am_b}{2c^2\mathbf{r}_{ab}}\left\{3(v_a^2+v_b^2) - \left[7\boldsymbol{v}_a\cdot\boldsymbol{v}_b + (\boldsymbol{v}_a\cdot\boldsymbol{n}_{ab})(\boldsymbol{v}_b\cdot\boldsymbol{n}_{ab})\right] - \frac{G(m_a+m_b)}{\mathbf{r}_{ab}}\right\}, && \text{(4.22d)}
\end{aligned}
$$

where $\mathbf{r}_{ab}$ is the separation vector between the two particles, $\boldsymbol{n}_{ab} = |\mathbf{r}|/r$ the normal along r_{ab}, and $\boldsymbol{v}_i$ is the particle velocity (for $i = a, b$).

4.1.2 2PN expansion for non-spinning compact binaries

In this section, we have derived the Lagrangian for non-spinning two-body dynamics using a computational tool, Mathematica. To get 2PN Lagrangian, we have to perturb the metric field and matter field till the order of $\mathcal{O}((v/c)^4)$. Therefore we will use the expansions 4.2 and 4.3 in the field equation 2.1. By using the results for first-order, as derived in the previous section, we can obtain the second-order metric correction terms $g_{00}^{(6)}$, $g_{0i}^{(5)}$, and $g_{ij}^{(4)}$ by equating the field equation 4.1 for second-order. In order to get it, we have used the harmonic gauge condition $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$ for the metric tensor given in 4.2 and obtained the Ricci tensors as shown in appendix A.5.

Now we have to equate the components Ricci tensor, A.5 with effective energy-momentum tensor using the Einstein field equations 2.1. Therefore expanding the expression for energy-momentum tensor 4.10 till the order of $\mathcal{O}(c^{-4})$ by imposing $d\tau_a/dt$ upto order $\mathcal{O}(v^4/c^4)$ as

$$\frac{d\tau_a}{dt} = 1 + \frac{1}{c^2}\left(\phi - \frac{v_a^2}{2}\right) + \frac{1}{c^4}(\phi^2 + \psi - 4V_iv_a^i + \phi v_{ai}v_a^i). \tag{4.23}$$

Using Mathematica we have obtained the second-order post-Newtonian, 2PN Lagrangian, which is given as

$$\begin{aligned}
\mathcal{L}_{\mathrm{2PN}} = &\frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2 + \frac{G m_a m_a}{r_{ab}} + \frac{1}{8} m_a v_a^4 + + \frac{1}{8c^2} m_b v_b^4 + \frac{G m_a m_b}{2c^2 r_{ab}} \left[3v_a^2 + 3v_b^2 \right. \\
&\left. -7 \left(v_a \cdot v_b\right) - \left(n \cdot v_a\right)\left(n \cdot v_b\right) \right] - \frac{G^2 m_a m_b}{2c^2 r_{ab}^2} (m_a + m_b) + \frac{1}{16c^2} m_a v_a^6 + \frac{1}{16c^2} m_b v_b^6 \\
&+ \frac{G m_a m_b}{c^4 r_{ab}} \left[\frac{7}{8} v_a^4 + \frac{7}{8} v_b^4 + \frac{11}{8} v_a^2 v_b^2 - \frac{7}{4} \left(v_a^2 + v_b^2\right) \left(v_a \cdot v_b\right) + \frac{1}{4} (v_a \cdot v_b)^2 - \frac{5}{8} (n \cdot v_a)^2 v_b^2 \right. \\
&- \frac{5}{8} (n \cdot v_b)^2 v_a^2 + \frac{3}{2} (n \cdot v_a)(n \cdot v_b)(v_a \cdot v_b) - \frac{1}{4} (v_a^2 + v_b^2) \left(n \cdot v_a\right) \left(n \cdot v_b\right) + \frac{3}{8} (n \cdot v_a)^2 (n \cdot v_b)^2] \\
&\left. - \frac{G^2 M m_a m_b}{4c^4 r_{ab}^2} \left[v_a^2 + v_b^2 - 2(v_a \cdot v_b) \right] \right] + \frac{G^2 m_a m_b^2}{8c^4 r_{ab}^2} [18 v_a^2 + 13 v_b^2 - 34 (v_a \cdot v_b) + 15 (n \cdot v_b)^2] \\
&+ \frac{\mathrm{G}^2 \mathrm{m}_a^2 \mathrm{m}_b}{8 \mathrm{c}^4 \mathrm{r_{ab}}^2} \left[13 v_a^2 + 18 v_b^2 - 34 \left(v_a \cdot v_b\right) + 15 \left(n \cdot v_a\right)^2 \right] + \frac{G^3 M^2 m_a m_b}{4c^4 r_{ab}^3} \left(1 + 3\nu\right),
\end{aligned} \tag{4.24}$$

where $M = (m_a + m_b)$ is the total mass, and $\nu := (m_a m_b)/M^2$ is the symmetric mass ratio. This has been previously derived in [23]. In the above Lagrangian 4.24, the subscripts a, b represent the objects, and n_{ab} (n) is the unit vector between a and b.

4.2 Post-Newtonian approximation method in scalar-tensor theory

The post-Newtonian approximation method for scalar-tensor theory was first formulated by T. Damour and G. Esposito-Farese [19]. This section considers the field equations as derived in the previous section, equation 3.8. To achieve the first-order post-Newtonian limit in scalar-tensor theory, we start by assuming that at spatial infinity, the metric becomes asymptotically flat, and the scalar field goes to φ_0. (Note: for some function f, $f^{(0)} \equiv f_0 \equiv f(\varphi_0) \equiv f^0$.) In order to analyze the scalar fields and metric in the near zone of matter source, we utilize an expansion about a parameter ϵ as

$$\varphi = \varphi_0 + \varphi_{\mathrm{lin}} + \left(c^{-4}\right). \tag{4.25}$$

where φ_{lin} itself represents $\epsilon^2 \varphi_{\mathrm{lin}}$. By using the metric expansion derived previously 4.19, we can derive potentials as a function of both the metric and scalar field. The metric will have additional terms given as

$$\tilde{g}_{00} = -1 + \frac{2}{c^2}V - 2(\tilde{\beta} - 1)\frac{V^2}{c^4} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.26a}$$

$$\tilde{g}_{0\boldsymbol{i}} = -\frac{2}{c^3}(\tilde{\gamma} + 1)V_i + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{4.26b}$$

$$\tilde{g}_{ij} = \delta_{ij} + \frac{2}{c^2}\tilde{\gamma}V + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.26c}$$

here $\tilde{\gamma}$ and $\tilde{\beta}$ are the additional parameters defined for the parametrized post-Newtonian metric as defined in section 3 of [19]. Due to the additional scalar field, equation 4.20 will get modified and require additional term

$$\Box_g V = -4\pi G\sigma \tag{4.27a}$$

$$\Box_g V^i = -4\pi G\sigma^i \tag{4.27b}$$

$$\Box_g \varphi = -4\pi\frac{G}{c^2}S \tag{4.27c}$$

here $\Box_g$ is the d'Alambertian, a scalar operator which represents $\Box_g = g^{\mu\nu}\partial_\mu\partial_\nu$

$$\sigma \equiv \frac{(T^{00} + T^{ii})}{c^2} = \frac{{}^{(0)}T^{(00)} + {}^{(2)}T^{(00)} + {}^{(2)}T^{(ii)}}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.28a}$$

$$\sigma^i \equiv \frac{T^{0i}}{c} = \frac{{}^{(1)}T^{0i}}{c} + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{4.28b}$$

$$S \equiv \alpha(\varphi)\sqrt{-g}\frac{T}{c^2} = -\alpha(\varphi)\frac{(T^{00} - T^{ii})}{c^2} = \frac{{}^{(2)}T^{(ii)} - {}^{(0)}T^{(00)} - {}^{(2)}T^{(00)}}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{4.28c}$$

To obtain the potentials from 4.27 we need to us the convolution by retarded greens function given as $\mathcal{R}$,

$$\mathcal{R}[\sigma](t, x) \equiv \int \mathrm{d}^3x' \frac{\sigma(t - |x - x'|/c, x')}{|x - x'|} \tag{4.29}$$

using 4.29 we can write the 4.27c as

$$\varphi = \varphi_0 + \frac{G}{c^2}\mathcal{R}\left[\alpha(\varphi)\sqrt{-g}\frac{T}{c^2}\right] + \mathcal{O}\left(\frac{1}{c^4}\right) \tag{4.30}$$

up to linear order of φ_{lin}. The lowest-order (linearized) solution of 4.27c is given as

$$\begin{aligned}\varphi_{\text{lin}} &= \varphi_0 + \frac{G}{c^2}\mathcal{R}\left[\alpha(\varphi_0)\sqrt{-g}\frac{T}{c^2}\right] + \mathcal{O}\left(\frac{1}{c^4}\right) \\ &= \varphi_0 - \frac{1}{c^2}\alpha(\varphi_0)V + \mathcal{O}\left(\frac{1}{c^4}\right).\end{aligned} \tag{4.31}$$

As defined in the previous section, the $A(\varphi^a)$ gives the coupling of the scalar to the matter field such that the metric can be given as (using the rescaling of the metric 3.5)

$$\begin{aligned}\mathrm{d}s^2 &= g_{\mu\nu}(x^\lambda)\mathrm{d}x^\mu\mathrm{d}x^\nu \\ \mathrm{d}\widetilde{s}^2 &= \widetilde{g}_{\mu\nu}(x^\lambda)\mathrm{d}x^\mu\mathrm{d}x^\nu = \Omega^2\mathrm{d}s^2,\end{aligned} \tag{4.32}$$

in which the matter is universally coupled such that we can define

$$T_{\mu\nu} \equiv \frac{2c}{\sqrt{g}}\frac{\delta S_m[\psi_m, \Omega^2 g_{\mu\nu}]}{\delta g^{\mu\nu}}. \tag{4.33}$$

Due to the motion of particles and gravitational wave generation by two self-gravitating objects, let's fix some auxiliary conditions

$$\alpha_a(\varphi) \equiv \frac{\partial \ln \Omega}{\partial \varphi^a} \equiv \alpha(\varphi_0) + \alpha(\varphi_{\text{lin}}) \tag{4.34a}$$

$$\beta_a(\varphi) \equiv \frac{d\alpha_a}{d\varphi}, \tag{4.34b}$$

$$\beta'_a(\varphi) \equiv \frac{d\beta_a}{d\varphi} \tag{4.34c}$$

$$M_a(z_a) \equiv \left(1 + \frac{v_A^2}{c^2}\right) m_a(\varphi_a)\,[g(z_a)]^{-1/2}\left[-g_{\mu\nu}(z_a)\frac{v_a^\mu}{c}\frac{v_a^\nu}{c}\right]^{-1/2} \tag{4.34d}$$

$$\sigma_a \equiv -\alpha_a(\varphi_a)m_a(\varphi_a)\left[-g_{\mu\nu}(z_a)\frac{v_a^\mu}{c}\frac{v_a^\nu}{c}\right]^{1/2} \tag{4.34e}$$

where $m_a(\varphi)$ represents the mass of particle a^{th} coupled with the scalar field, z_a denotes the coordinates of the centre of mass of body a, and $v_a \equiv dz_a/dt$ its velocity, and M_a is the auxiliary mass term. In general relativity, far from infinity, the metric $g_{\mu\nu}$ approaches $\eta_{\mu\nu}$, which means that the mass $m_a(\varphi_0) \equiv m_a^0$ remains constant. However, due to matter coupling with the scalar field in the Einstein frame, masses $m_a(\varphi)$ vary depending on φ. The equation 4.34a measures the effective coupling constant between the scalar fields and the a^{th} compact body. For the scalar-tensor case, let us consider m_a^0 as a known function.

At negligible self-gravity region

$$\alpha_a \to \alpha = \frac{d\ln\mathcal{A}}{d\varphi}, \quad \beta_a \to \beta \equiv \frac{d\alpha}{d\varphi}, \quad \beta'_a \to \beta' = \frac{d\beta}{d\varphi}, \tag{4.35}$$

the energy-momentum tensor is given as

$$T^{\mu\nu}(x^i,t) = \sum_a M_a(t)\frac{v_a^\mu v_a^\nu}{1+v_a^2/c^2}\delta^{(3)}\left(x^i - z_a^i(t)\right), \tag{4.36}$$

the scalar function S

$$S(x,t) = \sum_a \sigma_a(t)\delta^{(3)}\left(x - z_a(t)\right), \tag{4.37}$$

where $\delta^{(3)}$ represents the three-dimensional Dirac delta function. Thereby using σ-mainfold coordinates, which are geodesic at the spatial infinity limit φ_0, the potentials are given by

$$\begin{aligned}
\Box_g V &= -4\pi G\sum_a M_a(t)\delta^{(3)}(x-z_a(t)) + \mathcal{O}\left(\frac{1}{c^4}\right),\\
\Box_g V^i &= -4\pi G\sum_A M_a(t)v_a^i(t)\delta^{(3)}(x-z_a(t)) + \mathcal{O}\left(\frac{1}{c^5}\right),\\
\Box_g \varphi &= -4\pi\frac{G}{c^2}\sum_A \sigma_a(t)\delta^{(3)}(x-z_a(t)) + \mathcal{O}\left(\frac{1}{c^4}\right),
\end{aligned} \tag{4.38}$$

by taking the convolution, we get

$$\begin{aligned}
V &= V(t,x) = G\sum_a M_a(t)f_a(t,x) + \mathcal{O}\left(\frac{1}{c^4}\right),\\
V^i &= V^i(t,x) = G\sum_A M_a(t)v_a^i f_a(t,x) + \mathcal{O}\left(\frac{1}{c^5}\right),\\
\varphi_{\text{lin}} &= \varphi_{\text{lin}}(t,x) = \varphi_0 + \frac{G}{c^2}\sum_a \sigma_a(t)f_a(t,x) + \mathcal{O}\left(\frac{1}{c^4}\right).
\end{aligned} \tag{4.39}$$

Here $M_a(t)$ and $\sigma_a(t)$ from 4.34d 4.34e can be simplified up to order of $\mathcal{O}(c^{-2})$, that can be given as

$$\begin{aligned}
M_a(t) &= m_a(\varphi_a)\left(1 + \frac{3}{2}\frac{v_a^2}{c^2} - \frac{V_a(z_a)}{c^2}\right) + \mathcal{O}(c^{-4}),\\
\sigma_a(t) &= -\alpha_a(\varphi_a)m_a(\varphi_a)\left(1 - \frac{1}{2}\frac{v_a^2}{c^2} - \frac{V_a(z_a)}{c^2}\right) + \mathcal{O}(c^{-4}).
\end{aligned} \tag{4.40}$$

In these equations, $m_a(\varphi_a)$ can be expanded around the spatial infinity value $\varphi_0 = 0$. Therefore the expansion can be given as

$$m_a(\varphi_a) = m_a^{(0)}\left[1 + \alpha_a^{(0)}\varphi^a + \frac{1}{2}(\alpha_a^{(0)}\alpha_a^{(0)} + \beta_a^{(0)})\varphi^a\varphi^a\right] + \mathcal{O}\left(c^{-6}\right) \tag{4.41}$$

using 4.41 in the 4.40 we can get the simplified auxiliary mass term 4.34d and $\sigma_a(t)$ in 4.34e as

$$M_a(t) = m_a^{(0)}\left[1 + \frac{3}{2}\frac{v_a^2}{c^2} - \frac{G}{c^2}\sum_{b\neq a}\left(1 + \alpha_a^{(0)}\alpha_b^{(0)}\right)\frac{m_b^{(0)}}{r_{ab}}\right] + \mathcal{O}\left(c^{-4}\right), \tag{4.42a}$$

$$\sigma_a(t) = -m_a^{(0)}\left[\alpha_a^{(0)}\left(1 - \frac{1}{2}\frac{v_a^2}{c^2}\right) - \frac{G}{c^2}\sum_{b\neq a}\left\{\alpha_a^{(0)}\left(1 + \alpha_a^{(0)}\alpha_b^{(0)}\right) + \beta_a^{(0)}\alpha_b^{(0)}\right\}\frac{m_b^{(0)}}{r_{ab}}\right] + \mathcal{O}\left(c^{-4}\right). \tag{4.42b}$$

To write the potentials 4.39 in a simpler form we can simplify by equating scalar potential with the scalar mass. Similarly, in vector potential, 'vector masses' and tensorial potential with 'tensor masses'. That is the mass M_a can be split into three parts, $M_a = M_a^S + M_a^V + M_a^T$ where $M_a^S = M_a - \alpha^{(0)}\sigma_a(t)$, $M_a^V = M_a$, and $M_a^T = M_a + \alpha^{(0)}\sigma_a(t)$. Therefore, we get the masses as given in [24]

$$M_a^S = m_a^{(0)}\Bigg[\left(1 + \alpha_b^{(0)}\alpha_a^{(0)}\right) + \frac{1}{2c^2}\left(3 - \alpha_b^{(0)}\alpha_a^{(0)}\right)v_a^2 - \frac{G}{c^2}\sum_{b\neq a}\Bigg\{\left(1 + \alpha_b^{(0)}\alpha_a^{(0)}\right) \tag{4.43a}$$

$$\left(1 + \alpha_a^{(0)}\alpha_b^{(0)}\right) + \alpha_a^{(0)}\beta_a^{(0)}\alpha_b^{(0)}\Bigg\}\frac{m_a^{(0)}}{r_{ab}}\Bigg] + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.43b}$$

$$M_a^V = m_a^{(0)} + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{4.43c}$$

$$M_a^{\mathrm{T}} = m_a^{(0)}\left(1 - \alpha_b^{(0)}\alpha_a^{(0)}\right) + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{4.43d}$$

Hence, the potentials 4.39 can be rewritten in a simple form as

$$\tilde{V} = G \sum_a M^S f_a + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.44a}$$

$$\tilde{V}^i = G \sum_a M^V v_a^i f_a + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{4.44b}$$

$$\widetilde{V}_{ij} = G \sum_a M_a^{\mathcal{T}} \delta_{ij} f_a + O\left(\frac{1}{c^2}\right), \tag{4.44c}$$

$$\widetilde{W} = G^2 \sum_a \sum_b \beta^{(0)} \alpha_a^{(0)} \alpha_b^{(0)} m_a^{(0)} m_b^{(0)} f_a f_b. \tag{4.44d}$$

Comparing the equations 4.44 with 4.26, we have absorbed a factor of $(\tilde{\gamma}+1)/2$ in the definition of vector potential 4.44b. Whereas the additional scalar potential $\widetilde{W}$ corresponds to the first-order post-Newtonian metric component correction of post-Newtonian metric expansion in 4.26 as $2(\tilde{\beta}-1)V^2 = W$. Therefore, the metric will become

$$\tilde{g}_{00} = -A_0^2 \exp\left[-\frac{2}{c^2}\widetilde{V} + \frac{1}{c^4}\widetilde{W}\right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.45a}$$

$$\tilde{g}_{0i} = -\frac{4}{c^3} A_0^2 \widetilde{V}_i + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{4.45b}$$

$$\tilde{g}_{ij} = A_0^2 \left[\delta_{ij} + \frac{2}{c^2}\widetilde{V}_{ij}\right] + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{4.45c}$$

Lagrangian for the conformally transformed scalar field to Einstein frame can be given as

$$L = -m_a(\varphi(r_a))c\sqrt{-\tilde{g}(r_a)_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \tag{4.46a}$$

$$L = -m_a(\varphi(r_a))c\sqrt{-\tilde{g}(r_a)_{\mu\nu} v_a^\mu v_a^\nu} \tag{4.46b}$$

$$= -m_a(\varphi(r_a))c^2\sqrt{-\tilde{g}_{00} - 2\tilde{g}_{0j}v^j/c - \tilde{g}_{jk}v^j v^k/c^2} \tag{4.46c}$$

$$= -m_a(\varphi(r_a))c^2\sqrt{A_0^2 \exp\left[-\frac{2}{c^2}\widetilde{V} + \frac{1}{c^4}\widetilde{W}\right] + \frac{4}{c^3}A_0^2\widetilde{V}_i\frac{v^i}{c} - A_0^2\left[\delta_{ij} + \frac{2}{c^2}\widetilde{V}_{ij}\right]\frac{v^i v^j}{c^2}} \tag{4.46d}$$

We should note that the scaling factor between Einstein and Jordan frame goes to 1 at infinity, i.e. $A_0 \equiv A(\varphi_0) = 1$, so the Lagrangian will be

$$L = -m_a(\varphi(r_a))c^2 \sqrt{\exp\left[-\frac{2}{c^2}\widetilde{V} + \frac{1}{c^4}\widetilde{W}\right] + \frac{8}{c^3}\widetilde{V}_i \frac{v^i}{c} - \left[\delta_{ij} + \frac{2}{c^2}\widetilde{V}_{ij}\right]\frac{v^i v^j}{c^2}} \tag{4.47a}$$

$$= -m_a(\varphi(r_a))c^2 \left(1 - \frac{1}{c^2}\widetilde{V} + \frac{1}{2c^4}\widetilde{W} + \frac{4}{c^4}\widetilde{V}_i v^i - \delta_{ij}\frac{v^i v^j}{2c^2} - \frac{1}{c^4}\widetilde{V}_{ij} v^i v^j\right) \tag{4.47b}$$

