Cuspidal Cohomology for GL(N) over Number Fields

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

A thesis submitted in partial fulfillment of the
requirements of the degree of Doctor of Philosophy

द्वारा / By
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IISER PUNE

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INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

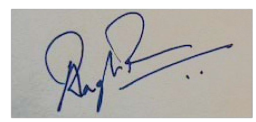
2024

Certificate

This is to certify that this dissertation entitled Cuspidal Cohomology for GL_N over Number Fieldstowards the partial fulfilment of the degree of Doctor in Philosophy at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Nasit Darshan Prafulbhaiat Indian Institute of Science Education and Research under the supervision of Dr. Chandrasheel Bhagwat, Associate Professor at IISER Pune, and Prof. A. Raghuram, Professor at Fordham University, during the academic years 2019-2024 .

चं.रा. भागवत

Dr. Chandrasheel Bhagwat



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Declaration

I hereby declare that the matter embodied in the report entitled Cuspidal Cohomology for GL_N over Number Fields, are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Chandrasheel Bhagwat, Prof. A. Raghuram and the same has not been submitted elsewhere for any other degree.

Darshan

Nasit Darshan Prafulbhai

Acknowledgments

As I am on the edge of finishing my graduate studies, I am immensely grateful to the many individuals and institutions whose unwavering support and guidance have been instrumental in bringing this work to fruition.

First and foremost, I express my deepest gratitude to my mentors, Raghuram and Chandrasheel, whose exceptional guidance, scholarly prowess, and unwavering commitment have served as a constant source of motivation during this journey. Their insightful advice, constructive critiques, and continual support have significantly enriched the development and enhancement of this thesis. I would also like to express my sincere appreciation to my thesis committee members, Baskar Balasubramanyam and C.S. Rajan, for their valuable questions and suggestions. Special thanks to Institute for Advanced Study (IAS), where a portion of this thesis was undertaken during the summer collaboration program. I also thank the other members of this summer collaboration program: Baskar Balasubramanyam, Chandrasheel Bhagwat, P. Narayanan, and Freydoon Shahidi, who acted as a sounding board at various times.

This endeavor would have been impossible without the funding that supported my academic pursuits. I am deeply grateful to the MHRD for awarding me the Prime Minister's Research Fellowship (PMRF) throughout my doctoral journey. Additionally, I extend my heartfelt thanks to the numerous reviewers for their invaluable suggestions during the review process. I am also thankful to several local institutes for enriching my teaching experiences. I would also like to thank all the research institutes that have provided me with the opportunities to attend summer schools, workshops, and conferences across India.

I wish to present my special thanks to Dr. Amit Hogadi, Dr. Debargha Banerjee, and Dr. Kaneenika Sinha for several mathematical discussions. I would also like to thank the chair and co-chair of the department. I am also grateful to the administrative staff, especially to Mrs. Suvarna Bhardwaj, Mr. Yogesh Kolap, and Mrs. Sayalee Damle. I am deeply indebted to my school teachers for their motivation to pursue my research career.

My heartfelt gratitude goes out to my friends, particularly Hiren, Pranjal, Hitendra, Ajinkya, Dhruv, Narayanan, Shubham, Tushar, Bittu, and Kunal, for numerous academic and non-academic discussions. I also wish to extend my sincere appreciation to the Independent Student Body (ISB) and the Dean of Student and Campus Activities for their role in enhancing my personal development. Additionally, I am thankful for being a part of the 'Zeher' sports club. Lastly, I am indebted to my school friends and others whose names I have not mentioned for their invaluable support throughout this journey.

Finally, words cannot express my gratitude to my parents. I could not have undertaken this journey without your support. I would like to thank Ayushi and Raj for emotionally supporting me and fulfilling my responsibilities. I would like to extend my gratitude to other family members for supporting me to overcome the hurdles in my life.

Darshan Nasit

Abstract

Let GL_N be the algebraic group over a number field F. We are interested in a subspace (known as cuspidal cohomology) of the sheaf cohomology of locally symmetric adèlic space in the coefficient system of a finite-dimensional representation $\mathcal{M}_\lambda$ of $\mathrm{Res}_{F/\mathbb{Q}} GL_N$ with the highest weight λ. Our study focuses on establishing a non-vanishing property of cuspidal cohomology. We prove the non-vanishing of a Lefschetz number to prove the non-vanishing of cuspidal cohomology for SL_N when F is Galois over maximal totally real subfield and the highest weight is strongly pure. It also proves the non-vanishing of cuspidal cohomology for GL_N.

Given an irreducible representation of $SL_2(\mathbb{F}_q)$ for an odd prime q, we find the dimension of the space of cusp forms with respect to the full modular group taking values into certain representation spaces. The dimension equals the multiplicity of the representation in the space of classical cusp forms with respect to the principal congruence subgroup of level q.

The arXiv links for papers:

- Cuspidal cohomology for $GL(n)$ over a number field,
  https://arxiv.org/abs/2407.10859
- On the dimensions of certain spaces of vector-valued cusp forms,
  https://arxiv.org/abs/2407.21664

Contents

Chapter 1

Introduction

1.1 Cuspidal Cohomology

The question of the existence of cusp forms is related to the non-vanishing of cuspidal cohomology in the context of algebraic groups over number fields. Consider the algebraic group $G_0 = GL_N/F$ over a number field F, and let $\mathcal{M}_\lambda$ be an algebraic irreducible finite-dimensional complex representation of G_0 with the highest weight λ. The cuspidal cohomology of G_0 in the coefficient system $\mathcal{M}_\lambda$ is defined in Section 8.3.2. The key question then is whether we can find an open compact subgroup $K_f \subset G_0(\mathbb{A}_{F,f})$ such that the cuspidal cohomology does not vanish, i.e., $H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \neq 0$.

For instance, when $N = 2$ and $F = \mathbb{Q}$, any finite-dimensional algebraic irreducible representation of G_0 is $V_{a,d} \cong Sym^{a-1}(\mathbb{C}^2) \otimes (\det^d)$ for $a \geq 1$ and $d \in \mathbb{Z}$. In this case, cuspidal cohomology for open compact subgroup K_f is isomorphic to the parabolic or Eichler cohomology $H^\bullet_{cusp}(\Gamma, V_{a,0})$ where $\Gamma = K_f \cap GL_2^+(\mathbb{Q})$ is a discrete subgroup of $GL_2^+(\mathbb{R})$. The latter cohomology is isomorphic to $\mathcal{S}_{a+1}(\Gamma) \oplus \overline{\mathcal{S}_{a+1}(\Gamma)}$ via the Eichler-Shimura Isomorphism. Thus, the existence of cusp forms is equivalent to the non-vanishing of the cohomology.

Similarly, for $N = 2$ and F a totally real number field of degree d, then any finite-dimensional algebraic irreducible representation of G_0 is $V_{(a_i,d_i)} \cong \otimes[Sym^{a_i-1}(\mathbb{C}^2) \otimes (\det^{d_i})]$, the question about the existence of cusp forms translates to the existence of Hilbert cusp forms. The cuspidal cohomology for congruence subgroup Γ is isomorphic to $\oplus_{J\subset\Sigma_F}\mathcal{S}^J_k(\Gamma)$

through the Eichler-Shimura Isomorphism.

1.1.1 Literature Survey

In the context of a totally imaginary quadratic extension $F = \mathbb{Q}(\sqrt{D})$ where $D < 0$, consider the ring of algebraic integers $\mathcal{O}_F$. Grunewald and Schwermer proved the non-vanishing of cuspidal cohomology for the trivial coefficient system in [16].

Theorem 1.1.1. ***(F. Grunewald, J. Schwermer)*** *Let $\Gamma \subset SL_2(\mathcal{O}_F)$ be a subgroup of finite index, and let n be a positive integer. There exists a subgroup $\Gamma' \subset \Gamma$ with the finite index such that the dimensions of the cuspidal cohomology groups satisfy $\dim H^q_{\text{cusp}}(\Gamma', \mathbb{C}) \geq n$ for $q = 1, 2$.*

Rohlfs showed the following result by estimates for the trace of the linear map induced by complex conjugation on the first cohomology group, in [32].

Theorem 1.1.2. ***(J. Rohlfs)*** *For $\Gamma \subset SL_2(\mathcal{O}_F)/\pm\mathbf{1}$ with a finite index subgroup, it holds that $\dim H^1_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{C})/SU(2), \mathbb{C}) \geq \frac{1}{24}\phi(D) - \frac{1}{4} - \frac{1}{2}h_D$, where h_D is the class number and ϕ denotes Euler's totient function.*

Let K be a totally real number field and F be a number field such that F/K is a solvable extension, i.e., $K = F_0 \subset F_1 \subset \cdots \subset F_l = F$ such that F_i/F_{i-1} is an extension of the prime degree for $1 \leq i \leq l$. Then Labesse and Schwermer proved the following result in [23].

Theorem 1.1.3. ***(J.P. Labesse, J. Schwermer)*** *The cuspidal cohomology of SL_N/F with the trivial coefficient system is non-vanishing for $N = 2$ and $N = 3$.*

Clozel proved the following result in [10] by using the construction of Hecke-Maass-Jacquet-Langlands associating cusp forms to Grössencharakterer on a quadratic extension of F.

Theorem 1.1.4. ***(L. Clozel)*** *The cuspidal cohomology of SL_{2N} over a number field F with the trivial coefficient system is non-vanishing.*

Let F be a totally real number field. Let G_0 be the connected, semi-simple algebraic group over F and $G = \text{Res}_{F/\mathbb{Q}} G_0 \times E$ for the cyclic Galois extension E/F. Then Rohlfs and Speh proved the following theorem in [33].

Theorem 1.1.5. ***(J. Rohlfs, B. Speh)*** *If $G(\mathbb{R})$ contains a compact Cartan subgroup then there exists $Gal(E/F)$ stable subgroup Γ such that $H^{\bullet}_{\text{cusp}}(\Gamma, \mathbb{C}) \neq 0$.*

Borel, Labesse, and Schwermer proved the following result using the Lefschetz number and twisted trace formula in [5].

Theorem 1.1.6. ***(A. Borel, J.P. Labesse, J. Schwermer)*** *Let K be the totally real number field and F be a number field such that F/K is a solvable extension. Then the cuspidal cohomology of SL_N/F in the trivial coefficient system is non-vanishing.*

Let $G = SL_1(D)$ be the algebraic group over E, where D is quaternion division algebra over the number field E. Let M be a locally symmetric space associated with G. Let $q_0 = q_1 + q_2$, where q_1 (resp. $2q_2$) is the number of real (resp. complex) Archimedean places of E at which D splits. Rajan proved the following theorem in [30].

Theorem 1.1.7. ***(C.S. Rajan)*** *If E is a subfield of a solvable extension of some totally real number field then there exists a congruence subgroup $\Gamma \subset G(E \otimes \mathbb{R})$ such that the cohomology group $H^{q_0}(\Gamma \backslash M, \mathbb{C}) \neq 0$.*

Let $G = SL_2(D)$ be the algebraic group over $\mathbb{Q}$, where D is quaternion algebra over $\mathbb{Q}$ such that D does not split at the Archimedean place. Grobner proved the following theorem in [15].

Theorem 1.1.8. ***(H. Grobner)*** *Let $\mathcal{M}$ be a finite-dimensional, self-dual, irreducible, complex representation of G then the cuspidal cohomology $H^{\bullet}_{cusp}(G, \mathcal{M}) \neq 0$.*

Bhagwat and Raghuram proved the following result using Arthur's endoscopic classification and Mok's classification of discrete spectrums in [2].

Theorem 1.1.9. ***(C. Bhagwat, A. Raghuram)*** *Let F be either totally real or CM field extension of $\mathbb{Q}$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(GL_N/F)$ and λ be a pure weight (see Section 8.5) (with purity weight 0 if F is a CM field). Then the cuspidal cohomology with the coefficient system $\mathcal{M}_\lambda$ is non-vanishing.*

1.1.2 Our Contribution

Let λ be a strongly pure weight, and $\mathcal{M}_\lambda$ be the finite dimensional irreducible representation of $\mathrm{Res}_{F/\mathbb{Q}} GL_N/F$. Let $\mathcal{M}_{\lambda^1}$ is a restriction of representation to $\mathrm{Res}_{F/\mathbb{Q}} SL_N/F$. We show that the cuspidal cohomology for GL_N/F with the coefficient system $\mathcal{M}_\lambda$ does not vanish if the cuspidal cohomology for SL_N/F with the coefficient system $\mathcal{M}_{\lambda^1}$ does not vanish (see Theorem 8.4.1). We show that if F is a Galois over its maximal totally real subfield then there exists an admissible unitary irreducible representation $\mathbb{J}_{\lambda^1}$ such that the Lefschetz number does not vanish for the representation $\mathcal{M}_{\lambda^1} \otimes \mathbb{J}_{\lambda^1}$ (see Proposition 8.6.2). It implies that the cuspidal cohomology of SL_N/F does not vanish in the representation space $\mathcal{M}_{\lambda^1}$ (Theorem 8.6.3) and also for GL_N/F (Corollary 8.6.4).

1.2 Dimension of $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho)$

For an odd prime $q \geq 5$, let $\mathbb{F}$ denote the finite field of order q. Consider an irreducible representation (ρ, V_ρ) of the group $SL_2(\mathbb{F})$. This induces a natural irreducible representation of $SL_2(\mathbb{Z})$, which we also denote by (ρ, V_ρ):

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}) \to GL(V_\rho).$$

Now, considering the normal subgroup $\Gamma(q)$ of $\Gamma(1) = SL_2(\mathbb{Z})$, we have a natural homomorphism:

$$\sigma : SL_2(\mathbb{Z}) \to GL(\mathcal{S}_k(\Gamma(q))),$$

defined as $[\sigma(g)f](z) = j(g^{-1}, z)^{-k} f(g^{-1}z)$, where j is the automorphy factor. This representation extends naturally to $SL_2(\mathbb{F})$ since the kernel of σ is $\Gamma(q)$. Thus, we can express σ as a direct sum:

$$\sigma = \bigoplus m_\rho \rho,$$

where the sum runs over the inequivalent classes of irreducible representations ρ of $SL_2(\mathbb{F})$, and m_ρ denotes the multiplicity of ρ.

In Chapter 5, we establish that the dimension of $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho)$ is equal to the multiplicity of $\overline{\rho}$ for any irreducible representation ρ (Lemma 5.1.2). Moreover, we compute dimension formulas when ρ is a parabolically induced or a cuspidal representation, see Theorem 5.4.1 and Theorem 5.5.1.

1.3 Introduction to Thesis

Chapter 2 covers fundamental concepts in Lie algebras, including the classification of irreducible finite-dimensional representations of semisimple Lie algebras and their cohomology. In this chapter, we compute the root system and Cartan decomposition for $\mathfrak{sl}_N(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{sl}_N(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$. In Chapter 3, we discuss the affine algebraic groups, real reductive groups, torus, parabolic subgroups, and related definitions. Chapter 4 introduces the notion of automorphic forms and those on GL_N are induced from those on SL_N, setting the stage for the proof of Theorem 8.4.1.

Chapter 6 covers the notion of admissible representations and their differential cohomology. Specifically, we define the Lefschetz number and compute it in the case of $SL_N(\mathbb{C})$ (Theorem 6.6.10) and $SL_N(\mathbb{R})$ (Theorem 6.7.9). Those results are important to derive our main theorem 8.6.3.

Chapter 7 provides a comprehensive review of the seminal work of Borel, Labesse, and Schwermer. It covers the decomposition of uniformly moderate growth functions on $\Gamma\backslash G_S$ and cohomology groups of S-arithmetic groups, culminating in a proof of the non-vanishing of cuspidal cohomology. Terminology in Chapter 7 is defined in Chapters 2 and 3, allowing readers to navigate this chapter independently of Chapters 4 and 5.

Chapter 2

Lie Algebras

This chapter is an introduction to the basics of Lie algebras. We focus on the classification of finite-dimensional representations of semisimple Lie algebras and computations about their cohomology. Throughout this chapter, we assume that a Lie algebra is a finite-dimensional vector space over an algebraically closed field with characteristic 0. The discussion is based on the references [19], [21], and [38]. We will also compute the explicit Cartan decomposition of $\mathfrak{sl}_N(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{sl}_N(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ in Section 2.6 and 2.7, respectively.

2.1 Introduction

Consider a vector space L over a field F equipped with an operation $[\ ,\ ] : L \times L \to L$. This structure is called a **Lie Algebra** if the operation is bilinear, $[x, x] = 0$ for all $x \in L$, and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. The operation is called the **Lie bracket**.

Example 2.1.1. *Any associative F-algebra A admits a natural Lie algebra structure under a bracket operation defined as $[x, y] = x \cdot y - y \cdot x$, i.e., any algebra homomorphisms between associative algebras are Lie algebra homomorphisms.*

Example 2.1.2. *Consider a vector space V over a field F. The set of all endomorphisms of V, denoted as $End(V)$, forms an associative F-algebra. If we equip $End(V)$ with the Lie algebra structure defined in Example 2.1.1, it is denoted by $\mathfrak{gl}(V)$.*

Example 2.1.3. *The adjoint Lie algebra homomorphism* ad : $L \to \mathfrak{gl}(L)$ *is defined as* $\mathrm{ad}(x)(y) = [x, y]$.

The **derived Lie algebra**, denoted as $[L, L]$, of L is the Lie subalgebra generated by $[x, y]$ for all $x, y \in L$. Note that derived Lie algebra is an ideal of Lie algebra. The **derived series** of L is defined as $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. A **solvable Lie algebra** is a Lie algebra such that $L^{(n)} = 0$ for some $n \geq 0$. The **radical** of a Lie algebra, denoted as $\mathrm{Rad}(L)$, is the maximal solvable ideal of L. If $\mathrm{Rad}(L) = 0$, then L is called a **semisimple Lie algebra**.

A **descending central series** of L is defined by $L^0 = L$ and $L^i = [L, L^{i-1}]$. A Lie algebra is called a **nilpotent Lie algebra** if $L^n = 0$ for some $n \geq 0$. An element $x \in L$ is called **ad-nilpotent** if $\mathrm{ad}(x)$ (defined in Example 2.1.3) is a nilpotent endomorphism. Similarly, an element $x \in L$ is called **ad-semisimple** if $\mathrm{ad}(x)$ is a semisimple endomorphism.

For any Lie algebra L, the **Killing Form** is defined as $\kappa_L(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$. This gives rise to a symmetric bilinear form on L. Additionally, for any ideal I, the Killing form on I is $\kappa_I = \kappa_L|_{I\times I}$. The Killing form is called nondegenerate if, for every $x \in L$, there exists $y \in L$ such that $\kappa_L(x, y) \neq 0$.

Theorem 2.1.1. *A Lie algebra is semisimple if and only if the Killing form is nondegenerate.*

If L is a semisimple Lie algebra, it possesses a non-trivial subalgebra consisting of semisimple elements, known as a **toral subalgebra**. Let H be a maximal toral subalgebra. In this case, L undergoes a **root space decomposition** or **Cartan decomposition** expressed as

$$L = H \oplus \bigoplus_{\alpha\in\Phi(L,H)} L_\alpha,$$

where $L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$ for any $\alpha \in H^* := \mathrm{Hom}_F(H, F)$, and $\alpha \in \Phi(L, H)$ if $L_\alpha \neq 0$. The set $\Phi := \Phi(L, H)$ is called the **root system**.

A subset $\Delta \subset \Phi$ is called a **base** if Δ forms a basis for H^* and for each root $\alpha = \sum\limits_{\beta\in\Delta} k_\beta \beta$, the coefficients k_β are either all non-negative integers or all non-positive integers. When a base is fixed, we can define sets of positive and negative roots denoted by $\Phi^+(\Delta)$ and $\Phi^-(\Delta)$, respectively. Consequently, an order on Φ can be defined such that $\alpha \succ \beta$ if $\alpha - \beta \in \Phi^+(\Delta)$. Each $\alpha \in \Phi$ determines the hyperplane $P_\alpha = \{\beta \in H^* : (\alpha, \beta) = 0\}$. An open connected component of $H^* - \cup_{\alpha\in\Phi} P_\alpha$ is called a **Weyl Chamber**. Let σ_α represent a reflection on H^* for any $\alpha \in H^*$. The group generated by σ_α for all $\alpha \in \Phi$ is known as the **Weyl group**, denoted by $\mathcal{W}$.

Lemma 2.1.2. *The Weyl group is generated by $\{\sigma_\alpha : \alpha \in \Delta\}$ for any base $\Delta \subset \Phi$.*

A self-normalizing nilpotent subalgebra of L is called **Cartan Subalgebra**. When L is semisimple, Cartan subalgebras of L are the same as the maximal toral subalgebras of L.

Theorem 2.1.3. *Let L be a semisimple Lie algebra with Cartan subalgebra H and the corresponding root system Φ. Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be a base and $\alpha_i(h_j) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ with the standard set of generators $x_i \in L_{\alpha_i}$, $y_i \in L_{-\alpha_i}$, and $h_i = [x_i, y_i]$. Then L is generated as a Lie algebra by $\{x_i, y_i, h_i : 1 \le i \le l\}$ satisfying the following relations:*

- $[h_i, h_j] = 0$ *for any* $1 \le i, j \le l$.
- $[x_i, y_i] = h_i$ *and* $[x_i, y_j] = 0$ *for all* $1 \le i \ne j \le l$.
- $[h_i, x_j] = \alpha_j(h_i)x_j$ *and* $[h_i, y_j] = -\alpha_j(h_i)y_j$ *for all* $1 \le i, j \le l$.
- $(\operatorname{ad} x_i)^{-\alpha_j(h_i)+1}(x_j) = 0$ *and* $(\operatorname{ad} y_i)^{-\alpha_j(h_i)+1}(y_j) = 0$ *for all* $1 \le i \ne j \le l$.
- *The vector space S_i over $\mathbb{C}$ spanned by vectors x_i, y_i, h_i, is isomorphic to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.*

2.2 Representations of Lie Algebras

2.2.1 Universal Enveloping Algebra

A universal enveloping algebra of L, denoted as $\mathfrak{U}L$, is an associative algebra equipped with a Lie algebra homomorphism $\iota : L \to \mathfrak{U}L$, such that for any associative algebra A and any Lie algebra homomorphism $j : L \to A$, there exists a unique algebra morphism $J : \mathfrak{U}L \to A$ such that $J \circ \iota = j$. Recall that the algebra $\mathfrak{U}L$ naturally inherits a Lie algebra structure (see Example 2.1.1). Let $\mathfrak{T}L$ be the tensor algebra over the vector space L then every morphism $j : L \to A$ can be extended to the algebra morphism $\mathfrak{T}L \to A$. Let I be the two-sided ideal of $\mathfrak{T}L$ generated by $x \cdot y - y \cdot x - [x, y]$ for all $x, y \in L$. Since j is a Lie algebra homomorphism, the extended morphism factors through $\mathfrak{T}L/I \to A$. Hence, the quotient algebra $\mathfrak{T}L/I$ serves as a universal enveloping algebra, equipped with a natural map $\iota : L \to \mathfrak{T}L \xrightarrow{\pi} \mathfrak{T}L/I$.

Theorem 2.2.1. *(**Poincaré-Birkhoff-Witt**) Consider an ordered basis $(x_1, x_2, \ldots)$ for a Lie algebra L. Then, the set $\{\pi(x_{i(1)} \otimes \cdots \otimes x_{i(m)}) : 1 \le i(1) \le \cdots \le i(m)$ and $m > 0\} \cup \{1\}$ forms a basis for the universal enveloping algebra $\mathfrak{U}L$. Additionally, for any subalgebra H of L, the natural map $\mathfrak{U}H \to \mathfrak{U}L$ induced by the inclusion map $H \to L$ is an injection. Moreover, the map $L \to \mathfrak{U}L$ is a natural inclusion.*

Let H be a Cartan subalgebra of a semisimple Lie algebra L. So, we can identify H^* as H, and the Weyl group acts on H such that $(\sigma \cdot \alpha)(h) = \alpha(\sigma^{-1} \cdot h)$ for all $\sigma \in \mathcal{W}$, $\alpha \in H^*$ and $h \in H$. Then the Weyl group acts on $\mathfrak{U}H$, and let $\mathfrak{U}H^{\mathcal{W}}$ be the set consisting of all $\mathcal{W}-$invariant elements.

Theorem 2.2.2. *(**Harish-Chandra Isomorphism** in [21]) Let $Z(\mathfrak{U}L)$ be the center of $\mathfrak{U}L$. Then $Z(\mathfrak{U}L) = \mathfrak{U}H^{\mathcal{W}}$, considering $\mathfrak{U}H^{\mathcal{W}} \subset \mathfrak{U}H \subset \mathfrak{U}L$.*

2.2.2 Weights

Let L be a semisimple Lie algebra with a Cartan subalgebra H and the corresponding root system Φ. The **weight space** is denoted by $\Lambda = \{\lambda \in H^* : \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi\}$. An element in the weight space is referred to as a **weight**. It is important to note that Λ forms a subgroup of H^* and includes the set of roots, Φ. By fixing a base $\Delta \subset \Phi$, the weights $\{\lambda_1, \ldots, \lambda_l\}$ are called **fundamental weights** if $\frac{2(\lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{i,j}$ for $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. A weight λ is called a **dominant weight** with respect to Δ if $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \ge 0$. Any dominant weight is a non-negative integer linear combination of fundamental weights.

2.2.3 Representation Space

For any representation V of L, the action of H on V is semisimple since H is a toral subalgebra. Let $V_\lambda = \{v \in V : h \cdot v = \lambda(h)v \ \forall h \in H\}$ be the eigenspace for $\lambda \in H^*$.

Lemma 2.2.3. *If V is any Lie algebra representation of L then L_α maps V_λ into $V_{\lambda+\alpha}$ for $\alpha \in \Phi$ and $V' = \sum V_\lambda$ is a direct sum. Moreover, $V = V'$ if V is finite-dimensional.*

A non-zero vector $v^+ \in V_\lambda$ is called a **maximal vector** of weight λ if $x \cdot v^+ = 0$ for all $x \in L_\alpha$ with $\alpha \succ 0$. Then $\mathfrak{U}L \cdot v^+$ is called a **standard cyclic module** of L with the highest weight λ. Let $I(\lambda)$ be the left ideal generated by $\{x_\alpha : \alpha \in \Phi^+\} \cup \{h_\alpha - \lambda(h_\alpha) \cdot 1 : \alpha \in \Phi\}$, then the quotient $\mathfrak{U}L/I(\lambda)$ is a cyclic module with the highest weight λ.

Lemma 2.2.4. *For every weight λ, there exists a cyclic module with the highest weight λ and it is unique up to isomorphism.*

In a finite-dimensional irreducible representation V of the Lie algebra L, there exists a maximal vector $v^+ \in V$ with the highest weight λ. Since V is irreducible, the highest weight is unique. Furthermore, the action of the Lie subalgebra $B = H + \sum_{\alpha \succ 0} L_\alpha$ on v^+ is such that $\mathfrak{U}B \cdot v^+ = Fv^+$. For each $\alpha_i \in \Delta$, a copy of $\mathfrak{sl}(2, \mathbb{C})$ denoted as S_i (as per Theorem 2.1.3) exists in L. Consequently, V can be viewed as a finite-dimensional representation of each S_i. Therefore for the highest weight, $\lambda(h_i) = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$ must be non-negative integers, emphasizing that the highest weight must be dominant.

Let λ be a dominant weight then $m_i := \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$ are non-negative integers for all $\alpha_i \in \Delta$. Let $J(\lambda)$ be the left ideal generated by $I(\lambda)$ together with the elements $\{y_i^{m_i+1} : 1 \leq i \leq l\}$. Then $\mathfrak{U}L/J(\lambda)$ is a finite-dimensional representation of L with the highest weight λ.

Theorem 2.2.5. *For the dominant weight λ, there exists a unique (up to an isomorphism) finite-dimensional irreducible representation with the highest weight λ.*

The Killing form $\kappa_L|_H$ is nondegenerate for the Cartan subalgebra H of a semisimple Lie algebra L. Let $e_1, \ldots, e_r$ be a basis of H and $e^1, \ldots, e^r$ be a dual basis, i.e., $\kappa_L(e_i, e^j) = \delta_{i,j}$. Now consider $C_L = \sum e_i e^i$ as an element in $\mathfrak{U}L$, known as the **Casimir element**. Then C_L belongs to the center of $\mathfrak{U}L$. Moreover, the action of C_L on any finite-dimensional representation is nontrivial unless it is a trivial representation.

Let $Z(\mathfrak{U}L)$ be the center of the universal enveloping algebra $\mathfrak{U}L$. Then $Z(\mathfrak{U}L)$ also acts on V for any $V \in \mathfrak{U}L\text{-}Mod$. For any algebra homomorphism $\chi : Z(\mathfrak{U}L) \to F$, consider the subspace $V_\chi = \{v \in V : z \cdot v = \chi(Z)v \text{ for all } z \in Z(\mathfrak{U}L)\}$. Note that V_χ is stable under the action of $\mathfrak{U}L$. If V is an irreducible representation, then there exists χ such that $V_\chi = V$. Such χ is called an **infinitesimal character** of V, denoted as χ_V.

Lemma 2.2.6. *(Lecture 5 by Anthony Knapp and Peter Trapa in [22]) Let V and W be finite-dimensional irreducible representations of L with the highest weights λ and μ, respectively.*

Let χ_λ and χ_μ are infinitesimal characters of V and W, respectively. Then, $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same orbit for the action of the Weyl group.

Proof. Theorem 2.2.2 implies that $Z(\mathfrak{U}L) = \mathfrak{U}H^{\mathcal{W}} \subset \mathfrak{U}H$. Since H is a toral subalgebra, $\mathfrak{U}H$ is isomorphic to the polynomial algebra in $\dim H$ many variables. So, any character λ on H uniquely extends to an algebra homomorphism $\widetilde{\lambda}$ of $\mathfrak{U}H$.
Any characters λ and μ of H are in the same orbit if and only if extended algebra homomorphisms agree on $\mathfrak{U}H^{\mathcal{W}}$. So, infinitesimal characters are equal.
Let λ and μ be characters that are not in the same orbit, then there exists a multivariable polynomial f such that $\widetilde{\lambda}(f) = 1$ and $\widetilde{\mu}(f) = 0$ and $(\sigma \cdot \widetilde{\lambda})(f) = (\sigma \cdot \widetilde{\mu})(f) = 0$ for all non-trivial $\sigma \in \mathcal{W}$. Let g be the sum of all $\mathcal{W}$ conjugates of f. Then $g \in \mathfrak{U}H^{\mathcal{W}}$ and $\widetilde{\lambda}(g) = 1$, $\widetilde{\mu}(g) = 0$. So, $\chi_\lambda \neq \chi_\mu$. Hence, Lemma follows. □

2.3 Lie Algebra Cohomology

Let L be a Lie algebra over a field F. Consider the abelian category $L\text{-}Mod$ of L-modules or Lie algebra representations of L. For any $\phi : L \to \mathfrak{gl}(V)$, there exists a natural algebra homomorphism $\overline{\phi} : \mathfrak{U}L \to \mathrm{End}(V)$. Any $\mathfrak{U}L$-module M naturally becomes an L-module, given the inclusion $L \to \mathfrak{U}L$ (see Theorem 2.2.1). This establishes a natural equivalence of categories between $L\text{-}Mod$ and $\mathfrak{U}L\text{-}Mod$.

Let V be an L-module then define the 'trivial' subspace of V as

$$V^L = \{v \in V : x \cdot v = 0 \text{ for all } x \in L\}.$$

If $0 \to M \to N \to P \to 0$ is an exact sequence of Lie algebra representations, then $0 \to M^L \to N^L \to P^L$ is exact. Therefore, $V \mapsto V^L$ is a left exact functor. The Lie algebra cohomology is defined as the right-derived functors of the above left exact functor.

Consider F as a trivial L-module, i.e., $x \cdot a = 0$ for all $x \in L$ and $a \in F$. Since a map $\mathrm{Hom}_{\mathfrak{U}L}(F, V) \to V^L$ defined as $f \mapsto f(1)$ is an isomorphism, the Lie algebra cohomology is isomorphic to $\{Ext^q\}$ functors. In summary,

Theorem 2.3.1. *For any L-module V, the Lie algebra cohomology $H^q(L, V) \cong Ext^q_{\mathfrak{U}L}(F, V)$. Moreover, $H^0(L, V) = V^L \cong \mathrm{Hom}_{\mathfrak{U}L}(F, V)$.*

Theorem 2.3.2. *(**Wigner's Lemma**) Let V and W be $L-$modules with infinitesimal character χ_V and χ_W respectively. If $\chi_V \neq \chi_W$ then $Ext^q_{\mathfrak{U}L}(V, W) = 0$ for $q \geq 0$.*

Corollary 2.3.3. *Let V be a finite-dimensional irreducible non-trivial representation of a semisimple Lie algebra L. Then the Lie algebra cohomology vanishes for all degrees.*

Proof. Since F is a trivial representation of L, the highest weight λ for F is zero. So, any $\mathcal{W}-$conjugate of λ is also zero. If V is a non-trivial finite-dimensional irreducible representation then the highest weight λ of V is non-zero. So, μ and λ are not in the same orbit. Then the corollary follows from Lemma 2.2.6, Theorem 2.3.1 and Theorem 2.3.2. □

Theorem 2.3.4. *Let V and W be finite-dimensional irreducible representations of L. If $H^q(L, V \otimes W) \neq 0$ for some q then V is contragredient representation of W.*

Proof. Note that $V \otimes W \cong \mathrm{Hom}_F(W^*, V)$ as a representation of L if the action defined as $(x \cdot T)(f) := x \cdot T(f) - T(x \cdot f)$ for any $x \in L$, $f \in W^*$ and $T : W^* \to V$. Since $V \otimes W$ is a finite-dimensional representation, it is a direct sum of irreducible representations. The hypothesis and Corollary 2.3.3 implies $V \otimes W$ has a trivial subrepresentation. So, there exists $T \in \mathrm{Hom}(W^*, V)$ such that $x \cdot T = 0$ for all $x \in L$ implying $T \in \mathrm{Hom}_L(W^*, V)$. So, the theorem follows since V and W are irreducible representations and Schur's Lemma. □

Let $\Lambda^q L$ denote the q^{th} exterior product of L for $q \geq 0$. Then $\mathfrak{U}L \otimes_F \Lambda^q L$ are free $\mathfrak{U}L$-modules, since $\Lambda^q L$ are free F vector spaces. This forms the **Chevalley-Eilenberg Complex** $\mathfrak{U}L \otimes \Lambda^* L \xrightarrow{\epsilon} F$ with $\epsilon : \mathfrak{U}L \otimes \Lambda^0 L \to F$ as the natural map and the differential map

$$\begin{aligned} d(u \otimes x_1 \wedge \cdots \wedge x_q) = &\sum_{1 \leq i \leq q} (-1)^i u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q \\ &+ \sum_{1 \leq i < j \leq q} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_q. \end{aligned}$$

The Chevalley-Eilenberg complex is a projective resolution of F as Theorem 7.7.2 in [38]. For any $V \in L$-Mod, Lie algebra cohomology is the cohomology of the cochain complex: $\mathrm{Hom}_{\mathfrak{U}L}(\mathfrak{U}L \otimes \Lambda^* L, V) \cong \mathrm{Hom}_F(\Lambda^* L, V)$. Let $C^q(L, V) = \mathrm{Hom}_F(\Lambda^q L, V)$, and the differential map d on the cochain complex is given by:

$$(df)(x_0 \wedge \cdots \wedge x_q) = \sum (-1)^i x_i \cdot f(x_0 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_q)$$
$$+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_0 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_q),$$
$$(df)(x) = x \cdot f(1) - f(1) \text{ if } q = 0.$$

Hence the Lie algebra cohomology is isomorphic to the cohomology of the cochain complex, i.e., $H^q(L, V) \cong H^q(C^*(L, V))$.

2.4 Relative Lie Algebra Cohomology

For any $x \in L$, define $\theta_x : C^q(L, V) \to C^q(L, V)$ and $i_x : C^q(L, V) \to C^{q-1}(L, V)$ as:

$$(\theta_x f)(x_1, \ldots, x_q) = \sum f(x_1 \wedge \cdots \wedge [x_i, x] \wedge \cdots \wedge x_q) + x \cdot f(x_1, \ldots, x_q),$$
$$(i_x f)(x_1, \ldots, x_{q-1}) = f(x, x_1, \ldots, x_{q-1}).$$

Let H be any subalgebra of L. Define $C^q(L, H, V)$ as the subspace of $C^q(L, V)$ consisting of elements annihilated by both θ_x and i_x for all $x \in H$. This subspace remains stable under the differential map since $\theta_x = d \circ i_x + i_x \circ d$. The cohomology of the resulting cochain complex, denoted as $H^\bullet(L, H, V)$, is called the **relative Lie algebra cohomology**. Considering L/H as an H-module through the adjoint action, we can establish an identification: $C^q(L, H, V)$ corresponds to $\mathrm{Hom}_H(\Lambda^q(L/H), V)$ (a set of the morphisms between representation spaces).

2.5 Cartan Involution

Let L denote a finite-dimensional real semisimple Lie algebra. The Killing form on L is defined as $B(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$. Let θ be an involution on L (i.e., $\theta^2 = \mathrm{id}$) is called **Cartan involution** if the form $B_\theta(X, Y) = -B(X, \theta Y)$ is symmetric and positive definite.

