Stochastic Partial Differential Equations

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by

Sudheesh Surendranath

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Dr. Anup Biswas

Certificate

This is to certify that this dissertation entitled Stochastic Partial Differential Equations towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sudheesh Surendranathat Indian Institute of Science Education and Research under the supervision of Dr. Anup Biswas, Associate Professor, Department of Mathematics , during the academic year 2020-2021.

Dr. Anup Biswas

Committee:

Dr. Anup Biswas

Dr. Mathew Joseph

This thesis is dedicated to my parents and my sister

Declaration

I hereby declare that the matter embodied in the report entitled Stochastic Partial Differential Equations are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anup Biswas and the same has not been submitted elsewhere for any other degree.

Sudheesh Surendranath

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Abstract

In this thesis, we study the random-field approach for solving a class of parabolic stochastic partial differential equations and study some properties exhibited by their solutions, some of which include intermittency, chaotic character and fractal behaviour.

Contents

Chapter 1

Introduction

Stochastic Partial Differential Equations, SPDEs for short, are used to model dynamics of systems in a random environment. They generalize partial differential equations by adding random forcing terms. It has deep applications in various fields such as statistical physics, where it is used to model the random growth of surfaces (see KPZ [26, 31]), fluid dynamics (stochastic Navier -Stokes equation [12]) and mathematical finance just to name a few.

The study of SPDEs is an interesting field which combines methods from probability theory, functional analysis and the theory of partial differential equations. The approach taken here would be the random-field one, wherein the solutions are random elements in a suitable space of random-fields. The other approaches to solving SPDEs are: the infinite-dimensional method of Da Prato and Zabczyk [13] and the functional analytic approach developed by Krylov [29]. Each of these approaches has it's own advantages. The random-field approach discussed here allows us to study the probabilistic behaviour of the solution to the SPDE simultaneously in space and time. See [16] for a comparison between the random-field and the infinite dimensional approach.

In this thesis, we will be focusing on parabolic SPDEs of the form:

$$\frac{\partial}{\partial t}u_t(x) = \left(\mathcal{L}u_t\right)(x) + b\left(u_t(x)\right) + \sigma\left(u_t(x)\right)\dot{F}_t(x) \quad \text{for } t > 0,\, x \in \mathbb{R}^n. \tag{1.1}$$

Here, $\mathcal{L}$ is the generator of a Lévy process with transition densities, and the random forcing term $\dot{F}_t(x)$ is a *(spatially) colored noise.* Informally speaking, this is a Gaussian random field indexed by time and space such that is if $(t,x) \neq (s,y)$, then the random variables $F_t(x)$ and $F_s(y)$ are correlated in space but not in time. If the random field $\dot{F}_t(x)$ were sufficiently smooth, then we can use Duhamel's principle from the theory of partial differential equations to write the mild solution as follows:

$$u_t(x) = (P_t u_0)(x) + \int_0^t \int_{\mathbb{R}^n} p_{t-s}(y-x) b\left(u_s(y)\right) dsdy + \int_0^t \int_{\mathbb{R}^n} p_{t-s}(y-x) \sigma\left(u_s(y)\right) F(dsdy) \tag{1.2}$$

where $\{P_t\}_{t\geq 0}$ is the transition semigroup of the Lévy process generated by $\mathcal{L}$ and $\{p_t\}$ are the transition densities. Unfortunately, this is not the case. The noise is of a very rough nature and therefore, it is not possible to make sense of the third integration on the right hand side of (1.2) in the usual sense. To circumvent this technical difficulty, Walsh introduced a method of stochastic integration in his 1984 lectures [33] at the Saint-Flour summer school. This is a natural extension of the Itô integral to higher dimensions.

Once the existence and uniqueness of the mild solution is determined, we study various properties of the solutions. Among these are intermittency, chaotic behaviour and the fractal nature of the solution. Intermittency refers to the fact that the solution exhibits large peaks at large times, atypical of the expected behaviour of the process. The chaotic character describes the sensitive dependence of the global behaviour of the solution to the nature of the initial function.

We now briefly give the organization of the various chapters. Chapter 1 introduces the necessary preliminaries required for the thesis. We assume the reader is familiar with advanced probability theory including stochastic integration and will only be reviewing some important definitions and results. In Chapters 2 and 3 we introduce the notion of Walsh integral and find a unique random-field solution for (1.1) using a Picard iterative scheme. Chapter 4 introduces a very important SPDE, the Anderson model, and we study some well known results regarding its solution. From Chapter 5 onward, we look at and study some results of various properties of the random-field solution including intermittency, chaotic character and multifractality of the solutions.

This thesis is a survey of the existing literature on SPDEs as expounded in [8, 10, 14, 15, 19, 18, 20, 25, 30].

Chapter 2

Preliminaries

Throughout this thesis, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $\mathbf{E}$ the expectation operator with respect to the probability measure $\mathbf{P}$. Whenever there is a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of $\mathcal{F}$ involved, we assume it to be right continuous and complete.

2.1 Stochastic Integration

In what follows, $\{M_t\}_{t\geq 0}$ is a continuous square-integrable martingale on Ω with $M_0 = 0$. We review some standard results regarding $\{M_t\}_{t\geq 0}$. For the proofs, the reader is referred to [32].

Lemma 2.1.1 (Burkholder-Davis-Gundy) *For any $p \geq 2$, there exists a constant z_p such that*

$$\mathbf{E}(|M_t^*|^p) \leq \mathrm{z}_p \, \mathbf{E}(\langle M \rangle_t^{\frac{p}{2}})$$

where $M_t^* = \sup_{s\leq t} M_s$ and $\langle M \rangle_t$ is the quadratic variation process of M. Moreover $\mathrm{z}_k \leq 2\sqrt{k}$ (see [6]).

The next result due to Lévy characterizes the Brownian motion among all continuous square-integrable martingales.

Lemma 2.1.2 (Lévy) *If $\langle M \rangle_t = t$, then M is the standard Brownian motion.*

Lemma 2.1.3 (Itô's formula) *Let $f \in C^2(\mathbb{R})$. Then, $f(M_t)$ is a continuous (local) semi-martingale and*

$$f(M_t) = f(M_0) + \int_0^t \frac{\partial f(M_s)}{\partial s} dM_s + \frac{1}{2} \int_0^t \frac{\partial^2 f(M_s)}{\partial s^2} d\langle M \rangle_s \tag{2.1}$$

An important application of Itô's formula gives stochastic representation of solutions to some partial differential equations.

Lemma 2.1.4 (Feynman-Kac formula) *Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a constant function and $G : (0, \infty) \times \mathbb{R} \to \mathbb{R}_+$ be continuous. Suppose that $u \in C([0, \infty) \times \mathbb{R}) \cap C^{1,2}((0, \infty) \times \mathbb{R})$ solves the Cauchy problem*

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u_t(x) + u_t(x) G_t(x)$$

Then u admits a stochastic representation given by

$$u_t(x) = \mathbf{E}\Big(u_0(x + B_t) \mathrm{e}^{\int_0^t G_{t-s}(B_s + x) ds} \Big)$$

where $\{B_t\}_{t \geq 0}$ is a standard Brownian motion on the real line.

Now we present some important results in the theory of local times. These will be used in Chapter 6.

Theorem 2.1.5 (Tanaka's formula) *For any $x \in \mathbb{R}$, there exists an increasing process $L^x(M)$ called the* local time *of M at x such that*

$$|M_t - x| = |x| + \int_0^t \mathrm{sgn}(M_s - x) dM_s + L_t^x(M) \quad a.s. \tag{2.2}$$

Lemma 2.1.6 *For every even integer $k \geq 2$, there exists a constant $C_k > 0$ such that for all $x, y \in \mathbb{R}$, $s, t \geq 0$, $T > 0$,*

$$\sup_{x \in \mathbb{R}} \mathbf{E} \left(|L_t^z(M) - L_s^z(M)|^k \right) \leq C_k |t - s|^{\frac{k}{2}} \tag{2.3}$$

$$\sup_{r \in [0,T]} \mathbf{E} \left(|L_r^x(M) - L_r^y(M)|^k \right) \leq C_k T^{\frac{k}{4}} |x - y|^{\frac{k}{2}} \tag{2.4}$$

Thus, the process $L(M)$ has a modification that is locally Hölder continuous of index $< 1/2$ in both time and space.

Lemma 2.1.7 (Occupation density formula) *For every $\varphi \in C_c^\infty(\mathbb{R})$ and $t \geq 0$, the following holds:*

$$\int_0^t \varphi(M_s)ds = \int_{-\infty}^{\infty} \varphi(x) L_t^x(M) dx \quad a.s. \tag{2.5}$$

Using the above integration by parts formula with a sequence $\{\varphi_n\}_{n=1}^\infty$ of smooth mollifiers approximating the indicator function $\mathbf{1}_{[x-\varepsilon,x+\varepsilon]}$, the following representation of $L_t^x(M)$ is evident:

$$L_t^x(M) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[x-\varepsilon,x+\varepsilon]}(M_s)ds \tag{2.6}$$

Thus, informally, $L_t^x(M)$ measures the amount of time in $[0,t]$ that the process M spends at the point x. Hence the name *local time*.

2.2 Lévy Processes

Definition 2.2.1 *An $\mathbb{R}^n-$valued stochastic process $X = \{X_t\}_{t\geq 0}$ is a* Lévy Process *if*

1. $X_0 = 0$ *a.s.*
2. $X_t - X_s \sim X_{t-s}\ \forall\, s \leq t$.
3. $X_t - X_s$ *is independent of* $\sigma(X_r, r \leq s)\ \forall\, s \leq t$.
4. $t \to X_t$ *is a.s. càdlàg.*

Examples include:

- the standard Brownian motion on $\mathbb{R}$ where $X_t \sim \mathcal{N}(0,t)$.
- the compound Poisson process where $X_t = \sum_{k=1}^{N_t} \xi_k$, $N_t \sim Poisson(\lambda t)$ and $\{\xi_k\}_{k\geq 1}$ are i.i.d random variables independent of $\{N_t\}_{t\geq 0}$.

The following theorem characterizes Lévy processes. For a proof, see [4].

Theorem 2.2.1 (Lévy-Khinchtine) *If $X = \{X_t\}_{t\geq 0}$ is a $\mathbb{R}^n-$valued Lévy process, then there is a unique function $\Psi : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\mathbb{E}[e^{i\langle \xi, X_t\rangle}] = e^{-t\Psi(\xi)}$$

where

$$\Psi(\xi) = i\langle a, \xi\rangle + \frac{1}{2}\langle \xi, Q\xi\rangle + \int_{\mathbb{R}^n} \left(1 - \mathrm{e}^{i\langle \xi, x\rangle} + i\langle \xi, x\rangle \mathbf{1}_{(|x|<1)}\right) \Pi(dx) \text{ for every } \xi \in \mathbb{R}^n \ ,$$

where $a \in \mathbb{R}^n$, Q is a $n \times n$ positive semi-definite matrix, and Π is a $\sigma-$finite measure concentrated on $\mathbb{R}^n \backslash \{0\}$ satisfying $\int_{\mathbb{R}^n} (1 \wedge |x|^2)\,\Pi(dx) < \infty$.

The constant a corresponds to the drift term, Q to the Gaussian or diffusion process and Π gives the distribution of the jumps of the process. Moreover, the triplet (a, Q, Π) uniquely determines Ψ.

Examples:

- the standard Brownian motion: $\Psi(\xi) = c|\xi|^2$
- the compound Poisson process: $\Psi(\xi) = \mathrm{e}^{\lambda \int_{\mathbb{R}^n} (\mathrm{e}^{\langle \xi, x\rangle} - 1)\Pi(dx)}$

A *symmetric stable process* of index α $(0 < \alpha \leq 2)$ is the Lévy process with Lévy exponent $\Psi(\xi) = c|\xi|^\alpha$, where $c > 0$ is a fixed constant. The case $\alpha = 2$ is the Gaussian case, and $\alpha = 1$ corresponds to the Cauchy distribution.

2.3 Fourier Analysis

Definition 2.3.1 *The* Schwartz class $\mathcal{S}(\mathbb{R}^n)$ *is defined as the space of rapidly decreasing test functions on $\mathbb{R}^n$. That is,*

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C_c^\infty(\mathbb{R}^n) \text{ such that } \lim_{x\to\infty} |x^\alpha \partial^\beta \varphi| = 0 \ \forall \alpha, \beta \in \mathbb{N}\}$$

endowed with the semi-norm $\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi|$ *or equivalently* $\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |(1+x^2)^\alpha \partial^\beta \varphi|$ *for* $\alpha, \beta \in \mathbb{N}$.

For φ in $\mathcal{S}(\mathbb{R}^n)$, define the *Fourier transform* of ψ by:

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \varphi(x) dx$$

Theorem 2.3.1 (Plancherel) *The map* $\varphi \mapsto \widehat{\varphi}$ *is an isomorphism from* $\mathcal{S}(\mathbb{R}^n)$ *onto* $\mathcal{S}(\mathbb{R}^n)$ *which, by density of* $\mathcal{S}(\mathbb{R}^n)$ *in* $L^2(\mathbb{R}^n)$ *extends to an isometry from* $L^2(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$. *That is,*

$$\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\widehat{\varphi}\|_{L^2(\mathbb{R}^n)}$$

For a proof, see [22].

Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$, also known as the set of *tempered distributions* on $\mathbb{R}^n$. Examples include:

- a (tempered) function $g : \mathbb{R}^n \to \mathbb{R}$ which satisfies $\sup_{x\in\mathbb{R}^n} \frac{|g(x)|}{(1+|x|^2)^k} < \infty$ for some $k \in \mathbb{N}$. The distribution generated by g acts on $\mathcal{S}(\mathbb{R}^n)$ as: $T_g(\varphi) = \int_{\mathbb{R}^n} g\varphi dx$.
- a (tempered) measure μ which satisfies $\int_{\mathbb{R}^n} \frac{d\mu}{(1+|x|^2)^k} < \infty$ for some $k \in \mathbb{N}$. The distribution generated by μ acts on $\mathcal{S}(\mathbb{R}^n)$ as: $T_\mu(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu$.