After considering the initial order of mass $m_A = m_A^0$ the Lagrangian will become

$$L = -m_a^0 c^2 + m_a^0 \tilde{V} - \frac{1}{2c^2} m_a^0 \widetilde{W} - \frac{4}{c^2} m_a^0 \tilde{V}_{ai} v_a^i + \frac{1}{c^2} m_a^0 \tilde{V}_{ij} v^i v^j + \frac{1}{8c^2} m_a^0 v_a^4 + \frac{1}{2} m_a^0 v_a^2. \tag{4.48}$$

Now we can solve for the individual terms in 4.48, solving for the second term in 4.48 using 4.44a, solving for the third term in 4.48 using 4.44d, solving for the fourth term in eq4.48 using 4.44b, and solving for the fifth term in eq4.48 using eq4.44c are given in the appendix A.4. Using the individual expressions as obtained in A.29,A.30, A.31, and A.32 in eq4.48, we have obtained the Lagrangian for the n-body system, the total Lagrangian can be given as

$$L = \sum_a L_a^{(1)} + \frac{1}{2}\sum_a \sum_{b \neq a} L_{ab}^{(2)} + \frac{1}{2}\sum_a \sum_{b \neq a} \sum_{c \neq a} L_{bc}^{a(3)} + O\left(\frac{1}{c^4}\right). \tag{4.49}$$

Whereas we can write $(\alpha_a \alpha_b)_0 \equiv [\alpha_a \alpha_b]_{\varphi_0}$ and $(\alpha_b \beta_a \alpha_c)_0 \equiv [\alpha_b \beta_a \alpha_c]_{\varphi_0}$ at some the spatial infinity point as φ_0. We get the simplified form of expression 4.48 as [25]

$$L_a^{(1)} = -\, m_a^0 c^2 \sqrt{1 - \frac{v_a^2}{c^2}} \tag{4.50a}$$

$$= -\, m_a^{(0)} c^2 + \frac{1}{2} m_a^{(0)} v_a^2 + \frac{1}{8c^2} m_a^{(0)} v_a^4 + O\left(\frac{1}{c^4}\right) \tag{4.50b}$$

$$L_{ab}^{(2)} = \frac{G m_a^0 m_b^0}{r_{ab}} \left\{ [1 + (\alpha_a \alpha_b)_0] + \frac{1}{2c^2}[3 - (\alpha_a \alpha_b)_0](v_a^2 + v_b^2) \right. \tag{4.50c}$$

$$\left. - \frac{1}{2c^2}[7 - (\alpha_a \alpha_b)_0](v_a \cdot v_b) - \frac{1}{2c^2}[1 + (\alpha_a \alpha_b)_0](n_{ab} \cdot v_a)(n_{ab} \cdot v_b) \right\} \tag{4.50d}$$

$$L_{bc}^{a(3)} = -\, \frac{G^2 m_a^{(0)} m_b^{(0)} m_c^{(0)}}{c^2 r_{ab} r_{ac}} \left\{ [1 + (\alpha_a \alpha_b)_0][1 + (\alpha_a \alpha_c)_0] + (\alpha_b \beta_a \alpha_c)_0 \right\}. \tag{4.50e}$$

It is an interesting outcome that Newtonian gravitational coupling ($1/r^2$, matter coupling with spacetime is given by gravitational constant G) between two bodies with some inertial masses can be described by now an effective coupling constant in Einstein frame for $A_0 = 1$

as

$$G_{ab} = G[1 + (\alpha_a \alpha_b)_0]. \tag{4.51}$$

Hence, if we write the equations 4.50a, 4.50d, and 4.50e in terms of G_{ab} by taking $[1+(\alpha_A\alpha_B)_0]$ factor common we will get the first-order post-Newtonian Lagrangian in scalar-tensor theory [25] as

$$L = - m_a^0 - m_b^0 + L_{\mathrm{K}} + L_{1\mathrm{PK}} + L_{2\mathrm{PK}} + \cdots , \tag{4.52a}$$

$$L_{\mathrm{K}} = \frac{1}{2} m_a^0 v_a^2 + \frac{1}{2} m_b^0 v_b^2 + \frac{G_{ab} m_a^0 m_b^0}{r_{ab}}, \tag{4.52b}$$

$$L_{1PK} = \frac{1}{8} m_A^0 v_a^4 + \frac{1}{8} m_b^0 v_b^4 + \frac{G_{ab} m_a^0 m_b^0}{r_{ab}} \Bigg(\frac{3}{2}(v_a^2 + v_b^2) - \frac{7}{2} \vec{v}_a \cdot \vec{v}_b - \frac{1}{2}(\vec{n_{ab}} \cdot \vec{v}_a) \tag{4.52c}$$

$$(\vec{n_{ab}} \cdot \vec{v}_b) + \vec{\gamma}_{ab}(\vec{v}_a - \vec{v}_b)^2 \Bigg) - \frac{G_{ab}^2 m_a^0 m_b^0}{2 r_{ab}^2} \left(m_a^0 (1 + 2\vec{\beta}_b) + m_b^0 (1 + 2\bar{\beta}_a) \right) \tag{4.52d}$$

where the auxiliary terms are given as

$$\bar{\gamma}_{ab} = - \frac{2(\alpha_a \alpha_b)_0}{[1 + (\alpha_a \alpha_b)_0]}, \;\; \bar{\beta}_a \equiv \frac{1}{2} \frac{\beta_a^0 (\alpha_a^0 \alpha_b^0)}{(1 + \alpha_a^0 \alpha_b^0)^2}, \;\; \bar{\beta}_b \equiv \frac{1}{2} \frac{\beta_b^0 (\alpha_a^0 \alpha_b^0)}{(1 + \alpha_a^0 \alpha_b^0)^2}. \tag{4.53a}$$

4.3 Post-Newtonian approximation in $f(R)$ gravity

In this section, we will discuss the post-Newtonian limit of $f(R)$ for the harmonic gauge [26]. We will focus on the limit where the velocity is small, and the field is weak using the metric approach. It is important to check at any level our modification in the weak-field limit using $f(R)$ gravity should recover the outcome by GR. We can obtain the scalarized $f(R)$ field equation by using 3.12 as

$$H = g^{\mu\nu} H_{\mu\nu} = 3\Box f'(R) + f'(R)R - 2f(R) = \kappa^2 T \tag{4.54}$$

4.3.1 1PN expanded field equations and the metric tensor

In this section, we will consider the first-order post-Newtonian correction in the metric tenor $g_{\mu\nu}$ as 4.4. Such that we can write the Ricci scalar as $R \sim R^{(2)} + R^{(4)} + \cdots$ and n^{th} order derivative of $f(R)$ as

$$f^n(R) \sim f^n(R) \sim \sum_{i\geq 0} \frac{f^{n+i}(0)}{i!}[R]^i \tag{4.55a}$$

$$\sim f^n(0) + f^{n+1}(0)[R^{(2)} + R^{(4)} + \cdots] + \frac{f^{n+2}}{2!}[R^{(2)} + R^{(4)} + \cdots] + \cdots \tag{4.55b}$$

using 3.12 at the lowest order $f(0) = 0$. If the Lagrangian is developable around a vanishing value of the Ricci scalar, the relation $f(0) = 0$ implies y that the cosmological constant contribution has to be zero for $f(R)$ gravity [26]. So we get simplified 4.55a as

$$f\left(R\right) \sim f\left(0\right) + f'\left(0\right)R^{(2)} + f'\left(0\right)R^{(4)} + \frac{1}{2}f''\left(0\right)\left(R^{(2)}\right)^2 + \cdots \tag{4.56a}$$

$$f'(R) \sim f'(0) + f''(0)R^{(2)} + f''(0)R^{(4)} + \frac{1}{2}f'''(0)(R^{(2)})^2 + \cdots \tag{4.56b}$$

$$f''(R) \sim f''(0) + \cdots \tag{4.56c}$$

Using 4.56 in 3.12, we can obtain the field equation for the Newtonian order, i.e., $H_{00}^{(2)}$ as [26]

$$H_{00}^{(2)} := f'(0)R_{00}^{(2)} - \frac{f'(0)R^{(2)}}{2}g_{00} - \partial_0\partial_0 f'(0)R^{(2)} + g_{00}\Box[f'(0) + f''(0)R^{(2)}] \tag{4.57a}$$

$$= f'(0)R_{00}^{(2)} + \frac{f'(0)}{2}R^{(2)} - f''(0)\nabla^2 R^{(2)} = \kappa^2 T_{00}^{(0)} \tag{4.57b}$$

similarly, we can obtain the field equation for first post-Newtonian order, that are, $H_{ij}^{(2)}$, $H_{00}^{(4)}$, and $H_{0i}^{(3)}$ as [26]

$$H_{ij}^{(2)} := \kappa^2 T_{ij}^{(0)} = f'(0)R_{ij}^{(2)} - \frac{f'(0)}{2}R^{(2)} + f''(0)\nabla^2 R^{(2)} + f''(0)R_{,ij}^{(2)} = 0, \tag{4.58a}$$

$$H_{0i}^{(3)} := \kappa^2 T_{0i}^{(1)} = f'(0)R_{0i}^{(3)} + f'(0)R_{,0i}^{(2)}, \tag{4.58b}$$

$$\begin{aligned} H_{00}^{(4)} =& f''(0)R^{(2)}R_{00}^{(2)} + f'(0)R_{00}^{(4)} + \frac{f'(0)}{2}R^{(4)} + \frac{1}{4}f''(0)R^{(2)2} - \frac{f'(0)}{2}g_{00}^{(2)}R^{(2)} \\ & - f''(0)\left\{g_{ij}^{(2)}R_{,ij}^{(2)} + g_{ij,i}^{(2)}R_{,j}^{(2)} - \frac{1}{2}\nabla g_{ii}^{(2)}\nabla R^{(2)} - \nabla^2 R^{(4)} + g_{00}^{(2)}\nabla^2 R^{(2)}\right\} \\ & + f'''(0)\left\{(\nabla R^{(2)})^2 + R^{(2)}\nabla^2 R^{(2)}\right\} = \kappa^2 T_{00}^{(2)}. \end{aligned} \tag{4.59}$$

If we scalarize the particular order we will get

$$H^{(2)} := 3f''(0)\nabla^2 R^{(2)} - f'(0)R^{(2)} = \kappa^2 T^{(0)}, \tag{4.60}$$

$$\begin{aligned} H^{(4)} &= 3f''(0)\nabla^2 R^{(4)} - f'(0)R^{(4)} + 3f'''(0)\left[|\nabla R^{(2)}|^2 + R^{(2)}\nabla^2 R^{(2)}\right] \\ &+ 3f''(0)\left[-R^{(2)}_{,00} - g^{(2)}_{ij}R^{(2)}_{,ij} - \frac{1}{2}\nabla(g^{(2)}_{00} - g^{(2)}_{ii})\cdot\nabla R^{(2)} - g^{(2)}_{ij,i}R^{(2)}_{,j}\right] = \kappa^2 T^{(2)} \end{aligned} \tag{4.61}$$

Note that the Ricci Scalar R and Ricci tensor are the same as the ones we have derived in GR. The simplified expression for components of Ricci tensor in the harmonic gauge ($g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0$) are given in 4.6. By introducing the Green function of the field operator $\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = \nabla^2 - \zeta^2$ the solution for the Ricci scalar $R^{(2)}$ 4.60 is given by

$$R^{(2)}(t, \boldsymbol{x}) = \frac{m^2\kappa^2}{f'(0)} \int d^3\boldsymbol{x}'\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}')T^{(0)}(t, \boldsymbol{x}') \tag{4.62}$$

where $\zeta := f'(0)/3f''(0)$, and $T^{(0)}_{00}$ as define in 4.11. Using 4.6a in field equation 4.57b we have obtain $g^{(2)}_{00}$ as

$$g^{(2)}_{00}(t, \boldsymbol{x}) = \frac{\kappa^2}{2\pi f'(0)} \int d^3\mathbf{x}' \frac{T^{(0)}_{00}(t, \mathbf{x}')}{|\boldsymbol{x} - \mathbf{x}'|} + \frac{1}{4\pi} \int d^3\mathbf{x}' \frac{R^{(2)}(t, \mathbf{x}')}{|\boldsymbol{x} - \mathbf{x}'|} - \frac{2}{3\zeta^2}R^{(2)}(t, \boldsymbol{x}) \tag{4.63}$$

similarly using 4.58b and 4.6c we get $g^{(3)}_{0i}$ as

$$g^{(3)}_{0i}(t, \boldsymbol{x}) = -\frac{\kappa^2}{2\pi f'(0)} \int d^3\mathbf{x}' \frac{T^{(1)}_{0i}(t, \mathbf{x}')}{|\boldsymbol{x} - \mathbf{x}'|} + \frac{1}{6\pi\zeta^2}\frac{\partial}{\partial t} \int d^3\mathbf{x}' \frac{\nabla_{i'}R^{(2)}(t, \mathbf{x}')}{|\boldsymbol{x} - \mathbf{x}'|} \tag{4.64}$$

similarly using 4.59 and 4.6d we get $g^{(2)}_{ij}$ as

$$\begin{aligned} g^{(2)}_{ij}(t, \boldsymbol{x}) = &\left[-\frac{1}{4\pi} \int d^3\boldsymbol{x}' \frac{R^{(2)}(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} + \frac{2}{3\zeta^2}R^{(2)}(t, \boldsymbol{x}) + \frac{1}{6\pi\zeta^2}\frac{1}{|\boldsymbol{x}|^3} \int_{\Omega_{|\boldsymbol{x}|}} d^3\boldsymbol{x}' R^{(2)}(t, \boldsymbol{x}')\right]\delta_{ij} \\ &- \left[\frac{1}{2\pi m^2|\boldsymbol{x}|^3} \int_{\Omega_{|\boldsymbol{x}|}} d^3\boldsymbol{x}' R^{(2)}(t, \boldsymbol{x}') + \frac{2}{3m^2}R^{(2)}(t, \boldsymbol{x})\right]\frac{x_i x_j}{|\boldsymbol{x}|^2} \end{aligned} \tag{4.65}$$

where $\Omega_{|\boldsymbol{x}|}$ represents the integration volume with radius $|\boldsymbol{x}|$. Now, to obtain the $g_{00}^{(4)}$ we require $R^{(4)}$, which is the solution of equation A.12 is given as

$$
\begin{aligned}
R^{(4)}(t,\mathbf{x}) = \int d^3\mathbf{x}'\mathcal{G}(\mathbf{x},\mathbf{x}') \Big\{ & \frac{\zeta^2\kappa}{f'(0)}T^{(2)}(t,\mathbf{x}') - g_{ij,i}^{(2)}(t,\mathbf{x}')R_{,j}^{(2)}(t,\mathbf{x}') - g_{ij}^{(2)}(t,\mathbf{x}')R_{,ij}^{(2)}(t,\mathbf{x}') \\
& - R_{,00}^{(2)}(t,\mathbf{x}') + \frac{\zeta^2}{\xi^4}\left[|\nabla_{\mathbf{x}'}R^{(2)}(t,\mathbf{x}')|^2 + R^{(2)}(t,\mathbf{x}')\nabla^2_{\mathbf{x}'}R^{(2)}(t,\mathbf{x}')\right] \\
& -\frac{1}{2}\nabla_{\mathbf{x}'}\left[g_{00}^{(2)}(t,\mathbf{x}') - g_{ii}^{(2)}(t,\mathbf{x}')\right]\cdot\nabla_{\mathbf{x}'}R^{(2)}(t,\mathbf{x}')\Big\}
\end{aligned} \tag{4.66}
$$

where $\xi^4 := f'(0)/3f'''(0)$. Using 4.66 and 4.6b into 4.59 we can get $g_{00}^{(4)}$ as [26]

$$
\begin{aligned}
& g_{00}^{(4)}(t,\mathbf{x}) = \\
& = \int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\Big\{\frac{\kappa^2 T_{00}^{(2)}(t,\mathbf{x}')}{2\pi f'(0)} + \frac{1}{6\pi\zeta^4}\left[|\nabla R^{(2)}(t,\mathbf{x}')|^2 + R^{(2)}(t,\mathbf{x}')\nabla^2 R^{(2)}(t,\mathbf{x}')\right] \\
& + \frac{1}{6\pi\xi^2}\left[\frac{1}{4}{R^{(2)}}^2(t,\mathbf{x}') - \frac{1}{2}R^{(2)}(t,\mathbf{x}')\nabla^2 g_{00}^{(2)}(t,\mathbf{x}') + g_{ij,i}^{(2)}(t,\mathbf{x}')R^{(2)}{}_{,j}(t,\mathbf{x}') + \nabla^2 R^{(4)}(t,\mathbf{x}')\right. \\
& + \frac{1}{4\pi}\left[g_{ij}^{(2)}(t,\mathbf{x}')g_{00,ij}^{(2)}(t,\mathbf{x}') - g_{00,00}^{(2)}(t,\mathbf{x}') - |\nabla_{\mathbf{x}'}g_{00}^{(2)}(t,\mathbf{x}')|^2 - R^{(4)}(t,\mathbf{x}') - g_{00}^{(2)}(t,\mathbf{x}')R^{(2)}(t,\mathbf{x}')\right] \\
& \left. + g_{00}^{(2)}(t,\mathbf{x}')\nabla^2 R^{(2)}(t,\mathbf{x}') + g_{ij}^{(2)}(t,\mathbf{x}')R^{(2)}{}_{,ij}(t,\mathbf{x}') - \frac{1}{2}\nabla g_{ii}^{(2)}(t,\mathbf{x}')\cdot\nabla R^{(2)}(t,\mathbf{x}')\right]\Big\}
\end{aligned} \tag{4.67}
$$

From the solution of the metric components and curvature 4.63, 4.65, 4.64, 4.67, and 4.66 we can infer that it is possible to have non zero Ricci tensor even in a vacuum unlike GR because now curvature itself can mimic the presence of matter even when no matter is present. Therefore, in theories of gravity in which higher orders of Ricci scalar are included, the curvature can mimic the matter source even in the absence of ordinary matter.

For a sanity check, when $f(R) \to R$ the metric components 4.63, 4.65, 4.64, and 4.67 corresponds to the metric in GR.