Lemma 2.5.1. *A semisimple real Lie algebra admits a Cartan involution. Any two Cartan involutions are conjugate via an inner automorphism.*

Proof. Let $\mathfrak{g}$ be a semisimple real Lie algebra, and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification. The complexification can be expressed as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g} = \mathfrak{u} \oplus \mathbf{i}\mathfrak{u}$, where $\mathfrak{u}$ is a compact real form. The complex conjugation τ on $\mathfrak{g}$ with respect to $\mathfrak{u}$ ensures that $\mathfrak{g}$ is τ-stable. Furthermore, $\tau|_{\mathfrak{g}}$ satisfies the criteria of the Cartan involution. $\square$

2.6 Example $\mathfrak{sl}_N(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$

Let $\mathfrak{g}_0 = \mathfrak{sl}_N(\mathbb{C}) = \{X \text{ is } N \times N \text{ matrix with entries in } \mathbb{C} : \mathrm{Trace}(X) = 0\}$ be a real Lie algebra. Let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be a complexified Lie algebra, then $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}_0$ be the diagonal subalgebra of $\mathfrak{g}_0$ and $\mathfrak{h}$ be the corresponding subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{g}$ and the root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_{\alpha},$$

where $\Phi(\mathfrak{g},\mathfrak{h}) = \{(E_i^* - E_j^*) \pm (\overline{E}_i^* - \overline{E}_j^*) : 1 \leq i \neq j \leq N\}$. Here, $(E_i^* - E_j^*) + (\overline{E}_i^* - \overline{E}_j^*)$ is defined on $\mathfrak{h}$ as $\mathrm{diag}(x_1, \ldots, x_N) \otimes 1 \mapsto x_i - x_j$ and similarly, $(E_i^* - E_j^*) - (\overline{E}_i^* - \overline{E}_j^*)$ is defined on $\mathfrak{h}$ as $\mathrm{diag}(x_1, \ldots, x_N) \otimes 1 \mapsto \overline{x_i} - \overline{x_j}$. Let $\Delta = \{(E_i^* - E_{i+1}^*) \pm (\overline{E}_i^* - \overline{E}_{i+1}^*) : 1 \leq i \leq N-1\}$ is a base for $\Phi(\mathfrak{g},\mathfrak{h})$.

Let $\theta_0 : \mathfrak{g}_0 \to \mathfrak{g}_0$ be an automorphism of real Lie algebra defined as $\theta_0(X) = -{}^t\overline{X}$. Since θ_0 is an involution (i.e., $\theta_0^2 = \mathbf{1}_{\mathfrak{g}_0}$), $\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$, where $\mathfrak{p}_0$ and $\mathfrak{k}_0$ denotes the eigenspace of eigenvalues -1 and 1, respectively. Then

$$\begin{aligned}
\mathfrak{k}_0 &= {}^\circ\mathfrak{t}_0 \oplus \Bigg(\bigoplus_{1 \leq i < j \leq N} \mathbb{R} \cdot (E_{i,j} - E_{j,i}) \Bigg) \oplus \Bigg(\bigoplus_{1 \leq i < j \leq N} \mathbb{R} \cdot \sqrt{-1}(E_{i,j} + E_{j,i}) \Bigg), \\
\mathfrak{p}_0 &= \mathfrak{a}_0 \oplus \Bigg(\bigoplus_{1 \leq i < j \leq N} \mathbb{R} \cdot (E_{i,j} + E_{j,i}) \Bigg) \oplus \Bigg(\bigoplus_{1 \leq i < j \leq N} \mathbb{R} \cdot \sqrt{-1}(E_{i,j} - E_{j,i}) \Bigg),
\end{aligned}$$

where ${}^\circ\mathfrak{t}_0 = \oplus_{1 \leq i \leq N-1} \mathbb{R} \cdot \sqrt{-1}(E_{i,i} - E_{i+1,i+1})$, $\mathfrak{a}_0 = \oplus_{1 \leq i \leq N-1} \mathbb{R} \cdot (E_{i,i} - E_{i+1,i+1})$ and $E_{i,j}$ is the matrix with 1 in the (i,j)-th entry and 0 elsewhere. Note that $\mathfrak{k}_0$ is a Lie subalgebra, since $[\mathfrak{k}_0, \mathfrak{k}_0] \subset \mathfrak{k}_0$. Let $\mathfrak{p}, \mathfrak{k}, \mathfrak{a}$ and ${}^\circ\mathfrak{t}$ are complexified Lie algebra of $\mathfrak{p}_0, \mathfrak{k}_0, \mathfrak{a}_0$ and ${}^\circ\mathfrak{t}_0$, respectively. A map $\theta := \theta_0 \otimes \mathbf{1}_{\mathbb{C}}$ is also an automorphism of complex Lie algebra $\mathfrak{g}$. Moreover, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is the Cartan decomposition and

$$\mathfrak{k} = {}^{\circ}\mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Phi^+} \mathbb{C} \cdot X^k_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Phi^-} \mathbb{C} \cdot Y^k_\alpha \right),$$
$$\mathfrak{p} = \mathfrak{a} \oplus \left(\bigoplus_{\alpha \in \Phi^+} \mathbb{C} \cdot X^p_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Phi^-} \mathbb{C} \cdot Y^p_\alpha \right),$$

where $\Phi^+ = \{(\overline{E}^*_i - \overline{E}^*_j) : 1 \le i < j \le N\}$ and $\Phi^- = -\Phi^+$, and

$$\begin{aligned} X^k_{i,j} &= (E_{i,j} - E_{j,i}) \otimes \sqrt{-1} + \sqrt{-1}(E_{i,j} + E_{j,i}) \otimes 1, \\ Y^k_{i,j} &= (E_{i,j} - E_{j,i}) \otimes \sqrt{-1} - \sqrt{-1}(E_{i,j} + E_{j,i}) \otimes 1, \\ X^p_{i,j} &= (E_{i,j} + E_{j,i}) \otimes \sqrt{-1} + \sqrt{-1}(E_{i,j} - E_{j,i}) \otimes 1, \\ Y^p_{i,j} &= (E_{i,j} + E_{j,i}) \otimes \sqrt{-1} - \sqrt{-1}(E_{i,j} - E_{j,i}) \otimes 1. \end{aligned}$$

Observe that ${}^{\circ}\mathfrak{t}$ acts on itself and on $\mathfrak{a}$ by 0. On the elements $X^k_{i,j}$ and $X^p_{i,j}$, it acts by $(\overline{E}i^* - \overline{E}^*_j)$, while on Yi, j^k and Yi, j^p, it acts by $(\overline{E}^*_j - \overline{E}^*_i)$. Recall that $(\overline{E}^*_i - \overline{E}^*_j)$ is defined as $\mathrm{diag}(\sqrt{-1}x_1, \ldots, \sqrt{-1}x_N) \otimes 1 \mapsto \sqrt{-1}(x_i - x_j)$.

2.7 Example $\mathfrak{sl}_N(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$

Let $\mathfrak{g}_0 = \mathfrak{sl}_N(\mathbb{R}) = \{X \text{ is } N \times N \text{ matrix with entries in } \mathbb{R} : \mathrm{Trace}(X) = 0\}$ be a real Lie algebra. Let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be a complexified Lie algebra, then $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}_0$ be the diagonal subalgebra of $\mathfrak{g}_0$ and $\mathfrak{h}$ be the corresponding subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{g}$ and the root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha,$$

where $\Phi(\mathfrak{g}, \mathfrak{h}) = \{(E^*_i - E^*_j) : 1 \le i \ne j \le N\}$. Here, $(E^*_i - E^*_j)$ is defined on $\mathfrak{h}$ as $\mathrm{diag}(x_1, \ldots, x_N) \otimes 1 \mapsto x_i - x_j$. Let $\Delta = \{(E^*_i - E^*_{i+1}) : 1 \le i \le N-1\}$ is a base for $\Phi(\mathfrak{g}, \mathfrak{h})$.

Let $\theta_0 : \mathfrak{g}_0 \to \mathfrak{g}_0$ be an automorphism of real Lie algebra defined as $\theta_0(X) = -{}^tX$. Since θ_0 is an involution (i.e., $\theta_0^2 = \mathbf{1}_{\mathfrak{g}_0}$), $\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$, where $\mathfrak{p}_0$ and $\mathfrak{k}_0$ denotes the eigenspace of eigenvalues -1 and 1, respectively. Then

$$
\begin{aligned}
\mathfrak{k}_0 &= \bigoplus_{1\le i<j\le N} \mathbb{R}\cdot(E_{i,j}-E_{j,i}),\\
\mathfrak{p}_0 &= \Big(\bigoplus_{1\le i\le N-1} \mathbb{R}\cdot(E_{i,i}-E_{i+1,i+1})\Big)\oplus\Big(\bigoplus_{1\le i<j\le N} \mathbb{R}\cdot(E_{i,j}+E_{j,i})\Big),
\end{aligned}
$$

where $E_{i,j}$ is the matrix with 1 in the (i,j)-th entry and 0 elsewhere. Note that $\mathfrak{k}_0$ is a Lie subalgebra, since $[\mathfrak{k}_0,\mathfrak{k}_0]\subset\mathfrak{k}_0$. Let $\mathfrak{p}$ and $\mathfrak{k}$ are complexified Lie algebra of $\mathfrak{p}_0$ and $\mathfrak{k}_0$, respectively. A map $\theta := \theta_0\otimes\mathbf{1}_{\mathbb{C}}$ is also an automorphism of complex Lie algebra $\mathfrak{g}$. Moreover, $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$ is the Cartan decomposition. Let $\mathfrak{t}_0=\bigoplus\mathbb{R}\cdot(E_{2i-1,2i}-E_{2i,2i-1})$ be a subalgebra of $\mathfrak{k}_0$, then it is a maximal toral subalgebra of $\mathfrak{k}_0$. Let $\mathfrak{t}$ be the complexification of $\mathfrak{t}_0$, then we will discuss the action of $\mathfrak{t}$ on the $\mathfrak{g}$ in Subsection 2.7.1 if $N=2M$ and Subsection 2.7.2 if $N=2M+1$.

2.7.1 Case N=2M

The eigenspace decomposition for the action of $\mathfrak{t}$ on $\mathfrak{k}$ and $\mathfrak{p}$ are as following:

$$
\begin{aligned}
\mathfrak{k} &= \mathfrak{t}\oplus\Big(\bigoplus_{\alpha\in\Phi_1^+}\mathbb{C}\cdot X_\alpha^{1,k}\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi_1^-}\mathbb{C}\cdot Y_\alpha^{1,k}\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi_2^+}\mathbb{C}\cdot X_\alpha^{2,k}\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi_2^-}\mathbb{C}\cdot Y_\alpha^{2,k}\Big),\\
\mathfrak{p} &= \mathfrak{a}\oplus\Big(\bigoplus_{\alpha\in\Phi_1^+}\mathbb{C}\cdot X_\alpha^{1,p}\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi_1^-}\mathbb{C}\cdot Y_\alpha^{1,p}\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi_2^+}\mathbb{C}\cdot X_\alpha^{2,p}\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi_2^-}\mathbb{C}\cdot Y_\alpha^{2,p}\Big)\\
&\qquad\oplus\Big(\bigoplus_{\alpha\in\Phi_3^+}\mathbb{C}\cdot X_\alpha^{3,p}\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi_3^-}\mathbb{C}\cdot Y_\alpha^{3,p}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{t} &= \Big\{t := \sum_{1\le i\le M}(E_{2i-1,2i}-E_{2i,2i-1})\otimes t_i\sqrt{-1}:\ t_i\in\mathbb{C}\Big\},\\
\mathfrak{a} &= \Big\{\sum_{1\le i\le M}(E_{2i-1,2i-1}+E_{2i,2i})\otimes a_i:\ a_i\in\mathbb{C}\text{ such that }\sum a_i=0\Big\},\\
\Phi_1^+ &= \{(E_i^*-E_j^*) : 1\le i<j\le M\}\text{ and }\Phi_1^-=-\Phi_1^+,\\
\Phi_2^+ &= \{(E_i^*+E_j^*) : 1\le i<j\le M\}\text{ and }\Phi_2^-=-\Phi_2^+,
\end{aligned}
$$

$$
\begin{aligned}
&\Phi_3^+ = \{2E_i^* : 1 \le i \le M\} \text{ and } \Phi_3^- = -\Phi_3^+,\\
&X_{i,j}^{1,k} = (E_{2i,2j} - E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad - (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1,\\
&Y_{i,j}^{1,k} = (E_{2i,2j} - E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad + (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1,\\
&X_{i,j}^{2,k} = (E_{2i,2j} - E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad + (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1,\\
&Y_{i,j}^{2,k} = (E_{2i,2j} - E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad - (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1.\\
&X_{i,j}^{1,p} = (E_{2i,2j} + E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad - (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1,\\
&Y_{i,j}^{1,p} = (E_{2i,2j} + E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad + (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1,\\
&X_{i,j}^{2,p} = (E_{2i,2j} + E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad + (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1,\\
&Y_{i,j}^{2,p} = (E_{2i,2j} + E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1}\\
&\qquad\qquad - (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1,\\
&X_i^{3,p} = E_{2i,2i} \otimes 1 + E_{2i-1,2i} \otimes \sqrt{-1} + E_{2i,2i-1} \otimes \sqrt{-1} - E_{2i-1,2i-1} \otimes 1,\\
&Y_i^{3,p} = E_{2i,2i} \otimes 1 - E_{2i-1,2i} \otimes \sqrt{-1} - E_{2i,2i-1} \otimes \sqrt{-1} - E_{2i-1,2i-1} \otimes 1.
\end{aligned}
$$

Note that $\mathfrak{t}$ acts on itself and $\mathfrak{a}$ by 0. It acts on $X_{i,j}^{1,k}$ and $X_{i,j}^{1,p}$ by $(E_i^* - E_j^*)$, on $Y_{i,j}^{1,k}$ and $Y_{i,j}^{1,p}$ by $(E_j^* - E_i^*)$, on $X_{i,j}^{2,k}$ and $X_{i,j}^{2,p}$ by $(E_i^* + E_j^*)$, on $Y_{i,j}^{2,k}$ and $Y_{i,j}^{2,p}$ by $-(E_i^* + E_j^*)$. Additionally, it acts on $X_i^{3,p}$ by $2E_i^*$, and on $Y_i^{3,p}$ by $-2E_i^*$. Recall that E_i^* is defined as $t \mapsto t_i$ for all $t \in \mathfrak{t}$.

2.7.2 Case N=2M+1

The eigenspace decomposition for the action of $\mathfrak{t}$ on $\mathfrak{k}$ and $\mathfrak{p}$ are as following:

$$\begin{aligned}\mathfrak{k} = \mathfrak{t} &\oplus \left(\bigoplus_{\alpha\in\Phi_1^+} \mathbb{C}\cdot X_\alpha^{1,k}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_1^-} \mathbb{C}\cdot Y_\alpha^{1,k}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_2^+} \mathbb{C}\cdot X_\alpha^{2,k}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_2^-} \mathbb{C}\cdot Y_\alpha^{2,k}\right)\\ &\oplus \left(\bigoplus_{\alpha\in\Phi_3^+} \mathbb{C}\cdot X_\alpha^{3,k}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_3^-} \mathbb{C}\cdot Y_\alpha^{3,k}\right),\\ \mathfrak{p} = \mathfrak{a} &\oplus \left(\bigoplus_{\alpha\in\Phi_1^+} \mathbb{C}\cdot X_\alpha^{1,p}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_1^-} \mathbb{C}\cdot Y_\alpha^{1,p}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_2^+} \mathbb{C}\cdot X_\alpha^{2,p}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_2^-} \mathbb{C}\cdot Y_\alpha^{2,p}\right)\\ &\oplus \left(\bigoplus_{\alpha\in\Phi_3^+} \mathbb{C}\cdot X_\alpha^{3,p}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_3^-} \mathbb{C}\cdot Y_\alpha^{3,p}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_4^+} \mathbb{C}\cdot X_\alpha^{4,p}\right) \oplus \left(\bigoplus_{\alpha\in\Phi_4^-} \mathbb{C}\cdot Y_\alpha^{4,p}\right),\end{aligned}$$

where

$$\mathfrak{t} = \left\{t := \sum_{1\le i\le M} (E_{2i-1,2i} - E_{2i,2i-1}) \otimes t_i\sqrt{-1} :\ t_i \in \mathbb{C}\right\},$$

$$\mathfrak{a} = \left\{E_{N,N} \otimes a_{M+1} + \sum_{1\le i\le M} (E_{2i-1,2i-1} + E_{2i,2i}) \otimes a_i :\ a_i \in \mathbb{C} \text{ such that } a_{M+1} + 2\sum_{i=1}^{M} a_i = 0\right\},$$

$$\begin{aligned}\Phi_1^+ &= \{(E_i^* - E_j^*) : 1 \le i < j \le M\} \text{ and } \Phi_1^- = -\Phi_1^+,\\ \Phi_2^+ &= \{(E_i^* + E_j^*) : 1 \le i < j \le M\} \text{ and } \Phi_2^- = -\Phi_2^+,\\ \Phi_3^+ &= \{E_i^* : 1 \le i \le M\} \text{ and } \Phi_3^- = -\Phi_3^+,\\ \Phi_4^+ &= \{2E_i^* : 1 \le i \le M\} \text{ and } \Phi_4^- = -\Phi_4^+,\\ X_{i,j}^{1,k} &= (E_{2i,2j} - E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\ &\quad - (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1,\\ Y_{i,j}^{1,k} &= (E_{2i,2j} - E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\ &\quad + (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1,\\ X_{i,j}^{2,k} &= (E_{2i,2j} - E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\ &\quad + (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1,\\ Y_{i,j}^{2,k} &= (E_{2i,2j} - E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} - E_{2j,2i-1}) \otimes \sqrt{-1}\\ &\quad - (E_{2i,2j-1} - E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} - E_{2j-1,2i-1}) \otimes 1,\end{aligned}$$

$$
\begin{aligned}
X_i^{3,k} &= (E_{N,2i-1} - E_{2i-1,N}) \otimes 1 - (E_{N,2i} - E_{2i,N}) \otimes \sqrt{-1}, \\
Y_i^{3,k} &= (E_{N,2i-1} - E_{2i-1,N}) \otimes 1 + (E_{N,2i} - E_{2i,N}) \otimes \sqrt{-1}, \\
X_{i,j}^{1,p} &= (E_{2i,2j} + E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1} \\
&\quad - (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1, \\
Y_{i,j}^{1,p} &= (E_{2i,2j} + E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1} \\
&\quad + (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} + (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1, \\
X_{i,j}^{2,p} &= (E_{2i,2j} + E_{2j,2i}) \otimes 1 + (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1} \\
&\quad + (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1, \\
Y_{i,j}^{2,p} &= (E_{2i,2j} + E_{2j,2i}) \otimes 1 - (E_{2i-1,2j} + E_{2j,2i-1}) \otimes \sqrt{-1} \\
&\quad - (E_{2i,2j-1} + E_{2j-1,2i}) \otimes \sqrt{-1} - (E_{2i-1,2j-1} + E_{2j-1,2i-1}) \otimes 1 \\
X_i^{3,p} &= (E_{N,2i-1} + E_{2i-1,N}) \otimes 1 - (E_{N,2i} + E_{2i,N}) \otimes \sqrt{-1}, \\
Y_i^{3,p} &= (E_{N,2i-1} + E_{2i-1,N}) \otimes 1 + (E_{N,2i} + E_{2i,N}) \otimes \sqrt{-1}, \\
X_i^{4,p} &= E_{2i,2i} \otimes 1 + E_{2i-1,2i} \otimes \sqrt{-1} + E_{2i,2i-1} \otimes \sqrt{-1} - E_{2i-1,2i-1} \otimes 1 \\
Y_i^{4,p} &= E_{2i,2i} \otimes 1 - E_{2i-1,2i} \otimes \sqrt{-1} - E_{2i,2i-1} \otimes \sqrt{-1} - E_{2i-1,2i-1} \otimes 1.
\end{aligned}
$$

Observe that $\mathfrak{t}$ acts on itself and $\mathfrak{a}$ by 0. It acts on $X_{i,j}^{1,k}$ and $X_{i,j}^{1,p}$ by $(E_i^* - E_j^*)$, on $Y_{i,j}^{1,k}$ and $Y_{i,j}^{1,p}$ by $(E_j^* - E_i^*)$, on $X_{i,j}^{2,k}$ and $X_{i,j}^{2,p}$ by $(E_i^* + E_j^*)$, on $Y_{i,j}^{2,k}$ and $Y_{i,j}^{2,p}$ by $-(E_i^* + E_j^*)$. Moreover, it acts on $X_i^{3,k}$ and $X_i^{3,k}$ by E_i^*, on $Y_i^{3,p}$ and $Y_i^{3,p}$ by $-E_i^*$, on $X_i^{4,p}$ by $2E_i^*$, and on $Y_i^{4,p}$ by $-2E_i^*$. Recall that E_i^* is defined as $t \mapsto t_i$ for all $t \in \mathfrak{t}$.

Chapter 3

Algebraic Groups

This chapter contains a brief discussion of affine algebraic groups, with a particular emphasis on reductive groups. It is based on the references:[7],[24], [25], [26], [27], [35], and [37].

Throughout the chapter, K denotes a field of the characteristic 0, and $\overline{K}$ is an algebraic closure of K. Moreover, F denotes a subfield of K and E is a field extension of F in $\overline{K}$.

3.1 Affine Algebraic Varieties

The **affine** $n-$**space** $\mathbb{A}^n$ is identified with the Zariski topological space K^n. An **algebraic subset** V of $\mathbb{A}^n$ is defined as the zero set of a collection of polynomials in $K[x_1, \ldots, x_n]$. For a given algebraic set V, we denote $I(V)$ as the set of polynomials that vanish simultaneously on V, making $I(V)$ an ideal in $K[x_1, \ldots, x_n]$. Furthermore, $I(V)$ is a radical ideal, meaning $\sqrt{I(V)} = I(V)$. The **coordinate ring** of an algebraic set V is given by $K[V] = K[x_1, \ldots, x_n]/I(V)$.

If V is the union of smaller algebraic sets V_1 and V_2, then $I(V) = I(V_1) \cap I(V_2)$. In this case, we say that V is a **reducible** algebraic set. An algebraic set is **irreducible** if and only if $I(V)$ is a prime ideal. An **affine algebraic variety** is an irreducible algebraic set, implying that the coordinate ring of an affine algebraic variety is an integral domain.

The **dimension** of an affine algebraic variety is the supremum of the length of the chains of irreducible closed subsets, i.e.,

$$\dim V = \sup\{d : \exists\ V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_d \text{ for irreducible closed subspaces } V_n \text{ for } 0 \leq n \leq d\}.$$

Example 3.1.1. *The affine $n-$space $\mathbb{A}^n$ is an affine algebraic variety of dimension n, and the coordinate ring $K[x_1, \ldots, x_n]$.*

Example 3.1.2. *The multiplicative group $K^\times$ is an affine algebraic variety of dimension 1, and the coordinate ring $K[x, y]/(xy - 1)$.*

Let V and W be affine algebraic varieties. A map $T : V \to W$ is a morphism between algebraic varieties if and only if the induced map $\overline{T} : K(W) \to K(V)$ is a $K-$algebra homomorphism.

Consider affine algebraic varieties $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ with coordinate rings $K[X]$ and $K[Y]$, respectively. The product $X \times Y$ is also an affine algebraic variety with coordinate ring $K[X \times Y]$, which is isomorphic to $K[X] \otimes_K K[Y]$. Suppose $I(X) \subset K[x_1, \ldots, x_n]$ and $I(Y) \subset K[y_1, \ldots, y_m]$ are sets of polynomials, vanishing on X and Y respectively. Then, $X \times Y$ is a zero set for radical of ideal generated by $I(X)$ and $I(Y)$ in $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$.

3.2 Affine Algebraic Groups

An **affine algebraic group** G/K is an affine algebraic variety together with a group structure such that the multiplication map $\mu : G \times G \to G$ and the inverse map $\iota : G \to G$ are morphisms of affine algebraic varieties.

Example 3.2.1. *$\mathbb{G}_a/K$ is the affine algebraic variety over K with the coordinate ring $K[\mathbb{G}_a] = K[x]$. And, group operations are defined as $\mu(x, y) = x + y$ and $\iota(x) = -x$.*

Example 3.2.2. *$\mathbb{G}_m/K$ is the affine algebraic variety over K with the coordinate ring $K[\mathbb{G}_m] = K[x, y]/(xy - 1)$. And, the group operations are $\mu(x, y) = xy$ and $\iota(x) = x^{-1}$.*

Example 3.2.3. *The general linear group $GL_N/K = GL_N(K)$ is an affine algebraic variety over K with the coordinate ring $K[GL_N] = K[\{x_{i,j} : 1 \leq i, j \leq N\} \cup \{y\}]/(y \det([x_{i,j}]) - 1)$, and dimension N^2.*

For any commutative K-algebra A, the A-points of the algebraic group G are given by $G(A) = \mathrm{Hom}_{K\text{-Alg}}(K[G], A)$. Consequently, an algebraic group over K can be viewed as a representable functor from the category of K-algebras to the category of groups.

An algebraic group $G/K \subset \mathbb{A}^n_K$ is deemed to be defined over a subfield F if the coordinate ring $K[G] = K \otimes_F F[G]$, where $F[G]$ represents the coordinate ring of $G/F := G \cap \mathbb{A}^n_F$. In simpler terms, this implies the existence of a collection I of polynomials in $F[x_1, \ldots, x_n]$ such that G is the zero set of I in $\mathbb{A}^n_K$. In such cases, the group G/K maintains its definition over any extension E of F inside K. This persistence arises from the isomorphism

$$K \otimes_F F[G] \cong K \otimes_E E \otimes_F F[G] \cong K \otimes_E E[G].$$

Furthermore, a morphism $T : G/K \to H/K$ between affine algebraic groups is considered to be defined over F if the induced map $\bar{T} : K \otimes F[H] \to K \otimes F[G]$ is identical to $\mathrm{id}_K \otimes \bar{T}'$ for some F-algebra homomorphism $\bar{T}' : F[H] \to F[G]$.

An algebraic group G/K is called a **simple** if no proper normal subgroup exists. An algebraic group is called an **absolutely almost simple** if it does not have proper connected normal algebraic subgroups.

The **derived subgroup** $[G, G]$ of G is the subgroup generated by $\{xyx^{-1}y^{-1} : x, y \in G\}$. The **derived series** of G is a descending sequence of subgroups of G by setting $G^0 := G$ and $G^{n+1} := [G^n, G^n]$. An algebraic group is called **solvable** if $G^n = \{1\}$ for some $n \geq 0$. A connected component of the identity in the maximal solvable normal subgroup of G is called **radical** of G, denoted by $\mathrm{Rad}(G)$. An algebraic group is called a **semisimple** if the radical of G is trivial.

Lemma 3.2.1. *Let $G/\overline{K}$ be an algebraic group defined over K then $G/\mathrm{Rad}(G)$ is a semisimple algebraic group. Moreover, the derived subgroup of $G/\mathrm{Rad}(G)$ is equal to $G/\mathrm{Rad}(G)$.*

An **algebraic group homomorphism** is a morphism of varieties and a group homomorphism also. A surjective algebraic group homomorphism is called an **isogeny** if it has a finite kernel. An algebraic group G is called a **simply connected** if any isogeny $\phi : H \to G$ is an isomorphism given that H is connected.

Example 3.2.4. *The special linear group $SL_N/K = SL_N(K)$ is an affine algebraic variety over K with the coordinate ring $K[SL_N] = K[\{x_{i,j} : 1 \leq i, j \leq N\}]/(\det([x_{i,j}]) - 1)$, and* *dimension $N^2 - 1$. It is an absolutely almost simple algebraic group since only the proper normal subgroup is isomorphic to $\mu_N := \{x \in K : x^N = 1\}$. Moreover, it is a simply connected semisimple algebraic group (refer to Chapter 21 of [25]).*

3.3 Base Change and Weil Restriction of Scalars

3.3.1 Base Change

Let G/F be an affine algebraic group and consider a field extension E of F. The **base change** G_E or $G \times_F E$ is defined such that $G_E(A) = G(A)$ for any E-algebra A. The coordinate ring of G_E is $E[G_E] = E \otimes_F F[G]$. Therefore, the functor $- \times_F E$ maps algebraic groups over F to algebraic groups over E.

3.3.2 Weil Restriction of Scalars

Let G/E be an affine algebraic group, and let F be a subfield of E. The **Weil restriction of scalars** of G/E to F is denoted as $\operatorname{Res}_{E/F} G$ and is defined by $\operatorname{Res}_{E/F} G(A) = G(E \otimes_F A)$ for any F-algebra A. Consequently, $\operatorname{Res}_{E/F}$ is a functor from the category of E-algebraic groups to the category of F-algebraic groups.

The coordinate ring for the Weil restriction is a 'universal object' $F[\operatorname{Res}_{E/F} G]$ with an E-algebra homomorphism $i : E[G] \to E \otimes_F F[\operatorname{Res}_{E/F} G]$ such that for any F-algebra A and E-algebra homomorphism $j : E[G] \to E \otimes_F A$ there exists a unique F-algebra homomorphism $J : F[\operatorname{Res}_{E/F} G] \to A$ satisfying $j = (\mathrm{id}_E \otimes J) \circ i$.

Let $\{x_1, \ldots, x_d\}$ be a basis of E as F- vector space, and $\{x_1^*, \ldots, x_d^*\}$ is a dual basis of E^*. For any $B \in E - \mathrm{Alg}$, let $S(B) = \mathrm{Sym}(E^* \otimes_F B)$ be the symmetric algebra on a F-vector space $E^* \otimes_F B$. Let $I(B)$ be the ideal of S generated by

$$\begin{aligned}&\{y^* \otimes ab - \sum_{1 \le i,j \le d} y^*(x_i x_j)(x_i^* \otimes a)(x_j^* \otimes b) : y^* \in E^* \text{ and } a, b \in B\} \\ &\bigcup \{y^* \otimes x - y^*(x) \cdot 1 : y^* \in E^* \text{ and } x \in E\}.\end{aligned}$$

It is not necessary that for any $B \in E - \mathrm{Alg}$, $I(B) \neq S(B)$. We say $R(B) = S(B)/I(B)$ exists if $S(B) \neq I(B)$ or equivalently $R(B) \neq 0$. Note that $R(B)$ may not be E-algebra but it is F-algebra. If B is an affine algebra then $R(B)$ exists (see Corollary 11.4.3. in [37]). If $R(B)$ exists then it also satisfies a universal property mentioned in the second paragraph of Subsection 3.3.2. So, $R(E[G])$ exists and it is the coordinate ring for $\operatorname{Res}_{E/F} G$.

Example 3.3.1. *The coordinate ring of the Weil restriction of scalars to $\mathbb{R}$ of $G = SL_2/\mathbb{C}$ is*

$$\mathbb{R}[\operatorname{Res}_{\mathbb{C}/\mathbb{R}} G] \cong \frac{\mathbb{R}[x, y, z, w, \bar{x}, \bar{y}, \bar{z}, \bar{w}]}{(xw - yz - \bar{x}\bar{w} + \bar{y}\bar{z} - 1, x\bar{w} - y\bar{z} + w\bar{x} - z\bar{y})},$$

and the map $i : \mathbb{C}[SL_2] \cong \frac{\mathbb{C}[a,b,c,d]}{(ad-bc-1)} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[\operatorname{Res}_{\mathbb{C}/\mathbb{R}} G]$ is defined as $i(1) = 1 \otimes 1$, $i(\iota) = \iota \otimes 1$, $i(a) = 1 \otimes x + \iota \otimes \bar{x}$, $i(b) = 1 \otimes y + \iota \otimes \bar{y}$, $i(c) = 1 \otimes z + \iota \otimes \bar{z}$ and $i(d) = 1 \otimes w + \iota \otimes \bar{w}$.

Example 3.3.2. *Let F be a number field. Let $\{\omega_1 = 1, \ldots, \omega_d\}$ be an integral basis of the ring of integers of F, and so it is a basis for F over $\mathbb{Q}$ also. Let $G = SL_N/F$ be an algebraic group and $F[G] \cong \frac{F[x_{i,j}:1 \le i,j \le N]}{(\det[x_{i,j}]-1)}$ be the coordinate ring. Then the coordinate ring of Weil restriction of scalars to $\mathbb{Q}$ is*

$$\mathbb{Q}[\operatorname{Res}_{F/\mathbb{Q}} SL_N] \cong \frac{\mathbb{Q}[x^r_{i,j} : 1 \le i, j \le N \text{ and } 1 \le r \le d]}{(f_1 - 1, f_2, \ldots, f_d)},$$

where $f_s \in \mathbb{Z}[x^r_{i,j} : 1 \le i, j \le N \text{ and } 1 \le r \le d]$ such that $\det\left[\sum_{r=1}^d x^r_{i,j}\omega_r\right] = \sum f_r \omega_r$. The map $i : F[G] \to F \otimes_{\mathbb{Q}} \mathbb{Q}[\operatorname{Res}_{F/\mathbb{Q}} G]$ is defined as $x_{i,j} \mapsto \sum \omega_r \otimes x^r_{i,j}$.

Note that any E-algebraic group automorphism of G, induces an F-algebraic group automorphism of $\operatorname{Res}_{E/F} G$, but not all F-algebraic group automorphism of $\operatorname{Res}_{E/F} G$ are obtained from E-algebraic group automorphism of G.

Example 3.3.3. *Continue with the notations in Example 3.3.1. Let $\mathfrak{c}^*$ be an $\mathbb{R}$-algebra automorphism of $\mathbb{R}[\operatorname{Res}_{\mathbb{C}/\mathbb{R}} SL_2]$ defined as $x, y, z, w, \bar{x}, \bar{y}, \bar{z}, \bar{w}$ maps to $x, y, z, w, -\bar{x}, -\bar{y}, -\bar{z}, -\bar{w}$, respectively. Let $\mathfrak{c}$ be the algebraic group homomorphism of $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} SL_2$ induced by $\mathfrak{c}$ on $\mathbb{R}[\operatorname{Res}_{\mathbb{C}/\mathbb{R}} SL_2]$, then $\mathfrak{c}$ is not obtained by any $\mathbb{C}$-algebraic group automorphism of G.*

Example 3.3.4. *Continue with the notations in Example 3.3.2. Let $\tau_* : E \to E$ be an F-algebra automorphism and $\tau_*(\omega_r) = \sum a_{r,s}\omega_s$ for some $a_{r,s} \in \mathbb{Z}$ then $\tau^*(x^r_{i,j}) = \sum a_{s,r} x^s_{i,j}$ define a $\mathbb{Q}$-algebra automorphism on $\mathbb{Q}[\operatorname{Res}_{F/\mathbb{Q}} SL_N]$. Let τ be the algebraic group automorphism of $\operatorname{Res}_{F/\mathbb{Q}} SL_N$ induced by τ_* on $\mathbb{Q}[\operatorname{Res}_{F/\mathbb{Q}} SL_N]$, then τ is not obtained by any F-algebraic group automorphism of SL_N/F.*

Lemma 3.3.1. *The Weil restriction of scalars serves as the right adjoint functor to the base change functor, denoted as*

$$\mathrm{Hom}_{E-\mathrm{Alg.Grp}}(G \times_F E, H) \cong \mathrm{Hom}_{F-\mathrm{Alg.Grp}}(G, \mathrm{Res}_{E/F} H).$$

Proof. It is equivalent to show that

$$\mathrm{Hom}_{E-\mathrm{Alg}}(E[H], E[G \times_F E]) \cong \mathrm{Hom}_{F-\mathrm{Alg}}(F[\mathrm{Res}_{E/F} H], F[G]).$$

It follows from the universal property of $F[\mathrm{Res}_{E/F} H]$ and $E[G \times_F E] = E \otimes_F F[G]$. $\square$

3.4 Torus

An algebraic group T/K defined over F is called a **torus** if $T_{\overline{K}}$ is isomorphic to $\mathbb{G}_m^r/\overline{K}$ for some $r \geq 1$. This is expressed as $T(\overline{K}) \cong (\overline{K}^{\times})^r$, where r is called the **absolute rank of the torus**. A torus T is labeled as an E-**split torus** if $T/F \times_F E$ is isomorphic to $\mathbb{G}_m^r/E$ for the field extension E over F. The **splitting component** of T over E is maximal among the subgroups of $T/F \times_F E$ which splits over E. The E-rank of the torus T is defined as the maximum rank among subgroups of $T/F \times_F E$ that split over E. An **anisotropic torus** is characterized as a torus with K-rank 0, signifying the absence of any split subtorus.

A **character** on a torus T/K is an algebraic group homomorphism $\chi : T_{\overline{K}} \to \mathbb{G}_m/\overline{K}$. The collection of characters on a torus is an abelian group, denoted as $X(T)$ since the product and inverse of characters are also characters. Similarly, a **one-parameter subgroup** of a torus is given by $\rho : \mathbb{G}_m/\overline{K} \to T_{\overline{K}}$. The set $Y(T)$- consisting of one-parameter subgroups, is an abelian group. Let $X(T)_E$ be the character group consisting of the characters defined over E.

Example 3.4.1. *If $T \cong \mathbb{G}_m/K$, then $X(T) \cong Y(T) \cong \mathbb{Z}$ since any character or one-parameter subgroup can be defined as $x \mapsto x^r$ for any $r \in \mathbb{Z}$. Moreover, $X(T)_K = X(T)$. In general, if $T \cong \mathbb{G}_m^r/K$, then $X(T)_K = X(T) \cong X(\mathbb{G}_m)^r \cong \mathbb{Z}^r \cong Y(T)$.*

Lemma 3.4.1. *$X(T)_K = X(T_{spl})$ where T_{spl} is a splitting component of torus T/K.*

Example 3.4.2. *Let $G = GL_N(\overline{K})$ be the general linear group defined over K, and let T_N be a subgroup consisting of diagonal metrics. Then T_N is a maximal torus defined over K of absolute rank N.*

A **character** of an algebraic group G/K is an algebraic group homomorphism from the base change $G_{\overline{K}}$ to $\mathbb{G}_m/\overline{K}$. Similar to the torus case, the collection of characters of G also forms an abelian group, denoted by $X(G)$.

Lemma 3.4.2. *Let $G/\overline{K}$ be an algebraic group defined over K, then the natural restriction map $X(G) \to X(\mathrm{Rad}(G))$ is an injective homomorphism.*

Proof. Let $\chi : G \to \mathbb{G}_m$ be a character such that $\chi|_{\mathrm{Rad}(G)} = \mathbf{1}$. Lemma 3.2.1 implies $[G/\mathrm{Rad}(G), G/\mathrm{Rad}(G)] = G/\mathrm{Rad}(G)$, so the image $\chi(G/\mathrm{Rad}(G))$ is trivial. Hence, χ is trivial. $\square$

3.5 Linear Algebraic Groups

An affine algebraic group G is called a **linear algebraic group** if an injective homomorphism $\phi : G \to GL_N(K)$ exists for some positive integer N. This implies that G is a Zariski closed subgroup of the general linear group $GL_N(K)$.