The Fourier transform of a tempered distribution T is the tempered distribution $\widehat{T}$ defined by:

$$\widehat{T}(\varphi) = T(\widehat{\varphi}),\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We recall that a function $g : \mathbb{R}^n \to \mathbb{C}$ is called *positive-definite* if $\sum_{i,j=1}^n g(x_i - x_j)\xi_i\overline{\xi_j} \geq 0$ for all $x_1, x_2 \ldots x_n \in \mathbb{R}^n$ and $\xi_1, \xi_2 \ldots \xi_n \in \mathbb{C}$. One can extend this definition to tempered distributions: to say that $T \in \mathcal{S}'(\mathbb{R}^n)$ is positive-definite if $T(\varphi * \tilde{\varphi}) \geq 0\ \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$, where $\tilde{\varphi}(x) = \varphi(-x)$. The following theorem classifies all positive-definite tempered distributions. The proof can be found in [22].

Theorem 2.3.2 (Bochner-Schwartz) *A tempered distribution* T *is positive-definite if and only if it is the Fourier transform of a nonnegative tempered measure* μ.

2.4 Semigroup Theory

As before, assume $X = \{X_t\}_{t\geq 0}$ is a Lévy process on $\mathbb{R}^n$. It is easy to see that X is also a Markov process. Henceforth we shall assume for all the Lévy processes X, the distribution of X_t is absolutely continuos with respect to the Lebesgue measure with density given by p_t.

Definition 2.4.1 *The* transition semigroup *of X is the family of linear operators $\{P_t\}_{t\geq 0}$ on the space of bounded measurable functions defined as follows:*

$$\begin{aligned}(P_t\varphi)(x) = \mathbf{E}\left(\varphi\left(x + X_t\right)\right) &= \int_{\mathbb{R}^n} \varphi(y)p_t(y-x)dy \\ &= \left(\varphi * \tilde{p}_t\right)(x)\end{aligned}$$

where $\tilde{p}_t(x) = p_t(-x)$.

From the Markov property of Lévy processes, it is easy to see that $P_tP_s = P_{t+s}$, hence the name transition semigroup.

Definition 2.4.2 *The* resolvent operators *associated to $\{P_t\}_{t\geq 0}$ is defined as follows:*

$$R_\alpha = \int_0^\infty \mathrm{e}^{-\alpha s} P_s ds$$

If $\varphi : \mathbb{R} \to \mathbb{R}_+$ is Borel measurable, then

$$\left(R_\alpha\varphi\right)(x) = \int_0^\infty \varphi(y) r_\alpha(x-y)dy$$

$$= \left(\psi * \tilde{r}_\alpha\right)(x)$$

where $r_\alpha(x) = \int_0^\infty \mathrm{e}^{-\alpha t} p_t(x)dt$ for $\alpha \geq 0$ and $x \in \mathbb{R}$.

The *symmetrized Lévy process* $\overline{X}$ is defined as $\overline{X} = X + X^*$, where X^* is an independent copy of $-X$. The associated transition semigroup, transition densities and resolvents are denoted by $\{\overline{P}_t\}_{t\geq 0}, \{\overline{p}_t\}_{t\geq 0}$ and $\{\overline{R}_\alpha\}_{\alpha\geq 0}$ respectively. It is not hard to see that $\overline{P}_t = P_t^* P_t$, where P_t^* is the adjoint of P_t.

Definition 2.4.3 *The L^2-generator of the Lévy process X is defined as the linear operator $\mathcal{L}$ which acts as:*

$$\mathcal{L}\varphi = \lim_{t\to 0} \frac{P_t\varphi - \varphi}{t}$$

whenever the limit exists. The domain of $\mathcal{L}$ is defined as:

$$\mathrm{Dom}[\mathcal{L}] = \left\{ \varphi : \lim_{t\to 0} \frac{P_t\varphi - \varphi}{t} \text{ exists in } L^2 \right\}$$

The domain can also be defined as:

$$\mathrm{Dom}[\mathcal{L}] = \left\{ \varphi \in L^2(\mathbb{R}^n) : \Psi\widehat{\varphi} \in L^2(\mathbb{R}^n) \right\}$$

To see this, note that $\widehat{P_t\varphi} = \widehat{(\varphi * \tilde{p}_t)} = \widehat{\varphi}\widehat{\tilde{p}_t} = e^{-t\Psi}\widehat{\varphi}$. If $\lim_{t\to 0} \frac{P_t\varphi - \varphi}{t}$ exists in L^2, then so does $\lim_{t\to 0} \frac{\widehat{P_t\varphi} - \widehat{\varphi}}{t}$ due to Plancherel's theorem, and thus,

$$\lim_{t\to 0} \frac{\widehat{P_t\varphi} - \widehat{\varphi}}{t} = \lim_{t\to 0} \widehat{\varphi}\left(\frac{e^{-t\Psi} - 1}{t}\right) = -\Psi\widehat{\varphi}$$

Hence $\Psi\widehat{\varphi} \in L^2(\mathbb{R})$ and $\widehat{\mathcal{L}\varphi} = -\Psi\widehat{\varphi}$. Examples:

- the Laplacian $\frac{1}{2}\Delta$ is the generator of the standard Brownian motion.
- More generally, the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is the generator of the symmetric $\alpha-$stable process $(0 < \alpha < 2)$.

Chapter 3

Walsh Integration

3.1 Introduction

In this chapter, we will define the stochastic integral with respect to colored noise. The idea is in the same vein as the definition of the Itô integral: define it for a class of simple functions and then extend the definition to a larger class using an isometry. The material in this chapter is borrowed from [33].

3.2 Walsh Integral

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of all sets of finite measure in $\mathbb{R}^n$ and f be the Fourier transform of a nonegative tempered measure on $\mathbb{R}^n$.

Definition 3.2.1 *The* (spatially) colored noise F *on* $\mathbb{R}^n$ *is a mean zero Gaussian random field indexed by sets in* $\mathcal{B}(\mathbb{R}^n)$ *with covariance function given by*

$$\mathrm{Cov}(F(A), F(B)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{1}_A(x)\mathbf{1}_B(y) f(x-y) dx dy$$

It is easy to see that, f being positive-definite and symmetric implies $\mathrm{Cov}(\cdot, \cdot)$ being the same. Thus, by the general theory of Gaussian process, colored noise exists. In order to define stochastic integrals with respect to colored noise, we would require the *colored noise process* $\{F_t\}_{t\geq 0}$ defined as $F_t(A) = F([0,t]\times A)$, where $A \in \mathcal{B}(\mathbb{R}^n)$ and F is colored noise on $\mathbb{R}_+ \times \mathbb{R}^n$.

Let $\mathcal{F}$ denote filtration associated with the $\{F_t\}_{t\geq 0}$. That is, $\mathcal{F}_t$ is the $\sigma-$algebra generated by $\{F_s(A) : 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^n)\}$.

Lemma 3.2.1 $\{F_t(A)\}_{t\geq 0, A\in\mathcal{B}(\mathbb{R}^n)}$ *is a* martingale measure *in the sense that:*

- $\forall A \in \mathcal{B}(\mathbb{R}^n)$, $F_0(A) = 0$ *a.s.*
- $\forall t > 0$, *then* F_t *is a* σ*-finite,* $L^2(\mathbf{P})$*-valued signed measure*
- $\forall A \in \mathcal{B}(\mathbb{R}^n), \{F_t(A)\}_{t\geq 0}$ *is a mean-zero martingale.*

Proof. The proof is not hard and can be found in [33]. □

Henceforth, F will denote the martingale measure $\{F_t(A)\}_{t\geq 0, A\in\mathcal{B}(\mathbb{R}^n)}$.

Definition 3.2.2 *The covariance functional of the martingale measure* F *is defined as*

$$\overline{Q}_t(A, B) = \langle F(A), F(B)\rangle_t$$

where $\{\langle F(A), F(B)\rangle_t\}_{t\geq 0}$ *is the covariation process of the martingales* $\{F(A)_t\}_{t\geq 0}$ *and* $\{F(B)_t\}_{t\geq 0}$.

Now, define a random set function Q on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ as follows:

$$Q(A \times B \times (s,t]) = \overline{Q}_t(A, B) - \overline{Q}_s(A, B)$$

If $\{A_i \times B_i \times (s_i, t_i]\}_{i=1}^m$ are disjoint sets, then we can define

$$Q\Big(\bigcup_{i=1}^n (A_i \times B_i \times (s_i, t_i])\Big) = \sum_{i=1}^n Q\left(A_i \times B_i \times (s_i, t_i]\right)$$

In general, it is not possible to extend this definition of Q to all sets of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$. For that, we need the martingale measure to be *worthy*.

Definition 3.2.3 *A martingale measure* F *is* worthy *if there is a random* $\sigma-$*finite measure* $K(A\times B\times C, \omega)$ *on* $\mathcal{B}(\mathbb{R}^n)\times\mathcal{B}(\mathbb{R}^n)\times\mathcal{B}(\mathbb{R}_+)\times\Omega$ *called the* dominating measure, *such that:*

1. *the map $A \times B \mapsto K(A \times B \times C, \omega)$ is symmetric and nonnegative definite*

2. *$\forall t > 0$ and $A, B \in \mathcal{B}(\mathbb{R}^n)$ such that they are compact, we have $\mathbf{E}[K(A\times B\times(0,t])] < \infty$*

3. *$\forall t > 0$ and $A, B \in \mathcal{B}(\mathbb{R}^n)$, we have $|Q(A \times B \times (0,t])| \leq K(A \times B \times (0,t])$ a.s.*

Fortunately, all the martingale measures considered in this thesis will have $K = Q$.

Definition 3.2.4 *A function $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ is called* elementary *if there exists $0 \leq a < b$, $A \in \mathcal{B}(\mathbb{R}^n)$ and a bounded $\mathcal{F}_a-$measurable random variable X such that*

$$g(s, x, \omega) = \mathbf{1}_{(a,b]}(s)\mathbf{1}_A(x)X(\omega)$$

Denote the set of all finite linear combinations of elementary functions by $\mathcal{E}$. The $\sigma-$algebra generated by the elements of $\mathcal{E}$ is termed the *predictable* $\sigma-$algebra, and functions measurable with respect to this $\sigma-$algebra are called *predictable*, denoted by $\mathcal{P}$.

Definition 3.2.5 *For an elementary function g, the stochastic integral with respect to the martingale measure F is defined as,*

$$\int_0^t \int_B g dF = (g \cdot F)_t(B)(\omega) = X(\omega)[F_{t\wedge b}(A \cap B) - F_{t\wedge a}(A \cap B)](\omega) \tag{3.1}$$

This definition can be naturally extended to any $g \in \mathcal{E}$. It is easy to see that, if $g \in \mathcal{E}$, then $g \cdot F$ is a martingale measure.

Proposition 3.2.2 *For $g \in \mathcal{E}$,*

$$\mathbf{E}\Big(\big((g \cdot F)_t(B)\big)^2\Big) = \mathbf{E}\left(\int_0^t \int_B \int_B g(x,t)g(y,t)Q(dxdydt)\right) \tag{3.2}$$

Proof. First we do this when g is elementary. The case when $g \in \mathcal{E}$ follows easily.

$$\begin{aligned}\mathbf{E}\big((g \cdot F)_t^2(B)\big) &= \mathbf{E}\big(X^2 \left(F_{t\wedge b}(A \cap B) - F_{t\wedge a}(A \cap B)\right)^2\big) \\ &= \mathbf{E}\big(X^2 F_{t\wedge b}^2(A \cap B)\big) - 2\mathbf{E}\big(X^2 F_{t\wedge b}(A \cap B)F_{t\wedge a}(A \cap B)\big) + \mathbf{E}\big(X^2 F_{t\wedge a}^2(A \cap B)\big).\end{aligned}$$

Since X is $\mathcal{F}_a$ -measurable, we have by the definition of quadratic variation,

$$\mathbf{E}\big(X^2 \left(F_{t\wedge b}^2(A \cap B) - \langle F(A \cap B), F(A \cap B)\rangle_{t\wedge b}\right)\big) = \mathbf{E}\big(X^2 \left(F_{t\wedge a}^2(A \cap B) - \langle F(A \cap B), F(A \cap B)\rangle_{t\wedge a}\right)\big).$$

Using this in (3.2), gives

$$\mathbf{E}\big((g\cdot F)_t^2(B)\big)=\mathbf{E}\big(X^2\left(\langle F(A\cap B),F(A\cap B)\rangle_{t\wedge b}-\langle F(A\cap B),F(A\cap B)\rangle_{t\wedge a}\right)\big)$$

In other words, for elementary functions, we have

$$\mathbf{E}\left[\int_0^t\int_B gdF\right]^2=\mathbb{E}\left[\int_0^t\int_B\int_B g(s,x)g(s,y)Q(dxdyds)\right]$$

$\square$

In the present case, $Q(dxdydt)$ can be formally written as $f(x-y)dxdydt$. When F is space-time white noise, $Q(dxdydt)=\delta(x-y)dxdydt$. In that case, the noise will be denoted by the more standard notation W.

Here onwards, we shall fix a $T>0$ and study the processes in the time interval $[0,T]$. Before defining the stochastic integral for arbitrary predictable functions, define a norm on $\mathcal{P}$ as follows:

$$\|g\|_F=\left[\mathbf{E}\left(\int_0^T\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|g(x,t)g(y,t)|Q(dxdydt)\right)\right]^{1/2}.$$

Let $\mathcal{P}_F$ denote the set of predictable functions g which satisfies $\mathbf{E}\big(\|g\|_F\big)<\infty$. With this new notation, we can rewrite (3.2) as

$$\mathbf{E}\left[\int_0^T\int_{\mathbb{R}^n}gdF\right]^2\leq\|g\|_F^2. \tag{3.3}$$

Lemma 3.2.3 $\big(\mathcal{P}_F,\|\cdot\|_F\big)$ *is a Banach space and $\mathcal{E}$ is dense in it.*

Proof. See [33]. $\square$

Consequently, if $g\in\mathcal{P}_F$, then there exists a sequence $\{g_n\}_{n=0}^{\infty}$ in $\mathcal{E}$ such that $\|g-g_n\|_F\to 0$ as $n\to\infty$. From (3.3), the sequence $\int_0^T\int_{\mathbb{R}^n}g_ndF$ converges in L^2 as $n\to\infty$. We define this limit as $\int_0^T\int_{\mathbb{R}^n}gdF$.