4.3.2 2PN expanded field equations

In this section, we have derived the second-order, 2PN dynamics for compact binaries in $f(R)$ gravity. In order to derive that, first, we have to obtain the field equation 3.12 at 2PN order. Using the metric expansion 4.2, the Ricci scalar becomes

$$R \sim R^{(2)}(t,\mathbf{x}) + R^{(4)}(t,\mathbf{x}) + R^{(6)}(t,\mathbf{x}) + \dots \tag{4.68}$$

Therefore the $f(R)$ field equations 3.12 at $\mathcal{O}(c^{-4})$ becomes

$$\begin{aligned} H^{(4)}_{ij} &= f'(0)R^{(4)}_{ij} + f''(0)R^{(2)}R^{(2)}_{ij} - \frac{1}{2}f'(0)\left\{R^{(2)}g^{(2)}_{ij} + R^{(4)}\delta_{ij}\right\} - \frac{1}{4}f''(0)\left(R^{(2)}\right)^2\delta_{ij} \\ &- f''(0)\Bigg\{R^{(4)}_{,ij} - \frac{1}{2}\left(g^{(2)}_{ik,j} + g^{(2)}_{jk,i} - g^{(2)}_{ij,k}\right)R^{(2)}_{,k} + g^{(2)}_{ij}\nabla^2 R^{(2)} + \delta_{ij}\Bigg[-\frac{1}{2}g^{(2)}_{\alpha\alpha,k}R^{(2)}_{,k} \\ &+ g^{(2)}_{\alpha k,\alpha}R^{(2)}_{,k} + \nabla^2 R^{(4)} + g^{(2)\alpha\beta}R^{(2)}_{,\alpha\beta} - R^{(2)}_{,00}\Bigg]\Bigg\} - f'''(0)\delta_{ij}\left[\nabla(R^{(2)})^2 + R^{(2)}\nabla^2 R^{(2)}\right] \\ &= \kappa^2 T^{(2)}_{ij} \end{aligned} \tag{4.69}$$

where ∇ operator represents the gradient in the flat space, while at the order of $\mathcal{O}(c^{-5})$

$$\begin{aligned} H^{(5)}_{0i} =& f'(0)R^{(5)}_{0i} + f''(0)R^{(2)}R^{(3)}_{0i} - \frac{1}{2}f'(0)g^{(3)}_{0i}R^{(2)} - 2f'''(0)\left\{R^{(2)}_{,0}R^{(2)}_{,i} + R^{(2)}R^{(2)}_{,0i}\right\} \\ &- \frac{1}{2}f''(0)\left\{2R^{(4)}_{,0i} - g^{(3)}_{0i,i}R^{(2)}_{i} + g^{(2)}_{00,i}R^{(2)}_{,0} + 2g^{(3)}_{0i}\nabla^2 R^{(2)}\right\} = \kappa^2 T^{(3)}_{0i} \end{aligned} \tag{4.70}$$

the field equation 4.54 contracted with metric tensor at $\mathcal{O}(c^{-6})$ becomes

$$\begin{aligned} H^{(6)} =& 3f''(0)\nabla^2 R^{(6)} - f'(0)\,R^{(6)} + 3f'''(0)\Bigg[\nabla R^{(2)}\nabla R^{(4)} + R^{(4)}\nabla^2 R^{(2)} + R^{(4)}\nabla^2 R^{(4)} \\ &+ \frac{1}{2}R^{(2)}\left(\nabla R^{(2)}\right)^2 + \frac{1}{2}\left(R^{(2)}\right)^2\nabla^2 R^{(2)} + g^{(2)ij}\left(R^{(2)}R^{(2)}_{,ij} + R^{(2)}_{,i}R^{(2)}_{,j}\right)\Bigg] \\ &+ \frac{3f''(0)}{2}\left[g^{(2)ij}\left(g^{(t)}_{mi,j} + g^{(2)}_{mj,i} - g^{(t)}_{ij,m}\right)R^{(2)}_{,m} + 2g^{(2)ij}R^{(4)}_{,ij} + g^{ij}R^{(2)}_{,ij} - g^{(2)}_{00,0}R^{(2)}_{,0}\right. \\ &\left.+ \left(2g^{(2)}_{0\lambda,0} - g^{(4)}_{00,\lambda}\right)R^{(2)}_{,\lambda} + \left(2g^{(3)}_{0\lambda,0} - g^{(4)}_{00,\lambda}\right)R^{(2)}_{,\rho} + R^{(4)}_{,00}\right] \\ &- 3f'''(0)\left[\left(R^{(2)}_{,0}\right)^2 + R^{(2)}R^{(2)}_{,00}\right] - 3f''(0)\left[g^{(2)00}R^{(2)}_{,00} + g^{(3)0i}R^{(2)}_{,0i}\right] \\ &+ \frac{f'''(0)}{6}\left(R^{(2)}\right)^3 = \kappa^2 T^{(4)}. \end{aligned} \tag{4.71}$$

Now by introducing the Green function of the field operator $\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = (\nabla^2 - \zeta^2)$ the solution for the Ricci scalar $R^{(6)}$ using 4.71 is

$$\begin{aligned} R^{(6)}(t,\mathbf{x}) = \int d^3\mathbf{x}'\mathcal{G}(\mathbf{x},\mathbf{x}')\Bigg\{ & \frac{\zeta^2\kappa^2}{f'(0)}T^{(4)}(t,\mathbf{x}') + g^{(2)ij}(t,\mathbf{x}')R^{(4)}_{,ij}(t,\mathbf{x}') + \frac{1}{2}g^{ij}(t,\mathbf{x}')R^{(2)}_{,ij}(t,\mathbf{x}') \\ & - \frac{1}{2}g^{(2)}_{00,0}(t,\mathbf{x}')R^{(2)}_{,0}(t,\mathbf{x}') + \frac{1}{2}g^{(2)ij}(t,\mathbf{x}')\left(g^{(t)}_{mi,j}(t,\mathbf{x}') + g^{(2)}_{mj,i}(t,\mathbf{x}') - g^{(t)}_{ij,m}(t,\mathbf{x}')\right)R^{(2)}_{,m}(t,\mathbf{x}') \\ & + \frac{1}{2}\left(2g^{(2)}_{0\lambda,0}(t,\mathbf{x}') - g^{(4)}_{00,\lambda}(t,\mathbf{x}')\right)R^{(2)}_{,\lambda}(t,\mathbf{x}') + \frac{1}{2}\left(2g^{(3)}_{0\lambda,0}(t,\mathbf{x}') - g^{(4)}_{00,\lambda}(t,\mathbf{x}')\right)R^{(2)}_{,\rho}(t,\mathbf{x}') \\ & + \frac{1}{2}R^{(4)}_{,00}(t,\mathbf{x}') - g^{(2)00}(t,\mathbf{x}')R^{(2)}_{,00}(t,\mathbf{x}') - g^{(3)0i}(t,\mathbf{x}')R^{(2)}_{,0i}(t,\mathbf{x}') \\ & + \frac{\xi^2}{\zeta^4}\Bigg[\left(R^{(2)}_{,0}\right)^2(t,\mathbf{x}') + R^{(2)}(t,\mathbf{x}')R^{(2)}_{,00}(t,\mathbf{x}') + \nabla R^{(2)}(t,\mathbf{x}')\nabla R^{(4)}(t,\mathbf{x}') \\ & + R^{(4)}(t,\mathbf{x}')\nabla^2R^{(2)}(t,\mathbf{x}') + R^{(4)}(t,\mathbf{x}')\nabla^2R^{(4)}(t,\mathbf{x}') + \frac{1}{2}R^{(2)}(t,\mathbf{x}')\left(\nabla R^{(2)}(t,\mathbf{x}')\right)^2 \\ & + g^{(2)ij}(t,\mathbf{x}')\left(R^{(2)}(t,\mathbf{x}')R^{(2)}_{,ij}(t,\mathbf{x}') + R^{(2)}_{,i}(t,\mathbf{x}')R^{(2)}_{,j}(t,\mathbf{x}')\right) \\ & + \frac{1}{2}\left(R^{(2)}(t,\mathbf{x}')\right)^2\nabla^2R^{(2)}(t,\mathbf{x}')\Bigg]\Bigg\} \end{aligned} \tag{4.72}$$

whereas the field equation 3.12 at $\mathcal{O}(c^{-6})$ becomes

$$\begin{aligned} H^{(6)}_{00} =& f'(0)R^{(6)}_{00} + f''(0)R^{(2)}R^{(4)}_{00} + f''(0)R^{(4)}R^{(2)}_{00} + \frac{f'''(0)}{2}R^{(2)^2}R^{(2)}_{00} - \frac{1}{2}\Bigg\{f'(0)R^{(6)} \\ & + f''(0)R^{(2)}R^{(4)} + \frac{f'''(0)}{3!}\left(R^{(2)}\right)^3 + f'(0)g^{(2)}_{00}R^{(4)} + \frac{f'(0)}{2!}g^{(2)}_{00}\left(R^{(2)}\right)^2 + f'(0)g^{(4)}_{00}R^{(2)}\Bigg\} \\ & - \frac{f''(0)}{2}\Bigg\{ - g^{(2)}_{00,0}R^{(2)}_{,0} + \left(2g^{(2)}_{0k,0} - g^{(4)}_{00,k}\right)R^{(2)}_{,k} + \left(2g^{(2)}_{0k,0} - g^{(4)}_{00,k}\right)R^{(2)}_{,\rho} + 2R^{(4)}_{,00} \\ & - 2\nabla^2R^{(6)} - \left(2g^{(4)}_{mn,n} - g^{(4)}_{nn,m}\right)R^{(2)}_{,m} - g^{(2)mn}\left(2g^{(2)}_{mn,n} - g^{(2)}_{nn,m}\right)R^{(2)}_{,n} \\ & + \left(2g^{(3)}_{0n,n} - g^{(2)}_{nn,0}\right)R^{(2)}_{,0} - \left(2g^{(2)}_{mn,n} - g^{(2)}_{nn,m}\right)R^{(4)}_{m} + 2R^{(4)}_{,00} - 2g^{(2)nl}R^{(4)}_{,nl} \\ & - g^{(2)nl}\left(g^{(2)}_{mn,l} + g^{(2)}_{ml,n} - g^{(2)}_{nl,m}\right)R^{(2)}_{,m} - 2g^{(2)nl}\left(2R^{(2)}_{,n}R^{(2)}_{,l} + 2R^{(2)}R^{(2)}_{,nl}\right) - 2g^{(2)00}R^{(2)}_{,00} \\ & - 2g^{(4)nl}R^{(2)}_{,nl} - g^{(2)}_{00}\left(2g^{(2)}_{mn,n} - g^{(2)}_{nn,m}\right)R^{(2)}_{,m} - 2g^{(2)}_{00}\nabla^2R^{(4)} - 2g^{(2)nl}R^{(2)}_{,nl}\Bigg\} \end{aligned} \tag{4.73}$$

$$
\begin{aligned}
&+ f'''(0)\Bigg\{ \left(R^{(2)}_{,0}\right)^2 + R^{(2)}R^{(2)}_{,00} + R^{(4)}\nabla^2 R^{(2)} + R^{(2)}\nabla^2 R^{(4)} + 2\nabla R^{(2)}\nabla R^{(4)} \\
&+ R^{(2)}\left(\nabla R^{(2)}\right)^2 + \frac{1}{2}\left(R^{(2)}\right)^2\nabla^2 R^{(2)} - \left(R^{(2)}_{,0}R^{(2)}_{,0} + R^{(2)}R^{(2)}_{,00}\right) \\
&+ \frac{1}{4}\left(2g^{(2)}_{mn,n} - g^{(2)}_{nn,m}\right)R^{(4)}_{,m} + g^{(2)}_{00}\left(\nabla R^{(2)}\right)^2 + g^{(2)}_{00}R^{(2)}\nabla^2 R^{(2)}\Bigg\}.
\end{aligned}
\tag{4.74}
$$

Using these field equations 4.69, 4.70, and 4.73 we can obtain the metric correction at 2PN order, and by using these metric correction we can obtain the 2PN Lagrangian. To get the metric component for $g^{(4)}_{ii}$, we have to equate the $H^{(4)}_{ii}$ with $T^{(2)}_{ii}$ in the field equation 4.69. So we have obtained the $g^{(4)}_{ii}$ as

$$
\begin{aligned}
g^{(4)}_{ii}(t,\mathbf{x}) = &-\int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\Bigg\{\frac{\kappa^2 T^{(2)}_{ii}(t,\mathbf{x}')}{2\pi f'(0)} + \frac{1}{4\pi}\left[\mathcal{X}^{(4)}_{ii} + \frac{1}{2}\left(R^{(2)}g^{(2)}_{ii} + R^{(4)}\right)\right] \\
&- \frac{1}{6\pi\zeta^4}\left[|\nabla R^{(2)}(t,\mathbf{x}')|^2 + R^{(2)}(t,\mathbf{x}')\nabla^2 R^{(2)}(t,\mathbf{x}')\right] \\
&- \frac{1}{6\pi\xi^2}\Bigg[R^{(4)}_{,ii} - \frac{1}{2}\left(2g^{(2)}_{ik,i} - g^{(2)}_{ii,k}\right)R^{(2)}_{,k} + g^{(2)}_{ii}\nabla^2 R^{(2)} - \frac{1}{2}g^{(2)}_{mm,k}R^{(2)}_{,k} \\
&+ g^{(2)}_{mk,m}R^{(2)}_{,k} + \nabla^2 R^{(4)} + g^{(2)mn}R^{(2)}_{,mn} - R^{(2)}_{,00} + \frac{1}{4}\left(R^{(2)}\right)^2\Bigg]\Bigg\},
\end{aligned}
\tag{4.75}
$$

where $\mathcal{X}^{(4)}_{ii}$ is the expression appear in the second order $R^{(4)}_{ii}$ as shown in A.33 for $\mathcal{X}^{(4)}_{11}$, A.35 for $\mathcal{X}^{(4)}_{22}$, and A.37 for $\mathcal{X}^{(4)}_{33}$. Similarly, we have to equate the $H^{(5)}_{0i}$ with $T^{(3)}_{0i}$ in the field equation 4.70. We have obtained the $g^{(5)}_{0i}$ as

$$
\begin{aligned}
g^{(5)}_{0i}(t,\mathbf{x}) = &-\int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\Bigg\{\frac{\kappa^2 T^{(3)}_{0i}(t,\mathbf{x}')}{2\pi f'(0)} - \frac{1}{4\pi}\left[\mathcal{X}^{(5)}_{ii} + f'(0)g^{(3)}_{0i}(t,\mathbf{x}')R^{(2)}(t,\mathbf{x}')\right] \\
&+ \frac{1}{12\pi\xi^2}\Bigg[2R^{(4)}_{,0i}(t,\mathbf{x}') - g^{(3)}_{0i,i}(t,\mathbf{x}')R^{(2)}_{i}(t,\mathbf{x}') + g^{(2)}_{00,i}(t,\mathbf{x}')R^{(2)}_{,0}(t,\mathbf{x}') \\
&+ 2g^{(3)}_{0i}(t,\mathbf{x}')\nabla^2 R^{(2)}(t,\mathbf{x}') - 2f''(0)R^{(2)}R^{(3)}_{0i}(t,\mathbf{x}')\Bigg] \\
&+ \frac{1}{3\pi\zeta^4}\left[R^{(2)}_{,0}(t,\mathbf{x}')R^{(2)}_{,i}(t,\mathbf{x}') + R^{(2)}(t,\mathbf{x}')R^{(2)}_{,0i}(t,\mathbf{x}')\right]\Bigg\}
\end{aligned}
\tag{4.76}
$$

where $\mathcal{X}^{(5)}_{0i}$ is the expression appear in the second order $R^{(5)}_{0i}$ as shown in A.39 for $\mathcal{X}^{(5)}_{01}$, and A.41 for $\mathcal{X}^{(5)}_{01}$. Similarly, we can obtain the $g^{(6)}_{00}$ component by using the 4.73 field equation.

We have obtained it as

$$
\begin{aligned}
g_{00}^{(6)}(t,\mathbf{x}) = &-\int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\Bigg\{\frac{\kappa^2 T_{00}^{(4)}(t,\mathbf{x}')}{2\pi f'(0)} - \frac{1}{4\pi}\Bigg\{\mathcal{X}_{00}^{(6)}(t,\mathbf{x}') + R^{(6)}(t,\mathbf{x}') \\
&+ g_{00}^{(2)}(t,\mathbf{x}')R^{(4)}(t,\mathbf{x}') + g_{00}^{(2)}(t,\mathbf{x}')\left(R^{(2)}(t,\mathbf{x}')\right)^2 + 2g_{00}^{(4)}(t,\mathbf{x}')R^{(2)}(t,\mathbf{x}')\Bigg\} \\
&+ \frac{1}{6\pi\xi^2}\Bigg\{R^{(2)}(t,\mathbf{x}')R_{00}^{(4)}(t,\mathbf{x}') + R^{(4)}(t,\mathbf{x}')R_{00}^{(2)}(t,\mathbf{x}') - \frac{1}{2}\Bigg[R^{(2)}(t,\mathbf{x}')R^{(4)}(t,\mathbf{x}') \\
&- g_{00,0}^{(2)}(t,\mathbf{x}')R_{,0}^{(2)}(t,\mathbf{x}') + \left(2g_{0k,0}^{(2)}(t,\mathbf{x}') - g_{00,k}^{(4)}(t,\mathbf{x}')\right)R_{,k}^{(2)}(t,\mathbf{x}') + \Big(2g_{0k,0}^{(2)}(t,\mathbf{x}') \\
&- g_{00,k}^{(4)}(t,\mathbf{x}')\Big)R_{,\rho}^{(2)}(t,\mathbf{x}') + 2R_{,00}^{(4)}(t,\mathbf{x}') - 2\nabla^2 R^{(6)}(t,\mathbf{x}') - \big(2g_{mn,n}^{(4)}(t,\mathbf{x}') \\
&- g_{nn,m}^{(4)}(t,\mathbf{x}')\big)R_{,m}^{(2)}(t,\mathbf{x}') - g^{(2)mn}(t,\mathbf{x}')\left(2g_{mn,n}^{(2)}(t,\mathbf{x}') - g_{nn,m}^{(2)}(t,\mathbf{x}')\right)R_{,n}^{(2)}(t,\mathbf{x}') \\
&+ \left(2g_{0n,n}^{(3)}(t,\mathbf{x}') - g_{nn,0}^{(2)}(t,\mathbf{x}')\right)R_{,0}^{(2)}(t,\mathbf{x}') - \left(2g_{mn,n}^{(2)}(t,\mathbf{x}') - g_{nn,m}^{(2)}(t,\mathbf{x}')\right)R_{m}^{(4)}(t,\mathbf{x}') \\
&+ 2R_{,00}^{(4)}(t,\mathbf{x}') - 2g^{(2)nl}(t,\mathbf{x}')R_{,nl}^{(4)}(t,\mathbf{x}') - g^{(2)nl}(t,\mathbf{x}')\Big(g_{mn,l}^{(2)}(t,\mathbf{x}') + g_{ml,n}^{(2)}(t,\mathbf{x}') \\
&- g_{nl,m}^{(2)}(t,\mathbf{x}')\Big)R_{,m}^{(2)}(t,\mathbf{x}') - 2g^{(2)nl}(t,\mathbf{x}')\left(2R_{,n}^{(2)}(t,\mathbf{x}')R_{,l}^{(2)}(t,\mathbf{x}') + 2R^{(2)}(t,\mathbf{x}')R_{,nl}^{(2)}(t,\mathbf{x}')\right) \\
&- 2g^{(2)00}(t,\mathbf{x}')R_{,00}^{(2)}(t,\mathbf{x}') - 2g^{(4)nl}(t,\mathbf{x}')R_{,nl}^{(2)}(t,\mathbf{x}') - g_{00}^{(2)}(t,\mathbf{x}')\big(2g_{mn,n}^{(2)}(t,\mathbf{x}') \\
&- g_{nn,m}^{(2)}(t,\mathbf{x}')\big)R_{,m}^{(2)}(t,\mathbf{x}') - 2g_{00}^{(2)}(t,\mathbf{x}')\nabla^2 R^{(4)}(t,\mathbf{x}') - 2g^{(2)nl}(t,\mathbf{x}')R_{,nl}^{(2)}(t,\mathbf{x}')\Bigg]\Bigg\} \\
&+ \frac{1}{6\pi\zeta^4}\Bigg\{\frac{1}{2}R^{(2)}(t,\mathbf{x}')^2 R_{00}^{(2)}(t,\mathbf{x}') + \frac{1}{6}\left(R^{(2)}(t,\mathbf{x}')\right)^3 + \left(R_{,0}^{(2)}(t,\mathbf{x}')\right)^2 \\
&+ R^{(2)}(t,\mathbf{x}')R_{,00}^{(2)}(t,\mathbf{x}') + R^{(4)}(t,\mathbf{x}')\nabla^2 R^{(2)}(t,\mathbf{x}') + R^{(2)}(t,\mathbf{x}')\nabla^2 R^{(4)}(t,\mathbf{x}') \\
&+ 2\nabla R^{(2)}(t,\mathbf{x}')\nabla R^{(4)}(t,\mathbf{x}') + R^{(2)}(t,\mathbf{x}')\left(\nabla R^{(2)}(t,\mathbf{x}')\right)^2 + \frac{1}{2}\left(R^{(2)}(t,\mathbf{x}')\right)^2\nabla^2 R^{(2)}(t,\mathbf{x}') \\
&- \left(R_{,0}^{(2)}(t,\mathbf{x}')R_{,0}^{(2)}(t,\mathbf{x}') + R^{(2)}(t,\mathbf{x}')R_{,00}^{(2)}(t,\mathbf{x}')\right) + \frac{1}{4}R_{,m}^{(4)}(t,\mathbf{x}')\big(2g_{mn,n}^{(2)}(t,\mathbf{x}') \\
&- g_{nn,m}^{(2)}(t,\mathbf{x}')\big) + g_{00}^{(2)}(t,\mathbf{x}')\left(\nabla R^{(2)}(t,\mathbf{x}')\right)^2 + g_{00}^{(2)}(t,\mathbf{x}')R^{(2)}(t,\mathbf{x}')\nabla^2 R^{(2)}(t,\mathbf{x}')\Bigg\}\Bigg\},
\end{aligned}
\tag{4.77}
$$

where $\mathcal{X}_{00}^{(6)}$ are the expression appear in the second order $R_{00}^{(6)}$ as shown in A.33. These second-order post-Newtonian results are derived for the generic $f(R)$ gravity. We can introduce the energy-momentum tensors for the two-body system to get the two-body dynamics in $f(R)$ gravity. We conclude by showing the harmonic gauge condition satisfies the general solution of $f(R)$ field equations at Newtonian, first post-Newtonian, and second post-Newtonian limits. Such that we have successfully obtained the metric components $g_{ij}^{(4)}$, $g_{0i}^{(5)}$, and $g_{00}^{(6)}$ for harmonic gauge at 2PN order in $f(R)$ gravity.

Chapter 5

Effective-one-body formalism for non-spinning binaries

In this section, we study the conservative dynamics of non-spinning compact binaries in the EOB framework. The EOB framework maps the two-body motion onto the motion of a test mass particle in an effective external metric (deformed Schwarzschild metric). The total mass of the system is $M = m_1 + m_2$, which is the background mass, while the mass of the test body is the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$, with the deformation parameter $\nu = m_1 m_2/(m_1 + m_2)^2 = \mu/M$. The EOB approach has three parts: conservative dynamics, dynamics due to radiation reaction forces by accounting for the energy-momentum loss, and the gravitational waveform by resumming the EOB coefficients with the NR results. This work contains the description of the conservative EOB dynamics till 2PN order [27, 28].