An element $x \in G$ is called a **semisimple** if it is diagonalizable in G. An element $x \in G$ is called an **unipotent** if $(x - I)^n = 0$ for some n.

An element $x \in G$ is called a **elliptic** if it is semisimple and the maximal split subtorus of the center of the centralizer is equal to the maximal split subtorus of the center of G.

The **unipotent radical** $\mathrm{Rad}_u(G)$ is the set comprising of all unipotent elements within $\mathrm{Rad}(G)$. A linear algebraic group G is called a **reductive group** if the unipotent radical $\mathrm{Rad}_u(G)$ is trivial. If G is connected reductive then the radical is the connected component of identity in the center.

Lemma 3.5.1. *If G is a reductive algebraic group over $\overline{K}$ with the center $Z(G)$ and the derived subgroup $[G, G]$ then $G = Z(G) \cdot [G, G]$. Moreover, $Rad(G) = Z(G)$ and $[G, G]$ is a semisimple algebraic group.*

Let $G/\overline{K}$ be a reductive algebraic group defined over K. Let $T/\overline{K}$ be a maximal torus of G defined over K. Then a **rank** of G is a rank of a torus T. An algebraic group G is **split** over E if a maximal torus T splits over E. A **split component** of G over E is a split component of T over E.

An element $g \in G$ is called a **regular** if the dimension of the centralizer is equal to the rank of G.

Example 3.5.1. *Let $G = GL_2/\overline{K}$, then scalar matrices are not regular but any non-scalar matrix is regular.*

3.5.1 Lie Algebra of a Linear Algebraic Groups

The Lie algebra associated with a linear algebraic group G/K is defined as the set of derivatives on the coordinate ring $K[G]$, formally denoted by

$$\mathfrak{g} = \mathrm{Lie}(G) = \{\delta \in \mathrm{Hom}_{K\text{-Alg}}(K[G], K[G]) \mid \delta(fg) = \delta(f)g + f\delta(g)\}.$$

This Lie algebra, $\mathfrak{g}$, forms a K-vector space equipped with the bracket operation

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1.$$

A linear algebraic group homomorphism $\phi : G \to H$ induces a natural Lie algebra homomorphism $d\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$. Notably, as G is a closed subgroup of $GL_N(K)$, the Lie algebra $\mathrm{Lie}(G)$ is a Lie subalgebra of $M_N(K) = \mathrm{Lie}(GL_N(K))$.

For an element g in G, the map $\mathrm{Int}_g(x) = gxg^{-1}$ defines an algebraic group isomorphism, leading to the induced linear map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a group homomorphism since $\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1)\,\mathrm{Ad}(g_2)$; it is called the **adjoint representation**.

If a linear algebraic group G/K is defined over a subfield F then the Lie algebra of G is also defined over F. Moreover, the adjoint representation is also defined over F.

3.6 Root System and Parabolic Subgroups

Consider a connected reductive linear algebraic group $G/\overline{K}$ defined over the field K, and let T be a maximal torus in G. The centralizer $Z_G(T)$ of T in G is T. Let $N_G(T)$ be the normalizer of T in G. The **Weyl group** $\mathcal{W} := \mathcal{W}(G,T)$ is defined as the quotient $N_G(T)/Z_G(T)$. The Weyl group $\mathcal{W}(G,T)$ also acts on $X(T)$.

Under the adjoint representation $Ad : G \to GL(\mathfrak{g})$, $Ad(T)$ consists of commuting semisimple elements. For a character $\alpha \in X(T)$, we define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid Ad(t)X = \alpha(t)X \ \forall t \in T\}.$$

The set $\Phi = \Phi(G,T)$ comprises non-trivial characters such that $\mathfrak{g}_\alpha \neq 0$, referred to as the **roots of G relative to T**. Let $\langle\Phi\rangle$ be the subgroup of $X(T)$ and $V = \langle\Phi\rangle \otimes_{\mathbb{Z}} \mathbb{R}$.

- The Lie algebra of a reductive linear algebraic group G decomposes as $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{t}$ is the Lie algebra of the maximal torus T. Moreover, the dimension of each $\mathfrak{g}_\alpha$ is 1.

- Let T_α be the connected component of the identity of the kernel of the character α. Then $Z_\alpha := Z_G(T_\alpha)$ is a reductive group with Lie algebra $Lie(Z_\alpha) := \mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Furthermore, the group G is generated by the subgroups Z_α.

- There exists a unique connected T-stable subgroup $U_\alpha \subset Z_\alpha$ of G such that $Lie(U_\alpha) = \mathfrak{g}_\alpha$. Moreover, for $n \in N_G(T)$, the conjugation $nU_\alpha n^{-1} = U_{w(\alpha)}$ holds, where $n \mapsto w$ in $W(G,T) = N_G(T)/T$.

- If $\alpha \in \Phi$, then there exists a reflection s_α on V such that $s_\alpha(\Phi) \subset \Phi$.

- The abstract Weyl group generated by $\{s_\alpha : \alpha \in \Phi\}$ is isomorphic to $\mathcal{W}(G,T)$.

A subset $\Delta \subset \Phi$ is considered a **base** if Δ forms a basis of the vector space V, and each α is expressible as either a non-positive or non-negative linear combination of elements from Δ. The individual elements of the base are termed **simple roots**. Consequently, the group G is generated by Z_α for each $\alpha \in \Delta$. Let Φ^+ denote the set of positive roots, signifying those roots that can be represented as a non-negative linear combination of simple roots.

The **Borel subgroup** of a connected reductive algebraic group G is defined as a maximal solvable subgroup within G. Let T be the maximal torus in G, and it is included in some Borel subgroup B. Denoting the unipotent radical of B as U, a unique base Δ exists such that U can be generated by U_α for $\alpha \in \Phi^+$. Consequently, B takes the form $T \ltimes U$. Any two Borel subgroups of G are conjugate.

Lemma 3.6.1. *(**Bruhat Decomposition**) Let G be a connected reductive algebraic group with Borel subgroup B containing the torus T. Then G can be expressed as the disjoint union of double cosets: $G = \sqcup_{w \in \mathcal{W}} BwB$.*

A **parabolic subgroup** P of a connected reductive group G is defined as a closed subgroup containing a Borel subgroup. If P is a parabolic subgroup, it is self-normalizing. Moreover, when P and P' are conjugate parabolic subgroups both containing the Borel subgroup B, they are necessarily equal.

Now, consider a fixed Borel subgroup B in G along with the corresponding base Δ. Any parabolic subgroup containing B can be expressed as $P_I = B\mathcal{W}_I B$, where $I \subset \Delta$. Here, $\mathcal{W}_I$ denotes the subgroup of the Weyl group generated by $S_I = \{s_\alpha : \alpha \in I\}$. Let U_I be the unipotent radical of P_I, T_I be the connected component of identity in $\cap_{\alpha \in I} \operatorname{Ker} \alpha$, and $M_I := Z_G(T_I)$. Then M_I is also a reductive algebraic group. The decomposition $P_I = M_I \ltimes U_I$ is known as the **Levi decomposition**. Notably, P_I includes the Borel subgroup, and so, U_I is a subgroup of U. The set Φ_I^+ is the collection of roots α for which U_α generates U_I.

Example 3.6.1. *In the special cases where $I = \emptyset$, $P_\emptyset = B$, $U_\emptyset = U$, and $T_\emptyset = M_\emptyset = T$ since T is a maximal torus. When $I = \Delta$, $P_\Delta = G$, $U_\Delta = T_\Delta = 1$, and $M_\Delta = G$ due to the reductive nature of G.*

Let P be a parabolic subgroup of G defined over K, and $P = M \rtimes U$ be the Levi decomposition. Let Z_P be the center of M, and A_P be a maximal split subtorus of Z_P defined over K.

Lemma 3.6.2. *The character group $X(P)_K$ consisting of characters defined over K is isomorphic to $X(A_P)$.*

Since T (a maximal torus of G) is also a maximal torus of M, we define $\mathcal{W}_M := N_M(T)/T$. Let $\Phi(M,T)$ be the root system of M relative to T. Fixing a base Δ for G, we can fix a base $\Delta_M := \Delta \cap \Phi(M,T)$. Let $\mathcal{W}^P := \{w \in \mathcal{W} : w^{-1}(\alpha) > 0 \text{ for all } \alpha \in \Delta_M\} \approx \mathcal{W}/\mathcal{W}_M$.

Example 3.6.2. *Let $G = GL_3$ be a general linear group. Then T consisting of diagonals is a maximal torus, $\Delta = \{e_1 - e_2, e_2 - e_3\}$ is a base, and Weyl group is $\mathcal{W} \cong S_3$. Let $I = \{e_1 - e_2\}$ then $P_I = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$, $M_I = \left\{ \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix} \right\}$ and $U_I = \left\{ \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\}$. Moreover, $\mathcal{W}_M = \mathcal{W}_I \cong S_2$, $\Phi(M,T) = \{e_1 - e_2, e_2 - e_1\}$ and $\Phi_I^+ = \{e_1 - e_3, e_2 - e_3\}$.*

3.7 Real Reductive Groups

We define G as a **real reductive group** when there exists a reductive algebraic group $\mathcal{G}/\mathbb{R}$ and a group homomorphism $G \to \mathcal{G}(\mathbb{R})$ with finite kernel and cokernel. The torus T, Borel subgroups B, and parabolic subgroups P in G are mapped to the real points of a torus, Borel subgroups, and parabolic subgroups of $\mathcal{G}$ under this homomorphism. The **split component** A is the connected component of the identity of the maximal $\mathbb{R}$-split torus in the center of G. The character group of G consists of characters $\chi : G \to \mathbb{C}^\times$. We define ${}^\circ G = \cap \ker |\chi|$, where the intersection runs over all the characters. Thus, G is the product of the split component and ${}^\circ G$.

A ***p*-pair** (P, A) is characterized by a parabolic subgroup P with a split component A. The Levi decomposition of P is given by $P = M \ltimes U$, where $M = Z_G(A)$. The set $\Phi(P, A)$ represents the roots of P with respect to A, and $\Delta(P, A)$ is the collection of simple roots. The character ρ_P on A is defined as $\rho_P(a) = |\det(\mathrm{Ad}\, a|_{\mathfrak{u}})|^{1/2}$, where Ad denotes the adjoint action of M on the Lie algebra of U through the adjoint action. In this context, ρ_P can be considered as an element of $\mathfrak{a}^*$, then $\rho_P = \frac{1}{2}\sum_{\alpha \in \Phi_I^+} \alpha$ if $P = P_I$ for some $I \subset \Delta$.

Let K be a maximal compact subgroup of G. Let A_0 be a maximal $\mathbb{R}-$split subgroup of G whose Lie algebra is orthogonal to the Lie algebra of K. Let P_0 be a minimal parabolic subgroup with a split component A_0. A p-pair (P, A) is called a **standard *p*-pair** if $P_0 \subset P$ and $A \subset A_0$.

A **Cartan involution** θ of G is an order 2 automorphism whose fixed set is the maximal compact subgroup K of G. Let $d\theta : \mathfrak{g} \to \mathfrak{g}$ be its differential. Denoting the -1 eigenspace of $d\theta$ as $\mathfrak{s}$, the map $(k, x) \mapsto k \cdot \exp(x)$ is an isomorphism of $K \times \mathfrak{s}$ onto G. All Cartan involutions are conjugate under automorphisms.

Example 3.7.1. *Let $\mathcal{G} = GL_N/\mathbb{R}$ then $G = GL_N(\mathbb{R})$ is a real reductive group, and the Cartan involution is given by $\theta(x) = {}^t x^{-1}$.*

Example 3.7.2. *Let $\mathcal{G} = Res_{\mathbb{C}/\mathbb{R}}(GL_N/\mathbb{C})$ then $\mathcal{G}(\mathbb{R}) = GL_N(\mathbb{C})$ is a real reductive group, and the Cartan involution is given by $\theta(x) = {}^t\overline{x}^{-1}$.*

Chapter 4

Automorphic Forms

We begin by describing some basics of modular forms and their generalization to the theory of automorphic forms on GL_N over a number field F. Next, for later use, we show that the space of automorphic forms on GL_N/F is induced from automorphic forms on SL_N/F (see lemma 4.2.1). The discussion is based on the references: [4], [13],[14], [23],[34] and [36].

4.1 Automorphic Forms for $GL_2/\mathbb{Q}$

4.1.1 Holomorphic Modular Forms

Let $G = SL_2(\mathbb{R})$ and $X = \mathcal{H}$ denote the complex upper half plane. The action of G on X is through linear fractional transformations, i.e., $gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. Define $j : SL_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ as $(g, z) \mapsto cz + d$. Then j satisfies the cocycle condition:

$$j(g_1 g_2, x) = j(g_1, g_2 x) j(g_2, x).$$

The function $SL_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ satisfying cocycle condition is called **multiplier** or **type**.

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ such that $\Gamma \backslash SL_2(\mathbb{R})$ has finite volume. Let k be a positive integer. A **weak modular form** of weight k on Γ is defined as a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying the transformation property $f(\gamma z) = j(\gamma, z)^k f(z)$ for all $\gamma \in \Gamma$, where $j(\gamma, z)$ is the multiplier function described earlier. A weak modular form is called a **modular form** if $j(g, z)^{-k} f(gz)$ has a Fourier expansion around infinity for all $g \in SL_2(\mathbb{R})$, i.e., $j(g, z)^{-k} f(gz) = \sum_{n \geq 0} a_n e^{2n\pi \mathbf{i} z}$. A **cusp form** of weight k on Γ is a modular form satisfying the additional property that $a_0 = 0$ for all $g \in SL_2(\mathbb{R})$. The collection of modular forms and cusp forms of weight k on Γ is denoted by $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$, respectively.

4.1.2 Automorphic Forms on $GL_2^+(\mathbb{R})$

Let $G = GL_2^+(\mathbb{R})$ be a topological group and $K = SO(2) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$ be the standard maximal compact subgroup. Let $||\cdot||$ be a norm on $GL_2^+(\mathbb{R})$.

Theorem 4.1.1. *(**Iwasawa Decomposition**) $G = KAN$ where N is the subgroup of G comprising all upper triangular unipotent matrices and A is the subgroup of G comprising all diagonal matrices.*

Let $\omega : \mathbb{R}^* \to \mathbb{C}^*$ be a character. A function $f : GL_2^+(\mathbb{R}) \to \mathbb{C}$ is called a **slowly increasing function** if there exists $n > 0$ such that $|f(g)| \ll ||g||^n$. A slowly increasing smooth function $f : GL_2^+(\mathbb{R}) \to \mathbb{C}$ is an **automorphic form** of weight k, level $\Gamma \subset GL_2^+(\mathbb{Q})$ and central character ω if $f(z\gamma g k_\theta) = \omega(z) j(k_\theta, \mathbf{i})^{-k} f(g)$ for all $z \in Z(G)$, $\gamma \in \Gamma$ and $k_\theta \in K$. Let $\mathcal{A}(\Gamma, \omega, k)$ denote the set of all automorphic forms of weight k, level Γ and central character ω.

A **cuspidal automorphic form** is an automorphic form f satisfying the condition $\int_{(\Gamma \cap N)\backslash N} f(ng)dn = 0$ for all $g \in GL_2^+(\mathbb{R})$. The set of cuspidal automorphic forms is denoted by $\mathcal{A}_0(\Gamma, \omega, j, k)$.

Example 4.1.1. *If f belongs to the space $\mathcal{M}_k(\Gamma)$ for some subgroup $\Gamma \subset SL_2(\mathbb{Z})$, we can define $\tilde{f} : GL_2^+(\mathbb{R}) \to \mathbb{C}$ by setting $\tilde{f}(g) = \det(g)^{k/2} j(g, \mathbf{i})^{-k} f(g\mathbf{i})$. Consequently, $\tilde{f}$ lies in the space $\mathcal{A}(\Gamma, \mathbf{1}, j, k)$. Moreover, if $f \in \mathcal{S}_k(\Gamma)$ then $\tilde{f} \in \mathcal{A}_0(\Gamma, \mathbf{1}, j, k)$.*

4.2 Automorphic Forms for Reductive groups

Let F be a number field, G be a reductive group over F, and Z be the center of G. Let $G(F)$, $G(\mathbb{A}_F)$, $G(F_\infty)$ and $G(\mathbb{A}_{F,f})$ denote the F, $\mathbb{A}_F$, F_∞ and $\mathbb{A}_{F,f}$ points of G, where $\mathbb{A}_F$ and $\mathbb{A}_{F,f}$ denotes the adèle and finite adèle ring, and $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. Let $\mathfrak{g}_\infty$ be the Lie algebra of $G(F_\infty)$, $\mathcal{U}(\mathfrak{g}_\infty)$ be the universal enveloping algebra, and $\mathcal{Z}(\mathfrak{g}_\infty)$ be the center of $\mathcal{U}(\mathfrak{g}_\infty)$.

A function $f : G(\mathbb{A}) \to \mathbb{C}$ is called a **smooth automorphic form with the central character** $\omega : Z(\mathbb{A}_F) \to \mathbb{C}^\times$ if

- $f(zxg) = \omega(z)f(g)$ for all $z \in Z(\mathbb{A}_F)$ and $x \in G(F)$,
- $f_x(y) := f(x \times y)$ are smooth functions on $G(F_\infty)$ for all $x \in G(\mathbb{A}_{F,f})$ and $f_y(x) := f(x \times y)$ are locally constant functions on $G(\mathbb{A}_{F,f})$ for all $x \in G(F_\infty)$,
- $f_x(y)$ are slowly increasing functions on $G(F_\infty)$,
- there exists an ideal J of finite codimension of $\mathcal{Z}(\mathfrak{g}_\infty)$ such that J annihilates f,
- there exists $\xi_\infty \in C_c^\infty(G(F_\infty))$ and $\xi_f \in C_c^\infty(G(\mathbb{A}_{F,f}))$ such that $(\xi_\infty \otimes \xi_f) * f = f$.

The space of smooth automorphic forms with the central character ω is denoted by $\mathcal{A}(G(F)\backslash G(\mathbb{A}_F), \omega)^\infty$.

A **smooth cuspidal automorphic form** is a smooth automorphic form f such that $\int_{U(F)\backslash U(\mathbb{A}_F)} f(ug)du = 0$ for all $g \in G(\mathbb{A}_F)$ and for all unipotent radical U of any proper parabolic subgroups P. The set of smooth cuspidal automorphic forms with the character ω is denoted by $\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F), \omega)^\infty$.

Let ω be a unitary character, then f is called a **square-integrable automorphic form** if $\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A}_F)} |f(g)|^2 dg < \infty$. The collection of square-integrable, smooth automorphic forms is denoted $L^2(G(F)\backslash G(\mathbb{A}_F), \omega)^\infty$. Additionally, the subspace of square-integrable, smooth, and cuspidal automorphic forms is denoted $L_0^2(G(F)\backslash G(\mathbb{A}_F), \omega)^\infty$.

Let $G = GL_N/F$ and $G^1 = SL_N/F$ are algebraic groups, and let ω be a character on $Z(\mathbb{A}_F)$, and ω^1 be the restriction of ω to $Z(\mathbb{A}_F) \cap G^1(\mathbb{A}_F)$.

Lemma 4.2.1. *(Refer [23]) As $G(F_\infty)$-modules,*

$$L^2_{\text{cusp}}(\Gamma\backslash G(F_\infty), \omega|_\infty)^\infty \cong \text{Ind}^{G(F_\infty)}_{\Gamma Z(F_\infty)G^1(F_\infty)} L^2_{\text{cusp}}(\Gamma^1\backslash G^1(F_\infty), \omega^1|_\infty)^\infty.$$

Proof. Since $G(F_\infty) = Z(F_\infty) \cdot \prod_{v\in S_r} O(N) \cdot G^1(F_\infty)$, the subgroup $\Gamma Z(F_\infty)G^1(F_\infty)$ has finite index in $G(F_\infty)$. It operates on the space of automorphic cuspidal forms on $G^1(F_\infty)$ via $[(\gamma z x)\cdot f](y) = \omega|_\infty(z)f(\gamma^{-1}y\gamma x)$. Now, let us introduce the mapping

$$T : L^2_{\text{cusp}}(\Gamma\backslash G(F_\infty), \omega|_\infty)^\infty \longrightarrow \text{Ind}^{G(F_\infty)}_{\Gamma Z(F_\infty)G^1(F_\infty)} L^2_{\text{cusp}}(\Gamma^1\backslash G^1(F_\infty), \omega^1|_\infty)^\infty;$$

defined by $[T(f)(g)](x) = f(xg)$ for $x \in G^1(F_\infty)$ and $g \in G(F_\infty)$. It is easy to check $[T(f)(g)] \in L^2_{\text{cusp}}(\Gamma^1\backslash G^1(F_\infty), \omega^1|_\infty)^\infty$. The map is well defined since

$$[T(f)(\gamma zhg)](x) = f(\gamma^{-1}x\gamma zhg) = \omega(z)[T(f)(g)](\gamma^{-1}x\gamma h).$$

The map T is injective since $[T(f)(g)](1) = f(g)$. To establish the isomorphism of T, we define an inverse map S such that $T\circ S = \text{id}$ and $S \circ T = \text{id}$, given by $S(f)(g) = f(g)(1)$. For $z \in Z(F_\infty)$, $\gamma \in \Gamma$, and $g \in G(F_\infty)$, we have:

$$\begin{aligned} S(f)(z\gamma g) &= f(z\gamma g)(1) \\ &= \omega(z)f(g)(\gamma^{-1}\cdot 1\cdot\gamma\cdot 1) = \omega(z)S(f)(g). \end{aligned}$$

The function $S(f)$ is cuspidal since the unipotent radical U is a subgroup of G^1 for any parabolic subgroup P of G with the Levi decomposition $P = MU$. Moreover, $S(f)$ is square integrable since:

$$\begin{aligned} \int_{Z(F_\infty)\Gamma\backslash G(F_\infty)} |S(f)(g)|^2 dg &= \int_{Z(F_\infty)\Gamma\backslash G(F_\infty)} |f(g)(1)|^2 dx \\ &= \int_{Z(F_\infty)\Gamma G^1(F_\infty)\backslash G(F_\infty)} \left(\int_{\Gamma^1\backslash G^1(F_\infty)} |f(x^1 g)(1)|^2 dx^1\right) dg \\ &= \int_{Z(F_\infty)\Gamma G^1(F_\infty)\backslash G(F_\infty)} \left(\int_{\Gamma^1\backslash G^1(F_\infty)} |f(g)(x^1)|^2 dx^1\right) dg < \infty. \end{aligned}$$

So, the map S is well-defined. Therefore, the lemma follows. $\square$

Chapter 5

Dimension Formulas

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and (ρ, V_ρ) be a finite dimensional complex representation of Γ such that $\rho(\Gamma)$ is a compact and $\ker(\rho)$ has a finite index in Γ. For an integer $k \geq 0$, a holomorphic function $f : \mathcal{H} \to V_\rho$ is called a **modular form** of weight k on Γ if $f(\gamma z) = j(\gamma, z)^k \rho(\gamma) \cdot f(z)$ for all $\gamma \in \Gamma$ and $j(g, z)^{-k} f(gz)$ has a Fourier expansion around infinity for all $g \in SL_2(\mathbb{R})$, i.e., $j(g, z)^{-k} f(gz) = \sum_{n \geq 0} a_n e^{2n\pi \mathbf{i} z}$. A **cusp form** of weight k on Γ is a modular form satisfying the additional property that $a_0 = 0$ for all $g \in SL_2(\mathbb{R})$. The collection of modular forms and cusp forms of weight k on Γ is denoted by $\mathcal{M}_k(\Gamma, \rho)$ and $\mathcal{S}_k(\Gamma, \rho)$, respectively.

The chapter concerns dimension formulas for the space of cusp forms with values residing in the representation space of $SL_2(\mathbb{Z})$ that arise from an irreducible representation of $SL_2(\mathbb{F})$, where $\mathbb{F} = \mathbb{Z}/q\mathbb{Z}$ is a finite field of prime order $q \geq 5$.

5.1 Dimension and Multiplicity

We calculate the dimensions of $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho)$ by using the character of a certain representation of $SL_2(\mathbb{F})$ in the space of cusp forms $\mathcal{S}_k(\Gamma(q))$. Since $\Gamma(q)$ is a normal subgroup of $\Gamma(1) = SL_2(\mathbb{Z})$, there is a natural homomorphism

$$\sigma : SL_2(\mathbb{Z}) \to GL(\mathcal{S}_k(\Gamma(q))),$$

defined as $[\sigma(g)f](z) = j(g^{-1}, z)^{-k} f(g^{-1}z)$. Since σ is a representation with $\Gamma(q) \subset \ker(\sigma)$, it factors through a representation of $SL_2(\mathbb{F})$. Suppose σ decomposes as,

$$\sigma = \bigoplus m_\rho \rho,$$

where the direct sum runs over the inequivalent classes of irreducible representations of $SL_2(\mathbb{F})$, and m_ρ is the multiplicity of the irreducible representation ρ in σ.

Let $\mathcal{S}_k(\Gamma(q), \rho)$ is the space of classical cusp forms taking values in V_ρ, and

$$\theta : SL_2(\mathbb{Z}) \to GL(\mathcal{S}_k(\Gamma(q), \rho)),$$

defined as $[\theta(g)f](z) = j(g^{-1}, z)^{-k} \rho(g) \cdot f(g^{-1}z)$, is also a representation of $SL_2(\mathbb{Z})$.

Lemma 5.1.1. *$\mathcal{S}_k(\Gamma(q), \rho)$ is equivalent, as a representation of $SL_2(\mathbb{Z})$, to the tensor product of $\mathcal{S}_k(\Gamma(q)) \otimes V_\rho$, i.e., $\theta \sim \sigma \otimes \rho$. Moreover, $\chi_\theta = \chi_\sigma \cdot \chi_\rho$, where χ_θ, χ_σ and χ_ρ are the characters of θ, σ and ρ, respectively.*

Proof. Define a linear map $T : \mathcal{S}_k(\Gamma(q)) \otimes V_\rho \to \mathcal{S}_k(\Gamma(q), \rho)$ as $[T(f \otimes v)](z) = f(z)v$. The map is well-defined since $\Gamma(q) \subset \ker(\rho)$. Let $\{e_1, \ldots, e_n\}$ is a basis of V_ρ then for any $f \in \mathcal{S}_k(\Gamma(q), \rho)$, there exist holomorphic functions $f_1, \ldots, f_n : \mathcal{H} \to \mathbb{C}$ such that $f(z) = f_1(z)e_1 + \cdots + f_n(z)e_n$. Since $\Gamma(q) \subset \ker(\rho)$, $f_i \in \mathcal{S}_k(\Gamma(q))$. Then

$$T(f_1 \otimes e_1 + \cdots + f_n \otimes e_n) = f.$$

Hence, T is a surjective map. Since $\dim \mathcal{S}_k(\Gamma(q), \rho) = \dim \mathcal{S}_k(\Gamma(q)) \times \dim V_\rho$, the linear map T is an isomorphism. The lemma follows since for $f \in \mathcal{S}_k(\Gamma(q))$, $v \in V_\rho$ and $g \in SL_2(\mathbb{Z})$,

$$\begin{aligned}
[\theta(g) \circ T(f \otimes v)](z) &= j(g^{-1}, z)^{-k} \rho(g) \cdot [T(f \otimes v)](g^{-1}z) \\
&= j(g^{-1}, z)^{-k} \rho(g) \cdot [f(g^{-1}z)v] \\
&= [\sigma(g)f](z) \rho(g) \cdot v \\
&= [T \circ \sigma \otimes \rho(g)(f \otimes v)](z).
\end{aligned}$$

□

Lemma 5.1.2. *The dimension of $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho)$ is equal to the multiplicity of the complex conjugation $\overline{\rho}$ of the representation ρ in σ, i.e., $\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \rho) = m_{\overline{\rho}}$.*

Proof. Since $\mathcal{S}_k(SL_2(\mathbb{Z}),\rho) = \{f \in \mathcal{S}_k(\Gamma(q),V_\rho) : \theta(g)f = f,\ \forall g \in SL_2(\mathbb{Z})\}$, the dimension is equal to the multiplicity of the trivial representation in θ. Then

$$\dim \mathcal{S}_k(SL_2(\mathbb{Z}),V_\rho) = \langle \chi_\theta, \mathbf{1}\rangle = \langle \chi_\sigma\chi_\rho, \mathbf{1}\rangle = \langle \chi_\sigma, \chi_{\overline{\rho}}\rangle = m_{\overline{\rho}}.$$

Here, the second equality follows from Lemma 5.1.1. □

5.2 The Character Table for Representations

This section is a recall of the classification of irreducible representations of $SL_2(\mathbb{F})$ based on Section 4.1 of [8] and [18].

Let $\mathbb{E}$ be the quadratic extension of $\mathbb{F}$, then $\mathbb{E} = \mathbb{F}(\sqrt{\Delta})$ for $\Delta \notin \mathbb{F}^{*2}$. Let $z = x+\sqrt{\Delta}y \in \mathbb{E}$ then define $\overline{z} := x - \sqrt{\Delta}y$. The norm $N : \mathbb{E}^* \to \mathbb{F}^*$ is defined by $N(z) = z\overline{z} \in \mathbb{F}^*$. Let $\mathbb{E}_1^*$ denote the kernel of the norm map.

Remark 1. *We can view $\mathbb{E}_1^*$ as a subgroup of $SL_2(\mathbb{F})$ via the embedding*

$$z = x + \sqrt{\Delta}y \mapsto \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix}.$$

5.2.1 Principal Series Representations

Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbb{F}) \right\}$, $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{F} \right\}$ and $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}^* \right\}$ be the Borel subgroup, its unipotent radical, and the diagonal torus, respectively. For a character $\alpha : T \cong \mathbb{F}^* \to \mathbb{C}^*$, it can be extended to the character of B, whose restriction on T is α. Let $\rho(\alpha)$ denote the induced representation of B to $SL_2(\mathbb{F})$, which is called parabolically induced representation.

- If $\alpha = \mathbf{1}$ is the trivial character then $\rho(\mathbf{1}) = \mathbf{1} \oplus \mathbf{St}$, where $\mathbf{1}$ is a trivial representation, and $\mathbf{St}$ is the Steinberg representation of dimension q.
- If α is a quadratic character, denoted by ζ_e. Then $\rho(\zeta_e) = \rho(\zeta_e)^+ \oplus \rho(\zeta_e)^-$, where $\rho(\zeta_e)^\pm$

are irreducible representations of dimension $\frac{q+1}{2}$. Note that, a quadratic character exists and it is unique. Moreover, $\zeta_e(x) = 1$ for all $x \in \mathbb{F}^{*2}$ and $\zeta_e(x) = -1$ for all $x \in \mathbb{F}^{*}\backslash\mathbb{F}^{*2}$.

- If α is neither a trivial nor a quadratic character then $\rho(\alpha)$ is an irreducible representation of dimension $q+1$. Moreover, $\rho(\alpha) \sim \rho(\overline{\alpha})$.

Since $\left\{\epsilon_j = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} : j \in \mathbb{F}\right\} \cup \left\{\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$ forms a set of coset representatives of B in $SL_2(\mathbb{F})$, for any coset representative y,

$$f_y(x) = \begin{cases} \alpha(b) & x = by \in By \\ 0 & x \notin By \end{cases}$$

forms a basis of induced representations. So, for $g \in SL_2(\mathbb{F})$ the trace character is

$$\text{Trace}\, \rho(\alpha)(g) = \sum f_y(yg).$$

5.2.2 Cuspidal Representations

Let $\tau : \mathbb{E}_1^* \to \mathbb{C}^*$ be a non-trivial character and ψ be any non-trivial character of $\mathbb{F}$. Let

$$V_\tau = \left\{f : \mathbb{E} \to \mathbb{C} \text{ such that } f(zx) = \tau(z)^{-1} f(x), \forall z \in \mathbb{E}_1^*\right\}.$$

Let $\psi : \mathbb{F} \to \mathbb{C}^*$ be a non-trivial character. The group $SL_2(\mathbb{F})$ acts on V_τ as

$$\begin{aligned} [\omega(\tau)\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f](x) &:= f(ax) \\ [\omega(\tau)\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} f](x) &:= \psi(zN(x))f(x) \\ [\omega(\tau)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f](x) &:= -\frac{1}{q}\sum_{y \in \mathbb{E}} f(y)\psi(\text{trace}(\overline{x}y)). \end{aligned}$$

This action of group is called the special Weil representation (refer to [8]). Note that V_τ is stable under the above operations. The representation $(\omega(\tau), V_\tau)$ is called a cuspidal representation for the character τ.

- If τ is a quadratic character, denoted by ζ_0 then $\omega(\zeta_0) = \omega(\zeta_0)^+ \oplus \omega(\zeta_0)^-$, where $\omega(\zeta_0)^\pm$ are irreducible representations of dimension $\frac{q-1}{2}$.
- If τ is a nontrivial character such that $\tau \neq \zeta_0$ then $\omega(\tau)$ is an irreducible representation of dimension $q-1$. Moreover, $\omega(\tau) \sim \omega(\overline{\tau})$.

We can choose $\epsilon_y \in \mathbb{E}^*$ such that $N(\epsilon_y) = y$ for any $y \in \mathbb{F}^*$. For every $y \in \mathbb{F}^*$, define

$$f_y(x) = \begin{cases} \tau(x/\epsilon_y)^{-1} & N(x) = y, \\ 0 & N(x) \neq y. \end{cases}$$

Then $\{f_y : y \in \mathbb{F}^*\}$ forms a basis for V_τ and the trace character for any $g \in SL_2(\mathbb{F})$,

$$\operatorname{Trace} \omega(\tau)(g) = \sum [\omega(\tau)(g) f_y](y).$$

5.2.3 The Character Table

		Class Repre.	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$(-1)^\eta \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$
		Sizes	1	1	$\frac{q^2-1}{2}$	$q(q-1)$	$q(q+1)$
			1	1	4	$\frac{q-1}{2}$	$\frac{q-3}{2}$
Repr.	Dim.				$\eta \in \{0,1\}$ $\gamma \in \{1,\Delta\}$	$z \in \mathbb{E}_1^* - \{\pm 1\}$	$x \neq \pm 1 \in \mathbb{F}^*$
$\rho(\alpha)$	$q+1$	$\frac{q-3}{2}$	$(q+1)$	$(q+1)\alpha(-1)$	$\alpha(-1)^\eta$	0	$\alpha(x) + \overline{\alpha}(x)$
St	q	1	q	q	0	-1	1
1	1	1	1	1	1	1	1
$\omega(\tau)$	$q-1$	$\frac{q-1}{2}$	$(q-1)$	$(q-1)\tau(-1)$	$-\tau((-1)^\eta)$	$-\tau(z) - \overline{\tau}(z)$	0
$\rho(\zeta_e)^\pm$	$\frac{q+1}{2}$	2	$\frac{q+1}{2}$	$\frac{q+1}{2}\zeta_e(-1)$	$\zeta_e(-1)^\eta \zeta_e^\pm(\gamma)$	0	$\zeta_e(x)$
$\omega(\zeta_0)^\pm$	$\frac{q-1}{2}$	2	$\frac{q-1}{2}$	$\frac{q-1}{2}\zeta_0(-1)$	$\zeta_0(-1)^\eta \zeta_0^\pm(\gamma)$	$-\zeta_0(z)$	0

Table 5.1: Character Table

where $\zeta_e^\pm(1) = \frac{1}{2}(1 \pm \sqrt{q\epsilon})$, $\zeta_e^\pm(\Delta) = \frac{1}{2}(1 \mp \sqrt{q\epsilon})$, $\zeta_0^\pm(1) = \frac{1}{2}(-1 \pm \sqrt{q\epsilon})$, $\zeta_0^\pm(\Delta) = \frac{1}{2}(-1 \mp \sqrt{q\epsilon})$ and $\epsilon = \zeta_e(-1)$. (The computations left to the reader.)

Remark 5.2.1. *For $x \in \mathbb{F}^*$, the element $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is conjugate to either $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $x \in \mathbb{F}^{*2}$, or to $\begin{pmatrix} 1 & \Delta \\ 0 & 1 \end{pmatrix}$ if $x \notin \mathbb{F}^{*2}$.*

5.3 The Character of the Representation on Cusp Forms

Since $\Gamma_0(q)$, $\Gamma_1(q)$ and $\Gamma_{\mathbb{E}}(q)$ are subgroups of $SL_2(\mathbb{Z})$ such that they are the inverse image under the natural map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{F})$ of the Borel subgroup, unipotent subgroup, and $\mathbb{E}_1^*$ respectively. Note that $\Gamma_0(q)/\Gamma(q) \cong B$, $\Gamma_1(q)/\Gamma(q) \cong U$ and $\Gamma_{\mathbb{E}}(q)/\Gamma(q) \cong \mathbb{E}_1^*$.