Theorem 3.2.4 *Let F be a worthy martingale measure. Then $\forall g\in\mathcal{P}_F$, $\iint gdF$ is also a* *worthy martingale measure and*

$$Q_{g\cdot F}(dxdydt) = g(x,t)g(y,t)Q(dxdydt).$$

Moreover, for all $t \in (0,T]$ *and* $A, B \in \mathcal{B}(\mathbb{R}^n)$

$$\langle (g\cdot F)(A), (g\cdot F)(B)\rangle_t = \int_0^t \int_A \int_B g(x,s)g(y,s)Q(dxdyds)$$

$$\mathbf{E}\left[\int_0^T \int_{\mathbb{R}^n} g dF\right]^2 \leq \|g\|_F^2.$$

Proof. See [33]. □

Chapter 4

The Nonlinear Stochastic Heat Equation

4.1 Introduction

In this chapter, we derive an existence and uniqueness result for the solution to the following nonlinear stochastic heat equation:

$$\frac{\partial}{\partial t}u_t(x) = \left(\mathcal{L}u_t\right)(x) + b\left(u_t(x)\right) + \sigma\left(u_t(x)\right)\dot{F}_t(x) \quad \text{for } t > 0, x \in \mathbb{R}^n, \tag{4.1}$$

where $\mathcal{L}$ is the L^2-generator of a Lévy process with transition densities and $\dot{F}$ is a spatially colored and temporally white noise with covariance function $\delta(t-s)f(x-y)$. We also assume the initial data u_0 is a bounded measurable function and σ, b are globally Lipschitz functions on $\mathbb{R}^n$. Dalang, in [14] has given a sufficient condition for the existence and uniqueness of solutions to a large family of SPDEs with colored noise. In [18], Khoshnevisan and Foondun gave an equivalent potential theoretic condition for the same. While in [14] the author had assumed the initial function u_0 to be a constant, Khoshnevisan and Foondun have relaxed the assumption to u_0 just being bounded and measurable. This is the content of this chapter.

Recall from Chapter 3 the definition of the resolvent operators associated with the Lévy process generated by $\mathcal{L}$. Henceforth, we shall make the following assumption:

Assumption 4.1.1 $(\overline{R}_\alpha f)(0) < \infty$ *for all* $\alpha > 0$

Now we state the main result of this chapter:

Theorem 4.1.2 *Under Assumption 4.1.1,* (4.1) *has a.s. unique mild solution satisfying* $\sup_{[0,T]} \sup_{x\in\mathbb{R}^n} \mathbf{E}(|u_t(x)|^k) < \infty$ *for all* $T > 0$ *and* $k \geq 1$*, given by*

$$u_t(x) = (P_t u_0)(x) + \int_0^t \int_{\mathbb{R}^n} p_{t-s}(y-x) b\left(u_s(y)\right) dsdy + \int_0^t \int_{\mathbb{R}^n} p_{t-s}(y-x)\sigma\left(u_s(y)\right) F(dsdy) \tag{4.2}$$

where the second integral on the right hand side is to be understood in the Walsh sense and $\{P_t\}_{t\geq 0}$ *and* $\{p_t\}_{t\geq 0}$ *are the transition semigroup and densities of the Lévy process generated by* $\mathcal{L}$ *respectively.*

4.2 Proof Of Theorem 4.1.2

As is the case for many existence theorems for ODEs and PDEs, the proof follows by the Picard iterative scheme in a carefully chosen Banach space. In this vein, define for all predictable random fields u and real numbers $\beta, k > 0$

$$\|v\|_{\beta,k} = \sup_{t>0} \sup_{x\in\mathbb{R}^n} \left[\mathrm{e}^{-\beta t} \mathbf{E}\left(|v_t(x)|^k \right) \right]^{\frac{1}{k}}$$

Definition 4.2.1 *Let* $\mathbb{N}_{\beta,k}$ *denote the set of predictable random fields modulo indistinguishability such that* $\|v\|_{\beta,k} < \infty$*.*

It is easy to see that $\mathbb{N}_{\beta,k}$ with this norm is a Banach space. This will be the space for our Picard iteration procedure. Now define for all $v \in \mathbb{N}_{\beta,k}$,

$$(\mathcal{S}v)_t(x) = \int_0^t \int_{\mathbb{R}^n} p_{t-s}(y-x)\sigma\left(v_s(y)\right) F(dsdy)$$
$$(\mathcal{B}v)_t(x) = \int_0^t \int_{\mathbb{R}^n} p_{t-s}(y-x) b\left(v_s(y)\right) dsdy$$

The next two lemmas give estimates on $\|\mathcal{S}v\|_{\beta,k}$ and $\|\mathcal{B}v\|_{\beta,k}$ in terms of $\|v\|_{\beta,k}$.

Lemma 4.2.1 *Let u, v be predictable random fields. Then $\forall k \geq 2, \beta \in (0,\infty)$, we have:*

$$\|\mathcal{S}u\|_{\beta,k} \leq \mathrm{z}_k \left(|\sigma(0)| + L_\sigma \|u\|_{\beta,k}\right) \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)}$$
$$\|\mathcal{S}u - \mathcal{S}v\|_{\beta,k} \leq \mathrm{z}_p \, L_\sigma \|u - v\|_{\beta,k} \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)}$$

where L_σ is the Lipschitz constant of σ.

Proof. By the BDG inequality,

$$\begin{aligned}
\|(\mathcal{S}v)_t(x)\|_k^2 &\leq \mathrm{z}_k^2 \left\| \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_{t-s}(y-x) p_{t-s}(z-x) f(z-y) \sigma(u_s(y)) \sigma(u_s(z)) dydzds \right\|_{\frac{k}{2}} \\
&\leq \mathrm{z}_k^2 \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_{t-s}(y-x) p_{t-s}(z-x) f(z-y) \|\sigma(u_s(y)) \sigma(u_s(z))\|_{\frac{k}{2}} dydzds \\
&\leq \mathrm{z}_k^2 \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_{t-s}(y-x) p_{t-s}(z-x) f(z-y) \|\sigma(u_s(y))\|_k \|\sigma(u_s(z))\|_k dydzds,
\end{aligned}$$

where we have used Minkowski's integral inequality followed by Hölder's inequality in going from the first to third line. Now, $\|\sigma(u_s(y))\|_k, \|\sigma(u_s(z))\|_k \leq \mathrm{e}^{\frac{\beta s}{k}} \|\sigma(u)\|_{\beta,k} \leq \mathrm{e}^{\frac{\beta s}{k}} \left(|\sigma(0)| + L_\sigma \|u\|_{\beta,k}\right)$. Substituting this into the above equation and then doing a change of variables from $s \to t - s$

$$\|(\mathcal{S}v)_t(x)\|_k^2 \leq \mathrm{e}^{\frac{2\beta t}{k}} \left(|\sigma(0)| + L_\sigma \|u\|_{\beta,k}\right)^2 \mathrm{z}_k^2 \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{\frac{-2\beta s}{k}} p_s(y-x) p_s(z-x) f(z-y) dydzds$$

After some simple calculus, the second integral on the right hand side can be seen to be exactly $\overline{R}_{2\beta/k} f(0)$. Now raise both sides of the above equation to the power $\frac{k}{2}$, multiply by $\mathrm{e}^{-\beta t}$, raise again to the power $\frac{1}{k}$ and finally take the supremum over time and space to see that:

$$\|\mathcal{S}u\|_{\beta,k} \leq \mathrm{z}_k \left(|\sigma(0)| + L_\sigma \|u\|_{\beta,k}\right) \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)}$$

. □

Lemma 4.2.2 *Let u, v be predictable random fields. Then $\forall k \geq 2, \beta \in (0,\infty)$, we have:*

$$\|\mathcal{B}u\|_{\beta,k} \leq \frac{k}{\beta} \left(\frac{|b(0)|}{\mathrm{e}} + L_b \|u\|_{\beta,k} \right)$$
$$\|\mathcal{B}u - \mathcal{B}v\|_{\beta,k} \leq \frac{k L_b}{\beta} \|u - v\|_{\beta,k}$$

where L_b is the Lipschitz constant of b.

The proof of the above lemma is similar to that of the previouus one and we shall skip it.

We now begin the iterative procedure to prove the existence of the solution. Set $u_t^0(x) = u_0(x)$ and for all $n \geq 0$, define:

$$u_t^{n+1}(x) = (P_t u_0)(x) + (\mathcal{B}u^n)_t(x) + (\mathcal{S}u^n)_t(x)$$

Lemma 4.2.3 *Choose and fix a $\beta > 0$ and a $k \geq 2$ such that:*

$$\frac{kL_b}{\beta} + \mathrm{z}_p\, L_\sigma \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)} < 1$$

Then $\sup_{n\geq 0} \|u^n\|_{\beta,k} < \infty$.

Proof. Since u_0 is bounded, so is $P_t u_0$. By the triangle inequality,

$$\left\|u^{n+1}\right\|_{\beta,k} \leq \sup_{x\in\mathbb{R}^n} |u_0(x)| + \|\mathcal{B}u^n\|_{\beta,k} + \|\mathcal{S}u^n\|_{\beta,k}$$

By Lemmas 4.2.1 and 4.2.2, we have

$$\left\|u^{n+1}\right\|_{\beta,k} \leq \sup_{x\in\mathbb{R}^n} |u_0(x)| + \frac{k|b(0)|}{\beta \mathrm{e}} + \mathrm{z}_p\, |\sigma(0)| \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)} + \left(\frac{kL_b}{\beta} + \mathrm{z}_p\, L_\sigma \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)}\right) \|u^n\|_{\beta,k}$$

Repeating this procedure and using the fact that $\frac{kL_b}{\beta} + \mathrm{z}_p\, L_\sigma \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)} < 1$,

$$\sup_{k\geq 1} \left\|u^k\right\|_{\beta,k} \leq \frac{\sup_{x\in\mathbb{R}^n} |u_0(x)| + \frac{k|b(0)|}{\beta\mathrm{e}} + \mathrm{z}_p\, |\sigma(0)| \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)}}{1 - \frac{kL_b}{\beta} - \mathrm{z}_p\, L_\sigma \sqrt{\left(\overline{R}_{2\beta/k} f\right)(0)}} < \infty$$

Since $\|u^0\|_{\beta,k} \leq \sup_{x\in\mathbb{R}^n} |u_0(x)| < \infty$, the lemma follows. $\square$

The above lemma says that $u^n \in \mathbb{N}_{\beta,k}$ for all $n \geq 0$. Now we apply Lemmas 4.2.1 and

4.2.2 see that

$$\begin{aligned}\left\|u^{n+1}-u^{n}\right\|_{\beta,k} &\leq \left\|\mathcal{B}u^{n}-\mathcal{B}u^{n-1}\right\|_{\beta,k}+\left\|\mathcal{S}u^{n}-\mathcal{S}u^{n-1}\right\|_{\beta,k} \\ &\leq \left\|u^{n}-u^{n-1}\right\|_{\beta,k}\cdot\left(\frac{kL_b}{\beta}+\mathrm{z}_p\,L_\sigma\sqrt{\left(\overline{R}_{2\beta/k}f\right)(0)}\right).\end{aligned}$$

Because $\frac{kL_b}{\beta}+\mathrm{z}_p\,L_\sigma\sqrt{\left(\overline{R}_{2\beta/k}f\right)(0)}<1$, the above implies that:

$$\sum_{n=1}^{\infty}\left\|u^{n+1}-u^{n}\right\|_{\beta,k}\leq C\cdot\sum_{n=1}^{\infty}\left(\frac{kL_b}{\beta}+\mathrm{z}_p\,L_\sigma\sqrt{\left(\overline{R}_{2\beta/k}f\right)(0)}\right)^{n}<\infty,$$

where C is some constant. Since $\mathbb{N}_{\beta,k}$ is a Banach space, this implies that there exists a predictable $u\in\mathbb{N}_{\beta,k}$ such that

$$\lim_{n\to\infty}\|u^n-u\|_{\beta,k}=0.$$

Again, by Lemmas 4.2.1 and 4.2.2 we have,

$$\|\mathcal{B}u^n-\mathcal{B}u\|_{\beta,k}\leq\frac{kL_b}{\beta}\|u^n-u\|_{\beta,k}$$

$$\|\mathcal{S}u^n-\mathcal{S}u\|_{\beta,k}\leq \mathrm{z}_p\,L_\sigma\|u^n-u\|_{\beta,k}\sqrt{\left(\overline{R}_{2\beta/k}f\right)(0)}.$$

Taking limits as $n\to\infty$, we see that $\mathcal{S}u^n\to\mathcal{S}u$ and $\mathcal{B}u^n\to\mathcal{B}u$ in $\mathbb{N}_{\beta,k}$. Consequently, $u=P_tu_0+\mathcal{B}u+\mathcal{S}u$ and hence u solves (4.1). As for uniqueness, say u_1 and u_2 are two solutions to (4.1). Then, by Lemmas 4.2.1 and 4.2.2,

$$\|u_1-u_2\|_{\beta,k}\leq\|\mathcal{B}u_1-\mathcal{B}u_2\|_{\beta,k}+\|\mathcal{S}u_1-\mathcal{S}u_2\|_{\beta,k}\leq\left(\frac{kL_b}{\beta}+\mathrm{z}_p\,L_\sigma\sqrt{\left(\overline{R}_{2\beta/k}f\right)(0)}\right)\|u_1-u_2\|_{\beta,k}$$

As $\beta\to\infty$, $\left(\overline{R}_{2\beta/k}f\right)(0)\to 0$ and hence $\|u_1-u_2\|_{\beta,k}=0$ for large β implying $u_1=u_2$ in $\mathbb{N}_{\beta,k}$.