In the conservative regime, the effective metric is static and spherically symmetric deformation of the Schwarzschild metric. To get the effective metric, we have to use the *non-perturbative re-summation* technique. The mapping to the effective metric can be done separately for conservative dynamics (time-symmetric gravitational wave solution) and radiation reaction (time-antisymmetric gravitational wave solution). For conservative dynamics, the mapping can be interpreted in quantum terms. It is a mapping between the discrete energy spectrum of bound states, equivalent to the electromagnetically interacting quantum two-body system. This can be found in the work of Damour [29].

To obtain the 2PN Hamiltonian in EOB, the centre-of-mass frame Hamiltonian in the Arnwitt, Deser, and Misner (ADM) coordinate system must first be obtained. Further, the real two-body energy levels are mapped to the EOB energy levels. So that we can get an effective metric with the re-summed energy level, which allows us to define the dynamics of test mass, μ motion in an effective metric.

To associate the "real" two-body dynamics with the "effective" dynamics in the external spacetime, we can use the action given by [27]

$$S_{\text{eff}} = -\int m_0 c ds_0. \tag{5.1}$$

where m_0 represents the effective mass of the test particle in the effective metric, and ds_0 is the effective metric element. Here, we have discussed how the Hamiltonian for the EOB conservative dynamics can be obtained for simplicity in the 2PN order. The effective background metric is defined as

$$g^{\text{eff}}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -A(R)\mathrm{d}t^2 + B(R)\mathrm{d}r^2 + C(R)r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{5.2}$$

in the Schwarzchild coordinates ($C(R) = 1$), the metric functions are given by

$$A(R) = 1 + \frac{a_1}{c^2R} + \frac{a_2}{c^4R^2} + \frac{a_3}{c^6R^3} + \cdots, \tag{5.3a}$$

$$B(R) = 1 + \frac{b_1}{c^2R} + \frac{b_2}{c^4R^2} + \cdots. \tag{5.3b}$$

The metric element and the deformed mass-shell (energy constraint) constraint [30] are given as

$$\Rightarrow ds_0^2 = -g^{\text{eff}}_{\mu\nu}dx^{\mu}dc^{\nu} \tag{5.4a}$$

$$= -A(R)dt^2 - \frac{dr^2}{A(R)} - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{5.4b}$$

$$\Rightarrow \mu^2 = -g^{\mu\nu}_{\text{eff}}p_{\mu}p_{\nu}, \tag{5.4c}$$

where the motion is considered to be planar such that $d\theta^2 = 0$. By using 5.4 and 5.4c we can get the expression for the effective Hamiltonian as

$$-\frac{1}{A}\mathcal{H}_{\text{eff}}^2 + \frac{p_r^2}{B^2} + \frac{p_\phi^2}{r^2} = -\mu^2, \tag{5.5a}$$

$$\mathcal{H}_{\text{eff}} = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{r^2}\right)}. \tag{5.5b}$$

where $\mathcal{H}_{\text{eff}}$ is the Hamiltonian associated with the test mass motion in the effective framework, p_r is the redial momentum, and p_ϕ represents the angular momentum. In the limit $\nu \to 0$, $\mathcal{H}_{\text{eff}}$ reduced to the Schwarzchild Hamiltonan (where $A = 1 - 2GM/r$, $B = 1/A$) is given as

$$\mathcal{H}_{\text{Schw}} = \sqrt{\left(1 - \frac{2M}{r}\right)\left[\mu^2 + \left(1 - \frac{2M}{r}\right)p_r^2 + \frac{p_\phi^2}{r^2}\right]}. \tag{5.6}$$

The two-body Hamiltonian $\mathcal{H}_{\text{EOB}}$ can be defined by the energy map with the effective Hamiltonian $\mathcal{H}_{\text{eff}}$, one defines

$$\mathcal{H}_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}, \tag{5.7}$$

where $\nu := \mu/M$ is the symmetric mass ratio, as it is shown on [31] the EOB mapping can reproduce the 1PN two-body Hamiltonian when $\mathcal{H}_{\text{eff}} = \mathcal{H}_{Schw}$. To obtain the energy mapping for the order of 2PN, we first have to obtain the Hamiltonian at the 2PN order and its energy level.

5.1 Conservative Hamiltonian till 2PN

Let us recall the post-Newtonian dynamics of the two-body system. The 2PN truncated equation of motion defines the time-symmetric dynamics obtained from a generalized Lagrangian, as shown in [32], as a function of harmonic positions, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, velocities $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and acceleration $\boldsymbol{a}_1$, $\boldsymbol{a}_2$. In the De Donder coordinate (generic coordinate condition, $\partial_\nu h^{\mu\nu} = 0$) system, the Lagrangian will contain acceleration such that the whole dynamics stays invariant under the linear representation of the Poincare group: $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$. This Poincare symmetry has been explicitly checked at the 2PN level in [33], which shows the 2PN acceleration-dependent Lagrangian in harmonic condition changed only by total time derivative under an infinitesimal coordinate transformation.

The invariance leads to the construction of ten relativistic conserved quantities: linear momentum P, angular momentum L, energy E, and centre-of-mass constant $K = -Pt + G$, where G is the centre-of-mass position. Because of the freedom to perform the coordinate transformation, we can go to the centre-of-mass frame such that $P = G = K = 0$. However, for the existing restricted class of coordinates system where the 2PN Lagrangian only depends on relative positions and relative velocities but not the acceleration. In particular, the coordinates were introduced by Arnowitt, Deser and Misner (ADM). More precisely, we can find the coordinate transformation from De Donder to ADM by a canonical transformation [34]

$$\mathbf{q}_a(t) = \mathbf{x}_a(t) - \delta^* \mathbf{x}_a(x, \boldsymbol{v}), \tag{5.8}$$

where $\mathbf{q}_a(t)$ represents the motion in ADM coordinates and $\mathbf{x}_a(t)$ represents the motion in the De Donder (Harmonic) coordinates, $(a = 1, 2)$. It is not a point transformation (refers to a transformation that maps points in the configuration space of a system to other points in that space) but a higher-order transformation (contact transformation, it is a type of transformation that involves not only points but also higher-order quantities like velocities or momenta). Such that the action is invariant under this transformation.

To get the ADM Hamiltonian using the above transformation, we need the explicit expression of N, 1PN, and 2PN contribution to the Hamiltonian in an arbitrary reference frame, using the Lagrangian in harmonic gauge as a function of position $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, velocities, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and acceleration, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ of the two objects. In other words, we can not do Legendre transformation directly because the mapping between velocity and momentum is not one-to-one. Hence, we can use the Lagrange multipliers to make the mapping one-to-one so that the action becomes linear in the inverse metric (interaction of the test body is somewhat linear in $g^{\mu\nu}$ metric.) Back to our treatment using 5.8 for two-particle case (for $a = 1$). Therefore the 2PN truncated Lagrangian can be written as

$$\mathcal{L}(x, v, a) = \mathcal{L}_N(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) + \mathcal{L}_{1PN}(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) + \mathcal{L}_{2PN}(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{a}_1, \boldsymbol{a}_2). \tag{5.9}$$

The Lagrangian of order $\mathcal{L}_N(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) + \mathcal{L}_{1PN}(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$ equals 4.22d and the Lagrangian of order $\mathcal{L}_{2PN}(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{a}_1, \boldsymbol{a}_2)$, 2PN derived GR Lagrangian. To obtain the 1PN Hamiltonian, we need to perform the contact transformation. Using Noether's theorem, the conservation of the linear momentum of the system is given by (till 1PN order, $\mathcal{P}_{PN}$)

$$\mathcal{P}_{PN} = \frac{\partial(\mathcal{L}_N + \mathcal{L}_{1PN})}{\partial v_1} + \frac{\partial(\mathcal{L}_N + \mathcal{L}_{1PN})}{\partial v_2}, \tag{5.10}$$

and the relativistic centre of mass position is given by

$$G_{PN} = \sum_{a,b\neq a} \left(m_a + \frac{1}{2}\frac{m_a {v_a}^2}{c^2} - \frac{1}{2}\frac{G m_a m_b}{r c^2} \right) r_a, \tag{5.11}$$

where m_a is the mass, r_a is the position, and v_a is the velocity of particle a, $(a = 1, 2)$. Defining the instantaneous relative position as $\mathbf{r} = r_1 - r_2$, and instantaneous relative velocity as $\mathbf{v} = v_1 - v_2$. Define $\mu := m_1 m_2/(m_1 + m_2)$ as the reduced mass and $M = m_1 + m_2$ as the total mass. The post-Newtonian centre of mass $P_{PN} = K_{PN} = G_{PN} = 0$ by using Poincare transformation gives

$$r_a = \frac{\mu}{m_a}\mathbf{r} + \frac{\mu(m_a - m_b)}{2M^2c^2}\left(\boldsymbol{v}^2 - \frac{GM}{\mathbf{r}} \right)\mathbf{r}, \tag{5.12a}$$

$$r_b = -\frac{\mu}{m_b}\mathbf{r} + \frac{\mu(m_b - m_a)}{2M^2c^2}\left(\boldsymbol{v}^2 - \frac{GM}{\mathbf{r}} \right)\mathbf{r}. \tag{5.12b}$$

Such that the problem of two-body motion is now reduced to the *relative motion* of reduced mass in the centre-of-mass frame. Similarly, we can get the position and the velocities in the centre-of-mass frame using 5.12. To get the *non-relativistic* positions and velocity we will rescale them as

$$r_{aN} := \frac{\mu}{m_a}\mathbf{r}, \qquad v_{aN} := \frac{\mu}{m_a}\mathbf{v}, \quad r_{bN} := -\frac{\mu}{m_b}\mathbf{r}, \qquad v_{bN} := \frac{\mu}{m_b}\mathbf{v}. \tag{5.13}$$

By introducing the following linear change of spatial variables in the post-Newtonian Lagrangian ($\mathcal{L}_1 = \mathcal{L}_N + \mathcal{L}_{1PN}$), $\mathcal{L}_{1PN}(\mathbf{r}, dr_a/dt, dr_b/dt) : (r_a, r_b) \to (\mathbf{r}, \mathbf{R})$ with $\mathbf{R} := (m_a r_a + m_b r_b)/M$, we can write

$$r_a = r_{aN} + \mathbf{R}, \quad r_a = r_{aN} + \mathbf{R}, \quad v_a = v_{aN} + \mathbf{V}, \quad v_b = v_{bN} + \mathbf{V} \tag{5.14}$$

where $V := d\mathbf{R}/dt$. We can express $\mathcal{L}_1 = \mathcal{L}_N + \mathcal{L}_{1PN}$ as

$$\mathcal{L}_{PN} = \frac{1}{2}M\boldsymbol{v}^2 + \frac{1}{2}\mu\boldsymbol{v}^2 + \frac{G\mu M}{\mathbf{r}} + \frac{1}{c^2}\mathcal{L}_{1PN}\left(\mathbf{r}, \frac{\mu\mathbf{v}}{m_a} + \mathbf{V}, -\frac{\mu\mathbf{v}}{m_b} + \mathbf{V} \right) \tag{5.15}$$

we have discarded the $(1/2)M\boldsymbol{v}^2$ term as it does not give any contribution. Taking variation $(1/\mu)(\delta\mathcal{L}_{PN}/\delta\mathbf{r}) = 0$ one can see that the $\delta/\delta\mathbf{r}$ does not act on $\boldsymbol{v}$ at the order of $\mathcal{O}(c^{-2})$ but arise at the order of $\mathcal{O}(c^{-4})$ as a consequence we can get the relative Lagrangian as

$$\mathcal{L}^R_{PN}(\mathbf{r},\mathbf{v}) = \frac{1}{2}\mu\boldsymbol{v}^2 + \frac{G\mu M}{\mathbf{r}} + \frac{1}{c^2}\mathcal{L}_{1PN}\left(\mathbf{r}, \frac{\mu\mathbf{v}}{m_a}, -\frac{\mu\mathbf{v}}{m_b}\right). \tag{5.16}$$

Therefore, the full Lagrangian rescaled by the factor of $G\mu$ and defined in terms of symmetric mass ratio $\nu := \mu/M$ will be

$$\mathcal{L}^R_{PN} = \frac{1}{2}\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{v}} + \frac{\mu}{r} + \frac{1-3\nu}{8c^2}\left(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{v}}\right)^2 - \frac{1}{2c^2}\left(\frac{\mu}{r}\right)^2 + \frac{\mu}{2rc^2}\left[(3+\nu)\,\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{v}} + \frac{\nu}{r^2}\left(\mathbf{r}\cdot\dot{\boldsymbol{v}}\right)^2\right]. \tag{5.17}$$

Now let us do the contact transformation for 2PN Lagrangian using the 5.8 as given in [35]

$$\delta^*\boldsymbol{x}_1 = \frac{1}{m_a}\frac{\partial L_{2PN}}{\partial\boldsymbol{a}_1} + \frac{1}{m_1}\frac{\partial F(z,v)}{\partial\boldsymbol{v}_1}, \tag{5.18}$$

where

$$\frac{1}{m_1}\frac{\partial L_{2PN}}{\partial\boldsymbol{a}_1} = -\frac{Gm_2}{c^4}\left\{\frac{7}{4}\left(\boldsymbol{n}_{12}\cdot\boldsymbol{v}_2\right)\boldsymbol{v}_2 - \boldsymbol{n}_{12}\left[\frac{7}{8}{v_2}^2 - \frac{1}{8}\left(\boldsymbol{n}_{12}\cdot\boldsymbol{v}_2\right)^2\right]\right\}, \tag{5.19a}$$

$$F = \sum\frac{Gm_1m_2}{c^4}\left(\boldsymbol{n}\cdot\boldsymbol{v}_1\right)\left[-\frac{1}{4}\boldsymbol{v}_2^2 + \frac{7}{4}\frac{Gm_1}{R} + \frac{1}{4}\frac{Gm_2}{R}\right] \tag{5.19b}$$

therefore, we get $\delta^*\boldsymbol{x}_1$ as [34]

$$\delta^*\boldsymbol{x}_1 = \frac{Gm_2}{c^4}\left\{\boldsymbol{n}\left[\frac{5}{8}{v_2}^2 - \frac{1}{8}\left(\boldsymbol{n}\cdot\boldsymbol{v}_2\right)^2 + \frac{7}{4}\frac{Gm_1}{R} + \frac{1}{4}\frac{Gm_2}{R}\right] + \left(\frac{1}{2}\boldsymbol{v}_1 - \frac{7}{4}\boldsymbol{v}_2\right)\left(\boldsymbol{n}\cdot\boldsymbol{v}_2\right)\right\}. \tag{5.20}$$

This leads to the transformation where the total Lagrangian $L(x,v,a) \Rightarrow L'(x',v',a')$ such that the action remains numerically invariant. Now, we can use $\delta^*\boldsymbol{x_a}$ to transform away the acceleration terms. Therefore, the 2PN Lagrangian in the ADM gauge is given as

$$\mathcal{L}^{\mathrm{ADM}}[\boldsymbol{x},\boldsymbol{v}] = \mathcal{L}[\boldsymbol{x},\boldsymbol{v},\boldsymbol{a}] - \frac{\delta\mathcal{L}}{\delta x^i}\delta^* x_i + \frac{dF}{dt}, \tag{5.21}$$

The rescaled $\delta^* x_i$ can be given as

$$\frac{\delta^* x_i}{m} = \frac{1}{c^4}\left\{\left[\frac{v^2\nu}{8} - \frac{5\nu v^2}{8} + \frac{m}{r}\left(-\frac{1}{4} - 3\nu\right)\right]n + \frac{9v\nu}{4}(n\cdot v)\right\} \tag{5.22}$$

by using a formula

$$\begin{cases} n = 2,3 \Rightarrow x_1^n + x_2^n = 1 - n\nu, \\ n = 4,5 \Rightarrow x_1^n + x_2^n = 1 - n\nu + \dfrac{n(n-3)}{2}\nu^2. \end{cases} \tag{5.23}$$

Using the equations 5.22 and 5.19b to substituting in 5.21 we obtained [36]

$$\begin{aligned} \frac{\mathcal{L}^{\mathrm{ADM}}}{\mu} &= \frac{m}{R} + \frac{v^2}{2} + \frac{1}{c^2}\left\{\frac{v^4}{8} - \frac{3\nu v^4}{8} + \frac{m}{R}\left(\frac{\nu(n\cdot v)^2}{2} + \frac{3v^2}{2} + \frac{\nu v^2}{2}\right) - \frac{m^2}{2R^2}\right\} \\ &+ \frac{1}{c^4}\left\{\frac{v^6}{16} - \frac{7\nu v^6}{16} + \frac{13\nu^2 v^6}{16} + \frac{m}{R}\left(\frac{3\nu^2(n\cdot v)^4}{8} + \frac{\nu(n\cdot v)^2 v^2}{2} - \frac{5\nu^2(n\cdot v)^2 v^2}{4}\right.\right. \\ &+ \left.\frac{7v^4}{8} - \frac{3\nu v^4}{2} - \frac{9\nu^2 v^4}{8}\right) + \frac{m^2}{R^2}\left(\frac{3\nu(n\cdot v)^2}{2} + \frac{3\nu^2(n\cdot v)^2}{2} + 2v^2 - \nu v^2 + \frac{\nu^2 v^2}{2}\right) \\ &+ \left.\frac{m^3}{R^3}\left(\frac{1}{4} + \frac{3\nu}{4}\right)\right\}. \end{aligned} \tag{5.24}$$

Now using the rescaled Lagrangian 5.24, we obtain the Hamiltonian by simply performing

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}. \tag{5.25}$$

In the centre-of-mass frame ADM expression, the total linear momentum $\boldsymbol{P}$ is conserved, simply $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$, where $\boldsymbol{p}_2$ and $\boldsymbol{p}_1$ are the momentum of the particle 1 and particle 2. which implies

$$\boldsymbol{p}_1 \to \boldsymbol{P}, \quad \boldsymbol{p}_2 \to -\boldsymbol{P} \tag{5.26}$$

where $\boldsymbol{P} = \partial S/\partial \boldsymbol{R}$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_1$ are the positions of individual particle. $\boldsymbol{R}$ is the canonical momentum associated to the ADM position vector $\boldsymbol{R} = \boldsymbol{q}_1 + \boldsymbol{q}_2$. The *reduced centre-of-mass* 2PN Hamiltonian $\hat{\mathcal{H}}$ can be obtain by introducing the reduced variables

$$\begin{aligned} \boldsymbol{q} &:= \frac{R}{GM} = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{GM}, & \boldsymbol{p} &:= \frac{P}{\mu} \\ \hat{t} &:= \frac{t}{GM}, & \hat{\mathcal{H}} &:= \frac{\mathcal{H}}{\mu} \end{aligned} \tag{5.27}$$

The rescaled Hamiltonian $\widehat{\mathcal{H}}$ is given by

$$\widehat{\mathcal{H}}(\boldsymbol{q},\boldsymbol{p}) = \widehat{\mathcal{H}}_0(\boldsymbol{q},\boldsymbol{p}) + \frac{1}{c^2}\widehat{\mathcal{H}}_2(\boldsymbol{q},\boldsymbol{p}) + \frac{1}{c^4}\widehat{\mathcal{H}}_4(\boldsymbol{q},\boldsymbol{p}) \tag{5.28}$$

where [27]

$$\begin{aligned}
\widehat{\mathcal{H}}_0(\boldsymbol{q},\boldsymbol{p}) =& \frac{1}{2}\boldsymbol{p}^2 - \frac{1}{\boldsymbol{q}},\\
\widehat{\mathcal{H}}_2(\boldsymbol{q},\boldsymbol{p}) =& -\frac{1}{8c^2}\left(1-3\nu\right)\boldsymbol{p}^4 - \frac{1}{2c^2\boldsymbol{q}}\left[\left(3+\nu\right)\boldsymbol{p}^2 + \nu(\boldsymbol{n}\cdot\boldsymbol{p})^2\right] + \frac{1}{2c^2\boldsymbol{q}^2},\\
\widehat{\mathcal{H}}_4(\boldsymbol{q},\boldsymbol{p}) =& \frac{1}{16c^6}\left(1-5\nu+5\nu^2\right)\boldsymbol{p}^6\\
&+\frac{1}{8c^4\boldsymbol{q}}\left[\left(5-20\nu-3\nu^2\right)\boldsymbol{p}^4 - 2\nu^2\boldsymbol{p}^2(\boldsymbol{n}\cdot\boldsymbol{p})^2 - 3\nu^2(\boldsymbol{n}\cdot\boldsymbol{p})^4\right]\\
&+\frac{1}{2c^4\boldsymbol{q}^2}\left[\left(5+8\nu\right)\boldsymbol{p}^2 + 3\nu\left(\boldsymbol{n}\cdot\boldsymbol{p}\right)^2\right] - \frac{1}{4c^4\boldsymbol{q}^3}\left(1+3\nu\right),
\end{aligned} \tag{5.29}$$