Remark 5.3.1. *Since $\sigma(-1)f = (-1)^k f$ for all $f \in \mathcal{S}_k(\Gamma(q))$, $\chi_\sigma(-g) = (-1)^k \chi_\sigma(g)$ for all $g \in SL_2(\mathbb{Z})$.*

Let $\sigma_{\mathbb{E}}$ be the restriction of σ to $\Gamma_{\mathbb{E}}(q)$ or $\mathbb{E}_1^*$. Since $\mathbb{E}_1^*$ is an abelian group,

$$\mathcal{S}_k(\Gamma(q)) = \bigoplus_{\tau:\mathbb{E}_1^* \to \mathbb{C}^*} \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau),$$

where $\mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau) = \left\{ f \in \mathcal{S}_k(\Gamma(q)) : \ \sigma_{\mathbb{E}}(g) \cdot f = \tau(g) f \ \forall g \in \Gamma_{\mathbb{E}}(q) \right\}$. Since $\chi_\sigma(g) = \chi_{\sigma_{\mathbb{E}}}(g)$ for all $g \in \Gamma_{\mathbb{E}}(q)$,

$$\chi_\sigma(g) = \sum_{\tau:\mathbb{E}_1^* \to \mathbb{C}^*} \tau(g) \dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau). \tag{5.1}$$

Remark 5.3.2. *We will say τ is even (or odd) character if $\tau(-1) = 1$ (or $\tau(-1) = -1$ respectively). If k is even (or odd) then $\mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau) = 0$ for all odd (or even) τ, respectively. So,*

$$\sum_{\tau:\mathbb{E}_1^* \to \mathbb{C}^*} \tau(-1) \dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau) = (-1)^k \dim \mathcal{S}_k(\Gamma(q)).$$

Similarly, let σ_1 be the restriction of σ to $\Gamma_1(q)$ or U. Since $U \cong \mathbb{F}$ is an abelian group,

$$\mathcal{S}_k(\Gamma(q)) = \bigoplus_{n \in \mathbb{F}} \mathcal{S}_k(\Gamma_1(q), \psi_n),$$

where $\mathcal{S}_k(\Gamma_1(q), \psi_n) = \left\{ f \in \mathcal{S}_k(\Gamma(q)) : \sigma_1(g) \cdot f = \psi_n(g) f, \ \forall g \in \Gamma_1(q) \right\}$. Since $\chi_\sigma(g) = \chi_{\sigma_1}(g)$ for all $g \in \Gamma_1(q)$,

$$\chi_\sigma(g) = \sum_{n \in \mathbb{F}} \psi_n(g) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n). \tag{5.2}$$

Let σ_0 be the restriction of σ to $\Gamma_0(q)$.

Lemma 5.3.1. *If $f \in \mathcal{S}_k(\Gamma_1(q), \psi_n)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ then $\sigma_0(g) \cdot f \in \mathcal{S}_k(\Gamma_1(q), \psi_{nd^2})$.*

Proof. Let $\gamma = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma_1(q)$ then $\sigma_1(\gamma) \cdot f = \psi_n(y) f = e^{2\pi\iota \frac{ny}{q}} f$. Now,

$$\begin{aligned} \sigma_1(\gamma)\sigma_0(g) \cdot f = \sigma(\gamma g) \cdot f = \sigma_0(g)\sigma_1(g^{-1}\gamma g) \cdot f = \psi_n(g^{-1}\gamma g)\sigma_0(g) \cdot f \\ = \psi_n(d^2 y)\sigma_0(g) \cdot f = \psi_{nd^2}(y)\sigma_0(g) \cdot f. \end{aligned}$$

$\square$

Lemma 5.3.2. *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ such that $d \not\equiv \pm 1 (mod\ q)$. Then*

$$\chi_\sigma(g) = \sum_{\phi:\mathbb{F}^* \to \mathbb{C}^*} \phi(d) \dim \mathcal{S}_k(\Gamma_0(q), \phi). \tag{5.3}$$

Proof. Since $\mathcal{S}_k(\Gamma(q)) = \bigoplus \mathcal{S}_k(\Gamma_1(q), \psi_n)$, we have a basis of $\mathcal{S}_k(\Gamma(q))$ as an union of bases of $\mathcal{S}_k(\Gamma_1(q), \psi_n)$. By using Lemma 5.3.1, an element of basis of $\mathcal{S}_k(\Gamma_1(q), \psi_n)$ contributes to the trace of $\sigma(g) = \sigma_0(g)$ if and only if $n = 0 \in \mathbb{F}$. So, $\chi_{\sigma_0}(g) = \chi_{\widetilde{\sigma_0}}(g)$ where $\widetilde{\sigma_0} : \Gamma_0(q) \to GL(\mathcal{S}_k(\Gamma_1(q)))$. Since $\Gamma_1(q) \subset \ker(\widetilde{\sigma_0})$ and $\Gamma_0(q)/\Gamma_1(q) \cong \mathbb{F}^*$ is an abelian group,

$$\mathcal{S}_k(\Gamma_1(q)) = \bigoplus_{\phi:\mathbb{F}^* \to \mathbb{C}^*} \mathcal{S}_k(\Gamma_0(q), \phi),$$

where $\mathcal{S}_k(\Gamma_0(q), \phi) = \left\{ f \in \mathcal{S}_k(\Gamma_1(q)) : \sigma_0(x) \cdot f = \phi(x) f,\ \forall x \in \Gamma_1(q) \right\}$. So, the lemma follows. $\square$

Remark 5.3.3. *We will say ϕ is even (or odd) character if $\phi(-1) = 1$ (or $\phi(-1) = -1$ respectively). If k is even (or odd) then $\mathcal{S}_k(\Gamma_0(q), \phi) = 0$ for all odd (or even) ϕ, respectively. So,*

$$\sum_{\phi:\mathbb{F}^* \to \mathbb{C}^*} \phi(-1) \dim \mathcal{S}_k(\Gamma_0(q), \phi) = (-1)^k \dim \mathcal{S}_k(\Gamma_1(q)).$$

The character function of the representation σ is:

	Class Representative	size	χ_σ
1	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	1	$\dim \mathcal{S}_k(\Gamma(q))$
1	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	1	$(-1)^k \dim \mathcal{S}_k(\Gamma(q))$
1	$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$	$\frac{q^2-1}{2}$	$\sum_{n\in\mathbb{F}} \psi_n(1) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n)$
1	$\begin{pmatrix}1 & \Delta\\ 0 & 1\end{pmatrix}$	$\frac{q^2-1}{2}$	$\sum_{n\in\mathbb{F}} \psi_n(\Delta) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n)$
1	$\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}$	$\frac{q^2-1}{2}$	$(-1)^k \sum_{n\in\mathbb{F}} \psi_n(1) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n)$
1	$\begin{pmatrix}-1 & -\Delta\\ 0 & -1\end{pmatrix}$	$\frac{q^2-1}{2}$	$(-1)^k \sum_{n\in\mathbb{F}} \psi_n(\Delta) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n)$
$\frac{q-3}{2}$	$\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix}$	$q(q+1)$	$\sum_{\phi:\mathbb{F}^*\to\mathbb{C}^*} \phi(x) \dim \mathcal{S}_k(\Gamma_0(q), \phi)$
$\frac{q-1}{2}$	$\begin{pmatrix}x & \Delta y\\ y & x\end{pmatrix}$	$q(q-1)$	$\sum_{\tau:\mathbb{E}_1^*\to\mathbb{C}^*} \tau\left(\begin{pmatrix}x & \Delta y\\ y & x\end{pmatrix}\right) \dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau).$

Table 5.2: Character of σ

5.4 Cusp forms with values in principal series rep.

We will find the dimension of $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\alpha_0))$, where $\rho(\alpha_0)$ is the parabolically induced representation for a character $\alpha_0 : \mathbb{F}^* \to \mathbb{C}^*$.

Theorem 5.4.1. *For the principal series representation $\rho(\alpha)$,*

$$\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\alpha)) = \dim \mathcal{S}_k(\Gamma_0(q), \alpha).$$

Proof. Let $\mathcal{F} : \mathcal{S}_k(\Gamma_0(q), \alpha) \to \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\alpha))$ be a linear map defined as

$$[\mathcal{F} f(z)](x) = j(\tilde{x}, z)^{-k} f(\tilde{x} z)$$

for any $z \in \mathcal{H}$ and $\tilde{x} \in SL_2(\mathbb{Z})$ is a pre-image of $x \in SL_2(\mathbb{F})$. The map is independent of the choice of $\tilde{x}$ since for any x_1, x_2 be pre-images of x then $x_1 x_2^{-1} \in \Gamma(q)$ and

$$j(x_1, z)^{-k} f(x_1 z) = j(x_1 x_2^{-1}, x_2 z)^{-k} j(x_2, z)^{-k} f(x_1 x_2^{-1} x_2 z) = j(x_2, z)^{-k} f(x_2 z).$$

So, we will use x instead of $\tilde{x}$. To show the map is well-defined we need to show that $\mathcal{F}f(z) \in \rho(\alpha)$ and $\mathcal{F}(f)$ satisfies the automorphy condition. For $b \in B$, $x \in SL_2(\mathbb{F})$, $z \in \mathcal{H}$ and $g \in SL_2(\mathbb{Z})$,

$$\begin{aligned}
[\mathcal{F}f(z)](bx) &= j(bx,z)^{-k}f(bxz) = j(bx,z)^{-k}\alpha(b)j(b,xz)^{k}f(xz) \\
&= \alpha(b)j(x,z)^{-k}f(xz) = \alpha(b)[\mathcal{F}f(z)](x), \\
[\mathcal{F}f(gz)](x) &= j(x,gz)^{-k}f(xgz) = j(g,z)^{k}j(xg,z)^{-k}f(xgz) \\
&= j(g,z)^{k}[\mathcal{F}f(z)](xg) = j(x,z)^{k}[\rho(\alpha)(g)\cdot \mathcal{F}f(z)](x).
\end{aligned}$$

Let $\mathcal{G} : \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\alpha)) \to \mathcal{S}_k(\Gamma_0(q), \alpha)$ be a linear map defined as $\mathcal{G}f(z) := [f(z)](1)$. The map is well-defined since for $b \in \Gamma_0(q)$,

$$\begin{aligned}
\mathcal{G}f(bz) &= f(bz)(1) = j(b,z)^{k}[\rho(\alpha)(b)\cdot f(z)](1) \\
&= j(b,z)^{k}[f(z)](b) = j(b,z)^{k}\alpha(b)[f(z)](1) = j(b,z)^{k}\alpha(b)\mathcal{G}f(z).
\end{aligned}$$

To complete a proof of the theorem, we need to show that $\mathcal{G}$ is an inverse of $\mathcal{F}$, which is easy to see. This proof is an effect of Shapiro's lemma on parabolic group cohomology, isomorphic to the space of cusp forms via the Eichler-Shimura isomorphism. ∎

Remark 5.4.1. *If $\alpha(-1) \neq (-1)^k$ then $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\alpha)) \cong \mathcal{S}_k(\Gamma_0(q), \alpha) = 0$.*

Corollary 5.4.2. *For the Steinberg representation,*

$$\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \mathbf{St}) = \dim \mathcal{S}_k(\Gamma_0(q)) - \dim \mathcal{S}_k(SL_2(\mathbb{Z})).$$

Proof. Let $\alpha = \mathbf{1}$ then $\rho(\mathbf{1}) = \mathbf{1} \oplus \mathbf{St}$. So, $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\mathbf{1})) = \mathcal{S}_k(SL_2(\mathbb{Z}), \mathbf{St}) \oplus \mathcal{S}_k(SL_2(\mathbb{Z}))$. So, $\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \mathbf{St}) = \dim \mathcal{S}_k(\Gamma_0(q)) - \dim \mathcal{S}_k(SL_2(\mathbb{Z}))$ for all k. ∎

For the quadratic character ζ_e on $\mathbb{F}^*$, the parabolically induced representation $\rho(\zeta_e)$ is the direct sum of two irreducible subrepresentation $\rho(\zeta_e)^{\pm}$. So,

$$\mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)) = \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{+}) \oplus \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{-}).$$

Proposition 5.4.3. *For the quadratic character ζ_e on $\mathbb{F}^*$, if $\zeta_e(-1) \neq (-1)^k$ then*

$\mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{\pm}) = 0$, *and if* $\zeta_e(-1) = (-1)^k$ *then*

$$\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{\pm}) = \frac{1}{2}\dim \mathcal{S}_k(\Gamma_0(q), \zeta_e) \pm \frac{\zeta_e(-1)}{(q-1)} \sum_{n \in \mathbb{F}^*} \zeta_e(n) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n).$$

Proof. Since $\rho(\zeta_e) = \rho(\zeta_e)^+ \oplus \rho(\zeta_e)^-$. So,

$$\mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)) = \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^+) \oplus \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^-).$$

If $\zeta_e(-1) \neq (-1)^k$ then by using Remark 5.4.1, $\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{\pm}) = 0$.
So assume $\zeta_e(-1) = (-1)^k$. We will use the character formula to find the dimensions of $\mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{\pm})$. By using the character tables 5.1, 5.2, Equation 5.2 and Remark 5.2.1,

$$\begin{aligned}
\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{\pm}) = {} & \frac{\zeta_e(1)}{2q(q-1)} \dim \mathcal{S}_k(\Gamma(q)) + \frac{\zeta_e(-1)}{2q(q-1)} (-1)^k \dim \mathcal{S}_k(\Gamma(q)) \\
& + \frac{1}{q(q-1)} \sum_{x \in \mathbb{F}^*} \frac{1}{2}(1 \pm \zeta_e(x)\sqrt{q\zeta_e(-1)}) \sum_{n \in \mathbb{F}} \psi_n(x) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n) \\
& + \frac{1}{q(q-1)} \sum_{x \in \mathbb{F}^*} \Bigg(\zeta_e(-1)\frac{1}{2}(1 \pm \zeta_e(x)\sqrt{q\zeta_e(-1)}) \\
& \qquad (-1)^k \sum_{n \in \mathbb{F}} \psi_n(x) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n) \Bigg) \\
& + \frac{1}{2(q-1)} \sum_{x \neq \pm 1} \zeta_e(x) \sum_{\phi : \mathbb{F}^* \to \mathbb{C}^*} \phi(x) \dim \mathcal{S}_k(\Gamma_0(q), \phi) \\
\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \rho(\zeta_e)^{\pm}) = {} & \frac{1}{q(q-1)} \dim \mathcal{S}_k(\Gamma(q)) + \frac{1}{q(q-1)} \sum_{n \in \mathbb{F}} \Bigg(\sum_{x \in \mathbb{F}^*} \psi_n(x) \Bigg) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n) \\
& \pm \frac{\sqrt{q\zeta_e(-1)}}{q(q-1)} \sum_{n \in \mathbb{F}} \Bigg(\sum_{x \in \mathbb{F}^*} \zeta_e(x)\psi_n(x) \Bigg) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n) \\
& + \frac{1}{2(q-1)} \sum_{\phi : \mathbb{F}^* \to \mathbb{C}^*} \Bigg(\sum_{x \in \mathbb{F}^*} \zeta_e(x)\phi(x) \Bigg) \dim \mathcal{S}_k(\Gamma_0(q), \phi) \\
& - \frac{1}{2(q-1)} \sum_{x = \pm 1} \zeta_e(x) \sum_{\phi : \mathbb{F}^* \to \mathbb{C}^*} \phi(x) \dim \mathcal{S}_k(\Gamma_0(q), \phi).
\end{aligned}$$

For a fixed character ϕ on $\mathbb{F}^*$, the sum $\sum_{x \in \mathbb{F}^*} \zeta_e(x)\phi(x)$ is equal to 0 unless $\phi = \zeta_e$. Similarly, for a fixed $n \in \mathbb{F}$, the sum $\sum_{x \in \mathbb{F}} \psi_n(x)$ is equal to 0 unless $n = 0$. Note that the sum $\sum_{x \in \mathbb{F}^*} \zeta_e(x)\psi_n(x)$ is the quadratic Gauss sum if $n \neq 0$, and so it is equal to $\zeta_e(n)\sqrt{q\zeta_e(-1)}$ (Chapter 6 in [20]). Note that we are following the usual convention that the sign of $\sqrt{x}$ is positive. We conclude the proof of the proposition by using Remark 5.3.3. □

5.5 Cusp forms with values cuspidal representations

We will find the dimension of $\mathcal{S}_k(SL_2(\mathbb{Z}), \omega(\tau_0))$ for a character $\tau_0 : \mathbb{E}_1^* \to \mathbb{C}^*$.

Theorem 5.5.1. *For cuspidal representation $\omega(\tau)$, if $\tau(-1) \neq (-1)^k$ then $\mathcal{S}_k(SL_2(\mathbb{Z}), \omega(\tau)) = 0$, and if $\tau(-1) = (-1)^k$ then*

$$\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \omega(\tau)) = \frac{k-1}{12}(q^2-1) - \frac{1}{2}(q-1) - \frac{1}{2}\dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \overline{\tau}) - \frac{1}{2}\dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau).$$

Proof. We will use the character formula to prove this. By using Character tables 5.1,5.2 and Remark 5.2.1,

$$\begin{aligned}
\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \omega(\tau_0)) &= \frac{\tau_0(1)}{q(q+1)} \dim \mathcal{S}_k(\Gamma(q)) + \frac{\tau_0(-1)}{q(q+1)}(-1)^k \dim \mathcal{S}_k(\Gamma(q)) \\
&+ \frac{1}{q(q-1)} \sum_{x \in \mathbb{F}^*} (-\tau_0(1)) \sum_{n \in \mathbb{F}} \psi_n(x) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n) \\
&+ \frac{1}{q(q-1)} \sum_{x \in \mathbb{F}^*} (-\tau_0(-1))(-1)^k \sum_{n \in \mathbb{F}} \psi_n(x) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n) \\
&+ \frac{1}{2(q+1)} \sum_{z \neq \pm 1} -(\tau_0(z) + \overline{\tau_0}(z)) \sum_{\tau : \mathbb{E}_1^* \to \mathbb{C}^*} \tau(z) \dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau) \\
\dim \mathcal{S}_k(SL_2(\mathbb{Z}), \omega(\tau_0)) &= \frac{1 + \tau_0(-1)(-1)^k}{q(q+1)} \dim \mathcal{S}_k(\Gamma(q)) \\
&- \frac{1 + \tau_0(-1)(-1)^k}{q(q-1)} \sum_{n \in \mathbb{F}} \left(\sum_{x \in \mathbb{F}^*} \psi_n(x) \right) \dim \mathcal{S}_k(\Gamma_1(q), \psi_n) \\
&- \frac{1}{2(q+1)} \sum_{z \in \mathbb{E}_1^*} (\tau_0(z) + \overline{\tau_0}(z)) \sum_{\tau : \mathbb{E}_1^* \to \mathbb{C}^*} \tau(z) \dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau) \\
&+ \frac{1}{2(q+1)} \sum_{z = \pm 1} 2\tau_0(z) \sum_{\tau : \mathbb{E}_1^* \to \mathbb{C}^*} \tau(z) \dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q), \tau).
\end{aligned}$$

For a fixed character τ on $\mathbb{E}_1^*$, the sum $\sum_{z \in \mathbb{E}_1^*} \tau(z)\tau_0(z)$ is equal to 0 unless $\tau = \overline{\tau_0}$. Similarly for any $n \in \mathbb{F}$, the sum $\sum_{x \in \mathbb{F}} \psi_n(x)$ is equal to 0 unless $n = 0$. We get the following equality by using Remark 5.3.2.

$$\begin{aligned}\dim \mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\tau_0)) = &\ \frac{1+\tau_0(-1)(-1)^k}{q-1}\dim \mathcal{S}_k(\Gamma(q)) - \frac{1+\tau_0(-1)(-1)^k}{q-1}\dim \mathcal{S}_k(\Gamma_1(q)) \\ &- \frac{1}{2}\dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q),\overline{\tau_0}) - \frac{1}{2}\dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q),\tau_0).\end{aligned}$$

If $\tau_0(-1) \neq (-1)^k$ then $\mathcal{S}_k(\Gamma_{\mathbb{E}}(q),\overline{\tau_0}) = \mathcal{S}_k(\Gamma_{\mathbb{E}}(q),\tau_0) = 0$ since $-\mathbf{1} \in \Gamma_{\mathbb{E}}(q)$, and so $\mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\tau_0)) = 0$.
Now assume $\tau_0(-1) = (-1)^k$ then We complete a proof of the theorem by using the following explicit formula of dimensions of $\mathcal{S}_k(\Gamma(q))$ and $\mathcal{S}_k(\Gamma_1(q))$ (Figure 3.4 in [13]). For $k \geq 3$,

$$\begin{aligned}\dim \mathcal{S}_k(\Gamma_1(q)) &= \frac{k-1}{24}(q^2-1) - \frac{1}{2}(q-1) \\ \dim \mathcal{S}_k(\Gamma(q)) &= \frac{k-1}{24}q(q^2-1) - \frac{1}{4}(q^2-1).\end{aligned}$$

□

If $\tau_0 = \zeta_0$ is a quadratic character on $\mathbb{E}_1^*$ then the special Weil representation $\omega(\zeta_0)$ is the direct sum of irreducible representations $\omega(\zeta_0)^{\pm}$. So,

$$\mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\zeta_0)) = \mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\zeta_0)^+) \oplus \mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\zeta_0)^-).$$

Proposition 5.5.2. *If $\zeta_0(-1) \neq (-1)^k$ then $\mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\zeta_0)^{\pm}) = 0$, and if $\zeta_0(-1) = (-1)^k$ then*

$$\begin{aligned}\dim \mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\zeta_0)^{\pm}) = &\ \frac{k-1}{24}(q^2-1) - \frac{1}{4}(q-1) - \frac{1}{2}\dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q),\zeta_0) \\ &\pm \frac{\zeta_e(-1)}{q-1}\sum_{n\in\mathbb{F}^*}\zeta_e(n)\mathcal{S}_k(\Gamma_1(q),\psi_n).\end{aligned}$$

Proof. By using the character formula similar to Theorem 5.5.1, we get the following

$$\begin{aligned}\dim \mathcal{S}_k(SL_2(\mathbb{Z}),\omega(\zeta_0)^{\pm}) = &\ \frac{1}{q-1}\dim \mathcal{S}_k(\Gamma(q)) - \frac{1}{q-1}\dim \mathcal{S}_k(\Gamma_1(q)) - \frac{1}{2}\dim \mathcal{S}_k(\Gamma_{\mathbb{E}}(q),\zeta_0) \\ &\pm \frac{\zeta_e(-1)}{q-1}\sum_{n\in\mathbb{F}^*}\zeta_e(n)\mathcal{S}_k(\Gamma_1(q),\psi_n).\end{aligned}$$

□

Chapter 6

Admissible Representations

This chapter contains a brief discussion of admissible representations and their differential cohomology. It is based on the references [7] and [11]. We will compute certain Lefschetz numbers for the real semisimple Lie groups $SL_N(\mathbb{R})$ and $SL_N(\mathbb{C})$, which are essential to the proof of the main result of this thesis.

Introduction

Let k be any number field, and F be a completion of k. Let $\mathcal{G}$ be a connected reductive group over the number field k, and $G = \mathcal{G}(F)$. (We can view G as a F-points of an algebraic group $\mathcal{G} \times_k F$.) We assume that $G = \mathcal{G}(F)$ is a Lie group if F is an Archimedean completion.

Let V be a locally convex, metrizable, complete Hausdorff topological vector space over $\mathbb{C}$. Since V is a metrizable and complete vector space, one can talk about the smoothness of functions $G \to V$. Let (π, V) be a representation of G, i.e., $\pi : G \to GL(V)$ such that $G \times V \to V$ is a continuous map.

The representation space V is called **finitely generated** if there exists a finite set S such that the subspace generated by $\{g \cdot v : g \in G \text{ and } v \in S\}$ is dense in V. An element $v \in V$ is called G-finite if the subspace generated by $\{g \cdot v : g \in G\}$ is a finite-dimensional subspace. Let $V_{(G)}$ denote the set of **G-finite vectors**. Then $V_{(G)}$ is generated by the images of

$Hom_G(W,V) \otimes W \to V$ defined as $\tau \otimes w \mapsto \tau(w)$, for all finite dimensional representations W of G. A G-module is considered **locally finite** if every element is G-finite. For any compact subgroup H and any irreducible representation W of H, let $V_{(W)}$ be the image of $Hom_H(W,V) \otimes W \to V$. Then $V_{(W)}$ is the isotypic component of V of type W.

A representation V of G is called **admissible** if the isotypic components $V_{(W)}$ are finite-dimensional for all irreducible representation W of any compact subgroup H of G. If G possesses finitely many connected components, then the admissibility condition simplifies to finite dimensionality for isotypic subspaces $V_{(W)}$ associated with irreducible representations W of a maximal compact subgroup of G. Let $V_{(H)}$ denotes the collection of H-finite vectors in V, then $V_{(H)} = \oplus V_{(W)}$, where $V_{(W)}$ is finite-dimensional subspaces of V, and the summation is over all irreducible representations of H.

Let Z be the center of G. If (π, V) is an irreducible representation of G then the **central character** of π is a group homomorphism $\chi_\pi : Z \to \mathbb{C}^*$ such that $\pi(z)(v) = \chi_\pi(z) \cdot v$.

6.1 The Differential Cohomology

Let (π, V) be a representation of G. A vector $v \in V$ is called a **smooth vector** if the map $c_v : G \to V$ defined by $g \mapsto g \cdot v$ is smooth. The collection of smooth vectors in V is denoted as V^∞. A representation is called a **differentiable** or a **smooth representation** if $V = V^\infty$. Note that any finite-dimensional representation is a smooth representation.

The collection of smooth representations with morphisms between representations is an abelian category. Note that, this category has enough injective (refer to Chapter 9 in [7]). The group cohomology in the category of smooth representations is called the **differential cohomology**.

Since $G = \mathcal{G}(F)$ is a Lie group for an Archimedean completion case, then we can talk about the differential forms. Let K be a maximal compact subgroup of G then $M = G/K$ is a smooth manifold. Then G acts on M smoothly and properly. So for all $g \in G$, there is a linear isomorphism $Lg_p : T_p(M) \to T_{g \cdot p}M$. Let $A^q(M,V)$ be the space of smooth differential q forms then G acts on $A^q(M,V)$ by $(g \cdot \omega)(p,x) = g \cdot (\omega(g^{-1} \cdot p, g^{-1} \cdot x))$ for $p \in M$ and $x \in (T_pM)^q$.

Lemma 6.1.1. *(Chapter 9 in [7]) In the case of Archimedean completion, the differential cohomology is isomorphic to the cohomology of $A^*(M,V)^G$, i.e., $H_d^q(G,V) \cong H^q(A^*(M,V)^G)$.*

6.2 $(\mathfrak{g}, K)$- Cohomology

In this section, we assume that F is an Archimedean completion of k. Let K be a maximal compact subgroup of G modulo center. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the complexified Lie algebra of G and K, respectively.

Let (π, V) be an admissible representation of $G = \mathcal{G}(F)$. The Lie algebra of G acts on the space V^∞ (consisting smooth vectors) through the operation $d\pi(X)\cdot v = \left(\frac{d}{dt}\pi(exp(tX)) \cdot v\right)_{t=0}$. This action extends to the complexified Lie algebra as well. Let $V_0 := V_{(K)} \cap V^\infty$ be a subspace of smooth K-finite vectors. Since $\mathfrak{g}$ is finite-dimensional $K-$module via adjoint action, V_0 is stable under the action of $\mathfrak{g}$. This motivates the definition of $(\mathfrak{g}, K)$-module:

A **$(\mathfrak{g}, \mathbf{K})$-module** is a $\mathfrak{g}$-module (π, V), and a locally finite and semisimple K-module (ρ_π, V) such that $\pi|_{\mathfrak{k}} = d\rho_\pi$ and $\rho_\pi(k)\pi(X) = \pi(Ad(k)\cdot X)\rho_\pi(k)$ for all $k \in K$ and $X \in \mathfrak{g}$. An admissible $(\mathfrak{g}, K)$-module is an admissible as $K-$module.

The center of the universal enveloping algebra $\mathfrak{U}\mathfrak{g}$ is denoted by $Z(\mathfrak{g})$. For an admissible irreducible representation (π, V), an **infinitesimal character** is an algebra homomorphism $\chi_\pi : Z(\mathfrak{g}) \to \mathbb{C}$ such that $\pi(x) = \chi_\pi(x) \cdot \mathbf{1}_V$ on V.

The quotient $\mathfrak{g}/\mathfrak{k}$ is a K-module via the adjoint action since K stabilizes $\mathfrak{k}$ via adjoint action. So, the wedge product $\wedge^q(\mathfrak{g}/\mathfrak{k})$ are also K-modules. Let C^q be a cochain complex, where $C^q = C^q(\mathfrak{g}, K, V) := \mathrm{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), V)$ and coboundary map is

$$\begin{aligned}(df)(x_0 \wedge \cdots \wedge x_q) = &\sum (-1)^i x_i \cdot f(x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q)\\ &+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_q),\\ (df)(x) = &\ x \cdot f(1) - f(1) \text{ if } q = 0.\end{aligned}$$

The $(\mathfrak{g}, K)$ cohomology of V (denoted by $H^q(\mathfrak{g}, K, V)$) is the cohomology of above cochain complex, i.e., $H^q(\mathfrak{g}, K, V) = H^q(C^*(\mathfrak{g}, K, V))$.

Let K° be the connected component of identity. Then the Lie algebra of K and K° coincide. Then $(\mathfrak{g}, K^\circ)$ cohomology of coincide with the relative Lie algebra $(\mathfrak{g}, \mathfrak{k})$ cohomology (defined in Section 2.4). Since K/K° acts naturally on $Hom_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), V)$, it acts on the cohomology groups. Moreover,

$$H^q(\mathfrak{g}, K, V) \cong H^q(\mathfrak{g}, K^\circ, V)^{K/K^\circ} = H^q(\mathfrak{g}, \mathfrak{k}, V)^{K/K^\circ}.$$

Lemma 6.2.1. *(Chapter 9 in [7]) The differential cohomology is isomorphic to $(\mathfrak{g}, K)$ cohomology, i.e., $H^q_d(G, V) \cong H^q(\mathfrak{g}, K, V)$ for any smooth, admissible G-module V.*

Proof. Let $\omega \in A^*(G/K, V)^G$ then ω is determined by $\omega([e], \cdot)$, where $[e]$ is a class of identity in G/K. Note that $T_e(G/K) = \mathfrak{g}/\mathfrak{k}$. The map $\omega([e], \cdot) \in C^q(\mathfrak{g}, K, V)$ since $\omega([e], \cdot)$ is alternating q form and $[e] = [k]$ for all $k \in K$. So, the lemma follows. □

Lemma 6.2.2. *If V is an admissible $(\mathfrak{g}, K)$ module then $H^q(\mathfrak{g}, K, V)$ are finite dimensional vector spaces.*

Proof. Since $\mathfrak{g}$ is a finite dimensional Lie algebra, $\wedge^q(\mathfrak{g}/\mathfrak{k})$ are finite dimensional representations of K. So, there are finitely many $K-$types, which appear in $\wedge^q(\mathfrak{g}/\mathfrak{k})$ for each $q \geq 0$. Let $W_1, \ldots, W_{m_q}$ are classes of inequivalent irreducible representations of K, which appears in $\wedge^q(\mathfrak{g}/\mathfrak{k})$. Then Schur's Lemma and admissibility of the representation imply that $Hom_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), V_{(W_{m_i})})$ are finite-dimensional. So, the cochain complex is also finite-dimensional. Since cohomology is a subquotient of the chain complex, the lemma follows. □

Let $\{V_\alpha, v_{\alpha,\beta}\}$ be a filtered system of $(\mathfrak{g}, K)$-modules, i.e., $\{V_\alpha\}$ is a collection of $(\mathfrak{g}, K)$-modules with the collection of $(\mathfrak{g}, K)$- module homomorphisms $\{v_{\alpha,\beta} : V_\alpha \to V_\beta\}$ such that $v_{\alpha,\beta} \circ v_{\gamma,\alpha} = v_{\gamma,\beta}$, and for any α and β there exists γ such that $v_{\alpha,\gamma}$ and $v_{\beta,\gamma}$ exist. Then the colimit of the system is $V = \{(v_\alpha, V_\alpha) : v_\alpha \in V_\alpha\}/\sim$, where $(v_\alpha, V_\alpha) \sim (v_\beta, V_\beta)$ if there exists γ such that $v_{\alpha,\gamma}(v_\alpha) = v_{\beta,\gamma}(v_\beta)$. Let the closed bracket $[v_\alpha, V_\alpha]$ denote the equivalence class of (v_α, V_α). Note that V is also a $(\mathfrak{g}, K)$-module.

Remark 6.2.1. *The colimit (defined here) satisfies the universal definition of colimit in the category of $(\mathfrak{g}, K)$-modules.*

Lemma 6.2.3. *Let $\{M_\alpha, m_{\alpha,\beta}\}$, $\{N_\alpha, n_{\alpha,\beta}\}$ and $\{P_\alpha, p_{\alpha,\beta}\}$ be the filtered system of abelian groups such that $0 \to M_\alpha \xrightarrow{a_\alpha} N_\alpha \xrightarrow{b_\alpha} P_\alpha \to 0$ is an exact sequence for all α, and $n_{\alpha,\beta} \circ a_\alpha =$*

$a_\beta \circ m_{\alpha,\beta}$ and $p_{\alpha,\beta} \circ b_\alpha = b_\beta \circ p_{\alpha,\beta}$. Let M, N, and P denote the colimits, respectively then the induced sequence $0 \to M \xrightarrow{a} N \xrightarrow{b} P \to 0$ is an exact. Moreover, it is a natural.

Proof. Note that $a([m_\alpha, M_\alpha]) = [a_\alpha(m_\alpha), N_\alpha]$ and similarly b also. So, $b \circ a = 0$.
Let $a([m_\alpha, M_\alpha]) = 0$ then there exists β such that $n_{\alpha,\beta} \circ a_\alpha(m_\alpha) = 0$. Since square commutes, $a_\beta \circ m_{\alpha,\beta}(m_\alpha) = 0$. Since a_β is an injective, $m_{\alpha,\beta}(m_\alpha) = 0$. Hence, $[m_\alpha, M_\alpha] = 0$.
Let $[p_\alpha, P_\alpha] \in P$ then $[p_\alpha, P_\alpha] = b([n_\alpha, N_\alpha])$, since b_α is surjective.
Let $[n_\alpha, N_\alpha] \in N$ such that $b([n_\alpha, N_\alpha]) = 0$, then there exists β such that $p_{\alpha,\beta} \circ b_\alpha(n_\alpha) = 0$. Since square commutes, $b_\beta \circ n_{\alpha,\beta}(n_\alpha) = 0$. The exactness of sequences implies there exists $m_\beta \in M_\beta$ such that $a_\beta(m_\beta) = n_{\alpha,\beta}(n_\alpha)$. So $a([m_\beta, M_\beta]) = [a_\beta(m_\beta), N_\beta] = [n_\alpha, N_\alpha]$.
The naturality follows trivially. □

Remark 6.2.2. *The 'filtered' condition is necessary to check the well-definedness in the proof of Lemma 6.2.3.*

Lemma 6.2.4. *Let $\{V_\alpha, v_{\alpha,\beta}\}$ be a filtered system of $(\mathfrak{g}, K)$-modules then $H^q(\mathfrak{g}, K, V)$ is isomorphic to colimit of $H^q(\mathfrak{g}, K, V_\alpha)$.*

Proof. Since $\wedge^q(\mathfrak{g}/\mathfrak{k})$ are finite dimensional for each q and system is filtered, any linear map $\wedge^q(\mathfrak{g}/\mathfrak{k}) \to V$ factor through V_α for some α, i.e., it equal to $\wedge^q(\mathfrak{g}/\mathfrak{k}) \to V_\alpha \to V$. Let Z^q_α and B^{q+1}_α denotes the kernel and image of coboundary maps for V_α, respectively. Let Z^q and B^{q+1} denotes the kernel and image of coboundary maps for V, respectively then Z_q and B^{q+1} are colimit of filtered systems $\{Z^q_\alpha, v_{\alpha,\beta}\}$ and $\{B^{q+1}_\alpha, v_{\alpha,\beta}\}$, respectively. The lemma follows by applying Lemma 6.2.3 to $0 \to B^{q-1}_\alpha \to Z^q_\alpha \to H^q(\mathfrak{g}, K, V_\alpha) \to 0$. □

6.3 Lefschetz Number

Let α be a finite-order automorphism of $\mathcal{G}$ defined over k. Then α induces an isomorphism of $G = \mathcal{G}(F)$, denoted also by α. Let $L(F)^+ = \mathcal{G}(F) \rtimes \langle\alpha\rangle$ be the semi-direct product. Let $L(F) = \{x \rtimes \alpha \ : \ x \in G\}$ be the coset of α.

Let (π, V) be a representation of $L(F)^+$, then $T_x := \pi(x \rtimes \alpha)$ be an isomorphism of V for all $x \in G$. Let $\sigma := \pi|_G$ and $\sigma'_x := T_x \circ \sigma \circ T_x^{-1}$ be the representations of G on the space V, then $\sigma \sim \sigma'_x$, and $T_x \in Hom_G(\sigma, \sigma'_x)$. Moreover, T_x induces a linear isomorphism on the differential cohomology groups, denoted by T_x^q. Now define **Lefschetz Number** for $x \rtimes \alpha$ as

$$\mathrm{Lef}(x \rtimes \alpha, G, \pi) := \sum (-1)^q \operatorname{trace}(T_x^q | H_d^q(G, V)).$$

Let $y \rtimes \alpha \in L(F)$ is an another element then $T_y = \sigma(yx^{-1}) \circ T_x$. Since the induced maps on the cohomology groups by $\sigma(yx^{-1})$ are identity maps, $T_y^q = T_x^q$ for all q. So, the Lefschetz number does not depend on the choice of elements in $L(F)$. So, it is denoted by $\mathrm{Lef}(\alpha, G, \pi)$.

Let α and β are finite order automorphisms of $\mathcal{G}$ defined over k. Assume there exists $g \in \mathcal{G}$ such that $\beta = \alpha \circ \mathrm{Ad}(g)$. Let (π_α, V) be a representation of $G \rtimes \langle \alpha \rangle$ then $\pi_\alpha|_G \sim (\pi_\alpha|_G) \circ \alpha$. Note that $\pi_\alpha(1 \rtimes \alpha) \in \mathrm{Hom}_G(\pi_\alpha|_G, (\pi_\alpha|_G) \circ \alpha)$. It implies that $\pi_\alpha|_G \sim (\pi_\alpha|_G) \circ \beta$. So, $\pi_\alpha|_G$ can be extended to a representation (denoted by (π_β, V)) of $G \rtimes \langle \beta \rangle$. Similarly, let (π_β, V) be a representation of $G \rtimes \langle \alpha \rangle$ then we have a representation (π_α, V) of $G \rtimes \langle \alpha \rangle$.