Chapter 5

A Comparison Principle

5.1 Introduction

It is well known in the theory of partial differential equations that the heat equation satisfies the comparison principle. That is, if u and v are two functions that solve the heat equation with $u_0(x) \leq v_0(x)$, then $u \leq v$ at all times. A similar theorem exists for finite dimensional SDEs (see [24]). In this chapter, we prove the same theorem for white noise SPDEs, originally due to Mueller [30, 15]. Consider the following one-dimensional stochastic heat equation with a nonnegative, bounded and continuous initial function u_0:

$$\frac{\partial}{\partial t} u_t(x) = \frac{\partial^2}{\partial x^2} u_t(x) + \sigma\left(u_t(x)\right) \dot{W}_t(x) \quad \text{for } t > 0, x \in \mathbb{R}. \tag{5.1}$$

In addition to the usual Lipschitz condition, we assume $\sigma(0) = 0$.

The following theorem is the main result of this chapter:

Theorem 5.1.1 *Let $u_1(t,x) \equiv (u_1)_t(x)$ and $u_2(t,x) \equiv (u_2)_t(x)$ be solutions to* (5.1) *such that $u_1(0,x) \leq u_2(0,x)$. Then,*

$$u_1(t,x) \leq u_2(t,x) \quad a.s.$$

5.2 Proof Of Theorem 5.1.1

First, we prove the theorem when the initial data is of compact support. We then show how to relax this assumption. The main idea is to discretize the SPDE in time and space and then use the comparison theorem for SDEs.

First, we discretize space: for $n \in \mathbb{N}$, we divide the real line into intervals of the form $I_{n,k} = \left\{\frac{2k-1}{2n} \leq x < \frac{2k+1}{2n}\right\}$, where k runs over the integers. Let $\Delta^{(n)}$ be the discrete Laplacian operator defined as:

$$\Delta^{(n)}\varphi\left(\frac{k}{n}\right) = n^2\left[\varphi\left(\frac{k+1}{n}\right) - 2\varphi\left(\frac{k}{n}\right) + \varphi\left(\frac{k-1}{n}\right)\right]$$

for an appropriate class of functions and let $\bar{G}^{(n)}(t, \frac{j}{n}, \frac{k}{n})$ be the fundamental solution to the discrete heat equation:

$$\frac{\partial \bar{G}}{\partial t} = \Delta^{(n)}\bar{G}$$

with initial conditions:

$$\overline{G}^{(n)}(0, \frac{j}{n}, \frac{k}{n}) = n^2\delta_{j,k}$$

We extend $\overline{G}$ to arbitrary real numbers x, y by defining:

$$\overline{G}^{(n)}(t, x, y) = \overline{G}^{n}(t, \frac{k}{n}, \frac{l}{n}) \quad \text{if } \frac{2k-1}{2n} \leq x < \frac{2k+1}{2n} \text{ and } \frac{2l-1}{2n} \leq y \leq \frac{2l+1}{2n}$$

Next, we discretize time: for each $n \in \mathbb{N}$, we divide the time axis $t > 0$ into intervals of the form $T_{n,j} = \left\{\frac{j}{n^2} \leq t < \frac{j+1}{n^2}\right\}$, where j runs over the nonnegative integers. Set

$$G^{(n)}(t, s, x, y) = \overline{G}^{(n)}\left(\frac{[n^2t] - [n^2s]}{n^2}, x, y\right) \quad \text{for } s, t > 0$$

and let $u^{(n)}(t, x)$ be the solution to the integral equation:

$$u^{(n)}(t, x) = \int_{-\infty}^{\infty} G^{(n)}(t, 0, x, y)u^{(n)}(0, y)dy + \int_0^t \int_{-\infty}^{\infty} G^{(n)}(t, s, x, y)\sigma(u^{(n)}(s, y))W(dsdy) \tag{5.2}$$

with the initial condition

$$u^{(n)}(0,x) = n\int_{\frac{2k-1}{2n}}^{\frac{2k+1}{2n}} u(0,y)dy \quad \frac{2k-1}{2n} \leq x < \frac{2k+1}{2n}.$$

Given $t \in T_{n,j}$ we have $[n^2 t] = j$, thus $G^{(n)}(t,0,x,y) = \overline{G}^{(n)}\left(\frac{j}{n^2},x,y\right)$. In addition, for $0 \leq s \leq \frac{j}{n^2}$, we have $G(t,s,x,y) = G\left(\frac{j}{n^2},s,x,y\right)$. Hence we may re-write (5.2) as:

$$u^{(n)}(t,x) = u^{(n)}\left(\frac{j}{n^2},x\right) + \int_{\frac{j}{n^2}}^{t}\int_{\mathbb{R}} \overline{G}^{(n)}(t,s,x,y)\sigma\left(u^{(n)}(s,y)\right)W(dsdy) \tag{5.3}$$

Next, for $x \in I_{n,k}, t \in T_{n,j}$, and $\frac{j}{n^2} \leq s \leq t$, we have:

$$G^{(n)}(t,s,x,y) = \overline{G}^{(n)}\left(0,\frac{k}{n},y\right) = n\mathbf{1}_{I_{n,k}}(y)$$

Equation (5.3) then becomes

$$u^{(n)}(t,x) = u^{(n)}\left(\frac{j}{n^2},x\right) + n\int_{\frac{j}{n^2}}^{t}\int_{I_{n,k}} \sigma\left(u^{(n)}\left(s,\frac{k}{n}\right)\right)W(dsdy)$$

This can be written as

$$du^{(n)}(t,x) = \sqrt{n}\sigma\left(u^{(n)}(t,x)\right)dB_k \tag{5.4}$$

where

$$B_k(t) = \sqrt{n}\int_0^t\int_{I_{n,k}} W(dsdy)$$

is a standard Brownian motion by Lévy's Characterisation Theorem (2.1.2). By assumption, $u_1^n(0,x) \leq u_2^n(0,x)$. By the standard existence and uniqueness result for SDEs, we can solve (5.4) in $T_{n,j}$. Since, σ is globally Lipschitz, we can extrapolate this solution to all times. We then use the comparison theorem for finite dimensional diffusions [24] to yield $u_1^n(t,x) \leq u_2^n(t,x)$. The proof involves a lot of computations and can be found in [30]. Now, we use the following lemma to conclude that $u_1(t,x) \leq u_2(t,x)$.

Lemma 5.2.1 *Let $u(0,x) : \mathbb{R} \to \mathbb{R}_+$ be a continuous function which is compactly supported. Then,*

$$\lim_{k\to\infty}\left|u^{(n_k)}(t,x) - u(t,x)\right| = 0 \quad \forall t > 0, x \in \mathbb{R} \quad a.s.$$

for some subsequence $\{n_k\}$.

Now let $u_1(0,x)$ and $u_1(0,x)$ be two bounded continuous initial functions not necessarily of compact support. Introduce the following functions:

$$u_1^{[N]}(0,x) = \begin{cases} u_1(0,x), & \text{if } |x| \leq N \\ u_1(0,N)(-x+N+1), & \text{if } N < x < N+1, \\ u_1(0,-N)(x+N+1), & \text{if } -(N+1) < x < -N, \\ 0 & \text{if } |x| \geq N+1. \end{cases}$$

and an analogous one for u_2. Clearly, $u_1^{[N]}(0,x)$ and $u_2^{[N]}(0,x)$ have compact support for every N. By the comparison principle above, $u_1^{[N]}(t,x) \leq u_2^{[N]}(t,x)$. We can then use the following lemma to conclude that $u_1(t,x) \leq u_1(t,x)$. For a proof, see [10].

Lemma 5.2.2 *Let $u_1^{[N]}, u_2^{[N]}$ be as defined above. Then,*

$$\lim_{k\to\infty} |u_1^{[N_k]}(t,x) - u(t,x)| = 0 \quad \forall t > 0, x \in \mathbb{R} \quad a.s.$$

for some subsequence $\{N_k\}$. The same holds for u_2.

Chapter 6

The Parabolic Anderson Model

6.1 Introduction

In this chapter, we study a special case of the stochastic heat equation (3.2) called the parabolic Anderson model, given by:

$$\frac{\partial}{\partial t}u_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u_t(x) + \lambda u_t(x)\dot{W}_t(x) \quad \text{for } t > 0, x \in \mathbb{R},\ \lambda \neq 0. \tag{6.1}$$

with a bounded and measurable initial function u_0.

This equation arises in a number of quite different scientific scenarios, especially in statistical physics where it is connected to the KPZ equation (see [26, 31]). From the results of Chapter 4, we see that (6.1) has an a.s. unique mild solution whose $k-$th moment grows at most $\mathrm{e}^{k^3 t}$. In this chapter, we show that this growth rate is sharp. The following theorem due to Bertini and Cancrini [3] captures this phenomenon.

Theorem 6.1.1 *If u_0 is the constant function, then $\forall k \geq 1, t \geq 0$ and $x \in \mathbb{R}$,*

$$\mathbf{E}\left(|u_t(x)|^k\right) \geq c^k \mathrm{e}^{\left(\frac{\lambda^4 k(k^2-1)t}{24}\right)}.$$

6.2 Proof Of Theorem 6.1.1

Consider the situation when the white noise $\dot{W}_t(x)$ is replaced by a non-random function $G_t(x)$ which satisfies some growth conditions. To wit,

$$\frac{\partial}{\partial t}U_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}U_t(x) + \lambda U_t(x)G_t(x) \tag{6.2}$$

with initial condition U_0. Then, the Feynman-Kac formula [32] tells us that the solution to the above equation can be represented in terms of a standard Brownian motion. That is,

$$U_t(x) = \mathbf{E}\left(U_0\left(x + B_t\right) \cdot \mathrm{e}^{\left(\lambda \int_0^t G_{t-s}(B_s+x)ds\right)}\right). \tag{6.3}$$

Consequently, whenever $x_1, x_2 \cdots x_k \in \mathbb{R}$, we have

$$\prod_{j=1}^{k} U_t\left(x_j\right) = \mathbf{E}\left(\prod_{j=1}^{k} U_0\left(x_j + B_t^{(j)}\right) \cdot \mathrm{e}^{\lambda \sum_{j=1}^{k} \int_0^t G_{t-s}\left(B_s^{(j)}+x_j\right)ds}\right) \tag{6.4}$$

where $\{B^{(j)}\}_{j=0}^{k}$ are i.i.d standard Brownian motions.
Let p_ε, p_η be smooth approximations to identity. Consider a Gaussian random field G with covariance function given by

$$\mathbf{E}\left[G_t(x) \cdot G_s(y)\right] = p_\varepsilon(s-t) \cdot p_\eta(x-y) \quad \text{for all } s,t \geqslant 0 \text{ and } x,y \in \mathbb{R}. \tag{6.5}$$

As ε and $\eta \to 0$, p_ε and p_η both tend to the Dirac distribution. Therefore, intuitively, the random field G converges to the space-time white noise in distribution. This will be the approach in proving Theorem 6.1.1. Since the covariance function in (6.5) is sufficiently regular, by the general theory on Gaussian random fields, we can conclude that G is sufficiently regular too. Hence, by using conditional expectations and a Gaussian variance formula in (6.4), we get

$$\mathbf{E}\left(\prod_{j=1}^{k} U_t\left(x_j\right)\right) = \mathbf{E}\left(\prod_{j=1}^{k} U_0\left(x_j + B_t^{(j)}\right) \cdot \mathrm{e}^{\left(\frac{\lambda^2}{2} \cdot \mathrm{Var}\left(\sum_{i=1}^{k} \int_0^t G_{t-s}\left(B_s^{(j)}+x_j\right)ds\right)\right)} \mid \sigma\left(B_t^i : t \geq 0, a \leq i \leq k\right)\right) \tag{6.6}$$

$$\mathrm{Var}\left(\sum_{i=1}^{k} \int_0^t G_{t-s}\left(B_s^{(i)} + x_i\right) ds \mid \sigma\left(B_t^i : t \geq 0, a \leq i \leq k\right)\right) = \sum_{i=1}^{k}\sum_{j=1}^{k} \int_0^t ds \int_0^t dr p_\varepsilon(r-s) p_\eta\left(B_s^{(j)} - B_r^{(i)} - \right. \tag{6.7}$$

Letting $\varepsilon \to 0$, we see

$$\operatorname{Var}\left(\sum_{i=1}^{k}\int_0^t G_{t-s}\left(B_s^{(i)}+x_i\right)ds \mid \sigma\left(B_t^i : t \geq 0, a \leq i \leq k\right)\right) = \frac{tk}{\sqrt{2\pi\eta}} + 2\sum_{1\leqslant i<j\leqslant k}\int_0^t p_\eta\left(B_s^{(j)} - B_s^{(i)} - x_i + x\right. \tag{6.8}$$

Since $B^{(i)}$ and $B^{(j)}$ are independent, $X_s^{(j,i)} = B_s^{(j)} - B_s^{(i)}$ is also a Brownian motion. Therefore, the occupation density formula yields,

$$\operatorname{Var}\left(\sum_{i=1}^{k}\int_0^t G_{t-s}\left(B_s^{(i)}+x_i\right)ds \mid \mathcal{B}\right) \approx \frac{tk}{\sqrt{2\pi\eta}} + 2\sum_{1\leqslant i<j\leqslant k} L_t^{x_i-x_j}\left(X^{(i,j)}\right) \tag{6.9}$$

whenever $\eta \approx 0$. Therefore,

$$\mathbf{E}\left(\prod_{j=1}^{k} U_t\left(x_j\right)\right) \approx \mathrm{e}^{\lambda^2 tk/\sqrt{8\pi\eta}}\,\mathbf{E}\left(\prod_{\ell=1}^{k} U_0\left(x_\ell + B_t^{(\ell)}\right)\mathrm{e}^{\left(\lambda^2\sum_{1\leqslant i<j\leqslant k} L_t^{x_i-x_j}\left(X^{(i,j)}\right)\right)}\right). \tag{6.10}$$

Theorem 6.2.1 *For every integer $k \geqslant 1$, and for all real numbers $t > 0$ and $x_1, \ldots, x_k$, the solution u to (6.1) satisfies*

$$\mathbf{E}\left(\prod_{j=1}^{k} u_t\left(x_j\right)\right) = \mathbf{E}\left(\prod_{\ell=1}^{k} u_0\left(x_\ell + B_t^{(\ell)}\right)\cdot\exp\left[\lambda^2\sum_{1\leqslant i<j\leqslant k}\sum_{t}^{x_i-x_j}\left(B^{(j)} - B^{(i)}\right)\right]\right) \tag{6.11}$$

where $B^{(1)}, \ldots, B^{(k)}$ are i.i.d. standard Brownian motions.