Now, as we have the Hamiltonian, we can write the associated conserved quantities, which are the reduced centre-of-mass energy and the angular momentum of the binary system (calling it as real two-body system):

$$\widehat{\mathcal{H}}\left(\boldsymbol{q},\boldsymbol{p}\right) = \widehat{\mathcal{E}}^{\mathrm{NR}} \equiv \frac{\mathcal{E}^{\mathrm{NR}}_{\mathrm{c.m.}}}{\mu}, \quad \boldsymbol{q}\times\boldsymbol{p} = \boldsymbol{j} \equiv \frac{\mathcal{J}_{\mathrm{c.m.}}}{\mu GM}. \tag{5.30}$$

The $\mathcal{E}^{\mathrm{NR}}$ denotes the non-relativistic energy. As the motion of the binary is restricted to the plane of the relative trajectory so, it is preferred to use the polar coordinates (r,φ) such that

$$q^x = r\cos\varphi, \quad q^y = r\sin\varphi, \quad q^z = 0, \tag{5.31}$$

To do this the separation of time $\hat{t} \equiv t/(GM)$ and angular coordinates φ, we get the reduced action for "real" two-body system

$$\widehat{S} \equiv \frac{S}{\mu GM} = -\widehat{\mathcal{E}}^{\mathrm{NR}}\widehat{t} + j\,\varphi + \widehat{S}_r(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j). \tag{5.32}$$

The radial action variable $\widehat{S}_r(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j)$ is given by

$$\widehat{S}_r(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j) = \int dr\,\sqrt{\mathcal{R}(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j)} \tag{5.33}$$

where the radial 'effective potential' for the radial motion $\mathcal{R}(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j)$ can be obtained by solving 5.29 for $p_r^2 = (\boldsymbol{n}\cdot\boldsymbol{p})^2$. There $\widehat{\mathcal{H}} = E$ as the systems energy is conserved, and $\boldsymbol{p}^2 = (\boldsymbol{n}\cdot\boldsymbol{p})^2 + j^2/r^2$. We get the form of $\mathcal{R}(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j)$ using the Cauchy's integral formula as given in [36]

$$\left\{ R(r,E,j) = A_1 + \frac{2A_2}{r} + \frac{A_3}{r^2} + \frac{A_4}{r^3} + \frac{A_5}{r^4} + \frac{A_6}{r^5} \right. \tag{5.34}$$

where the A_i's are given by

$$\begin{cases} A_1 = 2E + \frac{1}{c^2}(1-3\nu)E^2 - \frac{1}{c^4}\nu(1-4\nu)E^3, \\ A_2 = 1 + \frac{1}{c^2}(4-\nu)E + \frac{1}{c^4}(2-2\nu+\nu^2)E^2, \\ A_3 = -j^2 + \frac{1}{c^2}(6+\nu) + \frac{1}{c^4}15E, \\ A_4 = -\frac{1}{c^2}\nu j^2 + \frac{1}{c^4}\left[\frac{17}{2} + \frac{5}{2}\nu + 2\nu^2 - \nu(1+\nu)Ej^2\right], \\ A_5 = -\frac{1}{c^4}\nu(1+3\nu)j^2, \\ A_6 = \frac{1}{c^4}\frac{3}{4}\nu^2 j^4. \end{cases} \tag{5.35}$$

the E's can be replaced by the $\widehat{\mathcal{E}}^{\mathrm{NR}}$. To obtain the radial action variable I_R and i_r we have to solve the integral of which is given by

$$i_r \equiv \frac{I_R}{\mu GM} \equiv \frac{2}{2\pi}\int_{r_{\min}}^{r_{\max}} dr\sqrt{\mathcal{R}(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j)}, \tag{5.36a}$$

$$I_R \equiv \frac{2}{2\pi}\mu GM\int_{r_{\min}}^{r_{\max}} dr\sqrt{\mathcal{R}(r,\widehat{\mathcal{E}}^{\mathrm{NR}},j)} = \frac{2}{2\pi}\int_{R_{\min}}^{R_{\max}} dR\frac{dS_R(R,\mathcal{E}^{\mathrm{NR}},\mathcal{J})}{dR}, \tag{5.36b}$$

where $\alpha \equiv \mu GM = Gm_1m_2$ and the rescaling factor are given as $\mathcal{E}^{\mathrm{NR}} = \mu\widehat{\mathcal{E}}^{\mathrm{NR}}$ and $J = \alpha j$. After solving 5.36b we get

$$\begin{aligned} I_R(\mathcal{E}^{\mathrm{NR}},\mathcal{J}) = {} & \frac{\alpha\mu^{1/2}}{\sqrt{-2\mathcal{E}^{\mathrm{NR}}}}\left[1 + \left(\frac{15}{4} - \frac{\nu}{4}\right)\frac{\mathcal{E}^{\mathrm{NR}}}{\mu c^2} + \left(\frac{35}{32} + \frac{15}{16}\nu + \frac{3}{32}\nu^2\right)\left(\frac{\mathcal{E}^{\mathrm{NR}}}{\mu c^2}\right)^2\right] \\ & -\mathcal{J} + \frac{\alpha^2}{c^2\mathcal{J}}\left[3 + \left(\frac{15}{2} - 3\nu\right)\frac{\mathcal{E}^{\mathrm{NR}}}{\mu c^2}\right] + \left(\frac{35}{4} - \frac{5}{2}\nu\right)\frac{\alpha^4}{c^4\mathcal{J}^3}. \end{aligned} \tag{5.37}$$

Now, using the action-angle coordinates, Delaunay variables (which are the constant action variable $\mathbf{N} \equiv I_R + \mathcal{J}$), and the constant radial action variables I_R to obtain the constant terms in the corresponding angle variables $\mathcal{J}$ such that we can obtain the Hamiltonian. The relativistic energy for "real" two-body system,the relativistic energy $\mathcal{E}^{\mathrm{R}}$ (superscript denote *relativistic*) is given as

$$\mathcal{E}^{\mathrm{R}}(\mathcal{N},\mathcal{J}) = Mc^2 - \frac{1}{2}\frac{\mu\alpha^2}{\mathcal{N}^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{6}{\mathcal{N}\mathcal{J}} - \frac{1}{4}\frac{15-\nu}{\mathcal{N}^2}\right)\right.$$
$$\left. + \frac{\alpha^4}{c^4}\left(\frac{5}{2}\frac{7-2\nu}{\mathcal{N}\mathcal{J}^3} + \frac{27}{\mathcal{N}^2\mathcal{J}^2} - \frac{3}{2}\frac{35-4\nu}{\mathcal{N}^3\mathcal{J}} + \frac{1}{8}\frac{145-15\nu+\nu^2}{\mathcal{N}^4}\right)\right], \tag{5.38}$$

where $\mathcal{N}$ denote the Delaunay action variable $\mathcal{N} \equiv I_R + \mathcal{J}$ [27]. Here we will set up the analogy with the quantum two-body system. For the corresponding quantum bound states, the classical action variables $\mathcal{J}$ and $\mathcal{I}_R$ approximately become quantized in the units of $\hbar$. $\mathcal{N} \equiv I_R/\hbar$ becomes the "principle quantum" number and $\mathcal{J}/\hbar$ as the "total angular momentum" quantum number. However, the degeneracy associated with the spherical symmetry of the problem implies the energy is independent of the "magnetic quantum number" but depends only on its magnitude.

5.2 2PN effective-one-body energy levels

In this section, we will map the energy level of the EOB metric with the effective Hamiltonian. From 5.1 and 5.2, given the energy relation $\mathcal{E}_0^{\mathrm{R}} = m_0c^2 + \mathcal{E}_0^{\mathrm{NR}}$ we can write the *non-relativistic* mass-shell constraint for the test-particle motion in the effective metric as

$$g_{\mathrm{eff}}^{\mu\nu}\frac{\partial S_{\mathrm{eff}}}{\partial x^\mu}\frac{\partial S_{\mathrm{eff}}}{\partial x^\nu} + m_0^2c^2 = 0, \tag{5.39}$$

the superscript 'NR' represents the *non-relativistic* and 'R' represents the *relativistic*. Whereas in this section, the '0' subscript/superscript denotes the quantities for the effective framework, so writing 'eff' and '0' ar eequivalant. As the dynamics of m_0 are planar, the effective action can be split into the action angle variable coordinates

$$S_{\mathrm{eff}} = -\mathcal{E}_0 t + \mathcal{J}_0\varphi + S_R^0(R,\mathcal{E}_0,\mathcal{J}_0). \tag{5.40}$$

Inserting 5.40 in 5.39 simply we get

$$-\frac{1}{A(R)}\frac{\mathcal{E}_0^2}{c^2} + \frac{1}{B(R)}\left(\frac{dS_R^0}{dR}\right)^2 + \frac{\mathcal{J}_0^2}{C(R)R^2} + m_0^2c^2 = 0, \tag{5.41}$$

we can get the radial action variable and radial 'effective potential' for m_0 as

$$S_R^0(R, \mathcal{E}_0, \mathcal{J}_0) = \int dR\sqrt{\mathcal{R}_0(R, \mathcal{E}_0, \mathcal{J}_0)}, \tag{5.42}$$

the radial motion variable can be written as

$$\begin{aligned}
\mathcal{R}_0(R, \mathcal{E}_0, \mathcal{J}_0) \equiv& \frac{B(R)}{A(R)}\frac{\mathcal{E}_0^2}{c^2} - B(R)\left(m_0^2c^2 + \frac{\mathcal{J}_0^2}{C(R)R^2}\right) \\
=& \frac{\left(1 + \frac{b_1}{c^2R} + \frac{b_2}{c^4R^2}\right)}{\left(1 + \frac{a_1}{c^2R} + \frac{a_2}{c^4R^2}\right)}\frac{\mathcal{E}_0^2}{c^2} - \left(1 + \frac{b_1}{c^2R} + \frac{b_2}{c^4R^2}\right)\left(m_0^2c^2 + \frac{\mathcal{J}_0^2}{R^2}\right) \\
=& \left(1 + \frac{b_1 - a_1}{c^2R} + \frac{-a_1b_1 + a_1^2 - a_2 + b_2}{c^4R^2}\right)\frac{\mathcal{E}_0^2}{c^2} - \left(1 + \frac{b_1}{c^2R} + \frac{b_2}{c^4R^2}\right)\left(m_0^2c^2 + \frac{\mathcal{J}_0^2}{R^2}\right) \\
=& \frac{\mathcal{E}_0^2}{c^2} + m_0^2c^2 + \frac{1}{R}\left(\frac{b_1 - a_1}{c^2}\frac{\mathcal{E}_0^2}{c^2} + m_0^2c^2\frac{b_1}{c^2}\right) + \frac{1}{R^3}\left(\frac{b_1}{c^2R}\mathcal{J}_0^2\right) + \frac{1}{R^4}\left(\frac{b_2}{c^4}\mathcal{J}_0^2\right) \\
&+ \frac{1}{R^2}\left(\frac{-a_1b_1 + a_1^2 - a_2 + b_2}{c^4}\frac{\mathcal{E}_0^2}{c^2} + \mathcal{J}_0^2 + m_0^2c^2\frac{b_2}{c^4}\right).
\end{aligned} \tag{5.43}$$

To derive the effective radial action variable I_R^0, which is defined as

$$I_R^0(\mathcal{E}_0, \mathcal{J}_0) \equiv \frac{2}{2\pi}\int_{R_{\min}}^{R_{\max}} dR\sqrt{\mathcal{R}_0(R, \mathcal{E}_0, \mathcal{J}_0)}. \tag{5.44}$$

To solve this we are using -

$$\begin{cases}
\begin{aligned}
I(A_1, .., A_6) =& \frac{1}{\pi}\int_{r_{\min}}^{r_{max}} \mathrm{d}r\left(A_1 + 2A_2/r + A_3/r^2 + \sum_{n\geqslant 4} A_n/r^{n-1}\right)^{1/2} \\
=& \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} \mathrm{d}r\left(A_1 + \frac{2B}{r} + \frac{A_3}{r^2} + \frac{A_4}{r^3} + \frac{A_5}{r^4}\right)^{1/2} \\
=& \frac{A_2}{\sqrt{-A_1}} - \sqrt{-A_3}\left\{1 - \frac{1}{2}\frac{A_2}{A_3^2}\left[A_4 - \frac{3}{2}\frac{A_5A_2}{A_3} + \frac{15}{8}\frac{A_4^2B}{A_3^2}\right] - \frac{1}{4}\frac{A_1}{A_3^2}\left[A_5 - \frac{3}{4}\frac{A_4^2}{A_3}\right]\right. \\
&\left.+ \frac{3}{4}\frac{A_2}{A_3^3}\left[A_1 - \frac{5}{3}\frac{A_2^2}{A_3}\right]A_6\right\} + \mathcal{O}(A_4^3 + A_5^2 + A_6^2 + A_4^2A_5 + ...)
\end{aligned}
\end{cases} \tag{5.45}$$

So, let's obtain the $I_R^0(\mathcal{E}_0, \mathcal{J}_0)$ by using 5.45 we get the coefficients A_i's as

$$
\begin{aligned}
A_3 &= \frac{-a_1 b_1 + a_1^2 - a_2 + b_2}{c^4}\frac{\mathcal{E}_0^2}{c^2} + \mathcal{J}_0^2 + m_0^2 c^2 \frac{b_2}{c^4}, & A_4 &= \frac{b_1}{c^2 R}\mathcal{J}_0^2 \\
A_2 &= \frac{b_1 - a_1}{c^2}\frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2 \frac{b_1}{c^2}, & A_1 &= \frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2, \quad A_5 = \frac{b_2}{c^4}\mathcal{J}_0^2.
\end{aligned} \tag{5.46}
$$

By using 5.46 and substituting into 5.45 we get

$$
\begin{aligned}
I_R^0(\mathcal{E}_0, \mathcal{J}_0) =& \frac{\left(\frac{b_1 - a_1}{c^2}\frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2 \frac{b_1}{c^2}\right)}{\sqrt{-\left(\frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2\right)}} - \sqrt{-\left(\frac{-a_1 b_1 + a_1^2 - a_2 + b_2}{c^4}\frac{\mathcal{E}_0^2}{c^2} + \mathcal{J}_0^2 + m_0^2 c^2 \frac{b_2}{c^4}\right)} \\
& \left\{1 - \frac{1}{2}\frac{\frac{b_1 - a_1}{c^2}\frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2 \frac{b_1}{c^2}}{\left(\frac{-a_1 b_1 + a_1^2 - a_2 + b_2}{c^4}\frac{\mathcal{E}_0^2}{c^2} + \mathcal{J}_0^2 + m_0^2 c^2 \frac{b_2}{c^4}\right)}\left[\frac{b_1}{c^2 R}\mathcal{J}_0^2 - \frac{3}{2}\frac{\left(\frac{b_2}{c^4}\mathcal{J}_0^2\right)\left(\frac{b_1 - a_1}{c^2}\frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2 \frac{b_1}{c^2}\right)}{\frac{-a_1 b_1 + a_1^2 - a_2 + b_2}{c^4}\frac{\mathcal{E}_0^2}{c^2} + \mathcal{J}_0^2 + m_0^2 c^2 \frac{b_2}{c^4}}\right.\right. \\
& \left.\left. + \frac{15}{8}\frac{\left(\frac{b_1}{c^2 R}\mathcal{J}_0^2\right)^2\left(\frac{b_1 - a_1}{c^2}\frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2 \frac{b_1}{c^2}\right)}{\left(\frac{-a_1 b_1 + a_1^2 - a_2 + b_2}{c^4}\frac{\mathcal{E}_0^2}{c^2} + \mathcal{J}_0^2 + m_0^2 c^2 \frac{b_2}{c^4}\right)^2}\right] - \frac{1}{4}\frac{\frac{\mathcal{E}_0^2}{c^2} + m_0^2 c^2}{\left(\frac{-a_1 b_1 + a_1^2 - a_2 + b_2}{c^4}\frac{\mathcal{E}_0^2}{c^2} + \mathcal{J}_0^2 + m_0^2 c^2 \frac{b_2}{c^4}\right)^2}\left[\frac{b_2}{c^4}\mathcal{J}_0^2\right]\right\}
\end{aligned} \tag{5.47}
$$

we will use a similar treatment as in 5.34 for the Schwarzschild gauge to derive the $I_R^0(\mathcal{E}_0, \mathcal{J}_0)$

$$
I_R^0(\mathcal{E}_0, \mathcal{J}_0) = \frac{m_0^{3/2}}{\sqrt{-2\mathcal{E}_0^{\mathrm{NR}}}}\left[A_1 + A_2 \frac{\mathcal{E}_0^{\mathrm{NR}}}{m_0 c^2} + A_3\left(\frac{\mathcal{E}_0^{\mathrm{NR}}}{m_0 c^2}\right)^2\right] - \mathcal{J}_0 + \frac{m_0^2}{c^2 \mathcal{J}_0}\left[A_4 + A_5 \frac{\mathcal{E}_0^{\mathrm{NR}}}{m_0 c^2}\right] + \frac{m_0^4}{c^4 \mathcal{J}_0^3}A_6, \tag{5.48}
$$

where the coefficients are given as

$$
\begin{aligned}
& A_1 = -\frac{1}{2}a_1, \quad A_2 = b_1 - \frac{7}{8}a_1, \quad A_4 = \frac{a_1^2}{2} - \frac{a_2}{2} - \frac{a_1 b_1}{4}, \quad A_5 = a_1^2 - a_2 - \frac{a_1 b_1}{2} - \frac{b_1^2}{8} + \frac{b_2}{2}, \\
& A_3 = \frac{b_1}{4} - \frac{19}{64}a_1, \quad A_6 = \frac{1}{64}\left[24 a_1^4 - 48 a_1^2 a_2 + 8 a_2^2 + 16 a_1 a_3 - 8 a_1^3 b_1 + 8 a_1 a_2 b_1 - a_1^2 b_1^2 + 4 a_1^2 b_2\right].
\end{aligned} \tag{5.49}
$$

One can check the coordinate independence of the coefficients A_i by solving $I_R^0(\mathcal{E}_0, \mathcal{J}_0)$ in Isotropic coordinate where $C_I(R) \equiv B_I(R)$ (such that the light cones appear round, only possible when all 3 spatial dimensions are treated same), subscript I denotes the Isotropic coordinate as shown in [?]. The energy can be obtain by solving 5.48 with respect to $\mathcal{E}_0^{\mathrm{NR}} \equiv \mathcal{E}_0^{\mathrm{R}} - m_0 c^2$ with the (2PN- accurate result) after rescaling the coefficients A_i's to $\widehat{A_i}$ as $\widehat{A_1} = -\frac{1}{2}\widehat{a_1}$, $\widehat{A_2} = \widehat{b_1} - \frac{7}{8}\widehat{a_1}$, similarly for others, where the scaling of the Schwarzschild gauge is given as

$$\widehat{a}_i \equiv a_i/(GM_0)^i, \quad \widehat{b}_i \equiv b_i/(GM_0)^i \tag{5.50}$$

So, we get

$$\mathcal{E}_0(\mathcal{N}_0, \mathcal{J}_0) = m_0c^2 - \frac{1}{2}\frac{m_0\alpha_0^2}{\mathcal{N}_0^2}\left[1 + \frac{\alpha_0^2}{c^2}\left(\frac{C_{3,1}}{\mathcal{N}_0\mathcal{J}_0} + \frac{C_{4,0}}{\mathcal{N}_0^2}\right) + \frac{\alpha_0^4}{c^4}\left(\frac{C_{3,3}}{\mathcal{N}_0\mathcal{J}_0^3} + \frac{C_{4,2}}{\mathcal{N}_0^2\mathcal{J}_0^2} + \frac{C_{5,1}}{\mathcal{N}_0^3\mathcal{J}_0} + \frac{C_{6,0}}{\mathcal{N}_0^4}\right)\right], \tag{5.51}$$

where the coefficients $C_{q,p}$ are [28]

$$\begin{aligned} C_{3,1} &= 2\widehat{A}_4, & C_{4,0} &= -\widehat{A}_2, & C_{3,3} &= 2\widehat{A}_6, \\ C_{4,2} &= 3\widehat{A}_4^2, & C_{5,1} &= -(4\widehat{A}_2\widehat{A}_4 + \widehat{A}_5), & C_{6,0} &= \frac{1}{4}\left(5\widehat{A}_2^2 + 2\widehat{A}_3\right) \end{aligned} \tag{5.52}$$

Now, we have to do the energy mapping between the real and effective framework.