Lemma 6.3.1. *The Lefschetz number is invariant under inner automorphism, i.e.,*

$$\mathrm{Lef}(\alpha, G, \pi_\alpha) = \mathrm{Lef}(\beta, G, \pi_\beta).$$

Proof. Let $\beta = \alpha \circ \mathrm{Ad}(g)$. Then $\pi_\beta(1 \rtimes \beta) = \pi_\alpha(\alpha(g) \rtimes 1)\pi_\alpha(1 \rtimes \alpha)$. Since the induced maps on the cohomology groups by $\pi_\alpha(\alpha(g) \rtimes 1)$ are identity maps, the lemma follows. $\square$

6.3.1 Lefschetz Number and Induced Representation

Let H be the subgroup of G such that $\alpha(H) \subset H$. Let $\alpha_H := \alpha|_H$. Consider a representation (σ, W) of H and let $\rho = \mathrm{Ind}_H^G \sigma$ be the induced representation.

Lemma 6.3.2. *If σ is α-invariant representation of subgroup H then the induced representation ρ is also α-invariant representation of G, i.e., ${}^{\alpha_H}\sigma \sim \sigma \implies {}^{\alpha}\rho \sim \rho$.*

Proof. Let $T : W \to W$ be the intertwining operator, i.e., $T \circ \sigma(h) = {}^{\alpha_H}\sigma(h) \circ T$ for all $h \in H$. Define $\bar{T}$ as $[\bar{T}(f)](g) = T(f(\alpha^{-1}(g)))$ on the induced representation space

$$W^G = \{f : G \to W \text{ smooth } : \ f(hg) = \sigma(h)f(g)\}.$$

This map is well-defined since

$$\begin{aligned}[\bar{T}(f)](hg) = T(f(\alpha^{-1}(h)\alpha^{-1}(g))) &= T \circ \sigma(\alpha^{-1}(h))(f(\alpha^{-1}(g))) \\ &= \sigma(h)T(f(\alpha^{-1}(g))) = \sigma(h)[\bar{T}(f)](g).\end{aligned}$$

Hence, $\bar{T}$ is an intertwining operator, as shown by

$$\begin{aligned}[\bar{T} \circ \rho(g)](f)(x) &= T((\rho(g)f)(\alpha^{-1}x)) \\ &= T(f(\alpha^{-1}(x\alpha(g)))) \\ &= [\bar{T}(f)](x\alpha(g)) \\ [\bar{T} \circ \rho(g)](f)(x) &= [{}^{\alpha}\rho(g) \circ \bar{T}](f)(x).\end{aligned}$$

□

Lemma 6.3.3. *(**Shapiro's Lemma**) Let (π, V) and (σ, W) be smooth representations of G and H, respectively. The mapping $f \mapsto \bar{f}(x) := f(x)(1)$, where $f \in Hom_G(V, W^G)$ and $\bar{f} \in Hom_H(V, W)$, induces an isomorphism between the* Ext *groups, i.e., $Ext^q_G(V, W^G)$ and $Ext^q_H(V, W)$ is isomorphic for all $q \geq 0$. Moreover, $H^{\bullet}_d(G, W^G) \cong H^{\bullet}_d(H, W)$.*

For a proof of Lemma 6.3.3, refer to Proposition 2.3 and Section 5.9 of Chapter 9 in [7].

Corollary 6.3.4. $\mathrm{Lef}(\alpha, G, \mathrm{Ind}_H^G \sigma) = \mathrm{Lef}(\alpha|_H, H, \sigma)$.

Proof. Let $0 \to W \to I^{\bullet}$ be an injective resolution in the category of smooth representations of H, and $\alpha_H^q : I^q \to I^q$ be an induced map. Then $0 \to \mathrm{Ind}_H^G W \to \mathrm{Ind}_H^G I^{\bullet}$ is an injective resolution in the category of smooth representations of G since Ind_H^G functor preserves injectives, and let $\alpha_G^q : \mathrm{Ind}_H^G I^q \to \mathrm{Ind}_H^G I^q$. Moreover, the following diagrams commute:

$$\begin{tikzcd} H_d^q(H, W) \arrow[r] \arrow[d, "\alpha_H^q"] & H_d^q(G, \mathrm{Ind}_H^G W) \arrow[d, "\alpha_G^q"'] \\ H_d^q(H, W) \arrow[r] & H_d^q(G, \mathrm{Ind}_H^G W) \end{tikzcd}$$

The horizontal arrows are isomorphism due to Shapiro Lemma (Lemma 6.3.3). So, $\mathrm{trace}\, \alpha_H^q = \mathrm{trace}\, \alpha_G^q$. So, the corollary follows. □

6.4 Admissible Representations and Differential Cohomology of $GL_N(F)$ for non-Archimedean Case

Let F be a non-Archimedean completion of k and $|\cdot|$ be the normalized absolute valuation on F. Then $G = GL_N(F)$ is a totally disconnected locally compact group. Note that every point has a neighborhood basis consisting of compact open neighborhoods, and $e \in G$ has a neighborhood basis consisting of compact open subgroups. For more details, we refer to Chapter 10 in [7].

6.4.1 Admissible Representations and Cohomology

Let $A_0 = \{x := \mathrm{diag}(x_1, \ldots, x_N) : \ x_i \in F^*\}$ be the $k-$split torus in G. Let $\mathcal{W}$ be the Weyl group $N_G(A_0)/Z_G(A_0)$. Let P_0 be the minimal parabolic subgroup of G consisting of upper triangular matrices and $P_0 = M_0N_0$ be the Levi decomposition then $M_0 = Z_G(A_0)$. The character group of M_0 consists of characters $\chi : M_0 \to F^\times$. We define ${}^\circ M_0 = \cap \ker|\chi|$, where the intersection runs over all the characters. Let $K = GL_N(\mathcal{O}_F)$ be a maximal compact subgroup where $\mathcal{O}_F$ is a ring of integers of F. Then ${}^\circ M_0 = M_0 \cap K$ and $G = K \cdot P$. We call a character χ an **unramified character** if $\chi|_{{}^\circ M_0}$ is the trivial.

Let (π, V) be any admissible representation of G. Let P be a parabolic subgroup and $P = MN$ be the Levi decomposition. Let (π, V) also denote the restriction to P. Let $V(N)$ is a $P-$submodule of V generated by $\{v - \pi(n)(v) : v \in V, n \in N\}$ then N acts trivially on $V/V(N)$. So, M acts naturally on V_N, the action is called the **Jacquet module** (π_N, V_N). Moreover, $(\pi, V) \mapsto (\pi_N, V_N)$ is an exact functor.

Let $\delta : P_0 \to \mathbb{R}^*$ be the modulus character defined as $\delta(p) = d_L(U \cdot p)/d_L(U)$, where d_L is a left Haar measure and U is any compact set. Let χ be an unramified character on M_0 then χ can be viewed as a character on P_0 considering trivial on N_0. Then $PS(\chi) = \text{n-Ind}_{P_0}^G \chi$ is the normalized induced representation of G on the following space:

$$PS(\chi) = \{f : G \to \mathbb{C} \text{ smooth } : f(pg) = \chi(p)\delta(p)^{1/2}f(g) \text{ for all } p \in P_0, g \in G\}.$$

Lemma 6.4.1. *(**W.Casselman** [9]) The semi-simplification of $PS(\chi)_N$ is isomorphic to $\oplus_{s\in\mathcal{W}}\delta^{1/2} \cdot {}^s\chi$ as a representation of M, where ${}^s\chi(x) = \chi(s^{-1}xs)$ for $s \in N_G(A_0)$.*

Let P be a parabolic subgroup of G, and $P = MN$ be the Levi decomposition. Let

$$I_P^G(\mathbf{1}_P) = \{f : G \to \mathbb{C} \text{ smooth } : \ f(pg) = f(g)\ \forall p \in P\}$$

denote the induced representation of G. Note that $PS(\chi) = I_{P_0}^G(\delta^{1/2}\chi)$. Let $P' \supset P$ then there is a natural injection $\pi_{P',P} : I_{P'}^G(\mathbf{1}_{P'}) \to I_P^G(\mathbf{1}_P)$. Let U_P denotes the submodule of $I_P^G(\mathbf{1}_P)$ spanned by the images of $\pi_{P',P}$ for all parabolic subgroups $P' \supsetneq P$. Let $V_P = I_P^G(\mathbf{1}_P)/U_P$. If $P = G$ then V_P is a trivial representation and $P = P_0$ then V_P is called the **Steinberg** representation.

Theorem 6.4.2. *(Chapter 10 in [7]) If V is an admissible irreducible representation of G such that $H_d^\bullet(G, V) \neq 0$ then the following holds:*

- *There exists a parabolic subgroup P such that V is isomorphic to V_P as $G-$module.*
- *Let $prk(P)$ denotes the rank of radical of $P/Z(G)$, where $Z(G)$ is the center of G then*

$$H_d^q(G, V_P) \cong \begin{cases} \mathbb{C} & q = prk(P), \\ 0 & else. \end{cases}$$

- *The irreducible representations V_P occurs in the composition series of $PS(\delta^{1/2})$.*

Theorem 6.4.3. *(Section 4, Chapter 11 in [7]) The admissible irreducible representation V_P is unitary if and only if $P = G$ or $P = P_0$.*

6.4.2 Lefschetz Number

Let α be an automorphism of $G = GL_N(F)$ defined over $\mathbb{Q}$ as $g \mapsto {}^*g = {}^tg^{-1}$. Since $\alpha(P_0) \subset P_0$, the Steinberg and trivial representations of $GL_N(F)$ can be extended to the representations of $G \rtimes \langle\alpha\rangle$. The following corollary follows by using Theorem 6.4.2:

Corollary 6.4.4. $\mathrm{Lef}(\alpha, G, \mathbf{1}_G) \neq 0$ *and* $\mathrm{Lef}(\alpha, G, St) \neq 0$*.*

Proposition 6.4.5. *(Proposition 8.2 in [5]) Let π be a representation of $G \rtimes \langle\alpha\rangle$ such that the restriction τ of π to G is a constituent of a representation* $\text{n-Ind}_P^G \sigma$ *parabolically induced from a unitary representation σ of a proper parabolic subgroup. Then* $\mathrm{Lef}(\alpha, G, \pi) = 0$*.*

Proof. The following proof is based on [5].

- Theorem 6.4.2 implies that τ is the subrepresentation of semi-simplification of $\text{n-Ind}_{P_0}^{G}\,\delta^{1/2}$.
- The Frobenius reciprocity, $Hom_G(\text{n-Ind}_{P_0}^{G}\,\delta^{1/2}, \text{n-Ind}_{P}^{G}\,\sigma) = Hom_M(\kappa, \delta_P^{1/2}\sigma)$ is non-trivial, where $\kappa := \text{Res}_P^G\,\text{n-Ind}_{P_0}^{G}\,\delta_{P_0}^{1/2}$. Therefore, σ is a subrepresentation of the semi-simplification of the Jacquet module of κ.
- Since the semi-simplification of Jacquet module of κ is sum of $\text{n-Ind}_{M\cap P_0}^{M}\,\delta_{P_0}^{1/2}\circ\omega$ for ω running through a subset of the Weyl group, σ is a subrepresentation of κ.
- The central character of $\text{n-Ind}_{M(F)\cap P_0(F)}^{M(F)}\,\delta_{P_0}^{1/2}\circ\omega$ is non-unitary unless $P = G$. This contradicts the fact that σ is unitary.

□

6.5 Admissible Representations of $GL_N(F)$ for Archimedean Case

Let $G = SL_N(F)$, it is a semisimple Lie group (see Section 3.7) for either $F = \mathbb{R}$ or $\mathbb{C}$, K be a maximal compact. Let T_0, A_0 and ${}^{\circ}T_0$ be a diagonal, $\mathbb{R}$-split and a compact torus, respectively, i.e., $T_0 = \{\text{diag}(x_1, \dots, x_N) \in G\}$, $A_0 = \{\text{diag}(a_1, \dots, a_N) \in G : \ a_i \in \mathbb{R}_+^*\}$ and $T_0 = A_0 \times {}^{\circ}T_0$. Let B_0 be a subgroup consisting of upper triangular matrices, then $B_0 = T_0 N$ is the Levi decomposition, A_0 is $\mathbb{R}$-split component of B_0, and (B_0, A_0) be the standard p-pair. Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{a}_0$ be the complexified Lie algebra of G, K and A_0, respectively.

Let $\underline{n} = (n_1, \dots, n_r)$ be a partition of N. Let $P_{\underline{n}}$ be a parabolic subgroup consisting of block upper diagonal matrices, and $A_{\underline{n}} \cong \prod_{i=1}^{r-1}\mathbb{R}^*$ be a maximal $\mathbb{R}$-split component of P. Let $M_{\underline{n}} = G \cap (GL_{n_1}(\mathbb{C}) \times \ldots \times GL_{n_r}(\mathbb{C}))$ be the subgroup consisting of block diagonal matrices, and $U_{\underline{n}}$ be a subgroup consisting of identity matrices on diagonal blocks. Then ${}^{\circ}M_{\underline{n}}$ (for definition see Section 3.7) is the subgroup consisting of block diagonal matrices where the modulus of the determinant of each block is 1 then $M_{\underline{n}} = A_{\underline{n}} \times {}^{\circ}M_{\underline{n}}$. Then $(P_{\underline{n}}, A_{\underline{n}})$ is a standard p-pair with the Levi decomposition $P_{\underline{n}} = M_{\underline{n}} \rtimes U_{\underline{n}}$. Let $\rho_{P_{\underline{n}}}$ be a character on $A_{\underline{n}}$ defined in Section 3.7.

Let (σ, H_σ) be a unitary tempered smooth representation of ${}^\circ M_{\underline{n}}$, and $v \in \mathfrak{a}^*_{\underline{n},\mathbb{C}}$ be the character on $A_{\underline{n}}$. Let $\mathcal{C}^\infty(G, H_\sigma)$ be the space of smooth functions on G taking values in H_σ. The **parabolically induced representation** of G is the right regular representation of G in the following subspace of functions:

$$I_{P_{\underline{n}},\sigma,v} = \big\{ f \in \mathcal{C}^\infty(G, H_\sigma) : f(maug) = (v + \rho_{P_{\underline{n}}})(a)\sigma(m)f(g) \ \forall m \in {}^\circ M_{\underline{n}}, a \in A_{\underline{n}}, u \in U_{\underline{n}} \big\}.$$

The induced representation $I_{P_{\underline{n}},\sigma,v}$ has a unique non-zero irreducible quotient, denoted by $J_{P_{\underline{n}},\sigma,v}$ (Corollary 4.6 of Chapter 4 in [7]).

6.6 $(\mathfrak{g}, K)$ Cohomology and Lefschetz Number for $SL_N(\mathbb{C})$

Let $G = SL_N(\mathbb{C})$ be a semisimple Lie group, and $K = SU(N)$ be a maximal compact. Let T_0, A_0 and ${}^\circ T_0$ be a diagonal torus, $\mathbb{R}$-split torus and a compact torus, respectively, i.e., $T_0 = \{\underline{x} = \mathrm{diag}(x_1, \ldots, x_N) \in G : \ x_i \in \mathbb{C}^*\}$, $A_0 = \{\underline{a} = \mathrm{diag}(a_1, \ldots, a_N) \in G : \ a_i \in \mathbb{R}^*_+\}$ and ${}^\circ T_0 = \{\underline{z} = \mathrm{diag}(z_1, \ldots, z_N) \in G : \ z_i = e^{\iota\theta_i}\}$. Let B_0 be a subgroup consisting of upper triangular matrices, then $B_0 = T_0 N$ be the Levi decomposition, A_0 is $\mathbb{R}$-split component of B_0, and (B_0, A_0) be the standard p-pair. Let $\Phi = \{(E^*_i - E^*_j) \pm (\bar{E}^*_i - \bar{E}^*_j) : 1 \le i \ne j \le N\}$ be the root system and $\Delta = \{(E^*_i - E^*_{i+1}) \pm (\bar{E}^*_i - \bar{E}^*_{i+1}) : 1 \le i \le N-1\}$ be a base (see Section 2.6 for more details). Let $\mathcal{W}$ be the Weyl group generated by the reflections by roots and isomorphic to $N_G(T_0)/T_0$. Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{a}_0$ be the complexified Lie algebra of G, K and A_0, respectively.

Let $\boldsymbol{\lambda} = (\lambda, \lambda^*) = \big((b_1, \ldots, b_N), (b^*_1, \ldots, b^*_N)\big)$ be a dominant weight for the real Lie group $GL_N(\mathbb{C})$ (for definition see Subsection 2.2.2), i.e., $b_i - b_{i+1}$ and $b^*_i - b^*_{i+1}$ are non-negative integers. It is a character on the diagonal torus of $GL_N(\mathbb{C})$ defined as $\mathrm{diag}(x_1 \ldots, x_N) \mapsto \prod x_i^{b_i} \bar{x}_i{}^{b^*_i}$. Let $(\rho_{\boldsymbol{\lambda}}, \mathcal{M}_{\boldsymbol{\lambda}})$ be the finite-dimensional irreducible representation of $GL_N(\mathbb{C})$ with the highest weight λ, i.e., $\mathcal{M}_{\boldsymbol{\lambda}} = \mathcal{M}_\lambda \otimes \mathcal{M}_{\lambda^*}$ and $\rho_{\boldsymbol{\lambda}}(g) = \rho_\lambda(g) \otimes \rho_{\lambda^*}(\bar{g})$ for any $g \in GL_N(\mathbb{C})$. In other words, there exists a non-zero vector $m_{\boldsymbol{\lambda}}$ such that the diagonal torus of $GL_N(\mathbb{C})$ acts according to the character $\boldsymbol{\lambda}$, and the unipotent subgroup N acts trivially on $m_{\boldsymbol{\lambda}}$. Note that the unipotent radical of the Borel subgroup consisting of the upper triangular matrices in $GL_N(\mathbb{C})$ is equal to the unipotent radical N of the Borel subgroup B_0 consisting of the upper triangular matrices in $SL_N(\mathbb{C})$. Let $(\rho_{\boldsymbol{\lambda}}, \mathcal{M}_{\boldsymbol{\lambda}})$ also denote the restriction of a representation to $G = SL_N(\mathbb{C})$.

If $H^\bullet(\mathfrak{g}, K, H_\pi \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \neq 0$ then $\boldsymbol{\lambda}$ satisfies the purity condition (see [11]), i.e., $b_i + b^*_{N-i+1} = w$ constant for all i. Let $(\alpha, \beta) = \big((\alpha_1, \ldots, \alpha_N), (\beta_1, \ldots, \beta_N)\big)$ be the cuspidal parameter for the pure weight $\boldsymbol{\lambda}$, i.e.,

$$\alpha_i = -b_{N-i+1} + \frac{N-(2i-1)}{2} \qquad \beta_i = -b^*_i - \frac{N-(2i-1)}{2}.$$

Let $\chi_{\boldsymbol{\lambda}}$ be a character on ${}^\circ T_0$ defined as $\mathrm{diag}(z_1, \ldots, z_N) \mapsto \prod z_i^{\alpha_i} \bar{z}_i^{\beta_i}$. Let $\mathbb{J}_{\boldsymbol{\lambda}} = \mathrm{Ind}_{B_0}^G \chi_{\boldsymbol{\lambda}}$ be a parabolically induced representation.

Theorem 6.6.1. *(Lemma 3.14 in [11]) The parabolically induced representation $\mathbb{J}_{\boldsymbol{\lambda}}$ is an admissible unitary irreducible representation of $SL_N(\mathbb{C})$. Moreover, if $t_N = \frac{N(N-1)}{2}$ then*

$$H^q(\mathfrak{g}, K, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \cong \wedge^{(q-t_N)} \mathfrak{a}_0^*.$$

Recall that $(\mathfrak{g}, K)$ cohomology groups are subquotient groups of $\mathrm{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$. By Proposition 9.4.3. in [37], the differential map is zero, and so

$$H^q(\mathfrak{g}, K, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \cong \mathrm{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}). \tag{6.1}$$

Lemma 6.6.2. $\mathrm{Ind}_{B_0}^G(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \cong (\mathrm{Ind}_{B_0}^G \chi_{\boldsymbol{\lambda}}) \otimes \mathcal{M}_{\boldsymbol{\lambda}}$.

Proof. The map $f \mapsto [\zeta(f)](x) := \rho_{\boldsymbol{\lambda}}(x)^{-1}\Big(f(x)\Big)$ for all $x \in G$ defines an isomorphism of representations. $\square$

Since $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$, we will use notation from Section 2.6. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, so $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$. Since $(\mathfrak{g}, K)$ cohomology depends only on K-types appearing in $\wedge^q \mathfrak{p}$ and $\mathrm{Ind}_{B_0}^G(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$ and $\mathrm{Res}_K^G \mathrm{Ind}_{B_0}^G(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \cong \mathrm{Ind}_{K \cap B_0}^K(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$,

$$H^q(\mathfrak{g}, K, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \cong \mathrm{Hom}_K(\wedge^q \mathfrak{p}, \mathrm{Ind}_{K \cap B_0}^K(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})),$$

i.e., the differential map is zero (Proposition 9.4.3. in [37]).

By using **Frobenius reciprocity**: $\mathrm{Hom}_G(\rho, \mathrm{Ind}_H^G \psi) \cong \mathrm{Hom}_H(\rho|_H, \psi)$ for every subgroup

H of any group G, and ρ and ψ be any representation of G and H, respectively,

$$H^q(\mathfrak{g}, K, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \cong \mathrm{Hom}_{K\cap B_0}(\wedge^q \mathfrak{p}, \chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}). \tag{6.2}$$

Note $K \cap B_0$ is the compact torus ${}^\circ T_0$. Let $\Phi^+ = \{(\overline{E}_i^* - \overline{E}_j^*) : 1 \leq i < j \leq N\}$ be a set of positive roots (see Section 2.6) and $\Phi^- = -\Phi^+$, and

$$\begin{aligned} X^p_{i,j} &= (E_{i,j} + E_{j,i}) \otimes \sqrt{-1} + \sqrt{-1}(E_{i,j} - E_{j,i}) \otimes 1, \\ Y^p_{i,j} &= (E_{i,j} + E_{j,i}) \otimes \sqrt{-1} - \sqrt{-1}(E_{i,j} - E_{j,i}) \otimes 1. \end{aligned}$$

Then $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{u}$ as ${}^\circ T_0-$module, where $\mathfrak{u} = \left(\bigoplus_{\alpha\in\Phi^+} \mathbb{C}\cdot X^p_\alpha\right) \oplus \left(\bigoplus_{\alpha\in\Phi^-} \mathbb{C}\cdot Y^p_\alpha\right)$ (see Section 2.6). Since ${}^\circ T_0$ acts trivially on $\mathfrak{a}_0$,

$$\begin{aligned} H^q(\mathfrak{g}, K, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) &\cong \bigoplus_{s+t=q} \mathrm{Hom}_{{}^\circ T_0}\left(\wedge^s \mathfrak{a}_0 \otimes_{\mathbb{C}} \wedge^t \mathfrak{u}, \chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}\right) & (6.3)\\ &\cong \bigoplus_{s+t=q} \mathrm{Hom}_{{}^\circ T_0}\left(\wedge^t \mathfrak{u}, \chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}\right) \otimes_{\mathbb{C}} \wedge^s \mathfrak{a}_0^* & (6.4) \end{aligned}$$

By computing the dimension and using Theorem 6.6.1, we can get the following lemma.

Lemma 6.6.3.

$$\mathrm{Hom}_{{}^\circ T_0}\left(\wedge^t \mathfrak{u}, \chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}\right) \cong \begin{cases} \mathbb{C} & t = t_N = \frac{N(N-1)}{2}, \\ 0 & t \neq t_N, \end{cases}$$

i.e., there is a unique character of the compact torus ${}^\circ T_0$ *denoted, say, the canonical character* χ_0*, that appears in* $\wedge^t \mathfrak{u}$ *and in* $\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}|_{{}^\circ T_0}$*; this character* χ_0 *appears in the unique degree* $t = t_N$*; furthermore,* χ_0 *appears with multiplicity one in* $\wedge^{t_N} \mathfrak{u}$ *and in* $\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}|_{{}^\circ T_0}$*.*

So, we will find the canonical character χ_0 by using the following Lemmas 6.6.4 and 6.6.5. For any compact Lie group $\mathcal{K}$ (such as K or ${}^\circ T_0$), and a $\mathcal{K}$-module V, the set of isomorphism classes of irreducible representations of $\mathcal{K}$ that appear in V is denoted $\mathrm{Spec}_{\mathcal{K}}(V)$. Let Λ^+ be the set of all dominant integral weights for $\mathcal{K}$, and let π_χ denote the irreducible representation of $\mathcal{K}$ with the highest weight χ. For any unitary admissible representation Π of $\mathcal{K}$, the map $v \otimes f \mapsto f(v)$ defines an isomorphism between representations

$$\bigoplus_{\chi\in\Lambda^+} \Big(\pi_\chi \otimes \mathrm{Hom}_{\mathcal{K}}(\pi_\chi, \Pi)\Big) \cong \Pi.$$

Lemma 6.6.4. *Let π be an irreducible representation of K that appears in $\mathrm{Ind}_{^\circ T_0}^K(\chi_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})$ and also in $\wedge^q\mathfrak{p}$, then $\mathrm{Spec}_{^\circ T_0}(\pi)\cap\mathrm{Spec}_{^\circ T_0}(\chi_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})=\{\chi_0\}$. Moreover,*

$$\mathrm{Hom}_K(\pi,\mathrm{Ind}_{^\circ T_0}^K\chi_0)\cong\mathrm{Hom}_K(\pi,\mathrm{Ind}_{^\circ T_0}^K(\chi_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})).$$

Proof. If $\pi=\mathbf{1}_K$ is the trivial representation then $\mathrm{Hom}_K(\wedge^0\mathfrak{p},\mathbb{J}_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})\neq 0$. Similarly, if the trivial character $\mathbf{1}_{^\circ T_0}\in\mathrm{Spec}_{^\circ T_0}(\chi_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})$ then $\mathrm{Hom}_K(\wedge^0\mathfrak{p},\mathbb{J}_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})\neq 0$ by Frobenius reciprocity. It contradicts Theorem 6.6.1. So, let π be a non-trivial representation.
Since π is an irreducible rep, it embeds in $\wedge^q\mathfrak{p}$. If $\chi\in\mathrm{Spec}_{^\circ T_0}(\pi)\cap\mathrm{Spec}_{^\circ T_0}(\chi_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})$, then $\chi\in\mathrm{Spec}_{^\circ T_0}(\wedge^q\mathfrak{p})$. Since χ is non-trivial character, $\chi\in\mathrm{Spec}_{^\circ T_0}(\wedge^q\mathfrak{u})$. So, the equality holds for the intersection because of Lemma 6.6.3. The second conclusion follows by using Frobenius reciprocity and the first conclusion. □

Lemma 6.6.5. *Let γ be a character of $^\circ T_0$ that appears in $\wedge^q\mathfrak{p}$. Then there exist subsets A and B of Φ^+, with $|A|+|B|\leq q$, and written additively,*

$$\gamma=\sum_{\alpha\in A}\alpha-\sum_{\beta\in B}\beta.$$

Furthermore, for the usual partial ordering on weights and $\rho=\frac{1}{2}\sum\limits_{\alpha\in\Phi^+}\alpha$, one has

$$-2\rho\leq\gamma\leq\ 2\rho.$$

Proof. Note that $\mathfrak{u}=\Big(\bigoplus_{\alpha\in\Phi^+}\mathbb{C}{\cdot}X_\alpha^p\Big)\oplus\Big(\bigoplus_{\alpha\in\Phi^-}\mathbb{C}{\cdot}Y_\alpha^p\Big)$ as $^\circ T_0$-modules. Let $\{H_1,\dots,H_{n-1}\}$ be a basis of $\mathfrak{a}_0$. Then the following set is a basis for $\wedge^q\mathfrak{p}$:

$$\left\{(H_{i_1}\wedge\cdots\wedge H_{i_r})\wedge\Big(\bigwedge_{\alpha\in A}X_\alpha^p\Big)\wedge\Big(\bigwedge_{\beta\in B}Y_\beta^p\Big):A\subset\Phi^+,B\subset\Phi^-,r=q-|A|-|B|\right\}.$$

The action of $^\circ T_0$ is by $\gamma=\sum\limits_{\alpha\in A}\alpha-\sum\limits_{\beta\in B}\beta$ on a basis vector. So, the lemma follows. □

Lemma 6.6.6. *The canonical character $\chi_0=2\rho=\sum\limits_{\alpha\in\Phi^+}\alpha=(N-1,N-3,\dots,1-N)$.*

Proof. From Lemma 6.6.5 one sees that the character 2ρ is realized by the weight vector $\bigwedge_{\alpha\in\Phi^+}X_\alpha^p$ in $\bigwedge^{t_N}\mathfrak{p}$. It is sufficient to show $2\rho\in\mathrm{Spec}_{^\circ T_0}(\chi_{\boldsymbol\lambda}\otimes\mathcal{M}_{\boldsymbol\lambda})$ due to Lemma 6.6.3. Since $\boldsymbol\lambda=\big((b_1,\dots,b_N),(b_1^*,\dots,b_N^*)\big)$ is the weight satisfying the purity, i.e., $b_i+b_{N-i+1}^*=w$

for all i. Let $v_{\boldsymbol{\lambda}} \in \mathcal{M}_{\boldsymbol{\lambda}}$ be the highest weight vector, i.e., $\rho_{\boldsymbol{\lambda}}(\underline{z}) \cdot v_{\boldsymbol{\lambda}} = \big(\prod z_i^{b_i} \bar{z}_i^{b_i^*}\big) v_{\boldsymbol{\lambda}}$, then for all $\underline{z} \in {}^\circ T_0$. Since the root system Φ is reducible, there exists $\mathbf{w_0}$ in the Weyl group $\mathcal{W}$ such that $\mathbf{w_0} \cdot \boldsymbol{\lambda} = \big((b_N, \ldots, b_1), (b_1^*, \ldots, b_N^*)\big)$. Since $\boldsymbol{\lambda}$ is the highest weight, $\mathbf{w_0} \cdot \boldsymbol{\lambda}$ is an extremal weight, and so there exists a non-zero vector $v_{\boldsymbol{\lambda}}^* \in \mathcal{M}_{\boldsymbol{\lambda}}$ such that $\rho_{\boldsymbol{\lambda}}(\underline{z}) \cdot v_{\boldsymbol{\lambda}}^* = \big(\prod z_i^{b_{N-i+1}} \bar{z}_i^{b_i^*}\big) v_{\boldsymbol{\lambda}}^*$, then for all $\underline{z} \in {}^\circ T_0$. Since for any $\underline{z} \in {}^\circ T_0$, $\bar{z}_i = z_i^{-1}$ and $\prod z_i = 1$,

$$(\chi_{\boldsymbol{\lambda}} \otimes \rho_{\boldsymbol{\lambda}})(\underline{z}) \cdot v_{\boldsymbol{\lambda}}^* = \Big(\prod z_i^{\alpha_i} \bar{z}_i^{\beta_i}\Big)\Big(\prod z_i^{b_{N-i+1}} \bar{z}_i^{b_i^*}\Big) v_{\boldsymbol{\lambda}}^* = \Big(\prod z_i^{N-(2i-1)}\Big) v_{\boldsymbol{\lambda}}^* = (2\rho)(\underline{z}) v_{\boldsymbol{\lambda}}^*.$$

So, $2\rho \in \mathrm{Spec}_{{}^\circ T_0}(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$. □

For χ a character of ${}^\circ T$, which is a dominant integral weight for the Lie group K, let π_χ denote the irreducible representation of K with highest weight χ.

Lemma 6.6.7. *The irreducible representation π_χ of K with the highest weight χ, appears in $\wedge^q \mathfrak{p}$ then $\chi = \chi_0$. Moreover, the embedding: $\pi_{\chi_0} \to \mathrm{Ind}_{{}^\circ T_0}^K(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$ induces equality, i.e.,*

$$\mathrm{Hom}_K(\wedge^q \mathfrak{p}, \pi_{\chi_0}) = \mathrm{Hom}_K(\wedge^q \mathfrak{p}, \mathrm{Ind}_{{}^\circ T_0}^K(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})). \tag{6.5}$$

Proof. Let π_χ be the representation of K with the highest weight χ, appears in $\wedge^q \mathfrak{p}$ then $-2\rho \leq \chi \leq 2\rho$ because of Lemma 6.6.5. If π_χ appears in $\mathrm{Ind}_{{}^\circ T_0}^K(\chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$ then π_χ also appears in $\mathrm{Ind}_{{}^\circ T_0}^K \chi_0$ due to Lemma 6.6.4. By using the Frobenius reciprocity, $\mathrm{Hom}_{{}^\circ T_0}(\pi_\chi, \chi_0) \neq 0$. So, χ_0 appears in $\pi_\chi|_{{}^\circ T_0}$. Since χ is the highest weight for π_χ, and $\chi_0 = 2\rho$, we have $\chi = \chi_0$. The second part of the lemma follows due to the first part and the multiplicity of π_{χ_0} in $\mathrm{Ind}_{{}^\circ T_0}^K \chi_0$ is equal to the dimension of $\mathrm{Hom}_K(\pi_\chi, \mathrm{Ind}_{{}^\circ T_0}^K \chi_0) \cong \mathrm{Hom}_{{}^\circ T_0}(\pi_{\chi_0}, \chi_0) \cong \mathbb{C}$. □

Let θ be a Cartan involution on $G = SL_N(\mathbb{C})$ defined as $\theta(g) = {}^t\bar{g}^{-1}$. Note that, θ also denotes the induced map on $\mathfrak{g}$. Then $\theta(X) = X$ for all $X \in \mathfrak{k}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$. We are interested to compute $\mathrm{Lef}(\theta, SL_N(\mathbb{C}), \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$ (defined in Section 6.3)

Lemma 6.6.8. *There exists an isomorphism $I_0 : \mathcal{M}_{\boldsymbol{\lambda}} \to \mathcal{M}_{\boldsymbol{\lambda}}$ with the intertwining property:*

$$I_0 \circ \rho_{\boldsymbol{\lambda}}(g) = \rho_{\boldsymbol{\lambda}}(\theta(g)) \circ I_0, \quad \forall g \in G = SL_N(\mathbb{C}).$$

Proof. Let $\theta'(g) = w_0 \theta(g) w_0^{-1}$ be an automorphism, where $w_0 = [(-1)^{i-1}\delta_{i,N-i+1}]$ is $N \times N$

matrix. Let $v_{\boldsymbol{\lambda}} \in \mathcal{M}_{\boldsymbol{\lambda}}$ be the highest weight vector. Then for any $\mathrm{diag}(x_1, \ldots, x_N) \in T_0$,

$$\begin{aligned}
\rho_{\boldsymbol{\lambda}}(\theta'(\mathrm{diag}(x_1, \ldots, x_N))) \cdot v_{\boldsymbol{\lambda}} &= \rho_{\boldsymbol{\lambda}}(\mathrm{diag}(\bar{x_N}^{-1}, \ldots, \bar{x_1}^{-1})) \cdot v_{\boldsymbol{\lambda}} \\
&= \left(\prod \bar{x}_{N-i+1}^{-b_i} x_{N_i+1}^{-b_i^*} \right) v_{\boldsymbol{\lambda}} \\
&= \left(\prod \bar{x_i}^{b_i^* - w} x_i^{b_i - w} \right) v_{\boldsymbol{\lambda}} \\
\rho_{\boldsymbol{\lambda}}(\theta'(\mathrm{diag}(x_1, \ldots, x_N))) \cdot v_{\boldsymbol{\lambda}} &= \boldsymbol{\lambda}(\mathrm{diag}(x_1, \ldots, x_N)) v_{\boldsymbol{\lambda}}
\end{aligned}$$

Since for any $u \in N$ unipotent radical of the Borel subgroup B_0, $\theta'(u) \in N$, a vector $v_{\boldsymbol{\lambda}}$ is also the highest weight vector for an irreducible representation $\rho_{\boldsymbol{\lambda}} \circ \theta'$ with the highest weight $\boldsymbol{\lambda}$. By Theorem 2.2.5, there exists $I_0' : \mathcal{M}_{\boldsymbol{\lambda}} \to \mathcal{M}_{\boldsymbol{\lambda}}$ such that $I_0' \circ \rho_{\boldsymbol{\lambda}}(g) = \rho_{\boldsymbol{\lambda}}(\theta'(g)) \circ I_0'$ for all $g \in G$. Then $I_0 := \rho_{\boldsymbol{\lambda}}(w_0^{-1}) \circ I_0'$ is satisfies the required intertwining property. $\square$

Lemma 6.6.9. *There exists an isomorphism* $I_\infty : \mathrm{Ind}_{B_0}^G \chi_{\boldsymbol{\lambda}} \to \mathrm{Ind}_{B_0}^G \chi_{\boldsymbol{\lambda}}$ *with the intertwining property:*

$$I_\infty \circ \psi_{\boldsymbol{\lambda}}(g) = \psi_{\boldsymbol{\lambda}}(\theta(g)) \circ I_\infty, \quad \forall g \in G.$$

Proof. Consider the operator $f \mapsto {}^\theta f$, where ${}^\theta f(g) = f(\theta(g))$ for $f \in \mathbb{J}_{\boldsymbol{\lambda}}$ which has the correct intertwining property except ${}^\theta f$ lives in the induced space where we induce from the opposing Borel subgroup of all lower triangular matrices; then appeal to the fact that a parabolically induced representation depends only on the representation of the Levi subgroup and is independent of the choice of parabolic subgroup with that Levi subgroup. $\square$

The operator $I = I_\infty \otimes I_0$ defines an isomorphism on $\mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}$ such that

$$I \circ \big(\psi_{\boldsymbol{\lambda}} \otimes \rho_{\boldsymbol{\lambda}}\big)(g) = \big(\psi_{\boldsymbol{\lambda}} \otimes \rho_{\boldsymbol{\lambda}}\big)(\theta(g)) \circ I, \quad \forall g \in G.$$

Since $\theta(k) = k$ for all $k \in K$, the operator I intertwines $\psi_{\boldsymbol{\lambda}} \otimes \rho_{\boldsymbol{\lambda}}$. The map induced by θ on the cohomology groups $H^q(\mathfrak{g}, K, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) \cong \mathrm{Hom}_K(\wedge^q \mathfrak{g}/\mathfrak{k}, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$ is:

$$f \mapsto \big(\theta^q f\big)(X) := I[f\big(\theta(X)\big)] \quad \forall X \in \mathfrak{g}/\mathfrak{k}.$$

To compute $\mathrm{Lef}(\theta, SL_N(\mathbb{C}), \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}})$, we need to compute $\mathrm{Trace}(\theta^q)$.