For a proof of the above theorem, see [3, 23, 9]. We will use this to prove Theorem 6.1.1. Before that, we state as simple combinatorial lemma without proof.

Lemma 6.2.2 *Fix an integer $k \geqslant 2$, and let $x_1, \ldots, x_k$ denote k distinct real numbers. Then,*

$$\sum_{j=1}^{k}\left[\sum_{\substack{1\leqslant i\leqslant k\\ i\neq j}}\left(2 1_{\{x_j > x_i\}} - 1\right)\right]^2 = \frac{k\left(k^2-1\right)}{3}.$$

Now we come to the proof of Theorem 6.1.1.
Define:

$$\mathcal{L}_t := \sum_{1\leqslant i<j\leqslant k} L_t^0\left(B^{(j)} - B^{(i)}\right) \quad (t \geqslant 0). \tag{6.12}$$

Substituting $x_1 = x_2 \cdots x_k = x$ in (6.11), we get

$$\mathbf{E}\left(\left(u_t(x)\right)^k\right) = c^k \mathbf{E}\left(\mathrm{e}^{\lambda^2 \mathcal{L}_t t}\right) \tag{6.13}$$

Now let

$$M_t = \sum_{1 \leqslant i < j \leqslant k} W_t^{(j,i)}, \text{ where } W_t^{(j,i)} = \int_0^t \operatorname{sgn}\left(B_s^{(j)} - B_s^{(i)}\right) d\left(B^{(j)} - B^{(i)}\right)_s. \tag{6.14}$$

Then by Tanaka's formula, it is not hard to see that $\mathcal{L}_t \geqslant -\frac{1}{2} M_t$ and hence $\mathbf{E}\left(\mathrm{e}^{\lambda^2 \mathcal{L}_t}\right) \geqslant \mathbf{E}\left(\mathrm{e}^{-\frac{\lambda^2 M_t}{2}}\right)$. Because $W_t^{(j,i)} = W_t^{(i,j)}$ and $W_t^{(i,i)} = 0$, we can write

$$M_t = \frac{1}{2} \sum_{j=1}^k \sum_{i=1}^k W_t^{(j,i)} = \sum_{j=1}^k \int_0^t \sum_{i=1}^k \operatorname{sgn}\left(B_s^{(j)} - B_s^{(i)}\right) dB_s^{(j)}.$$

M_t is seen to be continuous square-integrable martingale with quadratic variation

$$\langle M \rangle_t = \frac{k\left(k^2 - 1\right)}{3} t. \tag{6.15}$$

Therefore, by Levy's characterization theorem, $M_t = \frac{k\left(k^2-1\right)}{3} B_t$, for some standard Brownian motion. By raising both sides to the power of e and taking expectations, the result follows.

Chapter 7

Intermittency

7.1 Introduction

In this chapter, we focus on the concept of *intermittency* in the context of the stochastic heat equation. A random field $\{u_t(x)\}_{t>0,x\in\mathbb{R}^n}$ is said to be *physically intermittent* if, as t gets large, $u_t(x)$ develops large peaks, atypical of its expected behaviour.

Recall the (nonlinear) stochastic heat equation:

$$\frac{\partial}{\partial t}u_t(x) = (\mathcal{L}u_t)(x) + \sigma(u_t(x))\dot{F}_t(x) \quad \text{for } t > 0, x \in \mathbb{R}^n, \tag{7.1}$$

where $\dot{F}$ is a spatially-colored temporally-white noise, $\mathcal{L}$ is the generator of a Levy process with transition densities, $\sigma : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous, and the initial function u_0 is bounded and measurable. In order to translate the physical definition of intermittency into mathematical terms, we introduce the notion of *Lyapunov exponents*:

$$\overline{\gamma}_k(x) = \limsup_{t\to\infty} \frac{\log \mathbf{E}\big(|u_t(x)|^k\big)}{t}. \tag{7.2}$$

Definition 7.1.1 *A random field u is* (fully) intermittent *if, $\forall x \in \mathbb{R}^n$, the map $k \to \frac{\overline{\gamma}_k(x)}{k}$ is* strictly *increasing $\forall k \geq 2$.*

Note that by Jensen's inequality, the function $k \to \|u_t(x)\|_k$ and hence $k \to \frac{\overline{\gamma_k}(x)}{k}$ is always non-decreasing. So intermittency only occurs when it is *strictly* increasing. To see how this mathematical definition explains the physical intermittency, the reader is referred to [7].

The following proposition yields a useful method to prove intermittency:

Proposition 7.1.1 *If $\overline{\gamma}_k(x) < \infty$ for all k and $x \in \mathbb{R}^n$, then it is convex (w.r.t k on $(0,\infty)$). Furthermore, if $\overline{\gamma}_{k_0}(x) > 0$ and $\overline{\gamma}_1(x) = 0$ for some $k_0 > 1$ and all $x \in \mathbb{R}^n$, then the function $k \to \frac{\overline{\gamma}_k(x)}{k}$ is strictly increasing on $(0,\infty)$ for all $x \in \mathbb{R}^n$.*

Proof. For any $\alpha \in (0,1)$, $p > 1$ and $a, b > 0$, by Hölder's inequality,

$$\mathbf{E}\big(|u_t(x)|^{\alpha a+(1-\alpha)b}\big) \leq \Big(\mathbf{E}\big(|u_t(x)|^{p\alpha a}\big)\Big)^{\frac{1}{p}} \Big(\mathbf{E}\big(|u_t(x)|^{q(1-\alpha)b}\big)\Big)^{\frac{1}{q}},$$

where $q = \frac{p}{p-1}$ is the Hölder conjugate of p. Now substitute $p = \frac{1}{\alpha}$, take logarithms on both sides, divide by t and take the limit as $t \to \infty$ to see that:

$$\overline{\gamma}_{\alpha a+(1-\alpha)b}(x) \leq \alpha\overline{\gamma}_a(x) + (1-\alpha)\overline{\gamma}_b(x).$$

Hence the function $\overline{\gamma}_k(x)$ is convex on $(0,\infty)$. Now let $\overline{\gamma}_{k_0}(x) > 0$ for some $k_0 > 1$ and all $x \in \mathbb{R}^n$. Without any loss of generality, assume $k_1 > k_2 \geq k_0$. Then k_2 can be written as a convex linear combination of k_1 and 1. That is $k_2 = \lambda k_1 + (1-\lambda)$, where $\lambda = \frac{k_2-1}{k_1-1}$. As $\overline{\gamma}_1(x) = 0$, we use the convexity of $\overline{\gamma}$ to see that

$$\overline{\gamma}_{k_2}(x) \leq \frac{k_2-1}{k_1-1}\overline{\gamma}_{k_1}(x).$$

Since $\frac{k_2-1}{k_1-1} < \frac{k_2}{k_1}$, the conclusion follows. $\square$

Thus, we see that, in order to show that u is intermittent, it suffices to show the much easier condition $\overline{\gamma}_2(x) > 0$ for all $x \in \mathbb{R}^n$ provided that $\overline{\gamma}_k(x) < \infty$ for all k and $x \in \mathbb{R}^n$ and $\overline{\gamma}_1$ is identically zero. This brings us to the next defintion.

Definition 7.1.2 *A random field u is* weakly intermittent *if $\overline{\gamma}_2(x) > 0$ for all $x \in \mathbb{R}^n$.*

In [18], the authors have proved a sufficient condition for the solution to (7.1) to be weakly intermittent under some additional assumptions on the covariance function f of the noise. This will be the main content of this chapter.

Henceforth, we shall assume the following about the covariance function f:

Assumption 7.1.2 • *$\widehat{f}(\xi)$ is a function of $|\xi_1|, \ldots, |\xi_n|$*

- *For $j = 1, \ldots, n$, the map $|\xi_j| \mapsto \widehat{f}(\xi)$ is nonincreasing*

- *$\mathrm{Re}\,\Psi(\xi)$ is a function of $|\xi_1|, \ldots, |\xi_n|$.*

Some examples of covariance functions which satisfy this condition include:

- Riesz kernels: $f(x) = \frac{c_1}{|x|^\beta}$ and $\widehat{f}(\xi) = \frac{c_2}{|\xi|^{n-\beta}}$ where $\beta < n$
- Poisson kernels: $f(x) = \frac{c_1}{\left(|x|^2+c_2\right)^{\frac{n+1}{2}}}$ and $\widehat{f}(\xi) = c_3 \mathrm{e}^{-c_4|\xi|}$
- White noise: $f(x) = \delta(x)$ and $\widehat{f}(\xi) = c_1$

Now we state the main theorem of this chapter:

Theorem 7.1.3 *The solution to* (7.1) *satisfies*

$$\overline{\gamma}_k(x) \leq \inf\left\{\beta > 0 : \mathrm{z}_k\, L_\sigma \sqrt{\left(\overline{R}_{\frac{2\beta}{k}} f\right)(0)} < 1\right\}$$

for all $k \geq 2$ and $x \in \mathbb{R}^n$. Here, L_σ is the Lipschitz constant of σ.

Theorem 7.1.4 *Suppose Assumption* (7.1.2) *holds and $\alpha = \inf_{x\in\mathbb{R}^n} u_0(x) > 0$. Also, assume there exists $l_\sigma > 0$ such that $\sigma(x) \geq l_\sigma |x|$ for all $x \in \mathbb{R}^n$. Then,*

$$\inf_{x\in\mathbb{R}^n} \overline{\gamma}_2(x) \geq \sup\left\{\beta > 0 : \left(\overline{R}_\beta f\right)(0) \geq \frac{2^{n-1}}{l_\sigma^2}\right\}$$

Theorem 7.1.3 tells that $\overline{\gamma}_k(x) < \infty\ \forall k \geq 2$ and $x \in \mathbb{R}^n$. Theorem 7.1.4 tells that $\overline{\gamma}_2(x) > 0\ \forall x \in \mathbb{R}^n$. If $u_t(x) > 0\ \forall t > 0$ and $x \in \mathbb{R}^n$, then $\|u\|_{\beta,1} < \infty$ would imply $\overline{\gamma}_1(x) = 0\ \forall x \in \mathbb{R}^n$. Thus, Theorem 7.1.4 proves the solution to (7.1) is intermittent provided

$u_t(x) > 0\ \forall t > 0$ and $x \in \mathbb{R}^n$.. In the case of space-time white noise and $\mathcal{L} = \Delta$, this automatically follows by the Comparison Principle (Chapter 5) as $u_0(x) \geq \alpha > 0\ \forall x \in \mathbb{R}^n$.

7.2 Proof Of Theorem 7.1.4

By our assumption that $u_0 \geq \alpha$ and $\sigma(x) \geq l_\sigma |x|$, we have for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
\mathbf{E}(|u_t(x)u_t(y)|) &\geq \mathbf{E}(u_t(x)u_t(y)) \\
&\geq \alpha^2 + \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{E}\left(\sigma\left(u_s(x')\right)\sigma\left(u_s\left(y'\right)\right)\right) p_{t-s}(x'-x) p_{t-s}\left(y'-y\right) f\left(x'-y'\right) dx'dy'ds \\
&\geq \alpha^2 + l_\sigma^2 \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{E}(|u_s(x')u_s(y')|) p_{t-s}(x'-x) p_{t-s}\left(y'-y\right) f\left(x'-y'\right) dx'dy'ds
\end{aligned} \tag{7.3}$$

Define the following functions on $\mathbb{R}^n \times \mathbb{R}^n$:

$$H_\beta(x,y) = \int_0^\infty \mathrm{e}^{-\beta t} \mathbf{E}\left(|u_t(x)u_t(y)|\right) dt, \tag{7.4}$$

$$G_\beta(x,y) = \int_0^\infty \mathrm{e}^{-\beta t} p_t(x) p_t(y) dt, \tag{7.5}$$

$$F(x,y) = f(x-y). \tag{7.6}$$

Now consider a linear operator: $\mathbf{L}_\beta$ acting on H_β as

$$(\mathbf{L}_\beta H_\beta)(x,y) = (F H_\beta * \tilde{G})(x,y).$$

This can be explicitly calculated as:

$$\begin{aligned}
(\mathbf{L}_\beta\, H_\beta)(x,y) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(x',y') H_\beta(x',y') \tilde{G}_\beta(x-x', y-y') dx'dy' \\
&= \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{-\beta t} f(x'-y') \mathbf{E}\left(|u_t(x')u_t(y')|\right) p_t(x'-x) p_t(y'-y) dx'dy'dt.
\end{aligned}$$

This is exactly the second term in (7.3). Hence we can rewrite (7.3) as

$$H_\beta \geq \frac{\alpha^2}{\beta} + l_\sigma^2 \mathbf{L}_\beta H_\beta$$

Applying this inequality repeatedly,

$$H_\beta \geq \frac{\alpha^2}{\beta} \sum_{i=0}^{\infty} l_\sigma^{2i}\, \mathbf{L}_\beta^i\,.$$

The first term of this infinite series of functions is clearly $\frac{\alpha^2}{\beta}\times$ the constant function. The second term is much more involved. It is

$$\begin{aligned}
l_\sigma^2\, \mathbf{L}_\beta(x,y) &= \left(F * \tilde{G}_\beta\right)(x,y) \\
&= \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{-\beta t} f(x'-y') p_t(x'-x) p_t(y'-y) dx' dy' dt \\
&\geq \frac{1}{(2\pi)^n} \int_0^\infty \int_{\mathbb{R}^n} \mathrm{e}^{-\beta t} \widehat{f}(\xi) \mathrm{e}^{-2t \operatorname{Re}\Psi(\xi)} \mathrm{e}^{i\xi\cdot(x-y)} d\xi dt
\end{aligned}$$

where we have used Plancherel's theorem. Using Fubini's theorem yields

$$l_\sigma^2\, \mathbf{L}_\beta(x,y) \geq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{i\xi\cdot(x-y)} \widehat{f}(\xi)}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi.$$

By induction, it is not hard to see that the $k-$th term in the series is

$$l_\sigma^2\, \mathbf{L}_\beta^k(x,y) \geq \frac{1}{(2\pi)^{kn}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\prod_{i=1}^k \frac{\widehat{f}(\xi_i)}{(\beta + 2\operatorname{Re}\Psi(\xi_1 + \cdots + \xi_i))} \right) \mathrm{e}^{i\sum_{i=1}^k \xi_i\cdot(x-y)} d\xi_1 d\xi_2 \cdots d\xi_k$$

Now set $x = y$ to get

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty \mathrm{e}^{-\beta t} \mathbf{E}\left(|u_t(x)|^2\right) dt \geq \frac{\alpha^2}{\beta} \sum_{i=0}^\infty \frac{l_\sigma^{2i}}{(2\pi)^{in}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{i=1}^k \frac{\widehat{f}(\xi_j)}{(\beta + 2\operatorname{Re}\Psi\left(\xi_1 + \cdots + \xi_j\right))} d\xi_1 d\xi_2 \cdots d\xi_k$$

Setting $x_j = \xi_1 + \xi_2 \cdots \xi_j$ and using the change of variables formula,

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty \mathrm{e}^{-\beta t} \mathbf{E}\left(|u_t(x)|^2\right) dt \geq \frac{\alpha^2}{\beta} \sum_{l=0}^\infty \frac{l_\sigma^{2i}}{(2\pi)^{n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^k \frac{\widehat{f}(x_j - x_{j-1})}{(\beta + 2\operatorname{Re}\Psi(x_j))} dx_1 dx_2 \cdots dx_k$$

where $x_0 = 0$.