5.3 Relation between real and effective energy levels and the effective metric

In this section, we want to establish the mapping between the "real" and the "effective" energy levels. The most general requirement is that one-to-one mapping exists. Such that the mass of the effective test particle coincides with the reduced mass $m_0 = \mu$ and the *linearized* effective metric coincides with the Schwarzchild metric of mass $M = m_1 + m_2$, which gives $a_1 = -2GM$ and $b_1 = 2GM$. Therefore, there exists a function dependence for some function F as

$$\mathcal{E}_0(\mathcal{N}_0, \mathcal{J}_0) = f\left[\mathcal{E}_{\mathrm{real}}(\mathcal{N}, \mathcal{J})\right] \tag{5.53}$$

such that we can do a Taylor expansion as given in eq (4.11) of [27]

$$\frac{\mathcal{E}_0^{\mathrm{NR}}}{m_0c^2} = \frac{\mathcal{E}_{\mathrm{real}}^{\mathrm{NR}}}{\mu c^2}\left(1 + \alpha_1\frac{\mathcal{E}_{\mathrm{real}}^{\mathrm{NR}}}{\mu c^2} + \alpha_2\left(\frac{\mathcal{E}_{\mathrm{real}}^{\mathrm{NR}}}{\mu c^2}\right)^2 + \cdots\right) \tag{5.54}$$

inserting the equations for the energy in "real" 5.38 and the "effective" frame 5.51 in the relation 5.54 by replacing the $\mathcal{E}^{\mathrm{R}}$ with $\mathcal{E}_{\mathrm{NR}}$ using the relation $\mathcal{E}^R = \mathcal{E}^{\mathrm{NR}} + Mc^2$. That gives $\alpha_1 = \frac{\nu}{2}$ and $\alpha_2 = 0$. Such that we can obtain the relation between the metric coefficients $\hat{a}_1$,

$\hat{b}_1$, $\hat{c}_1$, $\hat{a}_3$, $\hat{b}_2$, $\hat{a}_3$ (rescaled coefficients) in terms of the symmetric mass ratio ν, that yields

$$\hat{a}_2 = 0, \quad \hat{a}_3 = 2\nu, \quad \hat{b}_2 = 4 - 6\nu \tag{5.55}$$

inserting these rescaled values in the 5.3 so that we can get the effective metric 5.2 as

$$A(R) = 1 - \frac{2GM}{c^2 R} + 2\nu \left(\frac{GM}{c^2 R}\right)^3 + \cdots , \tag{5.56a}$$

$$B(R) = 1 + \frac{2GM}{c^2 R} + (4 - 6\nu) \left(\frac{GM}{c^2 R}\right)^2 + \cdots . \tag{5.56b}$$

5.4 Description of dynamics of Innermost stable circular orbit

In this section, we briefly want to discuss the orbital motion of the relativistic two-body system and understand the innermost stable circular orbit (ISCO), which is a strong gravity effect. Here we consider the orbital motion to be planner such that the radial component remains constant, i.e., radial momentum $p_r^2 = 0$, $r =$ constant in the equation 5.4b. Therefore, the energy relation $\partial \mathcal{H}_{\text{eff}}/\partial r = 0 = -p_r$ or $\partial \mathcal{H}_{\text{eff}}^2/\partial r = 0$ satisfies the minima of the energy. Hence, we have the energy relation 5.6 as

$$\mathcal{H}_{\text{eff}}^2 = \left(1 - \frac{2M}{r} + \frac{2\nu}{r^3}\right)\left(\mu^2 + \frac{p_\phi^2}{r^2}\right) \tag{5.57}$$

where p_ϕ represents the angular momentum. At $r \to \infty$, the $1/r$ potential, which is the first expression on the RHS, dominates the $1/r^2$ angular momentum barrier in the second expression on RHS. As r decreases, $\mathcal{H}_{\text{eff}}^2$ decreases. We have shown this in the plot 5.4 at symmetric mass ratio $\nu = 1/4$ for different values of L. After a certain point, $1/r^2$ potential barrier dominates over the $1/r$ angular momentum barrier, creating a local minima (stable circular orbit) and a local maxima (unstable circular orbit) for some angular momentum p_ϕ. For some $p_\phi \gtrsim 3.4$, both local minima and local maxima exist together. However, during the inspiral due to gravitational wave emission, p_ϕ decreases. At $r = 6M$, the local minima and the local maxima overlap, giving a saddle point. We call this orbit the Innermost stable circular orbit (ISCO) because the orbital motion itself becomes unstable. Therefore, the motion transitions from an adiabatic inspiral to a plunge phase.

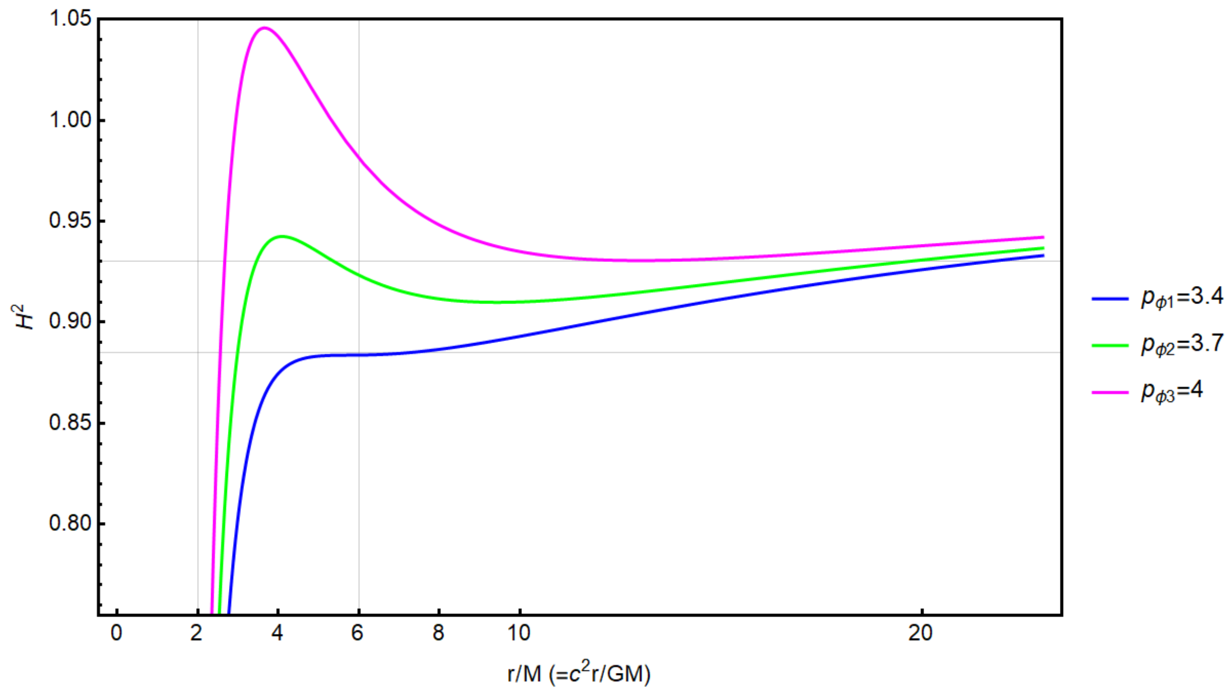

Figure 5.1: The square of effective Hamiltonian (for $\nu = 1/4$ at 2PN order) vs the dimensionless radial variable r/M (for $c = G = 1$) for three different values of angular momentum p_ϕ. The stable circular orbit is located at the minima due to the effective potential of each curve. The maxima represents the unstable orbit. Note that as the angular momentum decreases due to gravitational wave emission, at $r = 6M$, the local minima and maxima get overlap, which creates a saddle point at $p_\phi \sim 3.4$. Where $p_{\phi 1}$ corresponds to the innermost stable circular orbit of of the binary coalescence.

Chapter 6

Conclusion

The post-Newtonian approximation methods are essential to explain the inspiral of compact binary coalescence. Therefore, in this thesis, we have reproduced the post-Newtonian results till 2PN order in GR [34], given in section 4.1. The second-order post-Newtonian corrections are non-trivial, and it took a lot of time to evaluate each metric correction individually. Therefore, in order to make the lengthy calculation efficient, we used the computational tool Mathematica to obtain the 2PN Ricci curvature and the Lagrangian. The post-Newtonian methods are the basic construct to formulate effective-one-body dynamics. Using the PN results till the 2PN order, we have re-derived for GR, and we have reconstructed the conservative EOB dynamics in section 5. Extending the conservative dynamics during the thesis, we have also studied and understood the radiation reaction processes at 2.5PN order, which we want to study in future.

As general relativity needs to be tested, we have studied and re-derived the post-Newtonian dynamics in scalar-tensor theory 4.2 and $f(R)$ gravity till 1PN order 4.3.1. Along with this, we have reviewed and studied the post-Newtonian dynamics in Einstein-Maxwell-dilaton-Gauss-Bonnet theory. Because understanding the possible modification to theories available is necessary. By using the knowledge of post-Newtonian and EOB formalism in GR, we will construct the EOB dynamics of a compact binary system in $f(R)$ gravity. So, we have worked on the 2PN order of $f(R)$ gravity and derived the field equation and the metric correction at the second post-Newtonian order (see section 4.3.2). We can use these results now to construct the EOB dynamics in $f(R)$ gravity. So that one can synergise the numerical relativity results at late inspiral to generate a single waveform using EOB. Such that GR can be tested.

We also wanted to construct a conformal mapping between the scalar-tensor and $f(R)$ gravity, which we did for the exact form of the field equations and successfully being able to recover them using the conformal transformation. This establishes that the conformal mapping between scalar-tensor and $f(R)$ gravity is possible. Because the higher-order post-Newtonian calculation in scalar-tensor theory is already present in the literature. We wanted to recover the higher-order post-Newtonian field equation for $f(R)$ by conformal transformation. To make it plausible, first, we wanted to do a consistency check for the first-order post-Newtonian field equation that we have derived from both theories. Such that we can extend this mapping for higher post-Newtonian order. Our calculations and findings were not successful in being able to establish it. We need to do more work to understand and check the conformal mapping.

Appendix A

A.1 Convolution

>> **Convolution of some function f for gravitational potential ϕ :**

$$\nabla^2 f = \phi = -4\pi G \sum_a \frac{m_a}{|\boldsymbol{x} - \boldsymbol{x}_a(t)|} \Rightarrow f = -2\pi G \sum_a m_a |\boldsymbol{x} - \boldsymbol{x}_a(t)| \tag{A.1}$$

>> **Differentiation with respect to time t and spatial coordinate x^i of function f :**

$$f = -\frac{k}{2} \sum_a m_a |\mathbf{r} - \mathbf{r}_a| \Rightarrow \frac{df}{dx^i} = -\frac{k}{2} \sum_a m_a \frac{(x^i - x^i_a)}{|\mathbf{r} - \mathbf{r}_a|} \tag{A.2}$$

$$\begin{aligned}
\Rightarrow \frac{d^2 f}{dt dx^i} &= -\frac{k}{2} \sum_a \frac{m_a}{|\mathbf{r} - \mathbf{r}_a|} \Bigg[\frac{(v_i - v_{ia})}{|\mathbf{r} - \mathbf{r}_a|} - \frac{(x_i - x_{ia})[(x_i - x_{ia})(v_i - v_{ia}) + (x_j - x_{ja})(v_j - v_{ja})]}{|\mathbf{r} - \mathbf{r}_a|^3} \\
&\quad + \frac{(x_i - x_{ia})(x_k - x_{ka})(v_k - v_{ka})}{|\mathbf{r} - \mathbf{r}_a|^3} \Bigg] \\
\Rightarrow \frac{d^2 f}{dt dx^i} &= -\frac{k}{2} \sum_a \frac{m_a}{|\mathbf{r} - \mathbf{r}_a|} \left[(\mathbf{v_i} - \boldsymbol{v}_{ia}) - [(v_i - v_{ia}) \cdot n_a n_{ia}] \right] \\
\Rightarrow \frac{d^2 f}{dt dx^i} &= -\frac{k}{2} \sum_a \frac{m_a}{|\mathbf{r} - \mathbf{r}_a|} (\boldsymbol{v}_{ia} - (\boldsymbol{v} \cdot n_a) n_{ia})
\end{aligned} \tag{A.3}$$

[Note: this is due to a particle]

>> **Greens function:**

Whereas to make it simple we are writing $(\mathbf{r}-\mathbf{r}_a)$ and $(\boldsymbol{x}-\boldsymbol{x}_a(t))$ is equivalently (below K is some random function)

$$\frac{4\pi G}{c^3}\sum_a \frac{m_a v_a^i}{|\mathbf{r}-\mathbf{r}_a|} = \nabla^2 K \Longleftrightarrow K = -\frac{16\pi G}{c^3}\sum_a m_a v_a^i \delta(\mathbf{r}-\mathbf{r}_a) \tag{A.4a}$$

Where f_A is just

$$\begin{aligned} f_A(t,x) &= \frac{1}{|x-r_a(t)|} + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}|x-r_a(t)| + O\left(\frac{1}{c^4}\right) \\ f_B(t,r_a) &= \frac{1}{r_{ab}}\left\{1+\frac{1}{2c^2}\left[(v_a\cdot v_b)-(n_{ab}\cdot v_a)(n_{ab}\cdot v_b)\right]\right\} - \frac{1}{2c^2}\frac{\mathrm{d}}{\mathrm{d}t}(n_{ab}\cdot v_b) + \mathrm{O}\left(\frac{1}{c^4}\right) \end{aligned} \tag{A.5}$$

from the time-symmetric solution

$$\Box_g f_a(t,x) = -4\pi\delta_3(x-r_a(t)). \tag{A.6}$$

Here $r_{ab} \equiv |x_a(t)-x_b(t)|$ is the distance between particle A and B, and $n_{ab} \equiv (x_a(t)-x_b(t))/r_{ab}$ is the unit vector along ab. This can be seen in [A.23d] for general relativity.

A.2 Chritoffel, Ricci tensor and energy-momentum tensor

>> **Christoffel symbols are calculated for the metric expansion $g_{\mu\nu}$ given in 4.4**

$$\text{(i)}\Gamma^0_{00} = \frac{1}{2}g^{0m}\left(\partial_0 g_{0m} + \partial_0 g_{0m} - \partial_m g_{00}\right) \tag{A.7a}$$

$$= \frac{1}{2}g^{00}\left(\partial_0 g_{00}\right) + \frac{1}{2}g^{0i}\left(2\partial_0 g_{0i} - \partial_i g_{00}\right) \tag{A.7b}$$

$$= \left(-\frac{1}{2c}\frac{\partial {g^{(2)}}_{00}}{\partial t}\right) + \left(\frac{{g^{(2)}}^{00}}{2c}\frac{\partial {g^{(2)}}_{00}}{\partial t} - \frac{1}{2c}\frac{\partial g^{(4)}_{00}}{\partial t} - \frac{g^{(2)0i}}{2}\frac{\partial {g^{(2)}}_{00}}{\partial x^i}\right) \tag{A.7c}$$

$$+ \left(\frac{1}{c}g^{(3)0i}\frac{\partial g^{(3)}_{oi}}{\partial t} + \mathcal{O}(c^{-7})\right) \tag{A.7d}$$

$$= {\Gamma^{(3)0}}_{00} + {\Gamma^{(5)0}}_{00} + \mathcal{O}(c^{-7}) \tag{A.7e}$$

$$(\mathrm{ii})\Gamma^0_{0i} = \frac{1}{2}g^{0m}(\partial_i g_{m0} + \partial_0 g_{mi} - \partial_m g_{0i}) \tag{A.8a}$$

$$= \frac{1}{2}g^{00}\left(\partial_i g_{00}\right) + \frac{1}{2}g^{0i}\left(\partial_0 g_{ii}\right) \tag{A.8b}$$

$$= \left(-\frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial x^i}\right) + \left(-\frac{1}{2}\frac{\partial g^{(4)}_{00}}{\partial x^i} + \frac{g^{(2)00}}{2}\frac{\partial g^{(2)}_{00}}{\partial x^i}\right) + \mathcal{O}(c^{-6}) \tag{A.8c}$$

$$= \Gamma^{(2)0}{}_{0i} + \Gamma^{(4)0}{}_{0i} + \mathcal{O}(c^{-6}) \tag{A.8d}$$

$$(\mathrm{iii})\Gamma^0_{ij} = \frac{1}{2}g^{0m}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) \tag{A.9a}$$

$$= \frac{1}{2}g^{00}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}) + \frac{1}{2}g^{0k}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) \tag{A.9b}$$

$$= -\frac{1}{2}\left(\frac{\partial g^{(3)}_{0i}}{\partial x^j} + \frac{\partial g^{(3)}_{0j}}{\partial x^i} - \frac{1}{c}\frac{\partial g^{(2)}_{ij}}{\partial t}\right) + \mathcal{O}(c^{-5}) \tag{A.9c}$$

$$= \Gamma^{(3)0}{}_{ij} + \mathcal{O}(c^{-5}) \tag{A.9d}$$

$$(\mathrm{iv})\Gamma^i_{00} = \frac{1}{2}g^{im}\left(\partial_0 g_{m0} + \partial_0 g_{m0} - \partial_m g_{00}\right) \tag{A.10a}$$

$$= \frac{1}{2}g^{i0}\left(\partial_0 g_{00}\right) + \frac{1}{2}g^{ij}\left(2\partial_0 g_{j0} - \partial_j g_{00}\right) \tag{A.10b}$$

$$= \frac{1}{2}g^{i0}\left(\partial_0 g_{00}\right) + \frac{1}{2}\delta^j_i\left(2\partial_0 g_{j0} - \partial_j g_{00}\right) + \frac{1}{2}g^{ij}\left(2\partial_0 g_{j0} - \partial_j g_{00}\right) \tag{A.10c}$$

$$= \left(-\frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial x^i}\right) + \left(-\frac{1}{2}\frac{\partial g^{(4)}_{00}}{\partial x^i} - \frac{g^{(2)ij}}{2}\frac{\partial g^{(2)}_{00}}{\partial x^j} + \frac{1}{c}\frac{\partial g^{(3)}_{0i}}{\partial t}\right) \tag{A.10d}$$

$$+ \left(\frac{1}{2c}g^{(3)i0}\frac{\partial g^{(2)}_{00}}{\partial t} + \frac{1}{c}g^{(2)ij}\frac{\partial g^{(3)}_{0j}}{\partial t}\right) + \mathcal{O}(c^{-8}) \tag{A.10e}$$

$$= \Gamma^{(2)i}{}_{00} + \Gamma^{(4)i}{}_{00} + \mathcal{O}(c^{-6}) \tag{A.10f}$$

$$
\begin{aligned}
\text{(v)}\Gamma^i_{0j} &= \frac{1}{2}g^{im}\left(\partial_0 g_{mj} + \partial_j g_{m0} - \partial_m g_{0j}\right) && \text{(A.11a)}\\
&= \frac{1}{2}g^{i0}\left(\partial_j g_{00}\right) + \frac{1}{2}g^{ik}\left(\partial_0 g_{kj} + \partial_j g_{k0} - \partial_k g_{0j}\right) && \text{(A.11b)}\\
&= \frac{1}{2}g^{i0}\left(\partial_j g_{00}\right) + \frac{1}{2}\delta^k_i\left(\partial_0 g_{kj} + \partial_j g_{k0} - \partial_k g_{0j}\right) + \frac{1}{2}g^{(2)ik}\left(\partial_0 g_{kj} + \partial_j g_{k0} - \partial_k g_{0j}\right) && \text{(A.11c)}\\
&= \frac{1}{2}\left(\partial_0 g_{ij} + \partial_j g_{i0} - \partial_i g_{0j}\right) + \mathcal{O}(c^{-5}) && \text{(A.11d)}\\
&= \frac{1}{2}\left(\frac{1}{c}\frac{\partial g^{(2)}_{ij}}{\partial t} + \frac{\partial g^{(3)}_{0i}}{\partial x^j} - \frac{\partial g^{(3)}_{0j}}{\partial x^i}\right) + \mathcal{O}(c^{-5}) && \text{(A.11e)}\\
&= {\Gamma^{(3)i}}_{0j} + \mathcal{O}(c^{-5}) && \text{(A.11f)}
\end{aligned}
$$

$$
\begin{aligned}
\text{(vi)}\Gamma^i_{jk} &= \frac{1}{2}g^{im}\left(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}\right) && \text{(A.12a)}\\
&= \frac{1}{2}\delta^m_i\left(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}\right) + \frac{1}{2}g^{(2)im}\left(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}\right) && \text{(A.12b)}\\
&= \frac{1}{2}\left(\frac{\partial g^{(2)}_{ij}}{\partial x^k} + \frac{\partial g^{(2)}_{ik}}{\partial x^j} - \frac{\partial g^{(2)}_{jk}}{\partial x^i}\right) + \mathcal{O}(c^{-4}) && \text{(A.12c)}\\
&= \Gamma^{(2)i}_{jk} + \mathcal{O}(c^{-4}) && \text{(A.12d)}
\end{aligned}
$$