Let π_χ be the irreducible representation of K with the highest weight χ. Let V_χ be the isotypic component for π_χ in $\mathrm{Ind}_{B_0}^G \chi_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}$, i.e., each map $\pi_\chi \to \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}$; factors through

$\pi_\chi \to V_{\pi_\chi}$. For the canonical character χ_0 of ${}^\circ T_0$, π_{χ_0} has multiplicity 1 in $\mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}$ because of Lemma 6.6.7.

Theorem 6.6.10. $\mathrm{Lef}(\theta, SL_N(\mathbb{C}), \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}) = c_{\chi_0} 2^{N-1} \neq 0$.

Proof. Since an operator I intertwines $\mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}$ and π_{χ_0} has multiplicity one, $I(V_{\chi_0}) = V_{\chi_0}$. By Schur's Lemma, there exist $c_{\chi_0} \in \mathbb{C}$ such that $I(v) = c_{\chi_0} v$ for all $v \in V_{\chi_0}$. Since I is an isomorphism, $c_{\chi_0} \neq 0$. Since $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$,

$$\mathrm{Trace}(\theta^q) = (-1)^q c_{\chi_0} \dim H^q(\mathfrak{g}, K, \mathbb{J}_{\boldsymbol{\lambda}} \otimes \mathcal{M}_{\boldsymbol{\lambda}}).$$

The theorem follows by using Theorem 6.6.1. $\square$

6.7 $(\mathfrak{g}, K)$ Cohomology and Lefschetz Number for $SL_N(\mathbb{R})$

Let $G = SL_N(\mathbb{R})$, it is a Real reductive Lie group (Section 3.7), $K = SO(N)$ be a maximal compact subgroup. Let $T_0 = \{\mathrm{diag}(x_1, \ldots, x_N) \in G : \; x_i \in \mathbb{R}^*\}$ be a diagonal torus in G. Let $\Phi = \{E_i^* - E_j^* : 1 \leq i \neq j \leq N\}$ be the root system and $\Delta = \{E_i^* - E_{i+1}^* : 1 \leq i \leq N-1\}$ be a base. Let $\mathcal{W}$ be the Weyl group isomorphic to $N_G(T_0)/T_0$.

Let $\lambda = (b_1, \ldots, b_N)$ be a dominant weight for a real Lie group $GL_N(\mathbb{R})$ (for definition see Subsection 2.2.2), i.e., $a_i := b_i - b_{i+1}$ are non-negative integers. It is a character on the diagonal torus of $GL_N(\mathbb{R})$ defined as $\mathrm{diag}(x_1 \ldots, x_N) \mapsto \prod x_i^{b_i}$. Let $(\rho_\lambda, \mathcal{M}_\lambda)$ be the finite-dimensional irreducible representation of $GL_N(\mathbb{R})$ with the highest weight λ. Let $(\rho_\lambda, \mathcal{M}_\lambda)$ also denote the restriction of a representation to $G = SL_N(\mathbb{R})$.

For any integer $l \geq 1$, let D_l be the discrete series representation of $GL_2(\mathbb{R})$ (see Chapter 2 in [8] for more details) such that

$$\mathrm{Res}_{SO(2)}^{GL_2(\mathbb{R})} D_l \cong \bigoplus_{\substack{|k| \geq l+1 \\ k \equiv l+1 (\mathrm{mod}\, 2)}} V_k,$$

where $V_k \cong \mathbb{C}$ and $\theta \in SO(2)$ acts on V_k by $e^{\iota k \theta}$.

If $H^\bullet(\mathfrak{g}, K, H_\pi \otimes \mathcal{M}_\lambda) \neq 0$ then λ satisfy purity condition (see [11] or Chapter 3 in [17]), i.e., $b_i + b_{N-i+1} = w$ constant for all i.

Let's define the cuspidal parameter for the pure weight λ (see Section 3.1.4. in [17]):

$$l_i = -a_1 - \ldots - a_{i-1} + a_i + \ldots + a_{N-1}.$$

Note that $l_i = -l_{N-i+1} = b_i - b_{N-i+1} + (N - 2i + 1)$.

Let $\underline{N} := (2, 2, \ldots, 2)$ be the partition of N if $N = 2M$ and $\underline{N} := (2, 2, \ldots, 2, 1)$ be the partition of N if $N = 2M + 1$. Let $P_{\underline{N}}, A_{\underline{N}}, M_{\underline{N}}$ and ${}^\circ M_{\underline{N}}$ are subgroups defined in Section 6.5 corresponding to the partition $\underline{N}$. Let χ_λ be a representation of ${}^\circ M_{\underline{N}}$ defined as

$$\chi_\lambda = \begin{cases} \operatorname{Res}_{{}^\circ M_{\underline{N}}} D_{l_1} \otimes D_{l_2} \otimes \ldots \otimes D_{l_M} & N = 2M, \\ \operatorname{Res}_{{}^\circ M_{\underline{N}}} D_{l_1} \otimes D_{l_2} \otimes \ldots \otimes D_{l_M} \otimes \mathbf{1}_{\{\pm 1\}} & N = 2M + 1. \end{cases}$$

Let $(\psi_\lambda, \mathbb{J}_\lambda) = \operatorname{Ind}_{P_{\underline{N}}}^G \chi_\lambda$ be a parabolically induced representation. Note that χ_λ is an irreducible if $N = 2M+1$ and χ_λ is a direct sum of two inequivalent, irreducible representations $\chi_\lambda^\pm$ if $N = 2M$. Let $\mathfrak{a}_{\underline{N}}$ be complexified Lie algebra of $A_{\underline{N}}$.

Theorem 6.7.1. *(Lemma 3.14 in [11]) If $N = 2M$ then $\mathbb{J}_\lambda$ is a direct sum of two inequivalent, admissible, unitary, irreducible representations $\mathbb{J}_\lambda^\pm = \operatorname{Ind}_{{}^\circ M_{\underline{N}}}^G \chi_\lambda^\pm$, and if $N = 2M + 1$ then $\mathbb{J}_\lambda$ is an admissible, unitary, irreducible representation. Suppose $t_N = M^2$ if $N = 2M$, and $t_N = M(M + 1)$ if $N = 2M + 1$, then*

$$H^q(\mathfrak{g}, K, \mathbb{J}_\lambda^\pm \otimes \mathcal{M}_\lambda) \cong \wedge^{(q-t_N)} \mathfrak{a}_{\underline{N}}^* \cong \begin{cases} \wedge^{q-M^2} \mathbb{C}^{M-1} & N = 2M, \\ \wedge^{q-M(M+1)} \mathbb{C}^M & N = 2M + 1. \end{cases}$$

Remark 2. *Hereafter, we will use $\mathbb{J}_\lambda$ (respectively, χ_λ) to denote either $\mathbb{J}_\lambda^+$ (respectively, χ_λ^+) when $N = 2M$, unless otherwise specified.*

Recall that $(\mathfrak{g}, K)$ cohomology groups are subquotient groups of $\operatorname{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda)$. By Proposition 9.4.3. in [37], the differential map is zero, and so

$$H^q(\mathfrak{g}, K, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) \cong \operatorname{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda). \tag{6.6}$$

Since $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, we will use notation from Section 2.7. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Since $(\mathfrak{g}, K)$ cohomology depends only on K-types appearing in $\wedge^q \mathfrak{p}$ and $\mathbb{J}_\lambda$, and $\mathrm{Res}^G_K \left(\left(\mathrm{Ind}^G_{P_{\underline{N}}} \chi_\lambda \right) \otimes \mathcal{M}_\lambda \right) \cong \mathrm{Res}^G_K \left(\mathrm{Ind}^G_{P_{\underline{N}}} (\chi_\lambda \otimes \mathcal{M}_\lambda) \right) \cong \mathrm{Ind}^K_{K \cap P_{\underline{N}}} (\chi_\lambda \otimes \mathcal{M}_\lambda)$,

$$H^q(\mathfrak{g}, K, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) \cong \mathrm{Hom}_K(\wedge^q \mathfrak{p}, \mathrm{Ind}^K_{K \cap P_{\underline{N}}} (\chi_\lambda \otimes \mathcal{M}_\lambda)).$$

By using Frobenius reciprocity,

$$H^q(\mathfrak{g}, K, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) \cong \mathrm{Hom}_{K \cap P_{\underline{N}}}(\wedge^q \mathfrak{p}, \chi_\lambda \otimes \mathcal{M}_\lambda). \tag{6.7}$$

Note that $K_{\underline{N}} := K \cap P_{\underline{N}} = K \cap \widetilde{K}_{\underline{N}}$ is a compact non-abelian subgroup unless $N = 2$, where

$$\widetilde{K}_{\underline{N}} = \begin{cases} O(2) \times O(2) \times \ldots \times O(2) & N = 2M, \\ O(2) \times O(2) \times \ldots \times O(2) \times \{\pm 1\} & N = 2M+1. \end{cases}$$

Let $K^\circ_{\underline{N}}$ be the connected component of identity in $K_{\underline{N}}$ then it is a maximal abelian normal subgroup and the complexified Lie algebra of $K^\circ_{\underline{N}}$ is $\mathfrak{t}$. Moreover, $K^\circ_{\underline{N}}$ is a maximal torus in K also. Recall from Section 2.7,

$$\mathfrak{t} = \left\{ t := \sum_{1 \le i \le M} (E_{2i-1,2i} - E_{2i,2i-1}) \otimes t_i \sqrt{-1} : \ t_i \in \mathbb{C} \right\} \subset \mathfrak{k} \subset \mathfrak{g}.$$

The decomposition of $\mathfrak{p}$ as $\mathfrak{t}$-module (or equivalently $K^\circ_{\underline{N}}$-module) in Section 2.7, is also stable by the action of $K_{\underline{N}}$. Then $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{u}$ as $K_{\underline{N}}$-module, where $\mathfrak{u}$ is a sum of non-zero eigenspaces of $K^\circ_{\underline{N}}$. Since $K_{\underline{N}}$ acts trivially on $\mathfrak{a}$,

$$H^q(\mathfrak{g}, K, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) \cong \bigoplus_{s+t=q} \mathrm{Hom}_{K_{\underline{N}}} \left(\wedge^s \mathfrak{a} \otimes_{\mathbb{C}} \wedge^t \mathfrak{u}, \chi_\lambda \otimes \mathcal{M}_\lambda \right) \tag{6.8}$$

$$\cong \bigoplus_{s+t=q} \mathrm{Hom}_{K_{\underline{N}}} \left(\wedge^t \mathfrak{u}, \chi_\lambda \otimes \mathcal{M}_\lambda \right) \otimes_{\mathbb{C}} \wedge^s \mathfrak{a}^* \tag{6.9}$$

Since $\mathfrak{a}_{\underline{N}} = \mathfrak{a}$ as a subalgebra of $\mathfrak{g}$, we can get the following lemma by computing the dimension and using Theorem 6.7.1.

Lemma 6.7.2.

$$\mathrm{Hom}_{K_{\underline{N}}} \left(\wedge^t \mathfrak{u}, \chi_\lambda \otimes \mathcal{M}_\lambda \right) \cong \begin{cases} \mathbb{C} & t = t_N, \\ 0 & t \neq t_N, \end{cases}$$

i.e., there is a unique representation of $K_{\underline{N}}$ denoted, say, the canonical representation ϕ_{χ_0}, that appears in $\wedge^t\mathfrak{u}$ and in $\chi_\lambda \otimes \mathcal{M}_\lambda|_{K_{\underline{N}}}$; this representation ϕ_{χ_0} appears in the unique degree $t = t_N$; furthermore, it appears with multiplicity one in $\wedge^{t_N}\mathfrak{u}$ and in $\chi_\lambda \otimes \mathcal{M}_\lambda|_{K_{\underline{N}}}$. Here, ϕ_{χ_0} is an irreducible representation of $K_{\underline{N}}$ with the highest weight χ_0 for a torus $K^\circ_{\underline{N}}$.

So, we will find the canonical representation ϕ_{χ_0} or equivalently the character χ_0 of $K^\circ_{\underline{N}}$.

Lemma 6.7.3. *Let π be an irreducible representation of K that appears in $\mathrm{Ind}_{K_{\underline{N}}}^K(\chi_\lambda \otimes \mathcal{M}_\lambda)$ and also in $\wedge^q\mathfrak{p}$, then $\mathrm{Spec}_{K_{\underline{N}}}(\pi) \cap \mathrm{Spec}_{K_{\underline{N}}}(\chi_\lambda \otimes \mathcal{M}_\lambda) = \{\phi_{\chi_0}\}$. Moreover,*

$$\mathrm{Hom}_K(\pi, \mathrm{Ind}_{K_{\underline{N}}}^K \phi_{\chi_0}) \cong \mathrm{Hom}_K(\pi, \mathrm{Ind}_{K_{\underline{N}}}^K(\chi_\lambda \otimes \mathcal{M}_\lambda)).$$

Proof. If $\pi = \mathbf{1}_K$ is a trivial representation of K then $\mathrm{Hom}_K(\wedge^0\mathfrak{p}, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) \neq 0$. Similarly, if the trivial representation $\mathbf{1}_{K_{\underline{N}}} \in \mathrm{Spec}_{K_{\underline{N}}}(\chi_\lambda \otimes \mathcal{M}_\lambda)$ then $\mathrm{Hom}_K(\wedge^0\mathfrak{p}, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) \neq 0$ by Frobenius reciprocity. It contradicts Theorem 6.6.1. So, let π be a non-trivial representation. Since π is an irreducible rep, it embeds in $\wedge^q\mathfrak{p}$. If $\phi \in \mathrm{Spec}_{K_{\underline{N}}}(\pi) \cap \mathrm{Spec}_{K_{\underline{N}}}(\chi_\lambda \otimes \mathcal{M}_\lambda)$, then $\phi \in \mathrm{Spec}_{K_{\underline{N}}}(\wedge^q\mathfrak{p})$. Since ϕ is non-trivial representation, $\phi \in \mathrm{Spec}_{K_{\underline{N}}}(\wedge^q\mathfrak{u})$. So, the equality holds for the intersection because of Lemma 6.7.2. The second conclusion follows by using Frobenius reciprocity and the first conclusion. $\square$

Lemma 6.7.4. *Let γ be a character of $K^\circ_{\underline{N}}$ that appears in $\wedge^q\mathfrak{p}$. Let's assume the notations in Section 2.7. Then there exist subsets A and B of $\Phi^+ := \Phi_1^+ \cup \Phi_2^+ \cup \Phi_3^+$ if $N = 2M$ (or $\Phi^+ := \Phi_1^+ \cup \Phi_2^+ \cup \Phi_3^+ \cup \Phi_4^+$ if $N = 2M+1$, respectively), with $|A| + |B| \leq q$, and written additively,*

$$\gamma = \sum_{\alpha \in A} \alpha - \sum_{\beta \in B} \beta.$$

Furthermore, for the usual partial ordering on weights and $\rho = \frac{1}{2} \sum\limits_{\alpha \in \Phi^+} \alpha$, one has

$$-2\rho \leq \gamma \leq \ 2\rho.$$

Proof. Assume $N = 2M$. Note that $\mathfrak{u} = \bigoplus_{i=1,2,3}\Big(\big(\bigoplus_{\alpha\in\Phi_i^+} \mathbb{C}\cdot X_\alpha^p\big) \oplus \big(\bigoplus_{\alpha\in\Phi_i^-} \mathbb{C}\cdot Y_\alpha^p\big)\Big)$ as $K^\circ_{\underline{N}}$-modules. Let $\{H_1, \ldots, H_{M-1}\}$ be a basis of $\mathfrak{a}$. Then the following set is a basis for $\wedge^q\mathfrak{p}$:

$$\left\{(H_{i_1} \wedge \cdots \wedge H_{i_r}) \wedge \left(\bigwedge_{\alpha\in A} X_\alpha^p\right) \wedge \left(\bigwedge_{\beta\in B} Y_\beta^p\right) : A \subset \Phi^+, B \subset \Phi^-, r = q - |A| - |B|\right\}.$$

The action of $K_{\underline{N}}^{\circ}$ is by $\gamma = \sum\limits_{\alpha\in A} \alpha - \sum\limits_{\beta\in B} \beta$ on a basis vector. So, the lemma follows in the case $N = 2M$. For $N = 2M+1$, the similar proof works. $\square$

Lemma 6.7.5. *The canonical representation π_{χ_0} has the highest weight*

$$\chi_0 = 2\rho = \sum_{\alpha\in\Phi^+} \alpha = \begin{cases} (N, N-2, \ldots, 2) & N = 2M, \\ (N, N-2, \ldots, 3) & N = 2M+1. \end{cases}$$

Proof. From Lemma 6.7.4 one sees that the character 2ρ is realized by the weight vector $\bigwedge_{\alpha\in\Phi^+} X_\alpha^p$ in $\bigwedge^{t_N}\mathfrak{p}$. If we show $2\rho \in \mathrm{Spec}_{K_{\underline{N}}^{\circ}}(\chi_\lambda \otimes \mathcal{M}_\lambda)$ then $\pi_{2\rho}$ appears in $\wedge^{t_N}\mathfrak{p}$ and $\chi_\lambda \otimes \mathcal{M}_\lambda$. So the lemma follows due to Lemma 6.7.2.

Since $\lambda = (b_1, \ldots, b_N)$ is the weight satisfying the purity, i.e., $b_i + b_{N-i+1} = w$ for all i. Since the finite-dimensional irreducible group representations of $G = SL_N(\mathbb{R})$ have bijective correspondence with the finite-dimensional irreducible Lie algebra representations of $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$, we will view $\mathcal{M}_\lambda$ as a representation of Lie algebra. If $N = 2M$ then there exists $\mathbf{w_0}$ in the Weyl group $\mathcal{W}$ such that $\mathbf{w_0}\cdot\lambda = \big((b_1, b_N), (b_2, b_{N-1}), \ldots, (b_M, b_{M+1})\big)$. Similarly, if $N = 2M+1$ then there exists $\mathbf{w_0}$ in the Weyl group $\mathcal{W}$ such that $\mathbf{w_0}\cdot\lambda = \big((b_1, b_N), (b_2, b_{N-1}), \ldots, (b_M, b_{M+2}), b_{M+1}\big)$. Since λ is the highest weight, $\mathbf{w_0}\cdot\lambda$ is an extremal weight, and so there exists a non-zero vector $v_\lambda \in \mathcal{M}_\lambda$ such that

$$\rho_\lambda(\underline{\theta})\cdot v_\lambda = \big(\prod_{i=1}^{M} e^{\iota\theta_i(b_{N-i+1}-b_i)}\big)v_\lambda \quad \forall\underline{\theta} = (\theta_1, \ldots, \theta_M) \in K_{\underline{N}}^{\circ}.$$

Since the lowest non-negative $SO(2)$ type in the discrete series representation D_l is $l+1$, there exists a vector $w_\lambda \in \chi_\lambda$ such that

$$\chi_\lambda(\underline{\theta})\cdot w_\lambda = \big(\prod_{i=1}^{M} e^{\iota\theta_i(l_i+1)}\big)w_\lambda \quad \forall\underline{\theta} = (\theta_1, \ldots, \theta_M) \in K_{\underline{N}}^{\circ}.$$

So, $(\chi_\lambda \otimes \rho_\lambda)(\underline{\theta})\cdot w_\lambda \otimes v_\lambda = 2\rho(\underline{\theta}) w_\lambda \otimes v_\lambda$ for all $\underline{\theta} \in K_{\underline{N}}^{\circ}$. So, $2\rho \in \mathrm{Spec}_{K_{\underline{N}}^{\circ}}(\chi_\lambda \otimes \mathcal{M}_\lambda)$. $\square$

Lemma 6.7.6. *The irreducible representation π_χ of K with the highest weight χ, appears in $\wedge^q\mathfrak{p}$ then $\chi = \chi_0$. Moreover, the embedding: $\pi_{\chi_0} \to \mathrm{Ind}_{K_{\underline{N}}}^{K}(\chi_\lambda \otimes \mathcal{M}_\lambda)$ induces equality, i.e.,*

$$\mathrm{Hom}_K(\wedge^q\mathfrak{p}, \pi_{\chi_0}) = \mathrm{Hom}_K(\wedge^q\mathfrak{p}, \mathrm{Ind}_{K_{\underline{N}}}^{K}(\chi_\lambda \otimes \mathcal{M}_\lambda)). \tag{6.10}$$

Proof. Recall that $K_{\underline{N}}^{\circ}$ is a maximal torus in $K = SO(N)$. Let π_χ be the representation of

K with the highest weight χ, appears in $\wedge^q\mathfrak{p}$ then $-2\rho \le \chi \le 2\rho$ because of Lemma 6.7.4. If π_χ appears in $\mathrm{Ind}_{K_{\underline{N}}}^{K}(\chi_\lambda \otimes \mathcal{M}_\lambda)$ then π_χ also appears in $\mathrm{Ind}_{K_{\underline{N}}}^{K}\phi_{\chi_0}$ due to Lemma 6.7.3. By using the Frobenius reciprocity, $\mathrm{Hom}_{K_{\underline{N}}}(\pi_\chi, \phi_{\chi_0}) \neq 0$. So, ϕ_{χ_0} appears in $\pi_\chi|_{K_{\underline{N}}}$. Since χ is the highest weight for π_χ, and $\chi_0 = 2\rho$, we have $\chi = \chi_0$. The second part of the lemma follows due to the first part and the multiplicity of π_{χ_0} in $\mathrm{Ind}_{{}^\circ T_0}^{K}\chi_0$ is equal to the dimension of $\mathrm{Hom}_K(\pi_\chi, \mathrm{Ind}_{{}^\circ T_0}^{K}\chi_0) \cong \mathrm{Hom}_{{}^\circ T_0}(\pi_{\chi_0}, \chi_0) \cong \mathbb{C}$. $\square$

Let θ be a Cartan involution on $G = SL_N(\mathbb{R})$ defined as $\theta(g) = {}^t g^{-1}$. Note that, θ also denotes the induced map on $\mathfrak{g}$. Then $\theta(X) = X$ for all $X \in \mathfrak{k}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$. We are interested to compute $\mathrm{Lef}(\theta, SL_N(\mathbb{R}), \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda)$ (defined in Section 6.3)

Lemma 6.7.7. *There exists an isomorphism $I_0 : \mathcal{M}_\lambda \to \mathcal{M}_\lambda$ with the intertwining property:*

$$I_0 \circ \rho_\lambda(g) = \rho_\lambda(\theta(g)) \circ I_0, \quad \forall g \in G = SL_N(\mathbb{R}).$$

Proof. Let $\theta'(g) = w_0\theta(g)w_0^{-1}$ be an automorphism, where $w_0 = [(-1)^{i-1}\delta_{i,N-i+1}]$ is $N \times N$ matrix. Let $v_\lambda \in \mathcal{M}_\lambda$ be the highest weight vector. Then for any $\mathrm{diag}(x_1, \ldots, x_N) \in T_0$,

$$\begin{aligned}
\rho_\lambda(\theta'(\mathrm{diag}(x_1,\ldots,x_N)))\cdot v_\lambda &= \rho_\lambda(\mathrm{diag}({x_N}^{-1},\ldots,{x_1}^{-1}))\cdot v_\lambda \\
&= \left(\prod x_{N-i+1}^{-b_i}\right) v_\lambda \\
&= \left(\prod {x_i}^{b_i - w}\right) v_\lambda \\
\rho_\lambda(\theta'(\mathrm{diag}(x_1,\ldots,x_N)))\cdot v_\lambda &= \lambda(\mathrm{diag}(x_1,\ldots,x_N)) v_\lambda
\end{aligned}$$

Since for any $u \in N$ unipotent radical of the Borel subgroup B_0, $\theta'(u) \in N$, a vector v_λ is also the highest weight vector for an irreducible representation $\rho_\lambda \circ \theta'$ with the highest weight λ. By Theorem 2.2.5, there exists $I_0' : \mathcal{M}_\lambda \to \mathcal{M}_\lambda$ such that $I_0' \circ \rho_\lambda(g) = \rho_\lambda(\theta'(g)) \circ I_0'$ for all $g \in G$. Then $I_0 := \rho_\lambda(w_0^{-1}) \circ I_0'$ is satisfies the required intertwining property. $\square$

Lemma 6.7.8. *There exists an isomorphism $I_\infty : \mathrm{Ind}_{P_{\underline{N}}}^{G}\chi_\lambda \to \mathrm{Ind}_{P_{\underline{N}}}^{G}\chi_\lambda$ with the intertwining property:*

$$I_\infty \circ \psi_\lambda(g) = \psi_\lambda(\theta(g)) \circ I_\infty, \quad \forall g \in G.$$

Proof. Consider the operator $f \mapsto {}^\theta f$, where ${}^\theta f(g) = f(\theta(g))$ for $f \in \mathbb{J}_\lambda$ which has the correct intertwining property except ${}^\theta f$ lives in the induced space where we induce from the opposing Borel subgroup of all lower triangular matrices; then appeal to the fact that a parabolically induced representation depends only on the representation of the Levi subgroup and is independent of the choice of parabolic subgroup with that Levi subgroup. □

The operator $I = I_\infty \otimes I_0$ defines an isomorphism on $\mathbb{J}_\lambda \otimes \mathcal{M}_\lambda$ such that

$$I \circ \big(\psi_\lambda \otimes \rho_\lambda\big)(g) = \big(\psi_\lambda \otimes \rho_\lambda\big)(\theta(g)) \circ I, \quad \forall g \in G.$$

Since $\theta(k) = k$ for all $k \in K$, the operator I intertwines $\psi_\lambda \otimes \rho_\lambda$. The map induced by θ on the cohomology groups $H^q(\mathfrak{g}, K, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) \cong \mathrm{Hom}_K(\wedge^q \mathfrak{g}/\mathfrak{k}, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda)$ is:

$$f \mapsto \big(\theta^q f\big)(X) := I[f\big(\theta(X)\big)] \quad \forall X \in \mathfrak{g}/\mathfrak{k}.$$

To compute $\mathrm{Lef}(\theta, SL_N(\mathbb{R}), \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda)$, we need to compute $\mathrm{Trace}(\theta^q)$.

Let π_χ be the irreducible representation of K with the highest weight χ. Let V_χ be the isotypic component for π_χ in $\mathrm{Ind}_{B_0}^G \chi_\lambda \otimes \mathcal{M}_\lambda$, i.e., each map $\pi_\chi \to \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda$; factors through $\pi_\chi \to V_{\pi_\chi}$. For the canonical character χ_0 of $K_{\underline{N}}^\circ$, π_{χ_0} has multiplicity 1 in $\mathbb{J}_\lambda \otimes \mathcal{M}_\lambda$ because of Lemma 6.7.6.

Theorem 6.7.9. $\mathrm{Lef}(\theta, SL_N(\mathbb{R}), \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda) = \begin{cases} c_{\chi_0} 2^{M-1} \neq 0 & N = 2M, \\ c_{\chi_0} 2^M \neq 0 & N = 2M+1. \end{cases}$

Proof. Since an operator I intertwines $\mathbb{J}_\lambda \otimes \mathcal{M}_\lambda$ and π_{χ_0} has multiplicity one, $I(V_{\chi_0}) = V_{\chi_0}$. By Schur's Lemma, there exist $c_{\chi_0} \in \mathbb{C}$ such that $I(v) = c_{\chi_0} v$ for all $v \in V_{\chi_0}$. Since I is an isomorphism, $c_{\chi_0} \neq 0$. Since $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$,

$$\mathrm{Trace}(\theta^q) = (-1)^q c_{\chi_0} \dim H^q(\mathfrak{g}, K, \mathbb{J}_\lambda \otimes \mathcal{M}_\lambda).$$

The theorem follows by using Theorem 6.6.1. □

Chapter 7

Review of paper by Borel-Labesse-Schwermer

This chapter serves as a brief review of the seminal work of Borel-Labesse-Schwermer in [5], which is fundamental to this thesis. They proved the following result:

Theorem 7.0.1. *Let G be an almost absolutely simple connected algebraic group over a number field k of strictly positive $k-$rank, and E be a finite-dimensional irreducible representation of the Lie group G_∞. If there exists an algebraic automorphism α of finite order on G such that E extends to a representation of $G_\infty \rtimes \langle\alpha\rangle$, and there exists an admissible irreducible unitary representation $(\pi_\infty, H_{\pi_\infty})$ of G_∞ such that $\mathrm{Lef}(\alpha, G_\infty; H_{\pi_\infty} \otimes E) \neq 0$, then cuspidal cohomology does not vanish for any set S containing all infinite places, i.e.,*

$$H^\bullet_{cusp}(G, S, E) := H^\bullet_d(G_S; L^2_{cusp}(G(k)\backslash G(\mathbb{A}_k))^\infty \otimes E) \neq 0.$$

Notations

Let k be a number field, $\mathcal{O}$ be its ring of integers, and $\mathcal{V}$ (a union of $\mathcal{V}_\infty$ and $\mathcal{V}_f$) denote the set of places (infinite and finite, respectively) of k. For a place $v \in \mathcal{V}$, the completion of k at v is denoted by k_v, and $|\cdot|_v$ denote the normalized absolute value.

Let S be a finite set of places that includes $\mathcal{V}_\infty$, and $S_\infty := \mathcal{V}_\infty$ and $S_f := S \cap \mathcal{V}_f$. For any $x \in k_S := \prod_{v\in S} k_v$, define $|x|_S := \prod_{v\in S} |x_v|_v$. Let $\mathcal{O}_S := \{x \in k : |x|_v \leq 1 \text{ for all } v \notin S_f\}$ be the subring of k consisting $S-$integers. If $S = S_\infty$ then $\mathcal{O}_S = \mathcal{O}$.

Let G be an algebraic connected reductive group over k. Let $G(k_v)$ denotes k_v-points of G for $v \in \mathcal{V}$. Then $G_S := G(k_S)$, $G_{S_f} := G(k_{S_f})$ and $G_\infty := \prod_{v \in \mathcal{V}\infty} G(k_v)$. Let $\mathfrak{g}_\infty$ be the complexified Lie algebra of G_∞, and $\mathcal{U}(\mathfrak{g}_\infty)$ denote the universal enveloping algebra of $\mathfrak{g}_\infty$. Since G is a reductive group, it is isomorphic to the almost product of its center Z and its derived group $\mathcal{D}G$.

Let K_v be a compact subgroup of G_v for all $v \in \mathcal{V}$ such that K_v is a maximal open compact subgroup for $v \in \mathcal{V}_f \backslash S_f$, K_v is an open compact subgroup for $v \in S_f$, and K_v is a maximal compact subgroup for $v \in \mathcal{V}_\infty$. Then $K_\infty := \prod_{v \in S_\infty} K_v$ is a maximal compact subgroup of G_∞, $K_f := \prod_{v \in S_f} K_v$ is an open compact subgroup of G_{S_f}, and $K := K_\infty K_f$ is a compact subgroup of G_S.

Let Γ be an S-arithmetic subgroup of $G(k)$, i.e., it is a subgroup of finite index in $G(\mathcal{O}_S)$. We can view Γ as a subgroup of G_S via the diagonal embedding. Then, $\Gamma \backslash G_S$ has a finite volume. An arithmetic subgroup is a special case when $S = S_\infty$.

Example 7.0.1. *Let $G = SL_2/\mathbb{Q}$, and $S = \{\infty, p\}$ for some prime p. Then $SL_2(\mathbb{Z})$ is an arithmetic subgroup, and $SL_2(\mathbb{Z}[1/p])$ is $S-$arithmetic subgroup of $G(\mathbb{Q})$.*

If f and g are strictly positive functions on a set X, we say f is **essentially bounded** by g if there exists a constant $c > 0$ such that $f(x) \leq cg(x)$ for all $x \in X$, denoted by $f \prec g$.

Let (ρ, V) be a faithful finite-dimensional representation of G over k. The height $|| \cdot ||$ on G_S is defined as a product of local heights, i.e., $||x|| = \prod_{v \in S} ||\rho(x_v)||$ for $x \in G_S$. For Archimedean places, $||\rho(x_v)||$ is the Hilbert-Schmidt norm, and for non-Archimedean places, $||\rho(x_v)||$ is the supremum norm.

7.1 Decomposition of function space of uniform moderate growth on $\Gamma\backslash G_S$.

7.1.1 The vector space $\mathfrak{a}_P$ and the function $\hat{\tau}_P$.

Let P be a connected parabolic subgroup defined over k, with the Levi decomposition $P = MN$. The group $X(P) := \mathrm{Hom}(P, \mathbb{G}_m)$ consists of characters of P, and $X(P)_k$ is the subgroup of characters defined over k. If P is a split group over k, then $X(P)_k = X(P)$. Let A be a maximal $k-$split component of P, then there exists a bijection between $X(P)_k$ and $X(A)$. Let

$$\mathfrak{a}_P = \mathrm{Hom}_{\mathbb{Z}}(X(P)_k, \mathbb{R})$$

and its dual

$$\mathfrak{a}_P^* = X(P)_k \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $H_P : P_S \to \mathfrak{a}_P$ be the Harish-Chandra homomorphism defined such that

$$e^{\langle \lambda, H_P(x) \rangle} = |\chi(x)|_S^r$$

for all $x \in P_S$ and $\lambda = \chi \otimes r \in \mathfrak{a}_P^*$. Let P_S^1 denote the kernel of H_P. Then there exists a section $\mathfrak{a}_P \to A(\mathbb{R})^\circ \subset P_S$. Let a_P be the composition of the section and Harish-Chandra homomorphism. Then for all $x \in P_S$ there exists unique $y \in P_S^1$ such that $x = y \cdot a_P(x)$. The Langlands decomposition of P_S is $P_S = N_S A(\mathbb{R})^\circ M_S^1$, where M_S^1 is a kernel of the Harish-Chandra homomorphism on M_S. Additionally, $G_S = P_S K$. So, we can extend a_P to G_S by $a_P(pk) = a_P(p)$.

Lemma 7.1.1. *P_S^1 contains N_S, all compact subgroups and $S-$arithmetic subgroups of P_S.*

Proof.

- Let $x \in N_S$, then $\chi(x) = 1$ for all $\chi \in X(P)_k$. So, $x \in P_S^1$.
- Let K_S be a compact subgroup of P_S, then $\chi(K_S)$ is also a compact subgroup of k_S^* for all χ. By the structure of the multiplicative group of local fields, $|\chi(K_S)|_S = 1$. Hence, $K_S \subset P_S^1$.

- Let Γ be an $S-$arithmetic subgroup of P, and χ be a character. So,

$$\chi(\Gamma) \subset \chi(P(\mathcal{O}_S)) \subset \mathbb{G}_m(\mathcal{O}_S) = \mathcal{O}_S^*.$$

For $x \in \mathcal{O}_S^*$, the absolute value $|x|_v = 1$ if $v \notin S$. Then $|x|_S = 1$ since $\prod_{v\in\mathcal{V}} |x|_v = 1$. So, $|\chi(P(\mathcal{O}_S))|_S = 1$. Hence, $H_P(P(\mathcal{O}_S)) = 0$.

$\square$

Example 7.1.1. *Let $G = GL_N(k)$. Then $X(G)_k = \{\det^n\} \cong \mathbb{Z}$. Thus, $\mathfrak{a}_G \to \mathbb{R}$ is an isomorphism defined as $\alpha \mapsto \alpha(\det)$. Similarly, $\mathfrak{a}_G^* \to \mathbb{R}$ is an isomorphism defined as $\det^n \otimes r \mapsto n \cdot r$. Then $H_G(x) : n \mapsto n \log|\det x|_S$. So, the kernel of H_G is*

$$G_S^1 = \{x \in G_S : \det(x_v) \in \mathcal{O}_v^*, \ \forall v \in S\}.$$

Example 7.1.2. *Let P be a proper parabolic subgroup of $GL_N(k)$ of type $(m_1, \ldots, m_r)$. Then $X(P)_k = \{\det^{n_1} \otimes \cdots \otimes \det^{n_r}\} \cong \mathbb{Z}^r$. Hence, $\mathfrak{a}_P \to \mathbb{R}^r$ is an isomorphism defined as $\alpha \mapsto (\alpha(e_1), \ldots, \alpha(e_r))$, where $e_1, \cdots, e_r$ are standard basis of $\mathbb{Z}^r$.. Similarly, $\mathfrak{a}_P^* \to \mathbb{R}^r$ is an isomorphism defined as $(n_1, \ldots, n_r) \otimes t \mapsto (n_1 \cdot t, \ldots, n_r \cdot t)$. The kernel of H_P is a set of all matrices such that the determinant of each diagonal block is a unit for all $v \in S$.*

Since $P \hookrightarrow G$, there is a natural map $\mathfrak{a}_P \to \mathfrak{a}_G$. Let $\mathfrak{a}_P^G$ denote the kernel of this map. Let $\Delta(P, A)$ be the set of simple roots of P with respect to A (defined in 3.6), which can be identified with the elements of $\mathfrak{a}_P^*$.

7.1.2 The decomposition theorem

A function $f \in \mathcal{C}^\infty(G_S)$ is said to be of **moderate growth** if there exists $m \in \mathbb{N}$ such that $|f(x)| \prec ||x||^m$. Recall that a function $f : G_S \to \mathbb{C}$ is smooth if there exists an open compact subgroup K_f such that $f(xk) = f(x)$ for all $k \in K_f$ and $x \in G_S$, and $f_x(y) = f(x \times y)$ are smooth functions on G_∞ for all $x \in G_{S_f}$.