We now want to use Assumption 7.1.2. Before that, define

$$\mathbb{S} = \{z = (z_1, z_2 \ldots, z_n) \in \mathbb{R}^n : \operatorname{sign}(z_1) = \operatorname{sign}(z_2) \cdots = \operatorname{sign}(z_n)\}$$

Because $\widehat{f} \geq 0$ (Theorem 2.3.2) and $\operatorname{Re}\Psi \geq 0$, the following holds:

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty \mathrm{e}^{-\beta t}\mathbf{E}\left(|u_t(x)|^2\right) dt \geq \frac{\alpha^2}{\beta} \sum_{l=0}^{\infty} \frac{l_\sigma^{2i}}{(2\pi)^n} \int_{\mathbb{S}} \int_{\mathbb{S}} \cdots \int_{\mathbb{S}} \prod_{j=1}^{k} \frac{\widehat{f}(x_j - x_{j-1})}{(\beta + 2\operatorname{Re}\Psi(x_j))} dx_1 dx_2 \cdots dx_k. \tag{7.7}$$

Recall our assumption that $\widehat{f}(\xi)$ and $\operatorname{Re}\Psi(\xi)$ both only depend on $|\xi_1|, \ldots, |\xi_d|$. If $x_1, x_2 \cdots x_k \in \mathbb{S}$, then $|(x_j - x_{j-1})_l| \leq |(x_j)_l|$ for $l = 1, 2, \cdots n$. Hence, by the assumption on $\widehat{f}$, $\widehat{f}(x_j - x_{j-1}) \geq \widehat{f}(x_j)$. Consequently,

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty \mathrm{e}^{-\beta t}\mathbf{E}\left(|u_t(x)|^2\right) dt \geq \frac{\alpha^2}{\beta} \sum_{i=0}^{\infty} \left(\frac{l_\sigma^2}{(2\pi)^n} \int_{\mathbb{S}} \frac{\widehat{f}(z)}{(\beta + 2\operatorname{Re}\Psi(z))} dz \right)^i. \tag{7.8}$$

As there are 2^{n-1} quadrants of $\mathbb{R}^n$ where the sign of the coordinates are the same,

$$\int_{\mathbb{S}} \frac{\widehat{f}(z)}{(\beta + 2\operatorname{Re}\Psi(z))} dz = \frac{1}{2^{n-1}} \int_{\mathbb{R}^n} \frac{\widehat{f}(z)}{(\beta + 2\operatorname{Re}\Psi(z))} dz = \frac{1}{2^{n-1}} (\overline{R}_\beta f)(0).$$

Now, if there is a β such that

$$(\overline{R}_\beta f)(0) \geq \frac{1}{l_\sigma^2}$$

then the term in the parenthesis in (7.8) will be greater than 1 and the series will diverge to infinity. That is,

$$\int_0^\infty \mathrm{e}^{-\beta t}\mathbf{E}\left(|u_t(x)|^2\right) dt = \infty \qquad \text{for all } x \in \mathbb{R}^n.$$

This implies $\inf_{x\in\mathbb{R}^n} \overline{\gamma}_2(x) \geq \beta$. Indeed, assume $\inf_{x\in\mathbb{R}^n} \overline{\gamma}_2(x) < \beta$. Then it is not hard to see for some $x \in \mathbb{R}^n$, $\varepsilon \in (0, \beta)$ and $C > 0$ such that

$$\mathbf{E}\left(|u_t(x)|^2\right) \leq C\mathrm{e}^{(\beta-\varepsilon)t} \quad \forall t > 0.$$

This would mean $\int_0^\infty \mathrm{e}^{-\beta t}\mathbf{E}\left(|u_t(x)|^2\right) dt < \infty$, hence a contradiction.

Chapter 8

Strong Invariance And A Noise Comparison Principle

8.1 Introduction

We present in this chapter two theorems from the work [25]. The first shows that a system of interacting SDEs on the integer lattice, under suitable scaling, converges to a solution of the one dimensional stochastic heat equation driven by a fractional Laplacian. The second is a moment comparison principle for stochastic heat equations with different nonlinearities.

Consider the following SPDE on the real line

$$\frac{\partial}{\partial t}u_t(x) = \left(-(-\Delta)^{\alpha/2}u_t\right)(x) + \sigma\left(u_t(x)\right)\dot{W}_t(x) \tag{8.1}$$

with $u_0(x) = 1$ for all $x \in \mathbb{R}$ and σ as usual is assumed to be nonrandom and globally Lipschitz. Recall that the solution is given by

$$u_t(x) = 1 + \int_{(0,t)}\int_{\mathbb{R}} p_{t-s}(y-x)\sigma\left(u_s(y)\right)W(dsdy)$$

Let $\{X_t\}_{t\geq 0}$ a rate 1 continuous-time random walk on $\mathbb{Z}$ with $X_0 = 0$ and denote by $\mathcal{L}$

it's generator. That is,

$$(\mathcal{L}\varphi)(i) = \sum_{j\in\mathbb{Z}} q(i-j)\left(\varphi(i)-\varphi(j)\right)$$

for some probability measure q on $\mathbb{Z}$ and bounded measurable functions $\varphi : \mathbb{Z} \to \mathbb{R}$.

$\{X_t\}_{t\geq 0}$ can also be seen as a compound Poisson process:

$$X_t = \sum_{i=0}^{\Pi_t} J_i$$

where $\{J_i\}$ are i.i.d random variables with jump distribution ν and $\{\Pi_t\}_{t\geq 0}$ is a Poisson process of rate 1 independent of $\{J_i\}$. By the Lévy-Khinchtine formula, the characteristic function of X_t is:,

$$\mathbf{E}\mathrm{e}^{iwX_t} = \mathrm{e}^{-t(1-\phi(w))}$$

where ϕ is the characteristic function of ν. Henceforth, we shall assume the following about ν:

Assumption 8.1.1

- $\{w \in [-\pi, \pi] : \phi(w) = 1\} = \{0\}$
- *There is a constant $a > 0$ such that:*

$$1 - \phi(w) = |w|^\alpha + O\left(|w|^{a+\alpha}\right)$$

For every $\varepsilon > 0$, define the rescaled random walk on $\varepsilon\mathbb{Z}$ as $X_t^{(\varepsilon)} = \varepsilon X_{\frac{t}{\varepsilon^\alpha}}$. From the definition of the generator, it is easy to see that

$$\left(\mathcal{L}^{(\varepsilon)}\varphi\right)(x) = \varepsilon^{-\alpha}\left(\mathcal{L}\left(\pi_\varepsilon\varphi\right)\right)\left(\frac{x}{\varepsilon}\right) \quad \forall x \in \varepsilon\mathbb{Z},$$

where $\left(\pi_\varepsilon\varphi\right)(x) = \varphi(\varepsilon x)$ for all $\varphi : \varepsilon\mathbb{Z} \to \mathbb{R}$ and $x \in \mathbb{Z}$.

Consider the system of interacting SDEs on $\varepsilon\mathbb{Z}$:

$$dU_t^{(\varepsilon)}(x) = \left(\mathcal{L}^{(\varepsilon)}U_t^{(\varepsilon)}\right)(x)dt + \frac{1}{\sqrt{\varepsilon}}\sigma\left(U_t^{(\varepsilon)}(x)\right)dB_t\left(\frac{x}{\varepsilon}\right) \tag{8.2}$$

subject to $U_0^{(\varepsilon)}(x) = 1$.

It is possible to show that the characteristic function of the finite dimensional distributions of the scaled random walk converge to that of the symmetric α-stable process. An application of Lévy's continuity theorem then tells us that the scaled random walk converges in distribution to the α-stable process. One might imagine, as $\varepsilon \to 0$, $U_t^{(\varepsilon)}(x)$ approximates $u_t(x)$. This is the subject of the main theorem in this chapter.

Theorem 8.1.2 *Under Assumption 8.1.1, there is a coupling of* $\{U^{(\varepsilon)}\}_{0<\varepsilon<1}$ *and* u *such that:* $\left\{U^{(\varepsilon)}\right\}_{\varepsilon \in (0,1)}$ *and* u *such that* $\forall T > 0, k \geq 2$ *and* $0 < \rho < a \wedge \alpha - 1$*, the following holds:*

$$\sup_{t \in [0,T]} \sup_{x \in \varepsilon \mathbb{Z}} \mathbf{E}\left(\left|U_t^{(\varepsilon)}(x) - u_t(x)\right|^k\right) = o\left(\varepsilon^{\frac{\rho k}{2}}\right)$$

It is intuitively clear that the above theorem implies the following one:

Theorem 8.1.3 *For all* $K, T > 0$ *and* $\rho \in (0, a \wedge \alpha - 1)$*, the coupling in the previous theorem satisfies:*

$$\varepsilon^{\frac{-\rho}{2}} \sup_{t \in (0,T)} \sup_{\substack{|x \in \varepsilon \mathbb{Z}: \\ |x| \leq \varepsilon^{-K}}} \left|U_t^{(\varepsilon)}(x) - u_t(x)\right| \xrightarrow{\mathbf{P}} 0,$$

as $\varepsilon \to 0$.

An important corollary of Theorem 8.1.4 is that the normalized finite-dimensional distributions of $U^{(\varepsilon)}$ converge to that of u as $\varepsilon \to 0$.

Another important consequence of Theorem 8.1.2 is the following theorem about comparison of moments.

Theorem 8.1.4 *Let* $\bar{u}$ *be the solution to* (8.1) *with* $\bar{u}_0(x) = 1 \; \forall x \in \mathbb{R}$ *and a Lipschitz nonlinearity* $\bar{\sigma}$ *such that* $0 \leq \sigma(x) \leq \bar{\sigma}(x) \; \forall x \in \mathbb{R}$*. Then* $\forall t \geq 0$ *and* $x_1, x_2 \ldots x_n \in \mathbb{R}$*,*

$$\mathbf{E}\left(\prod_{j=1}^{n} u_t(x_j)\right) \leq \mathbf{E}\left(\prod_{j=1}^{n} \bar{u}_t(x_j)\right) \tag{8.3}$$

for all $t \geq 0$ *and* $x_1, \ldots, x_m \in \mathbb{R}$.

8.2 Proof Of Theorem 8.1.2

Before we move to the proof of Theorem 8.1.2, we first state a key lemma which will be used in the proof. We shall not prove this lemma here as it is very technical and instead guide the reader to [25] for the full details.

Lemma 8.2.1 *Let $\{P_t^{(\varepsilon)}\}_{t\geq 0}$ be the transition semigroup of the scaled random walk $X^{(\varepsilon)}$. Then, $\forall T > 0$,*

$$\sup_{x\in\varepsilon\mathbb{Z}} \left| \frac{P_t^{(\varepsilon)}(x)}{\varepsilon} - p_t(x) \right| \leq C \begin{cases} \frac{\varepsilon^a |\ln \varepsilon|^{\frac{(a+\alpha)}{\alpha}}}{t^{\frac{(a+1)}{\alpha}}} & \text{if } t \geq K\varepsilon^\alpha |\log \varepsilon|^{\frac{(a+\alpha)}{a}}, \\ t^{\frac{-1}{\alpha}} + \frac{1}{\varepsilon} & \text{otherwise}, \end{cases} \tag{8.4}$$

$\forall t \in (0,T]$ uniformly and $\varepsilon \in (0, \varepsilon_0)$.

The lemma basically says that $\frac{P_t^{(\varepsilon)}(x)}{\varepsilon} \approx p_t(x)$ for small ε. Moreover, this approximation is uniform for all $t \geq \varepsilon^{a+o(1)}$.

Let us say that $X \approx Y$ whenever $X^{(\varepsilon)} = \{X_t(x)\}$and $Y = \{Y_t^{(\varepsilon)}(x)\}$ satisfy (8.1.2). It is easy to see that if $X \approx Y$ and $Y \approx Z$, then $X \approx Z$.