>> **The Ricci tensors till 1PN order:**

$$
\begin{aligned}
\text{(i)}R_{00} =& \partial_i\Gamma^i_{00} - \partial_0\Gamma^i_{0i} + \Gamma^i_{ip}\Gamma^p_{00} - \Gamma^i_{0p}\Gamma^p_{i0} && \text{(A.13a)}\\
=& \partial_i\Gamma^i_{00} - \partial_0\Gamma^j_{0i} + \Gamma^i_{i0}\Gamma^0_{00} - \Gamma^i_{00}\Gamma^0_{i0} + \Gamma^i_{ij}\Gamma^j_{00} - \Gamma^i_{0j}\Gamma^j_{i0} && \text{(A.13b)}\\
=& \partial_i{\Gamma^{(2)i}}_{00} + \Bigg(\partial_i{\Gamma^{(4)i}}_{00} - \partial_0{\Gamma^{(3)i}}_{0i} + {\Gamma^{(2)i}}_{ij}{\Gamma^{(2)j}}_{00} - {\Gamma^{(2)i}}_{00}{\Gamma^{(2)0}}_{i0} && \text{(A.13c)}\\
& - {\Gamma^{(3)i}}_{0j}{\Gamma^{(3)j}}_{i0} + {\Gamma^{(3)i}}_{00}{\Gamma^{(3)0}}_{00}\Bigg) && \text{(A.13d)}\\
=& R^{(2)}_{00} + R^{(4)}_{00} + \mathcal{O}(c^{-6}) && \text{(A.13e)}
\end{aligned}
$$

$$R^{(2)}_{00} = \partial_i \Gamma^{(2)i}{}_{00} = -\frac{1}{2}\frac{\partial^2 g^{(2)}{}_{00}}{\partial x^{i2}} = -\frac{1}{2}\nabla^2 g^{(2)}_{00} \tag{A.14a}$$

$$R^{(4)}_{00} = \partial_i \Gamma^{(4)i}{}_{00} - \partial_0 \Gamma^{(3)i}{}_{0i} + \Gamma^{(2)i}{}_{ij}\Gamma^{(2)j}{}_{00} - \Gamma^{(2)i}{}_{00}\Gamma^{(2)0}{}_{i0} \tag{A.14b}$$

$$= -\frac{1}{2}\nabla^2 g^{(4)}_{00} - \frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial x^i}\frac{\partial g^{(2)}_{ij}}{\partial x^j} - \frac{g^{(2)ij}}{2}\frac{\partial^2 g^{(2)}{}_{00}}{\partial x^i \partial x^j} + \frac{1}{c}\frac{\partial^2 g^{(3)}_{0i}}{\partial t \partial x^i} - \frac{1}{2c^2}\frac{\partial^2 g^{(2)}_{ii}}{\partial t^2} \tag{A.14c}$$

$$- \frac{1}{4}\frac{\partial g^{(2)}_{00}}{\partial x^i}\frac{\partial g^{(2)}_{00}}{\partial x^i} + \frac{1}{4}\frac{\partial g^{(2)}_{jj}}{\partial x^i}\frac{\partial g^{(2)}_{00}}{\partial x^i} + \mathcal{O}(c^{-6}) \tag{A.14d}$$

$$\text{(ii)}\ R_{0i} = R^{(3)}_{0i} + R^{(5)}_{0i} \tag{A.15a}$$

$$R^{(3)}_{0i} = \partial_j \Gamma^j_{0i} - \partial_0 \Gamma^j_{ij} + \Gamma^j_{j\lambda}\Gamma^\lambda_{0i} - \Gamma^j_{0\lambda}\Gamma^\lambda_{ji} \tag{A.15b}$$

$$= \partial_j \Gamma^{(3)j}{}_{0i} + \partial_0 \Gamma^{(2)j}{}_{ij} + \mathcal{O}(c^{-5}) \tag{A.15c}$$

$$= \frac{1}{2}\partial_0 \frac{\partial g^{(2)}_{jj}}{\partial x^i} + \frac{1}{2}\partial_j \left(\frac{1}{c}\frac{\partial g^{(2)}_{ij}}{\partial t} + \frac{\partial g^{(3)}_{0i}}{\partial x^j} - \frac{\partial g^{(3)}_{0j}}{\partial x^i} \right) + \mathcal{O}(c^{-5}) \tag{A.15d}$$

$$= \frac{1}{2}\nabla^2 g^{(3)}_{0i} + \frac{1}{2c}\frac{\partial^2 g^{(2)}_{ij}}{\partial t \partial x^j} + \frac{1}{2c}\frac{\partial^2 g^{(2)}_{kk}}{\partial t \partial x^i} - \frac{1}{2}\frac{\partial^2 g^{(3)}_{0j}}{\partial x^i \partial x^j} + \mathcal{O}(c^{-5}) \tag{A.15e}$$

$$\text{(iii)}\ R_{ij} = R^{(2)}_{ij} + R^{(4)}_{ij} + \mathcal{O}(c^{-6}) \tag{A.16a}$$

$$R^{(2)}_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^0_{i0} - \partial_i \Gamma^k_{jk} \tag{A.16b}$$

$$= -\frac{1}{2}\nabla^2 g^{(2)}_{ij} + \frac{1}{2}\frac{\partial^2 g^{(2)}_{00}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2 g^{(2)}_{kk}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2 g^{(2)}_{ik}}{\partial x^j \partial x^k} + \frac{1}{2}\frac{\partial^2 g^{(2)}_{jk}}{\partial x^i \partial x^k} + \mathcal{O}(c^{-4}). \tag{A.16c}$$

>> The individual components of energy-momentum tensor:

$$\text{(i) } \mathbb{T}_{00} = g_{0\sigma} g_{0\tau} (T^{\sigma\tau} - \frac{1}{2} T^{\rho}_{\rho} g^{\sigma\tau}) = \mathbb{T}^{(0)}_{00} + \mathbb{T}^{(2)}_{00} \tag{A.17a}$$

$$(\sigma \to 0, \tau \to 0) \tag{A.17b}$$

$$= (T^{00} - \frac{1}{2} T^{\rho}_{\rho} g^{00}) \tag{A.17c}$$

$$= {}^{(0)}T^{00} + {}^{(2)}T^{00} - \frac{1}{2}({}^{(0)}T^{00} + {}^{(2)}T^{00}) - \frac{1}{2} g^{(0)}_{00} (2{}^{(0)}T^{00} + {}^{(2)}T^{00}) - \frac{1}{2} {}^{(2)}T^{ii} \tag{A.17d}$$

$$= \frac{1}{2}({}^{(0)}T^{00} + {}^{(2)}T^{00} - 2g^{2}_{00} {}^{(0)}T^{00} - {}^{(0)}T^{ii}) \tag{A.17e}$$

$$\mathbb{T}^{(0)}_{00} = \frac{1}{2} {}^{(0)}T^{00}, \tag{A.18a}$$

$$\mathbb{T}^{(2)}_{00} = \frac{1}{2}({}^{(2)}T^{00} - 2g^{2}_{00} {}^{(0)}T^{00} - {}^{(0)}T^{ii}), \tag{A.18b}$$

$$\text{(ii) } \mathbb{T}_{0i} = g_{0\sigma} g_{i\tau} (T^{\sigma\tau} - \frac{1}{2} T^{\rho}_{\rho} g^{\sigma\tau}) = \mathbb{T}^{(0)}_{0i} + \mathbb{T}^{(3)}_{0i} + \cdots \tag{A.19a}$$

$$(\sigma \to 0, \tau \to i) \tag{A.19b}$$

$$-(T^{0i} - \frac{1}{2} T g^{0i}) = -{}^{(1)}T^{0i} = \mathbb{T}_{0i} = {}^{(1)}\mathbb{T}_{0i} \tag{A.19c}$$

$$\text{(iii) } \mathbb{T}_{ij} = g_{i\sigma} g_{i\tau} (T^{\sigma\tau} - \frac{1}{2} T g^{\sigma\tau}) = {}^{0}\mathbb{T}_{ij} + {}^{2}\mathbb{T}_{ij} \tag{A.20a}$$

$$(\sigma \to i, \tau \to j) \tag{A.20b}$$

$$\Rightarrow \left(T^{ij} - \frac{1}{2} T g^{ij}\right) = {}^{(2)}T^{ij} - \frac{1}{2} T \left(\delta_{ij} + g^{(2)}_{ij}\right) \tag{A.20c}$$

$$= -\frac{1}{2}\delta_{ij}(-{}^{(0)}T^{00}) = \frac{1}{2}\delta_{ij} {}^{(0)}T^{00} = {}^{(0)}\mathbb{T}_{ij} = \mathbb{T}_{ij}. \tag{A.20d}$$

A.3 Equating the EFE to obtain the components of the metric tensor in GR

>> **Obtaining** $g_{00}^{(2)}$

To obtain the metric component $g_{00}^{(2)}$ we have use 4.1 at $\mathcal{O}(c^{-4})$. It can be evaluated as

$$R_{00}^{(2)} = -\frac{8\pi G}{c^4}\mathbb{T}_{00}^{(0)} \Rightarrow -\frac{1}{2}\nabla^2 g_{00}^{(2)} = -\frac{8\pi G}{c^4}\left(\frac{1}{2}\sum_a m_a c^2 \delta^{(3)}(\mathbf{r}-\mathbf{r}_a(t))\right) \Rightarrow g_{00}^{(2)} = \frac{2\phi}{c^2} \quad \text{(A.21)}$$

where ϕ is the potential as defined by the Poisson's equation $\nabla^2\phi = 4\pi G\sum_a m_a c^2\delta^{(3)}(\mathbf{r} - \mathbf{r}_a(t))$.

>> **Obtaining** $g_{ij}^{(2)}$

$$R_{ij}^{(2)} = -\frac{8\pi G}{c^4}\mathbb{T}_{ij}^{(0)} \Rightarrow -\frac{1}{2}\nabla^2 g_{ij}^{(2)} = -\frac{8\pi G}{c^4}\sum_a m_a v_a^i v_a^j \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) \Rightarrow g_{ij}^{(2)} = \delta_{ij}\frac{2\phi}{c^2} \quad \text{(A.22)}$$

>> **Obtaining** $g_{0i}^{(3)}$

$$R_{0i}^{(3)} = -\frac{8\pi G}{c^4}\mathbb{T}_{0i}^{(1)} = -\frac{8\pi G}{c^4}(-^{(1)}T^{0i}) \quad \text{(A.23a)}$$

$$\Rightarrow -\frac{1}{2}\nabla^2 g_{0i}^{(3)} - \frac{1}{2c}\frac{\partial^2 g_{ij}^{(2)}}{\partial t\partial x^j} = -\frac{8\pi G}{c^4}\mathbb{T}_{0i}^{(1)} \quad \text{(A.23b)}$$

$$\Rightarrow -\frac{1}{2}\nabla^2 g_{0i}^{(3)} = -\frac{8\pi G}{c^4}\mathbb{T}_{0i}^{(1)} + \frac{1}{2c^3}\frac{\partial^2\phi}{\partial t\partial x^j} \quad \text{(A.23c)}$$

$$\Rightarrow -\nabla^2 g_{0i}^{(3)} = -\frac{16\pi G}{c^3}\sum_a m_a v_a^i \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) + \frac{1}{c^3}\nabla^2\frac{\partial^2 f}{\partial t\partial x^j} \quad \text{(A.23d)}$$

using A.1 for the first term and A.1 for the second term.

$$\Rightarrow -g_{0i}^{(3)} = -\frac{4G}{c^3}\sum_a \frac{m_a v_a^i}{|\mathbf{r}-\mathbf{r}_a|} + \frac{G}{2c^3}\sum_a \frac{m_a}{|\mathbf{r}-\mathbf{r}_a|}(v_{ia} - (v_a\cdot n_a)n_{ia})$$

$$\Rightarrow g_{0i}^{(3)} = \frac{G}{2c^3}\sum_a \frac{m_a}{|\mathbf{r}-\mathbf{r}_a|}\left[7v_{ia} + (v_a\cdot n_a)n_{ia})\right] \quad \text{(A.24)}$$

$$\Rightarrow g_{0i}^{(3)} = -\frac{4V_i}{c^3}$$

>> **Obtaining $g_{00}^{(4)}$**

To solving for $g_{00}^{(4)}$ we have to use 4.1, so, we get $R_{00}^{(4)} = \frac{8\pi G}{c^4}\mathbb{T}_{00}^{(2)}$. Using A.1 we can simplify

$$\begin{aligned}
&\Rightarrow -\frac{1}{2}\nabla^2 g_{00}^{(4)} + \frac{1}{2c^2}\frac{\partial^2 g_{00}^{(2)}}{\partial t^2} - \frac{g^{(2)ij}}{2}\frac{\partial^2 g^{(2)}{}_{00}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial g_{00}^{(2)}}{\partial x^i}\frac{\partial g_{00}^{(2)}}{\partial x^i} = -\frac{8\pi G}{c^4}\mathbb{T}_{00}^{(2)} \\
&\Rightarrow -\frac{1}{2}\nabla^2 g_{00}^{(4)} + \frac{1}{2c^2}\frac{\partial^2 g_{00}^{(2)}}{\partial t^2} - \frac{g^{(2)ij}}{2}\frac{\partial^2 g^{(2)}{}_{00}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial g_{00}^{(2)}}{\partial x^i}\frac{\partial g_{00}^{(2)}}{\partial x^i} = -\frac{4\pi G}{c^4}\left(T^{(2)00} + T^{(2)ii} + 2g_{00}^{(2)}T^{(2)00}\right) \\
&\Rightarrow -\nabla^2 g_{00}^{(4)} = -\frac{1}{c^2}\frac{\partial^2 g_{00}^{(2)}}{\partial t^2} + g^{(2)ij}\frac{\partial^2 g^{(2)}{}_{00}}{\partial x^i \partial x^j} + \frac{\partial g_{00}^{(2)}}{\partial x^i}\frac{\partial g_{00}^{(2)}}{\partial x^i} - \frac{8\pi G}{c^4}\left(T^{(2)00} + T^{(2)ii} + 2g_{00}^{(2)}T^{(2)00}\right) \\
&\Rightarrow -\nabla^2 g_{00}^{(4)} = -\frac{2}{c^4}\frac{\partial^2 \phi}{\partial t^2} - \frac{4}{c^4}(\phi\nabla^2\phi - (\nabla\phi)^2) - \frac{8\pi G}{c^4}\left(T^{(2)00} + T^{(2)ii} + 2g_{00}^{(2)}T^{(2)00}\right) \\
&\Rightarrow -\frac{1}{2}\nabla^2 g_{00}^{(4)} = -\frac{1}{c^4}\frac{\partial^2 \phi}{\partial t^2} - \frac{2}{c^4}\nabla^2\phi^2 - \frac{4\pi G}{c^4}\left(T^{(2)00} + T^{(2)ii} + 2g_{00}^{(2)}T^{(2)00}\right) \\
&\Rightarrow \frac{1}{2}\nabla^2 g_{00}^{(4)} = \frac{1}{c^4}\frac{\partial^2 \phi}{\partial t^2} + \frac{2}{c^4}\nabla^2\phi^2 + \frac{4\pi G}{c^4}\left(T^{(2)00} + T^{(2)ii} + 2g_{00}^{(2)}T^{(2)00}\right).
\end{aligned} \tag{A.25}$$

where $g_{00}^{(4)}$ can be solve explicitly as

$$-\frac{1}{2}\nabla^2 g_{00}^{(4)} = -\frac{1}{c^4}\nabla^2\phi^2 - \frac{4\pi G}{c^4}\left(\sum_a m_a c^2\left(\frac{3v_a^2}{2c^2} + \frac{5\phi_a}{c^2}\right)\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t))\right) \tag{A.26a}$$

$$= -\frac{1}{c^4}\nabla^2\phi^2 - \frac{4\pi G}{c^2}\sum_a m_a\left(\frac{3v_a^2}{2c^2} + \frac{\phi_a'}{c^2}\right)\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a(t)) \tag{A.26b}$$

$$\Rightarrow g_{00}^{(4)} = \frac{2\phi^2}{c^4} - \frac{G}{c^4}\sum_a \frac{m_a(3(v_a)^2 + \phi')}{|\boldsymbol{x} - \boldsymbol{x}_a(t)|} \tag{A.26c}$$

here we have replaced ϕ_a by

$$\phi_a' = -G{\sum_b}' \frac{m_b}{|\mathbf{x_a} - \mathbf{x_b}|}. \tag{A.27}$$

A.4 Lagrangian for 1PN scalar-tensor theory

The Lagrangian for 1PN scalar-tensor theory takes the form

$$L = -m_a^0 c^2 + m_a^0\tilde{V} - \frac{1}{2c^2}m_a^0\widetilde{W} - \frac{4}{c^2}m_a^0\tilde{V}_{ai}v_a^i + \frac{1}{c^2}m_a^0\tilde{V}_{ij}v^i v^j + \frac{1}{8c^2}m_a^0 v_a^4 + \frac{1}{2}m_a^0 v_a^2. \tag{A.28}$$

In this expression, the solution for the individual terms is given by first solving for the second term in 4.48 using 4.44a, it gives

$$
\begin{aligned}
m_a^0\tilde{V} &= m_a^0\sum_b Gm_b^{(0)}\Bigg[\left(1+\alpha_a^{(0)}\alpha_b^{(0)}\right)+\frac{1}{2c^2}\left(3-\alpha_a^{(0)}\alpha_b^{(0)}\right)v_b^2 \\
&\quad -\frac{G}{c^2}\sum_{c\neq b}\left\{\left(1+\alpha_a^{(0)}\alpha_b^{(0)}\right)\left(1+\alpha_b^{(0)}\alpha_c^{(0)}\right)+\alpha_a^{(0)}\beta_b^{(0)}\alpha_c^{(0)}\right\}\frac{m_c^{(0)}}{r_{bc}}\Bigg] \\
&\quad \times\left[\frac{1}{r_{ab}}\left\{1+\frac{1}{2c^2}\left[(v_a\cdot v_b)-(n_{ab}\cdot v_a)(n_{ab}\cdot v_b)\right]\right\}-\frac{1}{2c^2}\frac{\mathrm{d}}{\mathrm{d}t}(n_{ab}\cdot v_b)+\mathrm{O}\left(\frac{1}{c^4}\right)\right]+\mathcal{O}\left(\frac{1}{c^4}\right) \\
&= \sum_b\frac{m_b^{(0)}m_a^{(0)}}{r_{ab}}\Bigg[\left(1+\alpha_a^{(0)}\alpha_b^{(0)}\right)\times\left\{1+\frac{1}{2c^2}\left[(v_a\cdot v_b)-(n_{ab}\cdot v_a)(n_{ab}\cdot v_b)\right]\right\} \\
&\quad +\frac{1}{2c^2}\left(3-\alpha_a^{(0)}\alpha_b^{(0)}\right)(v_b^2+v_a^2)\left\{1+\frac{1}{2c^2}\left[(v_a\cdot v_b)-(n_{ab}\cdot v_a)(n_{ab}\cdot v_b)\right]\right\} \\
&\quad -\sum_{c\neq b}\frac{G}{c^2}\left\{\left(1+\alpha_a^{(0)}\alpha_b^{(0)}\right)\left(1+\alpha_b^{(0)}\alpha_c^{(0)}\right)+\alpha_a^{(0)}\beta_b^{(0)}\alpha_c^{(0)}\right\}\frac{m_c^{(0)}}{r_{bc}}\Bigg]+\mathcal{O}\left(\frac{1}{c^4}\right),
\end{aligned}
\tag{A.29}
$$

solving for the third term in 4.48 using 4.44d gives

$$
\begin{aligned}
\frac{1}{2c^2}m_a^0\widetilde{W} &= \frac{1}{2c^2}m_a^0G^2\sum_c\sum_b\beta^{(0)}\alpha_b^{(0)}\alpha_c^{(0)}m_b^{(0)}m_c^{(0)}u_bu_c \\
&= \frac{1}{2c^2}\sum_c\sum_b\frac{G^2m_a^0\beta^{(0)}\alpha_b^{(0)}\alpha_c^{(0)}m_b^{(0)}m_c^{(0)}}{r_{ab}r_{ca}}+\mathrm{O}\left(\frac{1}{c^4}\right),
\end{aligned}
\tag{A.30}
$$

solving for the fourth term in eq4.48 using 4.44b gives

$$
\begin{aligned}
\frac{4}{c^2}m_a^0\tilde{V}_iv_a^i &= \frac{4}{c^2}m_a^0G\sum_b M_b^V v_{bi}f_bv_a^i \\
&= \sum_b\frac{4}{c^2}Gm_a^0m_b^0(v_a\cdot v_b)\times\frac{1}{r_{ab}}\left\{1+\frac{1}{2c^2}\left[(v_a\cdot v_b)-(n_{ab}\cdot v_a)(n_{ab}\cdot v_b)\right]\right\} \\
&= \sum_b\frac{4}{c^2}\frac{Gm_a^0m_b^0}{r_{ab}}(v_a\cdot v_b)+\mathrm{O}\left(\frac{1}{c^4}\right),
\end{aligned}
\tag{A.31}
$$

and solving for the fifth term in eq4.48 using eq4.44c gives

$$
\begin{aligned}
\frac{1}{c^2} m_a^0 \tilde{V}_{ij} v^i v^j &= \frac{1}{c^2} m_a^0 G \sum_a M_a^{\mathrm{T}} \delta_{ij} f_a v^i v^j \\
&= \frac{1}{c^2} m_a^0 G \sum_b m_b^{(0)} \left(1 - \alpha_a^{(0)} \alpha_b^{(0)}\right) v_a^2 \times \frac{1}{r_{ab}} \left\{1 + \frac{1}{2c^2} \left[(v_a \cdot v_b) - (n_{ab} \cdot v_a)(n_{ab} \cdot v_b)\right]\right\} \\
&= \sum_b \frac{G m_a^0 m_b^0}{r_{ab}} v_a^2 \left(1 - \alpha_a^{(0)} \alpha_b^{(0)}\right) + \mathcal{O}\left(\frac{1}{c^4}\right).
\end{aligned}
\tag{A.32}
$$

A.5 Ricci tensor at 2PN order

In this section, we have obtained the components of the Ricci tensor at 2PN order by using the harmonic gauge condition $g^{\mu}\Gamma^{\rho}_{\mu\nu} = 0$. [In this section, the subscripts '0', '1', '2', and '3' represent the time, x, y, and z components.]