A function $f \in \mathcal{C}^\infty(G_S)$ is called of **uniform moderate growth** if there exists $m \in \mathbb{N}$ such that $|Df(x)| \prec ||x||^m$ for every $D \in \mathcal{U}(\mathfrak{g})$ and $x \in G_S$. Let V_Γ denote a space consisting of smooth functions of uniformly moderate growth on $\Gamma\backslash G_S$. For an open compact subgroup $K_f \subset G_{S_f}$, let $V_\Gamma^{K_f}$ denotes K_f invariant subspace. Then V_Γ is an inductive limit of $V_\Gamma^{K_f}$.

A function $f \in \mathcal{C}^\infty(\Gamma\backslash G_S)$ is a **cusp form** if for all proper parabolic subgroup P of G and the Levi decomposition $P = MN$, the **constant term** f^P along P is $f^P(x) := \int_{\Gamma_N\backslash N_S} f(nx)dn$ vanishes for all $x \in G_S$, where dn is the normalized Haar measure on $\Gamma_N\backslash N_S$ (this means $\int_{\Gamma_N\backslash N_S} dn = 1$).

Let $\Gamma_P = \Gamma \cap P(k)$, $\Gamma_N = \Gamma \cap N(k)$, and $\Gamma_M = \Gamma_N\backslash\Gamma_P$. If f is a Γ_P-left invariant function, then define $f^P_x : m \mapsto f^P(mx)$ on M^1_S for any $x \in G_S$. Then f^P_x is Γ_M-left invariant. A function $f \in V_\Gamma$ is called **negligible along** P if f^P_x is orthogonal to the cusp forms on $\Gamma_M\backslash M^1_S$ for all $x \in G_s$. Let $\mathcal{A}$ be the set of classes of associate parabolic k-subgroups. For $\mathcal{P} \in \mathcal{A}$, denote $V_\Gamma(\mathcal{P})$ as the space of functions negligible along Q for every parabolic subgroup $Q \notin \mathcal{P}$.

Example 7.1.3. *Let $G = GL_2/\mathbb{Q}$ and $S = S_\infty$. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Let B be the Borel subgroup of G. For any cusp form f for level Γ, the constant term along B, denoted as f^B, vanishes. Hence, any cusp form is negligible along the Borel subgroup. Suppose f is an Eisenstein form for level Γ, then $f^G = f$. Since the Eisenstein forms are orthogonal to cusp forms on $GL_2/\mathbb{Q}$, Eisenstein forms are negligible along G.*

Lemma 7.1.2. *(Proposition 2.3 in [5]) If a function is negligible along all parabolic subgroups, then it is zero. If it is negligible along all proper parabolic subgroups, it is cuspidal.*

As a consequence of the lemma, we conclude that $V_\Gamma(\mathcal{A}) := \sum V_\Gamma(\mathcal{P})$ is a direct sum.

Theorem 7.1.3 (Langlands). $V_\Gamma = V_\Gamma(\mathcal{A}) = \bigoplus V_\Gamma(\mathcal{P})$.

Remark 7.1.1. *In the context of Example 7.1.3, Langlands's theorem asserts that every function with uniform moderate growth can be expressed as a linear combination of cusp forms and Eisenstein forms. Specifically, the space $V_\Gamma(G)$ corresponds to the set of cusp forms, while $V_\Gamma(\{xB_Sx^{-1} : x \in G_S\})$ consists of Eisenstein forms.*

7.2 Decomposition of Cohomology

Let E be a finite-dimensional irreducible complex representation of G_∞, which can be viewed as a G_S module by considering the trivial action of G_{S_f}.

For S-arithmetic subgroup Γ, we have $H^\bullet(\Gamma, E) = H^\bullet_d(G_S, C^\infty(\Gamma\backslash G_S) \otimes E)$.

Lemma 7.2.1. *(Section 3.4 in [3]) If $S = S_\infty$, then $H^\bullet(\Gamma, E) = H_d^\bullet(G_\infty, V_\Gamma \otimes E)$.*

Lemma 7.2.2. *The embedding $\iota : V_\Gamma \hookrightarrow C^\infty(\Gamma\backslash G_S)$ induces an isomorphism*

$$H_d^\bullet(G_S, V_\Gamma \otimes E) \to H_d^\bullet(G_S, C^\infty(\Gamma\backslash G_S) \otimes E).$$

Proof. The following proof is given by Borel, Labesse and Schwermer in [5].

Consider the spectral sequences

$$E_2^{p,q} = H_d^p(G_{S_f}, H_d^q(G_\infty, V_\Gamma \otimes E)) \text{ and } E_2'^{p,q} = H_d^p(G_{S_f}, H_d^q(G_\infty, C^\infty(\Gamma\backslash G_S) \otimes E)).$$

These sequences converge to $H_d^\bullet(G_S, V_\Gamma \otimes E)$ and $H_d^\bullet(G_S, C^\infty(\Gamma\backslash G_S) \otimes E)$, respectively. To establish the isomorphism, we need to prove that the induced map $E_2^{p,q} \to E_2'^{p,q}$ is an isomorphism of spectral sequences. Since ι is G_{S_f}-equivariant, it suffices to show the following map is an isomorphism:

$$\iota^* : H_d^q(G_\infty, V_\Gamma \otimes E) \to H_d^q(G_\infty, C^\infty(\Gamma\backslash G_S) \otimes E).$$

By restricting to the $K_\infty-$ finite vectors in V_Γ and $C^\infty(\Gamma\backslash G_S)$, we can replace differential cohomology with $(\mathfrak{g}_\infty, K_\infty)$ cohomology. Note that V_Γ and $C^\infty(\Gamma\backslash G_S)$ are inductive limits of $V_\Gamma^{K_f'}$ and $C^\infty(\Gamma\backslash G_S)^{K_f'}$ over compact open subgroups K_f' of G_{S_f}. Since $(\mathfrak{g}_\infty, K_\infty)$ cohomology commutes with inductive limits (refer Lemma 6.2.4), it suffices to prove that the following maps are isomorphisms:

$$H^\bullet(\mathfrak{g}_\infty, K_\infty, V_\Gamma^{K_f'} \otimes E) \to H^\bullet(\mathfrak{g}_\infty, K_\infty, C^\infty(\Gamma\backslash G_S/K_f') \otimes E).$$

Since $\Gamma\backslash G_S/K_f'$ is a finite disjoint union of $\Gamma_c\backslash G_\infty$, and $V_\Gamma^{K_f'}$ is a subspace of $C^\infty(\Gamma\backslash G_S/K_f')$, we have $V_\Gamma^{K_f'} = \oplus V_{\Gamma_c}$, where $V_{\Gamma_c} = C_{\text{umg}}^\infty(\Gamma_c\backslash G_\infty)$. So, it reduces to prove the lemma for $S = S_\infty$. So, the lemma follows by using Lemma 7.2.1. $\square$

For any $S_\infty \subset S$, the cohomology of Γ with coefficients in a finite-dimensional irreducible complex rational representation E of G_∞ can be decomposed as

$$H^\bullet(\Gamma, E) \cong \bigoplus_{\mathcal{P}} H_{\mathcal{P}}^\bullet(\Gamma, E),$$

where $H^\bullet_{\mathcal{P}}(\Gamma, E) \cong H^\bullet_d(G_S, V_\Gamma(\mathcal{P}) \otimes E)$.

Corollary 7.2.3. *The cuspidal cohomology of* Γ*, denoted as* $H^\bullet_{\text{cusp}}(\Gamma, E)$*, is defined as* $H^\bullet_d(G_S, V_\Gamma(\{G\}) \otimes E)$*, and it naturally embeds into* $H^\bullet(\Gamma, E)$*.*

Remark 7.2.1. *(Remark in [5]) Note that* $V_\Gamma(\{G\}) = L^2_{\text{cusp}}(\Gamma\backslash G_S)^\infty$*. So, the cuspidal cohomology* $H^\bullet_{\text{cusp}}(\Gamma, E) = H^\bullet_d(G_S, L^2_{\text{cusp}}(\Gamma\backslash G_S)^\infty \otimes E)$*.*

7.3 Cohomology with Coefficient in Discrete Spectrum

Let (π, H_π) be an admissible irreducible unitary representation of G_S then

$$(\pi, H_\pi) = (\pi_\infty, H_{\pi_\infty}) \otimes (\pi_f, H_{\pi_f}),$$

where $(\pi_\infty, H_{\pi_\infty})$ and (π_f, H_{π_f}) are representations of G_∞ and G_{S_f}, respectively.

Now onwards, we assume that G is semisimple, almost absolutely simple over k of strictly positive $k-$rank. Let $\widetilde{G}$ be the universal covering of G and $\tau : \widetilde{G} \to G$ the canonical isogeny. It induces a morphism $\tau_S : \widetilde{G}_S \to G_S$ with finite kernel and cokernel.

Lemma 7.3.1. *Let* (π, H_π) *be an irreducible admissible unitary representation of* G_S *occuring in* $L^2_{\text{disc}}(\Gamma\backslash G_S)$*. Suppose* π *has non-compact kernel. Then* π *is the trivial representation if either* G *is simply connected or* $H^\bullet_d(G_S; H^\infty_\pi \otimes E) \neq 0$*.*

Proof. The following proof is given by Borel, Labesse and Schwermer in [5].
Since G is almost absolutely simple over k, $G(k_v)$ is simple modulo its center. Then there exists $v_0 \in S$ such that $G(k_{v_0}) \subset \ker(\pi)$, since π has a non-compact kernel.

If we assume $G = \widetilde{G}$, then $G(k)$ is dense in $G(\mathbb{A}^S_k)$, where $\mathbb{A}^S_k$ be the restricted direct product $\prod'_{v\notin S} k_v$ with respect to $\mathcal{O}_v$ (Theorem 2.3 in [31]). Then $\Gamma \cdot G(k_{v_0})$ is dense in G_S. Since H^∞_π can be realized as a space of smooth functions on $\Gamma\backslash G_S/G(k_{v_0})$, π is trivial representation.

Now, we drop the assumption $G = \widetilde{G}$. Let $G' := \tau_S(\widetilde{G}_S)$ be the normal subgroup of finite index in G_S. Then H_π is the direct sum of finitely many irreducible $G'-$modules (π_i, H_{π_i}), which are permuted transitively by G_S. So, the kernel of H_i-modules is isomorphic, i.e., either all compact or all non-compact. Since G' is an open subgroup of finite index in G_S, $G' \cap \ker(\pi)$ is not compact. Hence, $\ker(\pi_i)$ are non-compact. So, (π_i, H_{π_i}) are trivial representations for all i. Consequently, (π, H_π) is an irreducible finite-dimensional representation of G_S. Since the cohomology groups are non-trivial, and $(\pi, H_\pi) = \otimes_{v \in S}(\pi_v, H_{\pi_v})$, we have

$$H_d^\bullet(G(k_v); H_{\pi_v} \otimes E_v) \neq 0 \text{ for all } v \in S_\infty \text{ and } H_d^\bullet(G(k_v); H_{\pi_s}) \neq 0 \text{ for all } v \in S_f.$$

If $v \in S_f$, then H_{π_v} is trivial. Now let $v \in S_\infty$, then $H_{\pi_v} \otimes E_v$ contain the trivial representation since differential cohomology is isomorphic to relative Lie algebra cohomology. By using Theorem 2.3.4, E_v is the contragradient representation to H_{π_v}. So, E_v also has a kernel of finite index, and so E_v is trivial. Hence, π_v is also trivial. □

Let $L^2(\Gamma\backslash G_S)$ be the space of square-integrable measurable functions on $\Gamma\backslash G_S$. Note that G_S acts on $L^2(\Gamma\backslash G_S)$ by right translation. Let $L^2_{\text{disc}}(\Gamma\backslash G_S)$ be the direct sum of G_S invariant subspaces of $L^2(\Gamma\backslash G_S)$. Let $L^2_{\text{cusp}}(\Gamma\backslash G_S)$ be the subspace consisting of the cuspidal functions on $\Gamma\backslash G_S$.

Let $\mathbf{St}$ denote the Steinberg representation of G_{S_f}. Then $\mathbf{St} \cong \otimes_{v \in S_f} St(G(k_v))$. Let $\hat{G}_{\text{cusp}}(S, \Gamma, \mathbf{St})$ (respectively $\hat{G}_{\text{disc}}(S, \Gamma, \mathbf{St})$) are the sets of equivalence classes of irreducible unitary admissible representations $\pi = \pi_\infty \otimes \pi_f$ of G_S that occur in the cuspidal (respectively discrete) spectrum in $L^2(\Gamma\backslash G_S)$ such that $\pi_f = \mathbf{St}$.

Remark 3. *(Refer [12] by **Clozel**) Let $\pi = \otimes \pi_v$ be an irreducible unitary admissible representation that occurs discretely in the L^2 spectrum. If π_v is tempered for some v, then π occurs with full multiplicity in the cuspidal spectrum.*

Note that the Steinberg representation of $G(k_v)$ is a tempered representation for $v \in S_f$. Therefore, if $S_f \neq \emptyset$, then $\hat{G}_{\text{cusp}}(S, \Gamma, \mathbf{St}) = \hat{G}_{\text{disc}}(S, \Gamma, \mathbf{St})$.

Theorem 7.3.2. *Let r_f be the sum of semisimple k_v ranks of G for $v \in S_f$. Let $I_\pi \otimes H_f$ denote the isotypic subspace of (π, H_π) in $L^2_{\text{cusp}}(\Gamma\backslash G_S)$. Then,*

$$H^q_{\text{cusp}}(\Gamma, E) \cong \bigoplus_{\pi \in \hat{G}_{\text{cusp}}(S, \Gamma, St)} H_d^{q - r_f}(G_\infty, I_\pi^\infty \otimes E).$$

If $S_f \neq \emptyset$, then $H^\bullet_{\text{disc}}(\Gamma, E) = H^\bullet_{\text{cusp}}(\Gamma, E) \oplus H^\bullet_d(G_\infty, E)$.

Proof. We assume that $H_d^\bullet(G_S, L^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E)$ and $H_d^\bullet(G_S, L^2_{\text{cusp}}(\Gamma\backslash G_S)^\infty \otimes E)$ are finite dimensional. By the Kunneth Rule,

$$H_d^q(G_S, H_\pi^\infty \otimes E) \cong \bigoplus_{m+n=q} H_d^m(G_\infty, H_\infty^\infty \otimes E) \otimes H_d^n(G_{S_f}, H_{\pi_f}^{\pi_\infty}).$$

If π is trivial, then $H_d^q(G_S, H_\pi^\infty \otimes E) \cong H_d^q(G_\infty, E)$.
If π is non-trivial and has non-vanishing cohomology, then π_f is isomorphic to **St**, since lemma 7.3.1. Moreover,

$$H_d^q(G_S, H_\pi^\infty \otimes E) \cong H_d^{q-r_f}(G_\infty, H_{\pi_\infty}^\infty \otimes E) \otimes H_d^{r_f}(G_{S_f}, St).$$

□

7.4 Lefschetz Functions for an Automorphism

Let α be an automorphism of G of finite order l. Let $\tilde{L} := G \rtimes \langle\alpha\rangle$ be the semi-direct product. Let L be the coset of α in $\tilde{L}$. For a $k-Algebra$ A, let $L(A)^+$ be the subgroup generated by $L(A)$.

Let F be a completion of k, and let E be a finite-dimensional representation (trivial if F is a non-Archimedean completion). Let (π, H_π) be an irreducible admissible representation of $L(F)^+$. For $\beta \in L(F)$, let β also denote the action $\pi(\beta)$ on $H_\pi \otimes E$. It induces isomorphisms of the cohomology groups $H_d^\bullet(G(F), H_\pi \otimes E)$. Recall the Lefschetz number (see Section 6.3) of β with respect to $H_\pi \otimes E$ is given by:

$$\mathrm{Lef}(\beta, G(F), H_\pi \otimes E) = \sum (-1)^n \,\mathrm{trace}(\beta^n : H_d^n(G(F), H_\pi \otimes E)).$$

Note that $H_d^\bullet(G(F), H_\pi \otimes E)$ is a subquotient of $\{x \in I^\bullet : g \cdot x = x \;\forall g \in G(F)\}$, where $0 \to H_\pi \otimes E \to I^\bullet$ is an injective smooth resolution. Thus, $G(F)$ acts trivially on cohomology groups. Let $\beta' \in L(F)$, then $\beta' = \beta x_0$ for some $x_0 \in G(F)$, then the induced maps on cohomology groups by β and β' agree. Consequently, the Lefschetz number is independent of elements in $L(F)$, so we can denote it by $\mathrm{Lef}(\alpha, G(F), H_\pi \otimes E)$. (For more detail, see Section 6.3)

Let P_0 be a minimal parabolic subgroup of G defined over F with Levi decomposition $P_0 = M_0N_0$. Choose $\beta_0 \in L(F)$ such that $\beta_0 P_0 \beta_0^{-1} = P_0$ and $\beta_0 M_0 \beta_0^{-1} = M_0$. Let P be a β_0-stable parabolic F-subgroup of G with β_0-stable Levi decomposition $P = MN$. Then, $P(F)^+$ and $M(F)^+$ are subgroups of $L(F)^+$ generated by β_0 with $P(F)$ and $M(F)$, respectively.

Let K be an α-stable maximal compact subgroup of $G(F)$. Consider $f \in C_c^\infty(G(F))$ such that it is K-finite. Define f^* on $L(F)^+$ as $x \rtimes \alpha^r \mapsto f(x)$. We say f^* is **cuspidal** if trace $\pi(f^*) = 0$ for all parabolically induced representations π on $L(F)^+$ from an irreducible representation τ on $M(F)^+$, where $P = MN$ is a β_0-stable proper parabolic subgroup.

We say f^* is **very cuspidal** if for any β_0-stable proper parabolic subgroup $P = MN$, and for all $m \in M(F)\beta_0$,

$$f_P^*(m) := \delta_P(m)^{1/2} \int_K \int_{N(F)} f^*(k^{-1}nmk)dndk = 0.$$

Lemma 7.4.1. *If f is very cuspidal then f is cuspidal.*

Proof.

$$\begin{aligned}
[\pi(f^*)\phi](y) &= \int_{L(F)^+} f^*(y^{-1}g)\phi(g)dg \\
&= \int_{K=P(F)^+\backslash L(F)^+} \int_{P(F)^+} f^*(y^{-1}pg)\delta_P(p)^{1/2}\tau(p)\phi(g)dpdg. \\
&= \int_{M(F)^+} \int_K \int_{N(F)} \delta_P(m)^{1/2} f^*(y^{-1}mng)\tau(m)\phi(g)dndgdm.
\end{aligned}$$

$$\begin{aligned}
\text{So, trace } \pi(f^*) &= \int_{M(F)^+} \int_K \int_{N(F)} \delta_P(m)^{1/2} f^*(g^{-1}mng)\tau(m)dndgdm \\
&= \int_{M(F)^+} f_P^*(m)\tau(m)dm.
\end{aligned}$$

□

Lemma 7.4.2. *(Proposition 8.4 in [5]) There exists K-finite, cuspidal and compactly sup-*

ported smooth function $f_{\alpha,E}$ on $G(F)$ such that for any admissible representation π of $L(F)^+$,

$$\text{trace}\ \ \pi(f^*_{\alpha,E}) = \text{Lef}(\alpha, G(F), H_\pi \otimes E).$$

Moreover, if F is an Archimedean, we can choose it very cuspidal.

Such functions are called **Lefschetz functions**. Let $\pi_S - \otimes_{v\in S}\pi_v$ be a representation of $L_S^+ = G_S \rtimes \langle\alpha\rangle$ and $f_{\alpha,E} = \prod f_{\alpha,E_v}$. Then

$$\begin{aligned}
\text{trace}\ \ \pi_S(f^*_{\alpha,E}) &= \prod_{v\in S} \text{trace}\ \ \pi_v(f^*_{\alpha,E_v}) \\
&= \prod \text{Lef}(\alpha, G(k_v), H_{\pi_v} \otimes E_v) \\
&= \prod \sum (-1)^n\, \text{trace}(\alpha | H^n_d(G(k_v), H_{\pi_v} \otimes E_v)) \\
&= \sum (-1)^n\, \text{trace}(\alpha | H^n_d(G_S, H_\pi \otimes E)) \\
\text{trace}\ \ \pi_S(f^*_{\alpha,E}) &= \text{Lef}(\alpha, G_S, H_\pi \otimes E).
\end{aligned}$$

7.5 Extension of Representations

Consider a locally compact topological group H^+ with a closed normal subgroup H such that the quotient group H^+/H is a compact abelian group. We define a representation of H^+ to be 'good' if it satisfies Schur's Lemma, meaning that any operator commuting with the image of H^+ is a scalar.

Let π be a good irreducible representation of H^+. Denote by $X(\pi)$ the group of characters χ of H^+/H such that $\chi \otimes \pi \cong \pi$. Let $\sigma = \text{Res}_H^{H^+}\pi$ be a restriction to H.

Lemma 7.5.1. *Let $U \in \text{Hom}_H(\sigma, \sigma)$. Define $U_\chi = \int_{H^+/H} \chi(x)\pi(x)U\pi(x)^{-1}\, dx$. This operator serves as an intertwining operator between π and $\pi \otimes \chi$.*

Proof. For any $g \in H^+$, consider $U_\chi \circ \pi(g) = \int_{H^+/H} \chi(x)\pi(x)U\pi(g^{-1}x)^{-1}\, dx$. By changing the variable to $y = g^{-1}x$, we obtain $U_\chi \circ \pi(g) = \int_{H^+/H} \chi(gy)\pi(gy)U\pi(y)^{-1}\, dy$. This expression verifies that U_χ acts as an intertwining operator. $\square$

Lemma 7.5.2. *Let π be an irreducible representation of H^+. Then*

- *The representation σ is reducible if and only if there exists a non-trivial character χ of H^+/H such that $\pi \cong \pi \otimes \chi$.*
- *If $X(\pi)$ is finite. The algebra $I(\sigma) = \mathrm{Hom}_H(\sigma, \sigma)$ is a finite-dimensional and $\dim I(\sigma) = |X(\pi)|$. Moreover, σ is a finite direct sum of irreducible representation.*
- *If $X(\pi)$ is a cyclic of order l then $I(\sigma)$ is commutative. Moreover, σ is a direct sum of l inequivalent irreducible representations.*

Proof. If $X(\pi)$ is a finite group of order l then we can choose I_χ such that $I_\chi^l = id$. Then by Fourier Inversion, $U = \sum c_\chi I_\chi$. So, the dimension of $I(\sigma) = |X(\pi)|$.

If $X(\pi)$ is cyclic then clearly $I(\sigma)$ is commutative. If σ is a direct sum of two equivalent irreducible representations then $I(\sigma)$ can't be commutative. $\square$

Let H^+/H be a cyclic group of order l. Consider an element $\alpha \in H^+$ such that its image in the quotient group generates H^+/H. Consequently, $\alpha^l \in H$. Define σ as a good representation of H. The transformation σ^α is given by $x \mapsto \sigma(\alpha x \alpha^{-1})$.

We say σ is α-invariant if σ and σ^α are equivalent. In the case of α-invariance, there exists an operator A such that $\sigma^\alpha(x) = A\sigma(x)A^{-1}$. Consequently, $\sigma^{\alpha^l}(x) = \sigma(\alpha^l)\sigma(x)\sigma(\alpha^{-l}) = A^l\sigma(x)A^{-l}$. Therefore, $\sigma(\alpha^{-l})A^l$ is a scalar, and we can choose A such that $\sigma(\alpha^l) = A^l$. Define π on H^+ by $\pi(x\alpha^i) = \sigma(x)A^i$. Clearly, π is an irreducible representation. Since σ is irreducible, $X(\pi)$ is trivial.

We say that π exists as a distribution if $\phi \mapsto \mathrm{tr}\ \pi(\phi)$ is a continuous linear form on $C_c^\infty(H^+)$.

Corollary 7.5.3. *If an irreducible π exists as a distribution such that σ is reducible to H. Then tr $\pi(\phi)$ vanishes on the subspace $C_c^\infty(\alpha H)$.*

Proof. Since $X(\pi)$ is non-trivial, there exists non-trivial χ such that $\chi \otimes \pi \cong \pi$.

$$\mathrm{tr}\ \ \pi(\phi) = \mathrm{tr}\ \ \chi \otimes \pi(\phi) = \chi(\alpha)\ \ \mathrm{tr}\ \ \pi(\phi).$$

□

7.6 Trace Formulas

Let $\mathbb{A}_k$ be the ring of adèles, and $\mathbb{A}_k^S$ be the restricted direct product $\prod'_{v \notin S} k_v$ with respect to $\mathcal{O}_v$. Then $L(\mathbb{A}_k)^+ = G(\mathbb{A}_k) \rtimes \langle\alpha\rangle$, $L_S^+ = G(k_S) \rtimes \langle\alpha\rangle$, and $L(\mathbb{A}_k^S)^+ = G(\mathbb{A}_k^S) \rtimes \langle\alpha\rangle$. Let ρ be the representation of $L(\mathbb{A}_k)^+$ in $L^2(G(k)\backslash G(\mathbb{A}_k))$ given by $\rho(x \rtimes \alpha^r)f(y) = f(\alpha^{-r}(yx))$. Let ρ_{disc} and ρ_{cusp} be the restrictions of this representation on $L^2_{\text{disc}}(G(k)\backslash G(\mathbb{A}_k))$ and $L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A}_k))$, respectively. For any $f \in C_c^\infty(L(\mathbb{A}_k)^+)$, the trace of $\rho_{\text{disc}}(f)$ is given by

$$\text{trace } \ \rho_{\text{disc}}(f) = \sum m_{disc}(\pi) \ \text{ trace } \ \pi(f).$$

Lemma 7.6.1. *(Theorem 7.1 in [1]) Let* $f = \prod f_v = f_{\alpha,E} \times h$, *where* $f_{\alpha,E}$ *is Lefschetz function for* S *and* $h \in C_c^\infty(\mathbb{A}_k^S)$. *Further, assume there exists* $w \notin S$ *such that* $\operatorname{supp}(f_w)$ *lies in the set of regular elements. Then*

$$\text{trace } \ \rho_{\text{disc}}(f) = \sum a_L(\gamma) J_L(\gamma, f),$$

where summation runs through regular elliptic $G(k)$ *conjugacy classes of* $L(k)^+$, $a_L(\gamma)$ *is measure of* $G_\gamma(k)\backslash G_\gamma(\mathbb{A}_k)$ *divided by index of* $G_\gamma(k)$ *in* $G(k)_\gamma$ *and* $J_L(\gamma, f)$ *is the orbital integral* $\int_{G_\gamma(\mathbb{A}_k)\backslash G(\mathbb{A}_k)} f(x^{-1}\gamma x)dx$.

According to the definition of the Lefschetz number,

$$\text{Lef}_{\text{disc}}(\alpha, h, G, E) = \sum (-1)^n \ \text{ trace } \ (\alpha \times h | H_d^n(G_S; L^2_{\text{disc}}(G(k)\backslash G(\mathbb{A}_k)) \otimes E)).$$

Since L^2_{disc} can be expressed as a sum of irreducible representations, the cohomology also admits a similar decomposition. By using Corollary 7.5.3, the trace of an irreducible representation π of $L(\mathbb{A}_k)^+$ is non-trivial if its restriction, denoted by σ, to $G(\mathbb{A}_k)$ remains irreducible. For such π, we can express $\sigma = \sigma_S \otimes \sigma^S$, where σ_S and σ^S are irreducible representations of G_S and $G(\mathbb{A}_k^S)$. Let π_S and π^S be the extended representations (as described in Section 7.5) of L_S^+ and $L(\mathbb{A}_k^S)^+$, respectively. Then, π is a restriction of $\pi_S \otimes \pi^S$ from

$L_S^+ \times L(\mathbb{A}_k^S)^+$ to $L(\mathbb{A}_k)^+$. Hence,

$$\begin{aligned}
\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G, E) &= \sum m_{\mathrm{disc}}(\pi) \sum (-1)^n \ \text{ trace } \ (\alpha \times h | H_d^n(G_S; H_\pi \otimes E)) \\
&= \sum m_{\mathrm{disc}}(\pi) \, \mathrm{Lef}(\alpha, G_S, \pi_S \otimes E) \ \text{ trace } \ \pi^S(h^*) \\
&= \sum m_{\mathrm{disc}}(\pi) \ \text{ trace } \ \pi_S(f^*_{\alpha,E}) \ \text{ trace } \ \pi^S(h^*) \\
\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G, E) &= \mathrm{trace}(\rho(f^*_{\alpha,E} \times h^*) | \ L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}_k)) \otimes E).
\end{aligned}$$

Therefore, by using Lemma 7.6.1, the discrete Lefschetz number is given by

$$\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G, E) = \sum_{\substack{\text{regular elliptic} \\ (L(k)^+)_{G(k)}}} a_L(\gamma) J_L(\gamma, f^*_{\alpha,E} \times h^*);$$

where $J_L(\gamma, f^*_{\alpha,E} \times h^*) = \int_{G_\gamma(k_S)\backslash G(k_S)} f^*_{\alpha,E}(x^{-1}\gamma x) dx \ \int_{G_\gamma(\mathbb{A}_k^S)\backslash G(\mathbb{A}_k^S)} h^*(x^{-1}\gamma x) dx$. Similarly, the cuspidal Lefschetz number is

$$\mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, G, E) = \mathrm{trace}(\rho(f^*_{\alpha,E} \times h^*) | \ L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k)) \otimes E).$$

7.7 Non-vanishing of Cuspidal Cohomology

We will say that the cuspidal cohomology of G over S with coefficients in E does not vanish if, for every $S-$arithmetic subgroup $\Gamma \subset G(k)$, there exists a finite-indexed subgroup $\Gamma_1 \subset \Gamma$ exhibiting non-trivial cuspidal cohomology with coefficients in E. Consequently, we define the cuspidal cohomology of G over S with coefficients in E as the inductive limit of the cuspidal cohomology of $S-$arithmetic subgroups of $G(k)$ with coefficients in E, i.e.,

$$H^\bullet_{\mathrm{cusp}}(G, S, E) := \varinjlim H^\bullet_{\mathrm{cusp}}(\Gamma, E) \cong H^\bullet_d(G_S, L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k))^\infty \otimes E).$$

The isomorphism follows from Remark 7.2.1.

7.7.1 Proof of Theorem 7.0.1

Proof. The following proof is by Borel, Labesse, and Schwermer in [5].

In the case where $S \subset T$ and includes all Archimedean places, the first part of Theorem 7.3.2 implies

$$H_d^{\bullet}(G_T, L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k))^{\infty} \otimes E) \hookrightarrow H_d^{\bullet}(G_S, L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k))^{\infty} \otimes E).$$

Now, let us consider the scenario where S contains at least one finite place. By utilizing Theorem 7.3.2, we have the relationship

$$\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G, E) = \mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, G, E) + \mathbf{1}(f_{\alpha,E} \times h),$$

for any $h \in C_c^{\infty}(G(\mathbb{A}_k^S))$, and $\mathbf{1}(f_{\alpha,E} \times h) = \mathrm{tr}\,\pi(f_{\alpha,E} \times h)$ where π is the trivial representation. To establish the result, we need to prove that the cuspidal Lefschetz number is non-zero for some $h \in C_c^{\infty}(G(\mathbb{A}_k^S))$. Given that $\mathrm{Lef}(\alpha, G_\infty, H_\infty \otimes E) \neq 0$ and the Lefschetz number $\mathrm{Lef}(\alpha, G(k_v); H_{\pi_v})$ is not identically zero for $v \in S_f$, there exists a unitary and irreducible representation (π_S, H_{π_S}) of G_S such that

$$\text{trace } \ \pi_S(f^*_{\alpha,E}) = \mathrm{Lef}(\alpha, G_S, \pi_S \otimes E) \neq 0.$$

By the trace formula, there exists a regular elliptic element $x_0 \in L(k)^+$ such that the orbital integral $J_L(x_0, f^*_{\alpha,E}) \neq 0$. By choosing a small enough support for h, we have

$$\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G, E) = \mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, G, E) + \mathbf{1}(f_{\alpha,E} \times h) = a_L(x_0) J_L(x_0, f^*_{\alpha,E}) J_L(x_0, h^*).$$

Since x_0 is a regular element and the rank of G is non-zero, the dimension of $G_{x_0}(k_v)\backslash G(k_v)$ is strictly lesser than the dimension of $G(k_v)$ for any finite place v. Choosing h_v for all $v \notin S$ except one place $w \notin S$, we fix h_w as the characteristic function of a decreasing sequence of open neighborhoods of x_0 such that

$$\begin{aligned} J_L(x_0, h^*) &= c_1 \int_{G_{x_0}(k_w)\backslash G(k_w)} h^*_w(x^{-1} x_0 x)\, dx \equiv 1, \\ \mathbf{1}(f_{\alpha,E} \times h) &= c_2 \int_{G(k_w)} h_w(x)\, dx \longmapsto 0 \text{ as the neighborhood decreases.} \end{aligned}$$

Since $a_L(x_0) \neq 0$, the cuspidal Lefschetz number is non-zero. Hence, the theorem follows.

$\square$

We say G admits Cartan-type automorphism if there exists an algebraic automorphism α and $x \in G_\infty$ such that $\alpha \circ \mathrm{Ad}(x)$ is a Cartan involution on G_∞ (defined in Section 3.7).

Theorem 7.7.1. *If G admits Cartan-type automorphism over k then $H^\bullet_{\mathrm{cusp}}(G, S, \mathbb{C}) \neq 0$.*

Proof. As the involution θ acts by -1 on $\mathfrak{g}/\mathfrak{k}$, the Lefschetz number of θ in the $(\mathfrak{g}, K)$ cohomology of the trivial representation does not vanish. Since $\mathrm{Lef}(\alpha, G, \mathbb{C}) = \mathrm{Lef}(\theta, G, \mathbb{C}) \neq 0$, the theorem follows. $\square$

Example 7.7.1. *Consider the algebraic group $G = SL_N/\mathbb{Q}$ or $G = \mathrm{Res}_{F/\mathbb{Q}}(SL_N/F)$ where F is an imaginary quadratic extension of $\mathbb{Q}$. Let α be an automorphism of G defined as $g \mapsto {}^t\overline{g}^{-1}$. The automorphism α is a Cartan involution since Examples 3.7.1 and 3.7.2. Hence, Theorem 7.7.1 implies $H^\bullet_{\mathrm{cusp}}(G, S, \mathbb{C}) \neq 0$.*

Chapter 8

Cuspidal Cohomology

Let F be a number field, and $\Sigma_F = \mathrm{Hom}(F, \mathbb{C})$ be the set of embeddings into $\mathbb{C}$. Then $\Sigma_F = \Sigma_r \cup \Sigma_c$, where Σ_r and Σ_c are the sets of real and complex embeddings, respectively. Let S_r and S_c denote the sets of real and complex places, respectively. There is a surjective map $\Sigma_F \to S$, defined as $\tau_v \mapsto v$ if v is real, and $\{\tau_v, \overline{\tau}_v\} \mapsto v$ if v is a complex place.

Since the composite of totally real subfields is also a totally real subfield, let F_0 be the maximal totally real subfield of F. Let F_1 denote a totally imaginary quadratic extension of F_0 in F. Note that F_1 may not exist, but if it does, it is unique. If F_1 does not exist, set $F_1 := F_0$.

8.1 Algebraic Group and Character group of Torus

Let $G_0 = GL_N/F$ be the algebraic group and $F[G_0]$ be the coordinate ring. Let $G = \mathrm{Res}_{F/\mathbb{Q}} G_0$ be the Weil restriction of scalars to $\mathbb{Q}$, and let $G \times \mathbb{C}$ be the base change of the algebraic group to $\mathbb{C}$. Since G_0 is defined over $\mathbb{Q}$, for any $A \in \mathbb{C}$-Alg, we have

$$\begin{aligned} G \times \mathbb{C}(A) &= G_0(F \otimes A) \\ &= \mathrm{Hom}_F(F \otimes_{\mathbb{Q}} \mathbb{Q}[GL_N/\mathbb{Q}], F \otimes_{\mathbb{Q}} A) \\ &= \prod_{F \to \mathbb{C}} \mathrm{Hom}_F(F \otimes_{\mathbb{Q}} \mathbb{Q}[GL_N/\mathbb{Q}], A) = \prod_{F \to \mathbb{C}} G_0(A). \end{aligned}$$

Let Z_0 be the center of G_0, B_0 be the Borel subgroup consisting of upper triangular matrices in G_0, and $T_0 \subset B_0$ be the diagonal torus in G_0. Let Z, B, and T denote the restrictions of scalars of Z_0, B_0 and T_0 to $\mathbb{Q}$, respectively. Also, let $Z \times \mathbb{C}$, $B \times \mathbb{C}$, and $T \times \mathbb{C}$ be the base changes of algebraic groups Z, B, and T to $\mathbb{C}$, respectively.

Let $X^*(T \times \mathbb{C})$ be the character group. Then

$$X^*(T \times \mathbb{C}) = \bigoplus_{\tau : F \to \mathbb{C}} X^*(T_0).$$

If $\lambda \in X^*(T \times \mathbb{C})$, then $\lambda = (\lambda^\tau)_{\tau : F \to \mathbb{C}}$, where $\lambda^\tau = \sum_{i=1}^{N-1} (a_i^\tau - 1) \cdot \gamma_i + d^\tau \cdot \delta_N = (b_1^\tau, \ldots, b_N^\tau)$. Here, γ_i are normalized fundamental weights on SL_N extended to GL_N such that they are trivial on the center, and δ_N is the determinant character of GL_N. Then $Nd^\tau = \sum b_i^\tau$ and $b_i^\tau - b_{i+1}^\tau = a_i^\tau$ for all $1 \leq i \leq N-1$.