Introduce the following random fields:

$$\begin{aligned}
u_t^{(\varepsilon)}(x) &= 1 + \int_{(0,t-\varepsilon^\alpha)} \int_{\mathbb{R}} p_{t-s}(y-x)\sigma\left(u_s\left(\varepsilon\left[\frac{y}{\varepsilon}\right]\right)\right) W(dsdy), \\
v_t^{(\varepsilon)}(x) &= 1 + \int_{(0,t-\varepsilon^\alpha)} \int_{\mathbb{R}} p_{t-s}\left(\varepsilon\left[\frac{y}{\varepsilon}\right] - \varepsilon\left[\frac{x}{\varepsilon}\right]\right)\sigma\left(u_s\left(\varepsilon\left[\frac{y}{\varepsilon}\right]\right)\right) W(dsdy) \\
V_t^{(\varepsilon)}(x) &= 1 + \frac{1}{\varepsilon}\int_{(0,t-\varepsilon^\alpha)} \int_{\mathbb{R}} P_{t-s}^{(\varepsilon)}\left(\varepsilon\left[\frac{y}{\varepsilon}\right] - \varepsilon\left[\frac{x}{\varepsilon}\right]\right)\sigma\left(u_s\left(\varepsilon\left[\frac{y}{\varepsilon}\right]\right)\right) W(dsdy) \\
W_t^{(\varepsilon)}(x) &= 1 + \frac{1}{\varepsilon}\int_{(0,t)} \int_{\mathbb{R}} P_{t-s}^{(\varepsilon)}\left(\varepsilon\left[\frac{y}{\varepsilon}\right] - \varepsilon\left[\frac{x}{\varepsilon}\right]\right)\sigma\left(u_s\left(\varepsilon\left[\frac{y}{\varepsilon}\right]\right)\right) W(dsdy) \\
\overline{U}_t^{(\varepsilon)}(x) &= 1 + \frac{1}{\varepsilon}\sum_{j=-\infty}^{\infty}\int_0^t \int_{(j\varepsilon,(j+1)\varepsilon)} P_{t-s}^{(\varepsilon)}(j\varepsilon - x)\sigma\left(\overline{U}_s^{(\varepsilon)}(j\varepsilon)\right) W(dsdy)
\end{aligned}$$

The aim would be to show the following assertions:

1. $u \approx u^{(\varepsilon)}$,

2. $u^{(\varepsilon)} \approx v^{(\varepsilon)}$,

3. $v^{(\varepsilon)} \approx V^{(\varepsilon)}$

4. $V^{(\varepsilon)} \approx W^{(\varepsilon)}$ and

5. $W^{(\varepsilon)} \approx \bar{U}^{(\varepsilon)}$.

8.2.1 $u \approx u^{(\varepsilon)}$

Using the BDG inequality, we have

$$\begin{aligned}\left\|u_t(x) - u_t^{(\varepsilon)}(x)\right\|_k^2 \leq 2(4k)^{\frac{1}{k}} \int_{t-\varepsilon^\alpha}^{t} \int_{-\infty}^{\infty} |p_{t-s}(y-x)|^2 \left\|\sigma\left(u_s(y)\right)\right\|_k^2 dsdy \\ + 2(4k)^{\frac{1}{k}} \int_{0}^{t-\varepsilon^\alpha} \int_{-\infty}^{\infty} |p_{t-s}(y-x)|^2 \left\|\sigma\left(u_s(y)\right) - \sigma\left(u_s(\varepsilon\left[\frac{y}{\varepsilon}\right])\right)\right\|_k^2 dsdy\end{aligned}$$

Now by Plancherel's theorem,

$$\int_{-\infty}^{\infty} |p_s(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2|\xi|^\alpha s} d\xi = \frac{C}{s^{\frac{1}{\alpha}}}$$

and hence

$$\int_0^{\varepsilon^\alpha} \int_{-\infty}^{\infty} |p_s(x)|^2 \, dsdx \leq C\varepsilon^{\alpha-1}.$$

As for the second term, by Walsh isometry and Plancherel's theorem, we see that whenever $|x - x'| < \varepsilon$,

$$\begin{aligned}\|u_s(x) - u_s\left(x'\right)\|_k^2 &\leq C \int_0^s \int_{-\infty}^{\infty} \left[p_{s-r}(y-x) - p_{s-r}\left(w - x'\right)\right]^2 drdy \\ &= C \int_0^s \int_{-\infty}^{\infty} \mathrm{e}^{-2(s-r)|\xi|^\alpha} \left|\mathrm{e}^{i(x-x')\xi} - 1\right|^2 drd\xi \\ &\leq C\mathrm{e}^{2s} \int_0^s \int_0^{\infty} \mathrm{e}^{-2r(1+\xi^\alpha)} \left(1 \wedge |x-x'|\,\xi\right)^2 drd\xi.\end{aligned}$$

This integral can be computed to be at most $C|x-x'|^{\alpha-1} < C\varepsilon^{\alpha-1}$ For more details, see [25].

8.2.2 $u^{(\varepsilon)} \approx v^{(\varepsilon)}$

By Walsh isometry and the fact that $\|\sigma(u_s(y))\|_k < \infty$ uniformly for all y $\in \mathbb{R}$ an $\in (0, T]$,

$$\left\| u_t^{(\varepsilon)}(x) - v_t^{(\varepsilon)}(x) \right\|_k^2 \leq C \int_{\varepsilon^\alpha}^t \int_{-\infty}^{\infty} \left(p_s(y) - p_s\left(\varepsilon \left[\frac{y}{\varepsilon} \right] \right) \right)^2 dsdy.$$

Now we use the following property of the stable process density

$$|p_1'(x)| \leq \frac{C}{1 + |x|^{\alpha+1}} \quad \text{for all } x \in \mathbb{R},$$

to see that

$$\left\| u_t^{(\varepsilon)}(x) - v_t^{(\varepsilon)}(x) \right\|_k^2 \leq C\varepsilon^{\alpha-1}$$

Again, $\rho < \alpha - 1$, the assertion follows.
We shall not prove the rest of the assertions here as they are quite technical and instead guide the reader to [25].

8.3 Proof Of Theorem 8.1.4

By a theorem of [11], the said comparison of moments principle holds for a system of interacting diffusions of the form (8.2). The rest follows from applying Theorem 8.1.2 and using Hölder's inequality.

Chapter 9

Chaotic Character Of The Stochastic Heat Equation

9.1 Introduction

In the previous chapters, we studied the intermittency property of the stochastic heat equation (9.1) wherein the focus is on the large $t-$behaviour of the solution to (9.1).

$$\frac{\partial}{\partial t}u_t(x) = \frac{\partial^2}{\partial x^2}u_t(x) + \sigma\left(u_t(x)\right)\dot{W}_t(x) \quad \text{for } t > 0, x \in \mathbb{R}. \tag{9.1}$$

In this chapter, we focus on the large x-behaviour of the solution at a fixed t as detailed in [10]. Foondun and Khoshnevisan in [17] showed that if $\sigma(0) = 0$, $\inf_{x\in\mathbb{R}}\left|\frac{\sigma(x)}{x}\right| > 0$ and the initial function u_0 is Lipschitz continuous and compactly supported, then $\forall t > 0$,

$$\sup_{x\in\mathbb{R}} u_t(x) < \infty \quad \text{a.s.}$$

By contrast, if $\inf_{x\in\mathbb{R}u_0(x)} > 0$, the solution is unbounded. Furthermore, the solution grows at a predescribed rate which depends on the nature of the nonlinearity of the equation. This sensitive dependence on the initial data is termed the *chaotic behaviour* of the stochastic heat equation. These results are encapsulated in the theorems stated below.

Theorem 9.1.1 *If* $0 < \inf_{x\in\mathbb{R}} \sigma(x) \le \sup_{x\in\mathbb{R}} \sigma(x) < \infty$ *and* $t > 0$, *then*

$$0 < \limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{\frac{1}{2}}} < \infty. \tag{9.2}$$

Theorem 9.1.2 *When* $\sigma(x) = \lambda(x)$ *for* $\lambda > 0$, *that is, in the case of the* parabolic Anderson model, *we have*

$$0 < \limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\frac{2}{3}}} < \infty. \tag{9.3}$$

Here we present a sketch of the proof of Theorem 9.1.2 only, as the actual proofs are extremely long and computational. First, we invoke the comparison principle to reduce to the constant initial function case. Then, we obtain some tail probability bounds using some moment estimates of the parabolic Anderson model followed by an application of Chebyshev's/ Paley-Zygmund. Theorem 9.1.2 follows from this and the Borel-Cantelli lemma, provided that we prove that if x_1 and x_2 are 'sufficiently far apart', then $u_t(x_1)$ and $u_t(x_2)$ are approximately independent. To achieve this, we construct a *localization* of the original stochastic heat equation. This localization has the property that, for two points sufficiently far apart, the random variables are exactly independent.

9.2 Tail Probability Estimates

Theorem 9.2.1 *For all* $t > 0$,

$$-\infty < \liminf_{z\to\infty} \frac{\log \mathbf{P}\left\{|u_t(x)| > z\right\}}{(\log z)^{\frac{3}{2}}} \le \limsup_{z\to\infty} \frac{\log \mathbf{P}\left\{|u_t(x)| > z\right\}}{(\log z)^{\frac{3}{2}}} < 0. \tag{9.4}$$

For the proof of the upper bound in the above theorem, we need the following lemma:

Lemma 9.2.2 *Let* X *be a random variable taking values in* $\mathbb{R}_+$ *such that there exists* $a, C > 0$ *and* $b > 1$ *for which*

$$\mathbf{E}(X^k) \le C^k \mathrm{e}^{ak^b}$$

for all $k \ge 1$. *Then*

$$\mathbf{E}\left(\mathrm{e}^{\alpha(\log_+ X)^{\frac{b}{b-1}}}\right) < \infty,$$

for all $\alpha \in \left(0, \frac{1-b^{-1}}{(ab)^{\frac{1}{b-1}}}\right)$.

Proof. With no loss of generality, we assume that $C = 1$. Let $\log_+(x) = \log(x \wedge \mathrm{e})$. If $z > \mathrm{e}$, then by Chebyshev's inequality,

$$\mathbf{P}\{\mathrm{e}^{\alpha\left(\log_+ X\right)^{\frac{b}{(b-1)}}} > z\} \leq \mathrm{e}^{\left(ak^b - k\left(\frac{\log z}{\alpha}\right)^{\frac{b-1}{b}}\right)}.$$

Now define:

$$g(k) = k(\frac{\log z}{\alpha})^{\frac{b-1}{b}} - ak^b.$$

Clearly,

$$\mathbf{P}\left\{\mathrm{e}^{\alpha\left(\log_+ X\right)^{\frac{b}{b-1}}} > z\right\} \leq \mathrm{e}^{\left(\sup_{k\geq 1} g(k)\right)}.$$

By differentiating, we find that

$$\max_{k\geq 1} g(k) = c \log z,$$

where $c = \frac{1-b^{-1}}{(ab)^{\frac{1}{b-1}}}$ Thus, it follows that

$$\mathbf{P}\left\{\mathrm{e}^{\alpha\left(\log_+ X\right)^{\frac{b}{b-1}}} > z\right\} = O\left(z^{-c}\right).$$

Integrating both sides with respect to dz, we find that the $\mathbf{E}\left(\mathrm{e}^{\alpha(\log_+ X)^{\frac{b}{b-1}}}\right) < \infty$ provided $c > 1$. This proves the lemma. $\square$

Recall from the existence and uniqueness result that the solution to (9.1) satisfies $\mathbf{E}(|u_t(x)|^k) \leq C^k \mathrm{e}^{ak^3}$. Hence, we can use the above lemma for $b = 3$ and Chebyshev's inequality to prove the upper bound in Theorem 9.2.1. As for the lower bound, we use the moment estimate for the Anderson model (see Chapter 6) combined with Paley-Zygmund inequality [27].

9.3 Localization

Consider the following random fields indexed by real numbers $\beta > 0$,

$$U_t^{(\beta)}(x) = 1 + \int_0^t \int_{[x-\sqrt{\beta t}, x+\sqrt{\beta t}]} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right) W(dsdy).$$

Lemma 9.3.1 *For a fixed $\beta > 0$, the above equation has an unique solution a.s. $U^{(\beta)}$ such that*

$$\sup_{\beta>0} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathbf{E}\left(\left|U_t^{(\beta)}(x)\right|^k\right) \leq C^k \mathrm{e}^{ak^3}.$$

The proof of the above lemma is very similar to the Picard iteration scheme used to derive the existence result in Chapter 4 and hence, we shall not do it here.

The next lemma shows that, if $\beta >> 1$, then $u \approx U^{(\beta)}$.

Lemma 9.3.2 *For $T > 0$, there are constants $C, D > 0$ such that for $\beta >> 1$ and all $k \geq 1$,*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathbf{E}\left(\left|u_t(x) - U_t^{(\beta)}(x)\right|^k\right) \leq C^k k^{\frac{k}{2}} \mathrm{e}^{Dk\left(k^2-\beta\right)}.$$

Now, us define $U_t^{(\beta,n+1)}(x)$ to be the nth Picard iteration to $U_t^{(\beta)}$. That is, $U_t^{(\beta,0} \equiv 1$ and

$$U_t^{(\beta,n+1)}(x) = 1 + \int_0^t \int_{[x-\sqrt{\beta t}, x+\sqrt{\beta t}]} p_{t-s}(y-x)\sigma\left(U_s^{(\beta,n)}(y)\right) W(dsdy). \tag{9.5}$$

From the existence proof, for large n, $U_t^{(\beta,n+1)}(x) \approx U_t^{(\beta}(x)$. Hence, for large n, β, $U_t^{(\beta,n+1)}(x) \approx u_t(x)$.