\>> **Obtaining** $R^{(4)}_{11}$

$$
R^{(4)}_{11} = -\frac{1}{2}\nabla^2 g^{(4)}_{11} + \mathcal{X}^{(4)}_{11} \tag{A.33}
$$

where the second term on the right hand side, $\mathcal{X}^{(4)}_{11}$ is

$$
\begin{aligned}
\mathcal{X}^{(4)}_{11} =& \frac{1}{4}\Big[2g^{(2)}_{33}g^{(2)}_{11,33} + 2(g^{(2)}_{12,3} - g^{(2)}_{13,2})^2 + g^{(2)}_{11,2}(g^{(2)}_{00,3} + g^{(2)}_{11,3} - g^{(2)}_{22,3} + g^{(2)}_{33,3} \\
&+ 2g^{(2)}_{23,2}) + g^{(2)}_{11,2}(2g^{(2)}_{23,3} + g^{(2)}_{00,2} + g^{(2)}_{11,2} + g^{(2)}_{22,2} - g^{(2)}_{33,2}) + 4g^{(2)}_{23}g^{(2)}_{11,23} + 2g^{(2)}_{22}g^{(2)}_{11,22} \\
&- (g^{(2)}_{00,1})^2 + g^{(2)}_{00,1}g^{(2)}_{11,1} + 2g^{(2)}_{11,2}g^{(2)}_{12,1} + 4(g^{(2)}_{12,1})^2 + 2g^{(2)}_{11,3}g^{(2)}_{13,1} + 4(g^{(2)}_{13,1})^2 - (g^{(2)}_{22,1})^2 \\
&+ 4g^{(2)}_{23,1}(g^{(2)}_{12,3} + g^{(2)}_{13,2}) - 2g^{(2)}_{23,2} - g^{(2)}_{33,1} - g^{(2)}_{11,1}(g^{(2)}_{22,1} + g^{(2)}_{33,1}) + 2g^{(2)}_{12,2}(g^{(2)}_{11,1} + 2g^{(2)}_{22,1}) \\
&+ 2g^{(2)}_{13,3}(g^{(2)}_{11,1} + 2g^{(2)}_{33,1}) + 2g^{(2)}_{12}(2g^{(2)}_{23,13} + g^{(2)}_{00,12} + g^{(2)}_{11,12} + g^{(2)}_{22,12} - g^{(2)}_{22,12} + 2g^{(2)}_{12,11}) \\
&+ 2g^{(2)}_{13}(g^{(2)}_{00,13} + g^{(2)}_{11,13} - g^{(2)}_{22,13} + 2(g^{(2)}_{23,12} + g^{(2)}_{13,11})) + 2g^{(2)}_{11,00}(2g^{(2)}_{13,13} + 2g^{(2)}_{12,12} \\
&+ g^{(2)}_{00,11} + 2g^{(2)}_{11,11} - g^{(2)}_{22,11} - g^{(2)}_{33,11}) + 2g^{(2)}_{11,00}\Big]
\end{aligned}
\tag{A.34}
$$

>> **Obtaining** $R^{(4)}_{22}$

$$R^{(4)}_{22} = -\frac{1}{2}\nabla^2 g^{(4)}_{22} + \mathcal{X}^{(4)}_{22} \tag{A.35}$$

where the second term on the right hand side, $\mathcal{X}^{(4)}_{22}$ is

$$\begin{aligned}
\mathcal{X}^{(4)}_{22} =& \frac{1}{4}\Big[g^{(2)}_{00,3}g^{(2)}_{22,3} - g^{(2)}_{11,3}g^{(2)}_{22,3} + (g^{(2)}_{22,3})^2 + g^{(2)}_{22,3}g^{(2)}_{33,3} + 2g^{(2)}_{33}g^{(2)}_{22,33} - (g^{(2)}_{00,2})^2 - (g^{(2)}_{11,2})^2 \\
&+ 4(g^{(2)}_{12,2})^2 + g^{(2)}_{00,2}g^{(2)}_{22,2} - g^{(2)}_{11,2}g^{(2)}_{22,2} + 4(g^{(2)}_{22,2})^2 + 2g^{(2)}_{22,3}g^{(2)}_{23,2} + 4(g^{(2)}_{23,2})^2 - g^{(2)}_{22,2}g^{(2)}_{33,2} \\
&- g^{(2)}_{33,2} + 2g^{(2)}_{23,3}(g^{(2)}_{22,2} + 2g^{(2)}_{33,2}) + 2g^{(2)}_{12,1}(2g^{(2)}_{11,2} + g^{(2)}_{22,2}) + 2g^{(2)}_{22,3}g^{(2)}_{13,1} + 2g^{(2)}_{12,2}g^{(2)}_{22,1} \\
&+ 2((g^{(2)}_{12,3})^2 - (g^{(2)}_{13,2})^2 + 2g^{(2)}_{22,1}g^{(2)}_{13,2} - 2g^{(2)}_{12,3}g^{(2)}_{23,1} + 2g^{(2)}_{13,2}g^{(2)}_{23,1} + (g^{(2)}_{23,1})^2) \\
&+ g^{(2)}_{22,1}(2g^{(2)}_{13,3} + g^{(2)}_{00,1} + g^{(2)}_{11,1} + g^{(2)}_{22,1} - g^{(2)}_{33,1}) + 2g^{(2)}_{22}(2g^{(2)}_{23,23} + g^{(2)}_{00,22} - g^{(2)}_{11,22} \\
&+ 2g^{(2)}_{22,22} - g^{(2)}_{33,22} + 2g^{(2)}_{12,12}) + 2g^{(2)}_{23}(2g^{(2)}_{00,23} - 2g^{(2)}_{12,22} + g^{(2)}_{22,23} + g^{(2)}_{33,23} + 2g^{(2)}_{23,22} \\
&+ 2g^{(2)}_{13,12}) + 2g^{(2)}_{12}(2g^{(2)}_{13,23} + 2g^{(2)}_{12,22} + g^{(2)}_{00,12} + g^{(2)}_{11,12} + g^{(2)}_{22,12} - g^{(2)}_{33,12}) + 4g^{(2)}_{13}g^{(2)}_{22,13} \\
&+ 2g^{(2)}_{11}g^{(2)}_{22,11} + 2g^{(2)}_{22,00}\Big]
\end{aligned} \tag{A.36}$$

>> **Obtaining** $R^{(4)}_{33}$

$$R^{(4)}_{33} = -\frac{1}{2}\nabla^2 g^{(4)}_{33} + \mathcal{X}^{(4)}_{33} \tag{A.37}$$

where the second term on the right hand side, $\mathcal{X}^{(4)}_{33}$ is

$$\begin{aligned}
\mathcal{X}^{(4)}_{33} =& \frac{1}{4}\Big[- (g^{(2)}_{00,3})^2 - (g^{(2)}_{11,3})^2 - 2(g^{(2)}_{12,3})^2 + 4(g^{(2)}_{13,3})^2 - (g^{(2)}_{22,3})^2 + 4(g^{(2)}_{23,3})^2 + g^{(2)}_{00,3}g^{(2)}_{33,3} \\
&- g^{(2)}_{11,3}g^{(2)}_{33,3} - g^{(2)}_{22,3}g^{(2)}_{33,3} + 4(g^{(2)}_{33,3})^2 + 4g^{(2)}_{22,3}g^{(2)}_{23,2} + 2g^{(2)}_{33,3}g^{(2)}_{23,2} + 2g^{(2)}_{23,3}g^{(2)}_{33,2} \\
&+ g^{(2)}_{33,2}(g^{(2)}_{00,2} - g^{(2)}_{11,2} + g^{(2)}_{22,2} + g^{(2)}_{33,2} + 2g^{(2)}_{12,1}) + 2g^{(2)}_{13,1}(2g^{(2)}_{11,3} + g^{(2)}_{33,3}) + 4g^{(2)}_{12,3}(g^{(2)}_{13,2} + g^{(2)}_{23,1}) \\
&+ g^{(2)}_{33,1}(2g^{(2)}_{13,3} + 2g^{(2)}_{12,2} + g^{(2)}_{00,1} + g^{(2)}_{11,1} - g^{(2)}_{22,1}) + (g^{(2)}_{33,1})^2 + 2g^{(2)}_{22}g^{(2)}_{33,22} + 2(g^{(2)}_{13,2} - g^{(2)}_{23,1})^2 \\
&+ 2g^{(2)}_{23}(2g^{(2)}_{23,33} + g^{(2)}_{00,23} - g^{(2)}_{11,23} + g^{(2)}_{22,23} + g^{(2)}_{33,23} + 2g^{(2)}_{12,13}) + 2g^{(2)}_{33}(g^{(2)}_{00,33} - g^{(2)}_{11,33} \\
&- g^{(2)}_{22,33} + 2g^{(2)}_{33,22} + 2g^{(2)}_{23,23} + g^{(2)}_{13,13}) + 2g^{(2)}_{13}(2g^{(2)}_{13,33} + 2g^{(2)}_{12,23} + g^{(2)}_{00,13} + g^{(2)}_{11,13} - g^{(2)}_{22,13} \\
&+ g^{(2)}_{33,13}) + 2g^{(2)}_{12}g^{(2)}_{33,23} + g^{(2)}_{11}g^{(2)}_{33,11} + g^{(2)}_{33,00}\Big]
\end{aligned} \tag{A.38}$$

>> **Obtaining** $R^{(5)}_{01}$

$$R^{(5)}_{01} = -\frac{1}{2}\nabla^2 g^{(5)}_{01} + \mathcal{X}^{(5)}_{01} \tag{A.39}$$

where the second term on the right hand side, $\mathcal{X}^{(5)}_{01}$ is

$$\begin{aligned}
\mathcal{X}^{(5)}_{01} =& \frac{1}{4}\Bigg[2g^{(2)}_{33}g^{(3)}_{01,33} + 2g^{(2)}_{23,3}g^{(3)}_{01,2} + g^{(3)}_{01,2}g^{(2)}_{11,2} + g^{(3)}_{01,2}g^{(2)}_{22,2} - g^{(3)}_{01,2}g^{(2)}_{33,2} + 4g^{(2)}_{23}g^{(3)}_{01,23} \\
&+ 2g^{(2)}_{22}g^{(3)}_{01,22} - 3g^{(2)}_{00m1}g^{(2)}_{01,1} - 2g^{(2)}_{11,2}g^{(3)}_{02,1} + 3g^{(3)}_{01,1}g^{(2)}_{11,1} + 2g^{(3)}_{01,3}(g^{(2)}_{11,3} - g^{(2)}_{22,3} \\
&+ g^{(2)}_{33,3}2(g^{(2)}_{23,2} + g^{(2)}_{13,1})) + 2g^{(3)}_{02,2}g^{(2)}_{22,1} - g^{(3)}_{01,1}g^{(2)}_{22,1} + 2g^{(3)}_{03,3}g^{(2)}_{33,1} - g^{(3)}_{01,1}g^{(2)}_{33,1} \\
&+ g^{(3)}_{02}(2g^{(2)}_{23,13} + g^{(2)}_{00,12} - g^{(2)}_{11,12} + g^{(2)}_{22,12} - g^{(2)}_{33,12} + 2g^{(2)}_{12,2}) + g^{(3)}_{03}(g^{(2)}_{00,13} - g^{(2)}_{11,13} \\
&- g^{(2)}_{22,13} + g^{(2)}_{33,13} + 2(g^{(2)}_{23,12} + g^{(2)}_{13,11})) + g^{(3)}_{01}(2g^{(2)}_{13,13} + 2g^{(2)}_{12,12} - g^{(2)}_{00,11} + g^{(2)}_{11,11} \\
&- g^{(2)}_{22,11} - g^{(2)}_{33,11}) + g^{(2)}_{00,1}g^{(2)}_{00,0} + 2g^{(2)}_{00,1}g^{(2)}_{11,0} + g^{(2)}_{11,1}g^{(2)}_{11,0} - g^{(2)}_{00,1}(g^{(3)}_{01,2} + 2g^{(3)}_{02,1} \\
&- 2g^{(2)}_{12,0}) + 2g^{(2)}_{11,2}g^{(2)}_{12,0} - g^{(2)}_{00,3}(g^{(2)}_{01,3} + 2g^{(3)}_{03,1} - 2g^{(2)}_{13,0}) + 2g^{(2)}_{11,3}g^{(2)}_{13,0} - g^{(2)}_{22,1}g^{(2)}_{22,0} \\
&+ 2g^{(2)}_{12,2}(g^{(2)}_{01,1} + g^{(2)}_{22,0}) + 2[g^{(3)}_{02,3}(g^{(2)}_{12,3} - g^{(2)}_{13,2} + g^{(2)}_{23,1}) + g^{(3)}_{03,2}(g^{(2)}_{13,2} - g^{(2)}_{12,3} + g^{(2)}_{23,1}) \\
&+ g^{(2)}_{23,0}(g^{(2)}_{12,3} + g^{(2)}_{13,2} - g^{(2)}_{23,1})] - g^{(2)}_{33,1}g^{(2)}_{33,0} + 2g^{(2)}_{13,3}(g^{(3)}_{01,1} + g^{(2)}_{33,0}) + g^{(2)}_{12}(4g^{(3)}_{01,12} \\
&+ 2g^{(2)}_{23,13} - g^{(2)}_{00,02} - g^{(2)}_{11,02} + g^{(2)}_{22,02} - g^{(2)}_{33,02} + 2g^{(2)}_{12,01}) + g^{(2)}_{13}(4g^{(3)}_{01,13} + g^{(2)}_{00,03} \\
&- g^{(2)}_{11,03} - g^{(2)}_{22,03} + g^{(2)}_{33,03} + 2(g^{(2)}_{23,02} + g^{(2)}_{13,01})) + g^{(2)}_{11}(2g^{(3)}_{01,11} + 2g^{(2)}_{13,03} + 2g^{(2)}_{12,02} \\
&+ g^{(2)}_{00,01} + g^{(2)}_{11,01} - g^{(2)}_{22,01} - g^{(2)}_{33,01}) + g^{(2)}_{00}(-2g^{(3)}_{03,13} - 2g^{(2)}_{02,12} - 2g^{(2)}_{01,11} + g^{(2)}_{00,01} \\
&+ g^{(2)}_{11,01} + g^{(2)}_{22,01} + g^{(2)}_{33,01}) + 2g^{(3)}_{01,00}\Bigg]
\end{aligned} \tag{A.40}$$

>> **Obtaining** $R^{(5)}_{02}$

$$R^{(5)}_{02} = -\frac{1}{2}\nabla^2 g^{(5)}_{02} + \mathcal{X}^{(5)}_{02} \tag{A.41}$$

where the second term on the right hand side, $\mathcal{X}_{02}^{(5)}$ is

$$\begin{aligned}\mathcal{X}_{02}^{(5)} =&\frac{1}{4}\Big[2g^{(2)}g_{02,33}^{(3)} - g_{02,2}^{(3)}g_{11,2}^{(2)} + 3g_{02,2}^{(3)}g_{22,2}^{(2)} - 2g_{03,3}^{(3)}(g_{22,3}^{(2)} - 2g_{23,2}^{(2)}) + 2g_{03,3}^{(3)}g_{33,2}^{(2)} - g_{02}^{(3)}g_{22,2}^{(2)}\\
&+ 2g_{11,2}^{(2)}g_{01,1}^{(3)} + 2g_{02,2}^{(3)}g_{12,1}^{(2)} + g_{02,3}^{(3)}(-g_{11,3}^{(2)} + g_{22,3}^{(2)} + g_{33,3}^{(2)} + 2(g_{23,2}^{(2)} + g_{13,1}^{(2)})) + g_{01,2}^{(3)}(4g_{12,2}^{(2)}\\
&- 2(g_{00,1}^{(2)} + g_{22,1}^{(2)})) + g_{02,1}^{(3)}(2g_{13,3}^{(2)} + 2g_{12,2}^{(2)} - g_{00,1}^{(2)} + g_{11,1}^{(2)} + g_{22,1}^{(2)} - g_{33,1}^{(2)}) + 4g_{13}^{(2)}g_{02,13}^{(3)}\\
&+ g_{02}^{(3)}(2g_{23,23}^{(2)} + g_{00,22}^{(2)} - g_{11,22}^{(2)} + g_{22,22}^{(2)} - g_{33,22}^{(2)} + 2g_{12,12}^{(2)}) + g_{03}^{(3)}(g_{00,23}^{(2)} - g_{11,23}^{(2)} - g_{22,23}^{(2)}\\
&+ g_{33,23}^{(2)} + 2(g_{23,22}^{(2)} + g_{13,12}^{(2)})) + g_{01}^{(3)}(2g_{13,23}^{(2)} + 2g_{12,22}^{(2)} + g_{00,12}^{(2)} + g_{11,12}^{(2)} - g_{22,12}^{(2)} - g_{33,12}^{(2)})\\
&- g_{11,2}^{(2)}g_{11,0}^{(2)} + 2g_{12,1}^{(2)}g_{11,0}^{(2)} + 2(g_{00,1}^{(2)} + g_{22,1}^{(2)})g_{12,1}^{(2)} + 2(g_{01,3}^{(2)}(g_{12,3}^{(2)} + g_{13,2}^{(2)} - g_{23,1}^{(2)})\\
&+ g_{03,1}^{(3)}(-g_{12,3}^{(2)} + g_{13,2}^{(2)} + g_{23,1}^{(2)}) + g_{13,0}^{(2)}(g_{12,3}^{(2)} - g_{13,2}^{(2)} + g_{23,1}^{(2)})) + g_{22,2}^{(2)}g_{22,0}^{(2)} + g_{00,2}^{(2)}(-3g_{02,2}^{(3)}\\
&+ g_{00,0}^{(2)} + 2g_{22,0}^{(2)}) - g_{00,3}^{(2)}(g_{02,3}^{(3)} + 2g_{03,2}^{(3)} - 2g_{23,0}^{(2)}) + 2g_{22,3}^{(2)}g_{23,0}^{(2)} - g_{33,2}^{(2)}g_{33,0}^{(2)} + 2g_{23,3}^{(2)}(g_{02,1}^{(3)}\\
&+ g_{33,1}^{(2)}) + g_{00}^{(2)}(-2g_{03,23}^{(3)} - 2g_{02,22}^{(3)} - 2g_{01,12}^{(3)} + g_{00,02}^{(2)} + g_{11,02}^{(2)} + g_{22,02}^{(2)} + g_{33,02}^{(2)})\\
&+ g_{22}^{(2)}(2g_{02,22}^{(3)} + 2g_{23,03}^{(2)} + g_{00,02}^{(2)} - g_{11,03}^{(2)} + g_{22,03}^{(2)} - g_{33,03}^{(2)} + 2g_{12,01}^{(2)}) + g_{23}^{(2)}(4g_{02,23}^{(3)}\\
&+ g_{00,03}^{(2)} - g_{11,03}^{(2)} - g_{22,03}^{(2)} + g_{33,03}^{(2)} + 2(g_{12,0,3}^{(2)} + g_{13,01}^{(2)})) + g_{12}^{(2)}(2g_{02,12}^{(3)} + 2g_{13,03}^{(2)} + 2g_{12,02}^{(2)}\\
&+ g_{00,01}^{(2)} + g_{11,01}^{(2)} - g_{22,01}^{(2)} - g_{33,01}^{(2)}) + 2(g_{11}^{(2)}g_{02,11}^{(3)} + g_{02,00}^{(3)})\Big]\end{aligned} \tag{A.42}$$

>> **Obtaining** $R_{00}^{(6)}$

$$R_{00}^{(6)} = -\frac{1}{2}\nabla^2 g_{00}^{(6)} + \mathcal{X}_{00}^{(6)} \tag{A.43}$$

where the second term on the right hand side, $\mathcal{X}_{00}^{(6)}$ are rest of the term appearing for $R_{00}^{(6)}$.

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