Define $X^*_{\mathbb{Q}}(T \times \mathbb{C}) = X^*(T \times \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$. A weight $\lambda \in X^*_{\mathbb{Q}}(T \times \mathbb{C})$ is called an integral weight if one of the following holds:

$$b_i^\tau \in \mathbb{Z} \Leftrightarrow \begin{cases} a_i^\tau \in \mathbb{Z}, \\ Nd^\tau \in \mathbb{Z}, \\ Nd^\tau \equiv \sum_{i=1}^{N-1} i(a_i^\tau - 1) \pmod{N}. \end{cases}$$

An integral weight λ is dominant for the Borel subgroup B_0 if $b_1^\tau \geq b_2^\tau \geq \ldots \geq b_N^\tau$ or $a_i^\tau \geq 1$. The collection of dominant integral weights is denoted by $X^+(T \times \mathbb{C})$.

8.2 Adèlic Locally Symmetric Space

The group $G(\mathbb{R})$ is isomorphic to $\prod_{S_r} GL_N(\mathbb{R}) \times \prod_{S_c} GL_N(\mathbb{C})$, and similarly, $Z(\mathbb{R})$ is isomorphic to $\prod_{S_r} \mathbb{R}^\times I_N \times \prod_{S_c} \mathbb{C}^\times I_N$, where I_N denotes the $N \times N$ identity matrix. Then $C_\infty = \prod_{S_r} O(N) \times \prod_{S_c} U(N)$ is the standard maximal compact subgroup. Let $K_\infty = C_\infty Z(\mathbb{R})$ then $S^G := G(\mathbb{R})/K_\infty^\circ$ is a symmetric space.

Let K_f be any open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},f})$ then the adèlic symmetric space is defined as follows:

$$G(\mathbb{A}_{\mathbb{Q}})/K_\infty^\circ K_f \cong S^G \times G(\mathbb{A}_{\mathbb{Q},f})/K_f.$$

Since $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A}_\mathbb{Q})$, it acts properly discontinuously $G(\mathbb{A}_\mathbb{Q})/K_\infty^\circ K_f$. The adèlic locally symmetric space for G of level K_f is denoted by

$$S^G_{K_f} := G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_\infty^\circ K_f.$$

Let $\pi : G(\mathbb{A}_\mathbb{Q})/K_\infty^\circ K_f \to S^G_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_\infty^\circ K_f$ be the natural surjection map.

Note that $K_0 = \prod G(\mathbb{Z}_p)$ is the standard maximal open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},f})$, where the product runs over primes and $\mathbb{Z}_p$ denote the ring of p-adic integers. Then, the strong approximation (Theorem 3.3.1 in [8]) implies that $G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q},f})/K_0$ is finite. Since K_f is an open compact subgroup, $K_f \cap K_0$ is an open subgroup with finite index in K_0 and K_f also. So, $G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q},f})/K_f$ is a finite discrete space. Let $\{\underline{g}_f^{(i)}\}$ be a set of representatives of this discrete space and $\Gamma_i := G(\mathbb{Q}) \cap \underline{g}_f^{(i)} K_f (\underline{g}_f^{(i)})^{-1}$ be the stabilizer of $\underline{g}_f^{(i)} K_f$ in $G(\mathbb{Q})$. Consequently, locally symmetric adèlic space is homeomorphic to the disjoint union of locally symmetric spaces, i.e.,

$$S^G_{K_f} = \bigsqcup \Gamma_i \backslash G(\mathbb{R})/K_\infty^\circ.$$

8.2.1 Sheaves on Locally Symmetric adèlic Spaces

Let λ^τ be a dominant integral weight, let $\mathcal{M}_{\lambda^\tau}$ be a finite-dimensional absolutely irreducible complex representation of $G_0 \times_{F,\tau} \mathbb{C}$ with the highest weight λ^τ. If λ is a dominant integral weight in $X^+(T \times \mathbb{C})$, then $\mathcal{M}_\lambda := \otimes_{\tau:F\to\mathbb{C}} \mathcal{M}_{\lambda^\tau}$ is an absolutely irreducible finite-dimensional complex representation of $G \times \mathbb{C}$ with the highest weight λ. The action of $G(\mathbb{Q})$ on $\mathcal{M}_\lambda$ is via diagonal embedding. Now, let us define the sheaf $\widetilde{\mathcal{M}_\lambda}$, where sections over an open set $V \subset S^G_{K_f}$ are given by

$$\{s : \pi^{-1}(V) \to \mathcal{M}_\lambda \text{ smooth } : \ s(ax) = \rho_\lambda(a)s(x) \text{ for all } a \in G(\mathbb{Q}), x \in V\}.$$

A sheaf is zero if $\rho_\lambda(a) \neq 1$ for some $a \in Z(\mathbb{Q}) \cap K_\infty^\circ K_f$. Thus, $(Nd^\tau)_{\tau:F\to\mathbb{C}}$ should be the infinity type of an algebraic Hecke character (defined in [29], Lemma 17):

- If $\Sigma_\mathbb{R} \neq \emptyset$, then there exists $w \in \mathbb{Z}$ such that $Nd^\tau = w$ for all $\tau \in \Sigma_F$.
- If $\Sigma_\mathbb{R} = \emptyset$, then there exists $w \in \mathbb{Z}$ such that $Nd^{\iota\circ\tau} + Nd^{\iota\circ\overline{\tau}} = w$ for all $\tau \in \Sigma_F$ and $\iota : \mathbb{C} \to \mathbb{C}$.

A dominant integral weight is called algebraic if the determinant character satisfies the above conditions. The set of algebraic dominant integral weights is denoted by $X^+_{\mathrm{alg}}(T \times \mathbb{C})$. If K_f is sufficiently small, then the stalk at any point is isomorphic to $\mathcal{M}_\lambda$. Thus, the sheaf forms a local system on the adèlic locally symmetric space $S^G_{K_f}$.

8.3 Cohomology of Sheaves

Let $\overline{S}^G_{K_f}$ be a Borel-Serre compactification of $S^G_{K_f}$ (refer to [6]). The boundary of $S^G_{K_f}$, denoted as $\partial S^G_{K_f}$, is the union of boundary components $\partial_P S^G_{K_f}$, where the union runs over $G(\mathbb{Q})$ conjugacy classes of proper parabolic subgroups defined over $\mathbb{Q}$. A sheaf $\widetilde{\mathcal{M}}_\lambda$ naturally extends to $\overline{S}^G_{K_f}$ and is denoted by $\widetilde{\mathcal{M}}_\lambda$. The inclusion $S^G_{K_f} \hookrightarrow \overline{S}^G_{K_f}$ is a homotopy equivalence, so it induces an isomorphism on cohomology groups: $H^\bullet(\overline{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda)$.

Let $\mathcal{H}^G_{K_f} = C^\infty_c(G(\mathbb{A}_{\mathbb{Q},f})//K_f)$ be the set of locally constant and compactly supported functions $f : G(\mathbb{A}_{\mathbb{Q},f}) \to \mathbb{Q}$ such that $f(gxh) = f(x)$ for all $g, h \in K_f$. The Haar measure on $G(\mathbb{A}_{\mathbb{Q},f})$ is a product of normalized local Haar measures. Then, $\mathcal{H}^G_{K_f}$ forms a Hecke algebra under convolution. There exists a long exact sequence of $\mathcal{H}^G_{K_f}$-modules:

$$\cdots \to H^m_c(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^m(\overline{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^m(\partial S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^{m+1}_c(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to \cdots$$

This sequence involves two key components:

- The inner cohomology: $H^m_!(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) = \mathrm{Image}(H^m_c(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^m(\overline{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda))$.
- Eisenstein cohomology: $\mathrm{Image}(H^m(\overline{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^m(\partial S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda))$.

8.3.1 Inner Cohomology

The inner cohomology of the symmetric space $S^G_{K_f}$ with coefficients in the sheaf $\widetilde{\mathcal{M}}_\lambda$ decomposes into a direct sum over absolutely irreducible representations (π_f, V_{π_f}) of the Hecke algebra $\mathcal{H}^G_{K_f}$ on $\mathbb{C}$-vector spaces:

$$H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) = \bigoplus_{\pi_f \in Coh_!(G, K_f, \lambda)} H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda)(\pi_f),$$

where $Coh_!(G, K_f, \lambda)$ is a collection of absolutely irreducible representations with strictly positive multiplicities and π_f is a representation of $\mathcal{H}^G_{K_f}$. The set $Coh_!(G, \lambda)$ denotes the inner spectrum of G with coefficients in λ and is defined as $Coh_!(G, \lambda) = \cup_{K_f} Coh_!(G, K_f, \lambda)$.

Note that there exists a finite set S of places such that $K_p = G(\mathbb{Z}_p)$ be the maximal compact subgroup for any $p \notin S$. So, the local Hecke algebra $\mathcal{H}^G_{K_p}$ is a commutative algebra. Since the irreducible representations for a commutative algebra, are one dimensional, let $\pi_p : \mathcal{H}^G_{K_p} \to \mathbb{C}$ be a local representation of π_f at place $p \notin \mathbb{C}$. The vector space V_{π_f} associated with the representation π_f decomposes as $V_{\pi_f} = \otimes_p V_{\pi_p} = V_{\pi_f,S} \otimes \mathbb{C}$. Here, $V_{\pi_f,S}$ represents the product of local representations at places in the set S.

8.3.2 Cuspidal Cohomology

Let $\mathfrak{g}_\infty$ and $\mathfrak{k}_\infty$ denote the Lie algebras of $G(\mathbb{R})$ and K°_∞, respectively. The cohomology of the sheaf $\widetilde{\mathcal{M}}_\lambda$ can be expressed as the cohomology of the De Rham complex associated with $(\mathfrak{g}_\infty, K^\circ_\infty)$ modules:

$$\Omega^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) = \mathrm{Hom}_{K^\circ_\infty}(\wedge^\bullet(\mathfrak{g}_\infty/\mathfrak{k}_\infty), \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f, \omega_\lambda^{-1}|_{Z(\mathbb{R})^\circ}) \otimes \mathcal{M}_\lambda),$$

where the function space consists of $\mathbb{C}$-valued smooth functions ϕ on $G(\mathbb{A}_\mathbb{Q})$ satisfying $\phi(agk_f z_\infty) = \omega_\lambda^{-1}(z_\infty)\phi(g)$ for $a \in G(\mathbb{Q}), k_f \in K_f$, and $z_\infty \in Z(\mathbb{R})^\circ$. Here, $\omega_\lambda : Z(\mathbb{R}) \to \mathbb{C}$ is a central character (defined in Introduction of Chapter 6). This implies that our interesting cohomology is $(\mathfrak{g}_\infty, \mathfrak{k}_\infty)$ cohomology, i.e.,

$$H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \cong H^\bullet(\mathfrak{g}_\infty, K^\circ_\infty; \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f, \omega_\lambda^{-1}|_{Z(\mathbb{R})^\circ}) \otimes \mathcal{M}_\lambda).$$

Since the space of smooth cusp forms is a subspace of smooth forms, i.e.,

$$\mathcal{C}^\infty_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f, \omega_\infty^{-1}) \subset \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f, \omega_\infty^{-1}),$$

it induces inclusion at relative Lie algebra cohomology (isomorphic to differential cohomology) as well (Corollary 7.2.3). Therefore,

$$H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \cong H^\bullet(\mathfrak{g}_\infty, K^\circ_\infty; \mathcal{C}^\infty_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f, \omega_\infty^{-1}) \otimes \mathcal{M}_\lambda).$$

Let $K_{f_1} \subset K_{f_2}$ are open compact subgroups of $G(\mathbb{A}_{\mathbb{Q},f})$. A natural map exists between the cohomology groups associated with these subgroups. Consequently, we define the cuspidal cohomology $H^\bullet_{\mathrm{cusp}}(S^G, \widetilde{\mathcal{M}}_\lambda)$ as the limit of all such cohomology groups. Therefore,

$$H^\bullet_{\mathrm{cusp}}(S^G, \widetilde{\mathcal{M}}_\lambda) \cong H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; C^\infty_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}))^\infty \otimes \mathcal{M}_\lambda).$$

Since the smooth cusp form has rapid decay, it is square integrable. So

$$H^\bullet_{\mathrm{cusp}}(S^G, \widetilde{\mathcal{M}}_\lambda) \cong H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}))^\infty \otimes \mathcal{M}_\lambda).$$

8.4 Cuspidal Cohomology for SL_N and GL_N

Let $G_0^1 = SL_N/F$ be the algebraic subgroup of $G_0 = GL_N/F$. Similarly, T_0^1 and B_0^1 are torus and Borel subgroups of G_0^1. Moreover, G^1, B^1, and T^1 are Weil restrictions of scalars of algebraic group to $\mathbb{Q}$, and $G^1 \times \mathbb{C}$, $B^1 \times \mathbb{C}$, and $T^1 \times \mathbb{C}$ are base change to $\mathbb{C}$. Then, $G^1(\mathbb{R}) \simeq \prod_{v \in S_r} SL_n(\mathbb{R}) \times \prod_{v \in S_c} SL_n(\mathbb{C})$; its Lie algebra is denoted $\mathfrak{g}^1_\infty$, and its maximal compact subgroup is $K^1_\infty \simeq \prod_{v \in S_r} SO(n) \times \prod_{v \in S_c} SU(n)$. Let $\lambda^1 = \lambda|_{T^1 \times \mathbb{C}}$ and let $(\rho_{\lambda^1}, \mathcal{M}_{\lambda^1})$ be the restriction of representation $(\rho_\lambda, \mathcal{M}_\lambda)$ to SL_N/F, i.e., $\mathcal{M}_{\lambda^1} = \mathcal{M}_\lambda$. Similar to GL_N/F case, for an open-compact subgroup K^1_f of $G^1(\mathbb{A}_{\mathbb{Q},f})$, the definitions for the locally symmetric space $S^{G^1}_{K^1_f} = G^1(\mathbb{Q})\backslash G^1(\mathbb{A}_{\mathbb{Q}})/C^1_\infty K^1_f$, the sheaf $\widetilde{\mathcal{M}}_{\lambda^1}$ of $\mathbb{C}$-vector spaces on $S^{G^1}_{K^1_f}$, its cohomology groups $H^\bullet(S^{G^1}_{K^1_f}, \widetilde{\mathcal{M}}_{\lambda^1})$, and cuspidal cohomology $H^\bullet_{\mathrm{cusp}}(S^{G^1}_{K^1_f}, \widetilde{\mathcal{M}}_{\lambda^1})$ works just the same for SL_N/F as for GL_N/F. So, the cuspidal cohomology for $G_0^1 = SL_N/F$ is

$$H^\bullet_{\mathrm{cusp}}(S^{G^1}, \widetilde{\mathcal{M}}_{\lambda^1}) \cong H^\bullet(\mathfrak{g}^1_\infty, K^1_\infty; L^2_{\mathrm{cusp}}(G^1(\mathbb{Q})\backslash G^1(\mathbb{A}_{\mathbb{Q}}))^\infty \otimes \mathcal{M}_{\lambda^1}).$$

Recall that, we defined the cuspidal cohomology in Section 7.2 and 7.7. Since the differential cohomology and relative Lie algebra cohomology are isomorphic, the two notions of cuspidal cohomology coincide. In other words,

$$H^\bullet_{\mathrm{cusp}}(G^1, S_\infty, \mathcal{M}_{\lambda^1}) = H^\bullet_{\mathrm{cusp}}(S^{G^1}, \widetilde{\mathcal{M}}_{\lambda^1}).$$

Theorem 8.4.1. *The cuspidal cohomology of $G = \mathrm{Res}_{F/\mathbb{Q}}\, GL_N/F$ in a finite-dimensional representation space $\mathcal{M}_\lambda$ does not vanish if the cuspidal cohomology of $G^1 = \mathrm{Res}_{F/\mathbb{Q}}\, SL_N/F$*

in a finite-dimensional representation space $\mathcal{M}_{\lambda^1}$ does not vanish, i.e.,

$$H^{\bullet}_{\text{cusp}}(S^{G^1}, \widetilde{\mathcal{M}}_{\lambda^1}) \neq 0 \Rightarrow H^{\bullet}_{\text{cusp}}(S^G, \widetilde{\mathcal{M}}_{\lambda}) \neq 0.$$

Proof. Since $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{\infty} \cong \varinjlim_{K_f \subset G(\mathbb{A}_f)} L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \omega)^{\infty}$, by using Lemma 6.2.4, we have

$$H^{\bullet}_{\text{cusp}}(S^G, \widetilde{\mathcal{M}}_{\lambda}) \cong \varinjlim_{K_f \subset G(\mathbb{A}_f)} H^{\bullet}_{d}(G(\mathbb{R}), L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_F)/K_f, \omega)^{\infty} \otimes \mathcal{M}_{\lambda}).$$

Since $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ is homeomorphic to finite disjoint union $\sqcup \Gamma_i \backslash G(\mathbb{R})$ (refer Section 8.2), $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \omega)^{\infty} \cong \oplus L^2_{\text{cusp}}(\Gamma_i\backslash G(\mathbb{R}), \omega|_{\infty})^{\infty}$ and the cohomology groups

$$H^{\bullet}_{d}(G(\mathbb{R}), L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \omega)^{\infty} \otimes \mathcal{M}_{\lambda}) \cong \oplus H^{\bullet}_{d}(G(\mathbb{R}), L^2_{\text{cusp}}(\Gamma_i\backslash G(\mathbb{R}), \omega|_{\infty})^{\infty} \otimes \mathcal{M}_{\lambda}).$$

We need to show that $H^{\bullet}_{d}(G(\mathbb{R}), L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}), \omega|_{\infty})^{\infty} \otimes \mathcal{M}_{\lambda}) \neq 0$ for some $\Gamma \subset G(\mathbb{Q})$ to get our result. By using Lemma 4.2.1 and Shapiro's Lemma 6.3.3, we can express the cuspidal cohomology for any $\Gamma \subset G(\mathbb{Q})$ as

$$H^{\bullet}_{d}(Z(\mathbb{R})\Gamma G^1(\mathbb{R}), L^2_{\text{cusp}}(\Gamma^1\backslash G^1(\mathbb{R}), \omega^1|_{\infty})^{\infty} \otimes \mathcal{M}_{\lambda}),$$

where $\Gamma^1 = \Gamma \cap G^1(\mathbb{Q})$. Since $Z(\mathbb{R})$ acts trivially on $L^2_{\text{cusp}}(\Gamma^1\backslash G^1(\mathbb{R}), \omega^1|_{\infty})^{\infty} \otimes \mathcal{M}_{\lambda}$, we further simplify it to

$$H^{\bullet}_{d}(\Gamma G^1(\mathbb{R}), L^2_{\text{cusp}}(\Gamma^1\backslash G^1(\mathbb{R}), \omega^1|_{\infty})^{\infty} \otimes \mathcal{M}_{\lambda}).$$

As $\Gamma G^1(\mathbb{R})/G^1(\mathbb{R}) \cong \Gamma/\Gamma^1$ is a finite abelian group, we can further refine this expression by using Proposition 2.5 in Chapter 9 of [7],

$$H^{\bullet}_{d}(G^1(\mathbb{R}), L^2_{\text{cusp}}(\Gamma^1\backslash G^1(\mathbb{R}), \omega^1|_{\infty})^{\infty} \otimes \mathcal{M}_{\lambda})^{\Gamma/\Gamma^1}.$$

By using hypothesis, there exists $\Gamma^1 \subset G^1(\mathbb{Q})$ such that

$$H^{q}_{d}(G^1(\mathbb{R}), L^2_{\text{cusp}}(\Gamma^1\backslash G^1(\mathbb{R}), \omega^1|_{\infty})^{\infty} \otimes \mathcal{M}_{\lambda}) \neq 0.$$

So, there exists an admissible irreducible representation $\pi \subset L^2_{\text{cusp}}(\Gamma^1\backslash G^1(\mathbb{R}), \omega^1|_{\infty})^{\infty}$ of

$G^1(\mathbb{R})$ such that

$$H_d^\bullet(G^1(\mathbb{R}), \pi \otimes \mathcal{M}_\lambda) \neq 0.$$

For $\gamma \in \Gamma/\Gamma^1$, we define $\pi^\gamma(x) := \pi(\gamma^{-1}x\gamma)$ as representations of $G^1(\mathbb{R})$. As a result, $\sum_{\gamma\in\Gamma/\Gamma^1} \pi^\gamma \in (L^2_{\text{cusp}}(\Gamma^1\backslash G^1(\mathbb{R}), \omega^1|_\infty)^\infty)^{\Gamma/\Gamma^1}$, establishing the theorem. $\square$

8.5 Strongly Pure weight

When an algebraic dominant integral weight $\lambda = (\lambda^\tau)$ supports cuspidal cohomology, it must satisfy the purity condition ([11]), i.e.,

$$a_i^\tau = a_{N-i}^{\overline{\tau}} \text{ for all } \tau : F \to \mathbb{C} \quad \Leftrightarrow \quad \text{there exists } w \in \mathbb{Z} \text{ such that } b_i^\tau + b_{N-i+1}^{\overline{\tau}} = w \text{ for all } \tau.$$

The integer w is called the purity weight of λ. A set of pure weights is denoted by $X_0^+(T\times\mathbb{C})$. Since the cuspidal cohomology of GL_N/F admits a rational structure ([11]), it is evident that for any $\xi \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, ${}^\xi\lambda$ also supports cuspidal cohomology, consequently, ${}^\xi\lambda = ({}^\xi\lambda^\tau)$ is also a pure weight, where ${}^\xi\lambda^\tau = \lambda^{\xi^{-1}\circ\tau}$. A pure weight is called **strongly pure** if ${}^\xi\lambda$ is pure for all $\xi \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and a set of strongly pure weights is denoted by $X_{00}^+(T \times \mathbb{C})$. The following statements are equivalent:

- $\lambda \in X_{00}^+(T \times \mathbb{C})$.
- $a_i^{\xi\circ\tau} = a_{N-i}^{\xi\circ\overline{\tau}}$ for all $\tau : F \to \mathbb{C}$ and $\xi \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.
- There exists an integer $w \in \mathbb{Z}$ such that $b_i^{\xi\circ\tau} + b_{N-i+1}^{\xi\circ\overline{\tau}} = w$ holds for all $\tau : F \to \mathbb{C}$ and $\xi \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. This integer w is referred to as the **purity weight** of λ.

Moreover, the following inclusions hold in the character group of $T\times\mathbb{C}$, with strict inclusions in general:

$$X_{00}^+(T \times \mathbb{C}) \subset X_0^+(T \times \mathbb{C}) \subset X_{alg}^+(T \times \mathbb{C}) \subset X^+(T \times \mathbb{C}) \subset X^*(T \times \mathbb{C}).$$

8.5.1 Examples $F = \mathbb{Q}(\sqrt{3}, \sqrt[3]{5})$

Consider the embeddings $\tau_{x,y} : F \to \mathbb{C}$ defined by $\tau_{x,y}(\sqrt{3}) = (-1)^x\sqrt{3}$ and $\tau_{x,y}(\sqrt[3]{5}) = \omega^y \sqrt[3]{5}$ for $x \in \{0,1\}$ and $y \in \{0,1,2\}$, where ω is the cube root of unity. The embeddings $\tau_{0,0}$ and $\tau_{1,0}$ are real, while $(\tau_{0,1}, \tau_{0,2})$ and $(\tau_{1,1}, \tau_{1,2})$ form pairs of complex embeddings and their conjugates. Note that, $F_1 = \mathbb{Q}(\sqrt{3})$ is the maximal totally real subfield. Furthermore, we observe that $\tau_{0,0}|_{F_1} = \tau_{0,1}|_{F_1} = \tau_{0,2}|_{F_1}$ and $\tau_{1,0}|_{F_1} = \tau_{1,1}|_{F_1} = \tau_{1,2}|_{F_1}$.

Let $\lambda^\tau = (a^\tau - 1) \cdot \gamma + d^\tau \cdot \delta_2$, let us denote it $[a^\tau, d^\tau]$. For strongly pure weight λ,

$$a^{\tau_{0,0}} = a^{\tau_{0,1} \circ \tau_{0,2}} = a^{\tau_{0,1} \circ \tau_{0,1}} = a^{\tau_{0,2}},$$
$$a^{\tau_{1,0}} = a^{\tau_{1,1} \circ \tau_{0,2}} = a^{\tau_{1,1} \circ \tau_{0,1}} = a^{\tau_{1,2}}.$$

Then any pure weights and strongly pure weights have the following structures.

Embeddings		Pure Weight	Strongly Pure Weight
$\tau_{0,0}$		[a,d]	[a,d]
$\tau_{1,0}$		[b,d]	[b,d]
$(\tau_{0,1}, \tau_{0,2})$		[x,d]	[a,d]
$(\tau_{1,1}, \tau_{1,2})$		[y,d]	[b,d]

8.5.2 Strongly Pure Weights as Base Change

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and let $\mathcal{G} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\mathfrak{c} \in \mathcal{G}$ be the complex conjugation. Let $\mathcal{N}'$ be a normal subgroup of $\mathcal{G}$ generated by $\{\gamma\mathfrak{c}\gamma^{-1}\mathfrak{c} : \gamma \in \mathcal{G}\}$.

Remark 8.5.1. *The fixed subfield of $\overline{\mathbb{Q}}$ by $\mathcal{N}'$ is the maximal CM subfield.*

Lemma 8.5.1. *Let $\lambda \in X^+_{00}(\mathrm{Res}_{F/\mathbb{Q}} T_0/F \times \mathbb{C})$ be an algebraic dominant strongly pure weight. Then there exists $\kappa \in X^+_{00}(\mathrm{Res}_{F_1/\mathbb{Q}} T_0/F_1 \times \mathbb{C})$ such that λ is a base change of κ from F_1 to F. In other words, for any $\tau : F \to \mathbb{C}$, we have $\lambda^\tau = \kappa^{\tau|_{F_1}}$. (Similar to Prop. 2.6 in [28]).*

Proof. Let F be a number field with F_1 being a subfield (defined as in the introduction of Chapter 8). Let $\lambda = (\lambda^\tau)_{\tau:F\to\mathbb{C}}$ be a strongly pure weight. We have $b_i^{\gamma\circ\mathfrak{c}\circ\tau} = b_i^{\mathfrak{c}\circ\gamma\circ\tau}$ for all $\tau \in \Sigma_F$, $\gamma \in \mathcal{G}$, and $1 \le i \le N$ due to the purity and strong purity conditions.
Consider two embeddings τ_1 and τ_2 of F into $\mathbb{C}$ such that $\tau_1|_{F_1} = \tau_2|_{F_1}$. This implies the existence of $\gamma \in \mathcal{N}'$ such that $\gamma\tau_1 = \tau_2$. Therefore, $b_i^{\tau_1} = b_i^{\tau_2}$, and subsequently, $\lambda^{\tau_1} = \lambda^{\tau_2}$. $\square$

8.6 Cuspidal Cohomology for SL_N

Let F be a totally real number field or a Galois number field over the maximum totally real subfield F_0 of F. Recall notations: let $G_0 = GL_N/F$ then $G_0^1 = SL_N/F$ and $G^1 = Res_{F/\mathbb{Q}} G_0^1$.

8.6.1 Finite Dimensional Representation

Let λ be a strongly pure weight, and let $\lambda^1 = \lambda|_{T^1 \times \mathbb{C}}$. Denote $(\rho_\lambda, \mathcal{M}_\lambda)$ as a finite-dimensional algebraic irreducible representation of $G(\mathbb{C})$ with the highest weight λ, and let $\rho_{\lambda^1} = \rho_\lambda|_{G^1(\mathbb{C})}$. Then

$$(\rho_{\lambda^1}, \mathcal{M}_{\lambda^1}) \cong \otimes_{\tau : F \to \mathbb{C}} (\rho_{\lambda^{1,\tau}}, \mathcal{M}_{\lambda^{1,\tau}}) \cong \otimes_{v \in S_\infty} (\rho_{\lambda^{1,v}}, \mathcal{M}_{\lambda^{1,v}}).$$

where $\rho_{\lambda^{1,v}} \cong \rho_{\lambda^{1,\tau_v}}$ if v is a real place, and $\rho_{\lambda^{1,v}} \cong \rho_{\lambda^{1,\tau_v}} \otimes \rho_{\lambda^{1,\bar{\tau}_v}}$ if v is a complex place.

Now, let $\iota : F \to \mathbb{C}$ be an embedding. Since F is a Galois over F_0, $\iota(F)$ is also a Galois over the maximal totally real subfield $\iota(F_0)$. So $\iota(F) = \overline{\iota(F)}$ and consequently, there exists an automorphism $\mathfrak{c} : F \to F$ such that $\iota \circ \mathfrak{c}(x) = \overline{\iota(x)}$ for $x \in F$. Note that $\mathfrak{c}|_{F_0} = \mathrm{id}$ and $\mathfrak{c}|_{F_1} \neq \mathrm{id}$ if F_1 is a quadratic totally imaginary extension. Let $\mathfrak{c}$ also denote the automorphism of the algebraic group $G = \mathrm{Res}_{F/\mathbb{Q}}(SL_N/F)$ induced by an automorphism $\mathfrak{c}$ (see Section 3.3 for more details). Let $\vartheta : SL_N/F \to SL_N/F$ be an algebraic group homomorphism defined as $g \mapsto {}^*g = {}^t g^{-1}$. Now, let $\theta := \left(\mathrm{Res}_{F/\mathbb{Q}} \vartheta\right) \circ \mathfrak{c}$. If F is totally real field then $\theta = \mathrm{Res}_{F/\mathbb{Q}} \vartheta$.

Lemma 8.6.1. *There exists $I_0 : \mathcal{M}_\lambda \to \mathcal{M}_\lambda$ such that*

$$I_0 \circ \rho_\lambda(g) = \rho_\lambda(\theta(g)) \circ I_0 \quad g \in G^1(\mathbb{R}).$$

Proof. Let $\theta'(g) = w_0 \theta(g) w_0^{-1}$ for a $N \times N$ matrix $w_0 = [(-1)^{i-1}\delta_{i,N+1-j}]$, where $\delta_{i,j} = 1$ if $i = j$ and 0 else. Note that $\det(w_0) = 1$. Let $x = (\mathrm{diag}(x_1^\tau, \ldots, x_N^\tau))_{\tau \in \Sigma_F} \in T^1(\mathbb{C})$, and $v_\lambda = \otimes_\tau v_{\lambda^\tau} \in \mathcal{M}_{\lambda^1} = \otimes_\tau \mathcal{M}_{\lambda^{1,\tau}}$ be a highest weight vector with weight λ^1. Then, ${}^{\theta'}\rho_{\lambda^1}$ acts on v_λ by:

$$\begin{aligned}
{}^{\theta'}\rho_{\lambda^1}(x)\cdot v_\lambda &= \rho_{\lambda^1}(\mathrm{diag}\left((x_N^{\bar\tau})^{-1},\ldots,(x_1^{\bar\tau})^{-1}\right))\cdot v_\lambda \\
&= \lambda(\mathrm{diag}\left((x_N^{\bar\tau})^{-1},\ldots,(x_1^{\bar\tau})^{-1}\right))v_\lambda \\
&= \bigotimes_{\tau\in\Sigma_F}\left(\prod_{1\le i\le N}(x_{N-i+1}^{\bar\tau})^{-b_i^\tau}\right)v_{\lambda^\tau} \\
&= \bigotimes_{\tau\in\Sigma_F}\left(\prod_{1\le i\le N}(x_{N-i+1}^{\bar\tau})^{-w+b_{N-i+1}^{\bar\tau}}\right)v_{\lambda^\tau} \\
{}^{\theta'}\rho_{\lambda^1}(x)\cdot v_\lambda &= \lambda^1(x)v_\lambda.
\end{aligned}$$

The second last equality holds due to the strong purity of λ^1.
Since θ' preserves the Borel subgroup, for $x\in B^1(\mathbb{C})$,

$${}^{\theta'}\rho_{\lambda^1}(x)\cdot v_\lambda = \rho_{\lambda^1}(\theta'(x))\cdot v_\lambda = v_\lambda.$$

Therefore, v_λ is also a highest weight vector in ${}^{\theta'}\rho_{\lambda^1}$ with the highest weight λ^1. As the highest weight characterizes finite-dimensional irreducible representations of semisimple connected Lie groups, there exists $I_0' : \mathcal{M}_\lambda \to \mathcal{M}_\lambda$ such that $I_0'\circ\rho_\lambda(g) = \rho_\lambda(\theta'(g))\circ I_0'$ for all $g\in G^1(\mathbb{R})$. Then $I_0 = I_0'\circ\rho_{\lambda^1}(w_0)$ has the required property. $\square$

Remark 8.6.1. *If λ^1 is not strongly pure, the extension of ρ_{λ^1} to $G(\mathbb{C})\rtimes\langle\alpha\rangle$ cannot be guaranteed. The extension property relies on the strong purity condition, and without it, the representation may not have a meaningful extension to the larger group.*

8.6.2 Unitary Admissible Representation

For a complex place v, consider the pair of complex conjugate embeddings τ_v and $\overline{\tau}_v$. Suppose we use τ_v to identify the completion at v as $F_v = \mathbb{C}$. Let B_0^1 denote the standard Borel subgroup of $G_0^1 = SL_N/F$ with Levi decomposition $B_0^1 = T_0^1 U_0^1$. Define the cuspidal parameter $(\alpha,\beta) = (\alpha^v,\beta^v)_{S_\infty} = \left((\alpha_1^v,\ldots,\alpha_N^v),(\beta_1^v,\ldots,\beta_N^v)\right)_{S_\infty}$ as

$$\alpha_i^v = -b_{N-i+1}^{\tau_v} + \frac{N-(2i-1)}{2}, \qquad \beta_i^v = -b_i^{\bar\tau_v} - \frac{N-(2i-1)}{2}.$$

Let $\chi_{\lambda^{1,v}}$ be a character on ${}^0T_0(F_{\tau_v})$ defined as $\mathrm{diag}(z_1,\ldots,z_N)\mapsto\prod z_i^{\alpha_i^v}\bar{z}_i^{\beta_i^v}$. Let $\mathbb{J}_{\lambda^{1,v}} = \mathrm{Ind}_{B_0^1(F_v)}^{G_0^1(F_v)}\chi_{\lambda^{1,v}}$ be a parabolically induced representation. (Note that $\mathbb{J}_{\lambda^{1,v}}$ is a representation of $SL_N(\mathbb{C})$ defined in Section 6.6.)

For a real place v, consider the real embeddings τ_v. Then the completion at v is $F_v=\mathbb{R}$. Let B_0^1 denote the standard Borel subgroup of $G_0^1 = SL_N/F$ with Levi decomposition $B_0^1 = T_0^1U_0^1$. Define the cuspidal parameter $l=(l^v)_{S_\infty}=(l_1^v,\ldots,l_N^v)_{S_\infty}$ as in Section 6.7. Let $\mathbb{J}_{\lambda^{1,v}}$ be the parabolically induced representation defined in Section 6.7.

Proposition 8.6.2. *Let $\mathbb{J}_{\lambda^1}=\otimes_{v\in S_\infty}\mathbb{J}_{\lambda^{1,v}}$ be an admissible irreducible unitary representation of $G^1(\mathbb{R})$ then $Lef(\theta, G^1(\mathbb{R}), \mathbb{J}_{\lambda^1}\otimes\mathcal{M}_{\lambda^1})\neq 0$.*

Proof. Since F is Galois over its maximal totally real subfield F_0, either F is totally real or totally imaginary. By the Kunneth theorem for cohomology, we have

$$H_d^\bullet(G^1(\mathbb{R}),\mathbb{J}_{\lambda^1}\otimes\mathcal{M}_{\lambda^1}) = \bigotimes_{v\in S_\infty} H_d^\bullet(G_0^1(F_v),\mathbb{J}_{\lambda^{1,v}}\otimes\mathcal{M}_{\lambda^{1,v}}).$$

As a consequence, $\mathrm{Lef}(\theta, G^1(\mathbb{R}),\mathbb{J}_{\lambda^1}\otimes\mathcal{M}_{\lambda^1}) = \prod'_{v\in S_\infty}\mathrm{Lef}(\theta, G_0^1(F_v),\mathbb{J}_{\lambda^{1,v}}\otimes\mathcal{M}_{\lambda^{1,v}})$. Now the proposition follows by using Theorem 6.6.10 if F is totally imaginary and by using Theorem 6.7.9 if F is totally real. □

8.6.3 Application of Theorem 7.0.1

Theorem 8.6.3. *Let F be a Galois extension over a maximal totally real subfield, and let $\mathcal{M}_{\lambda^1}$ be a finite-dimensional irreducible representation of $\mathrm{Res}_{F/\mathbb{Q}}SL_N/F(\mathbb{C})$ characterized by the highest weight λ^1. If $\lambda^1=\lambda|_{T^1\times\mathbb{C}}$ for some strongly pure weight λ and $N\geq 2$, then,*

$$H^\bullet_{\text{cusp}}(Res_{F/\mathbb{Q}}SL_N/F, S_\infty, \mathcal{M}_{\lambda^1})\neq 0.$$

Proof.

- The algebraic group $\mathrm{Res}_{F/\mathbb{Q}}SL_N/F$ is absolutely simple connected semisimple.
- An automorphism θ is finite order algebraic automorphism.
- A finite-dimensional representation ρ_{λ^1} is $\alpha-$invariant since the Lemma 8.6.1.

- The proposition 8.6.2 implies that there exists an admissile unitary irreducible representation of $G^1(\mathbb{R})$ such that Lefschetz number does not vanish.

Now, the theorem follows since all the hypotheses of Theorem 7.0.1 are satisfied. $\square$

By using Theorem 8.6.3 and Theorem 8.4.1, we get the following corollary.

Corollary 8.6.4. *Consider F, a Galois extension over a maximal totally real subfield, and let $\mathcal{M}_\lambda$ be a finite-dimensional irreducible representation of $\mathrm{Res}_{F/\mathbb{Q}}\, GL_N/F(\mathbb{C})$ characterized by the highest weight λ. If λ is a strongly pure weight then*

$$H^\bullet_{\text{cusp}}(S^G, \widetilde{\mathcal{M}}_\lambda) \cong H^\bullet_{\text{cusp}}(\mathrm{Res}_{F/\mathbb{Q}}\, GL_N/F, S_\infty, \mathcal{M}_\lambda) \neq 0.$$

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