Lemma 9.3.3 *Let $x_1, x_2, \ldots \in \mathbb{R}$ such that $|x_i - x_j| \geq 2n\sqrt{\beta t}$ whenever $i \neq j$. Then $\left\{\bar{U}_t^{(\beta,n)}(x_j)\right\}_{j\in\mathbb{Z}}$ is a collection of i.i.d. random variables.*

Proof. The proof is by induction on n. For $n = 1$, we have

$$U_t^{(\beta,1)}(x) = 1 + \sigma(1) \int_0^t \int_{[x-\sqrt{\beta t}, x+\sqrt{\beta t}]} p_{t-s}(y-x) W(dsdy).$$

If $\{x_i\}$ are such that $|x_i - x_j| \geq 2\sqrt{\beta t}$, the nonrandom integrands in the above equation have disjoint supports and hence have correlation zero. Since they are Gaussian random variables, they are hence independent. Now assume $\left\{\bar{U}_t^{(\beta,n)}(x_j)\right\}_{j\in\mathbb{Z}}$ are independent. Then from (9.5), we see that:

$$\mathbf{E}\Big(U_t^{(\beta,n+1)}(x_i)U_t^{(\beta,n+1)}(x_j)\Big)$$
$$= \int_0^t \int_{A_i\cap A_j} p_{t-s}(y-x_i)p_{t-s}(y-x_j)\mathbf{E}\Big(\sigma\left(U_s^{(\beta,n)}(y)\right)\sigma\left(U_s^{(\beta,n)}(y)\right)\Big)W(dsdy),$$

where $A_i\cap A_j = [x_i-\sqrt{\beta t}, x_i+\sqrt{\beta t}]\cap[x_j-\sqrt{\beta t}, x_j+\sqrt{\beta t}]$. If $|x_i - x_j| \geq 2n\sqrt{\beta t}$, then $A_i\cap A_j = \emptyset$ and so $\mathbf{E}\Big(U_t^{(\beta,n+1)}(x_i)U_t^{(\beta,n+1)}(x_j)\Big) = 0$. Again, since $U_t^{(\beta,n+1)}(x_i)$ and $U_t^{(\beta,n+1)}(x_j)$ are Gaussian variables, this implies that they are independent.
Hence the assertion is true for $n+1$. By induction, the lemma is proved. $\square$

The rest of the proof of Theorem 9.1.2 follows from a series of continuity estimates and Borel-Cantelli lemma.

Chapter 10

Boundedness Trichotomy Of The Stochastic Heat Equation

10.1 Introduction

In this chapter, we discuss the results of the paper [8] where the authors have proved that under some mild conditions, the rate of decay of the initial condition u_0 characterizes the boundedness of the solution $u_t(x)$ of the stochastic heat equation with white noise. To be specific, recall the stochastic heat equation:

$$\frac{\partial}{\partial t} u_t(x) = \frac{\partial^2}{\partial x^2} u_t(x) + \sigma\left(u_t(x)\right) \dot{W}_t(x) \quad \text{ for } t > 0, x \in \mathbb{R}, \tag{10.1}$$

with a bounded, measurable and non-negative initial condition u_0. Additionally, we will also make the following assumptions:

Assumption 10.1.1 • *there exists a $A \subset \mathbb{R}$ with $|A| > 0$ such that $u_0 > 0$ on A*

- $\lim_{x \to \infty} u_0(x) = 0$
- $u_0(x) = u_0(-x)$
- $u_0(x) \geq u_0(y)$ *if* $0 \leq x \leq y$

As for σ, in addition to the usual Lipschitz continuity, we shall assume $\sigma(0) = 0$ and $l_\sigma = \inf_{x\in\mathbb{R}} \left|\frac{\sigma(x)}{x}\right| > 0$

Define

$$\mathbf{\Lambda} = \lim_{|x|\to\infty} \frac{|\log u_0(x)|}{(\log|x|)^{\frac{2}{3}}},$$

and

$$M(t) = \sup_{x\in\mathbb{R}} u_t(x).$$

Theorem 10.1.2 *Then, under the above assumptions, the following trichotomy holds:*

1. *if* $\mathbf{\Lambda} = \infty$, *then* $\mathbf{P}(M(t) < \infty$ *for all* $t > 0) = 1$

2. *if* $\mathbf{\Lambda} = 0$, *then* $\mathbf{P}(M(t) = \infty$ *for all* $t > 0) = 1$

3. *if* $0 < \mathbf{\Lambda} < \infty$, *then there exists a random variable* $\mathbf{T}$ *and constants* $t_1, t_2 \in (0,\infty)$ *such that:*

   (a) $t_1 < \mathbf{T} < t_2$ *a.s.*

   (b)
   $$\mathbf{P}\left(M(t) < \infty \quad \forall t < \mathbf{T} \text{ and } M(t) = \infty \quad \forall\, t > \mathbf{T}\right) = 1.$$

10.2 Sketch Of Proof

The key step in the proof is the following tail bound:

Lemma 10.2.1 *There exist finite, positive constants* K, L *such that* $\forall\varepsilon > 0$,

$$-\frac{L\mathbf{\Lambda}^{3/2}}{\sqrt{t}} \le \liminf_{|x|\to\infty} \frac{\log \mathbf{P}\left\{u_t(x) > \varepsilon\right\}}{\log|x|} \le \limsup_{|x|\to\infty} \frac{\log \mathbf{P}\left\{u_t(x) > \varepsilon\right\}}{\log|x|} \le -\frac{K\mathbf{\Lambda}^{3/2}}{\sqrt{t}}$$

uniformly for all t *in every fixed compact subset of* $(0,\infty)$

Proof. We shall present the key ideas in the proof here, as it is quite technical and long. For the full details, see [8].

To establish the lower bound, we use the moment comparison principle (Chapter 8) to bound the moments of the solution to the SPDE (10.1) by the moments of the solutions corresponding to the Anderson model . We then use the Paley-Zygmund inequality [27] to bound the tail probability from below. As for the upper bound, the proof is similar except that we use Chebyshev's inequality to bound the tail probability from above. $\square$

Now we move to the proof of Theorem 10.1.2. We consider the three cases on $\mathbf{\Lambda}$ separately:

10.2.1 $\mathbf{\Lambda} > 0$ or $\mathbf{\Lambda} = \infty$

Fix $\lambda \in (0, \mathbf{\Lambda})$ and $\tau, \mathcal{T}$ such that $0 < \tau < \mathcal{T} = \frac{K^2 \lambda^3}{64}$ where K is the constant in Lemma 10.2.1. The proof begins by showing

$$\lim_{x \to \infty} \sup_{t \in (\tau, \mathcal{T})} u_t(x) = 0 \quad \text{a.s.} \tag{10.2}$$

The proof of this begins by showing the above holds over discrete intervals of $(\tau, \mathcal{T})$ using the Borel-Cantelli Lemma and then using some continuity estimates to extend to the whole interval $(\tau, \mathcal{T})$. Once this is done, due to symmetry, this also implies that $\lim_{x \to -\infty} \sup_{t \in (\tau, \mathcal{T})} u_t(x) = 0$ a.s. Since u is continuous,

$$\mathbf{P}\left(\sup_{x \in \mathbb{R}} u_t(x) < \infty \text{ for all } t \in (\tau, \mathcal{T}) \right) = 1.$$

If $\mathbf{\Lambda} = \infty$, then τ can be chosen as close to 0 as possible and $\mathcal{T}$ as close to ∞. This proves (1) of Theorem 10.1.2. Now if $\mathbf{\Lambda} < \infty$, then we can prove half of Theorem 10.1.2 (3). That is, choose $\mathcal{T}$ arbitrarily close to $\frac{K^2 \mathbf{\Lambda}^3}{64}$ and set $t_1 = \frac{K^2 \mathbf{\Lambda}^3}{64}$. This would imply $\sup_{x \in \mathbb{R}} u_t(x)$ $\forall t < t_1$ a.s.

10.2.2 $\mathbf{\Lambda} < \infty$ or $\mathbf{\Lambda} = 0$

Fix $\mathcal{T} > \tau > 4L^2\mathbf{\Lambda^3}$, where L is the constant in Lemma 10.2.1. The proof begins by showing that :

$$\inf_{t\in(\tau,\mathcal{T})} \sup_{x\in\mathbb{R}} u_t(x) = \infty \quad \text{a.s.} \tag{10.3}$$

The proof of the above is similar to (10.2). If $\mathbf{\Lambda} > 0$, then set $t_2 = 4L^2\mathbf{\Lambda}^3$ and (10.3) would imply $\sup_{x\in\mathbb{R}} u_t(x) = \infty$ a.s. for all $t \geq t_2 = 4L^2\mathbf{\Lambda}^3$, thus proving the other half of Theorem 10.1.2 (3). If $\mathbf{\Lambda} = 0$, then τ can be chosen arbitrarily close to 0 order to yield $\sup_{x\in\mathbb{R}} u_t(x) = \infty$ a.s. for all $t > 0$.

10.2.3 $0 < \mathbf{\Lambda} < \infty$

Let $0 < \mathbf{\Lambda} < \infty$. For every integer $N \geq 1$, define the stopping time:

$$\mathbf{T}_N = \inf\{t > 0 : M(t) \geq N\}.$$

Clearly, $\mathbf{T}_N \leq \mathbf{T}_{N+1}$ for all $N \geq 1$, and hence

$$\mathbf{T} = \lim_{N\to\infty} \mathbf{T}_N$$

exists. According to Parts (1) and (2) of Theorem 10.1.2,

$$0 < t_1 < \mathbf{T} < t_2 < \infty \quad \text{a.s.}$$

where t_1 and t_2 are non-random and depend only on $\mathbf{\Lambda}$. Now if there exists an ω such that $t < \mathbf{T}(\omega)$, then we can find an integer $N(\omega) > 0$ which satisfies $t \leq \mathbf{T}_{N(\omega)}(\omega) < \mathbf{T}(\omega)$. This would imply that $M(t)(\omega) \leq N(\omega) < \infty$.

On the other hand, if there exists an ω such that $t > \mathbf{T}(\omega)$ then we can find an integer $N_1(\omega) > 0$ such that $t \geq \mathbf{T}_n(\omega)\ \forall n \geq N_1(\omega_1)$. This would imply $M(t)(\omega) \geq n$ for all $n \geq N_1(\omega)$, and hence $M(t)(\omega) = \infty$. Thus (3) of Theorem 10.1.2 is proved.

Chapter 11

Moment Bounds For The Fractional Stochastic Heat Equation

11.1 Introduction

In Chapter 7, we saw that the second moment of the solution to

$$\frac{\partial}{\partial t}u_t(x) = (\mathcal{L}u_t)(x) + \sigma(u_t(x))\dot{F}_t(x) \quad \text{for } t > 0, x \in \mathbb{R}^n \tag{11.1}$$

grows exponentially with respect to t whenever u_0 is bounded below. In [20], the authors focus on the case where $\mathcal{L}$ is the fractional Laplacian and show that, under some additional assumptions on σ and u_0, the second moment does in fact grow exponentially with t even if the u_0 is not bounded below. The said assumptions are:

Assumption 11.1.1 *There exists l_σ, $L_\sigma > 0$ such that $\forall\ x \in \mathbb{R}^n$,*

$$l_\sigma|x| \leq \sigma(x) \leq L_\sigma(x).$$

Assumption 11.1.2 *$u_0 : \mathbb{R}^n \to \mathbb{R}_+$ is a bounded function such that there exists $A \subset \mathbb{R}^n$ with $|A| > 0$ and*

$$\int_A u_0(x)dx > 0.$$

Here $\dot{F}$ is space time colored noise with the Riesz kernel. That is,

$$\mathbf{E}(\dot{F}_t(x)\dot{F}_s(y)) = \delta_0(t-s)f(x-y)$$

where

$$f(x-y) = \frac{1}{|x-y|^\beta}$$

for some $\beta < n$.

Recall the mild solution to (11.4) given by

$$u_t(x) = \left(P_t u_0\right)(x) + \int_0^t \int_{\mathbb{R}^n} p_{t-s}(y-x)\sigma\left(u_s(y)\right) F(dsdy) \tag{11.2}$$

The existence of the above solution will impose $\beta < \alpha$ (see Chapter 4, Assumption 4.1.1).

Theorem 11.1.3 *There exist $c_1, c_2 > 0$ such that*

$$\sup_{x\in\mathbb{R}^n} \mathbf{E}\left|u_t(x)\right|^2 \leq c_1 \mathrm{e}^{\left(c_2\lambda^{\frac{2\alpha}{\alpha-\beta}}t\right)} \quad \text{for all } t > 0.$$

As for the lower bound, there exists c_3, c_4 and $T > 0$ such that $\forall t > T$,

$$\inf_{x\in B\left(0,t^{\frac{1}{\alpha}}\right)} \mathbf{E}\left|u_t(x)\right|^2 \geq c_3 \mathrm{e}^{\left(c_4'\lambda^{\frac{2\alpha}{\alpha-\beta}}t\right)}.$$

An immediate corollary of the above bounds is that, for any $x \in \mathbb{R}^n$

$$c_4\lambda^{\frac{2\alpha}{\alpha-\beta}} \leq \liminf_{t\to\infty} \frac{\log \mathbf{E}\left|u_t(x)\right|^2}{t} \leq \limsup_{t\to\infty} \frac{\log \mathbf{E}\left|u_t(x)\right|^2}{t} \leq c_2\lambda^{\frac{2\alpha}{\alpha-\beta}}.$$

11.2 Preliminaries

We start this section with some results concerning heat kernels of $\alpha-$stable processes on $\mathbb{R}^n$.

Lemma 11.2.1 *There exists $c_1, c_2 > 0$ such that:*

$$c_1\left(t^{\frac{-n}{\alpha}} \wedge \frac{t}{|x-y|^{n+\alpha}}\right) \leq p_t(x-y) \leq c_2\left(t^{\frac{-n}{\alpha}} \wedge \frac{t}{|x-y|^{n+\alpha}}\right).$$

A proof for this can be found in [5][21]. We then have the following proposition:

Proposition 11.2.2 *There exists* t_0 *such that* $\forall t > t_0$

$$(P_{t+t_0}u)(x) \geq c_1 t^{\frac{-n}{\alpha}} \qquad \textit{whenever } x \in B(0, (t+t_0)^{\frac{1}{\alpha}}).$$

Proof. Recall that $p_t(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-i\langle \xi, x\rangle} \mathrm{e}^{-t|\xi|^\alpha} d\xi$. Clearly, $p_t(0)$ decreases with t and by Dominated Convergence Theorem, $\lim_{t\to\infty} p_t(0) = 0$. Fix a large enough t_0 so that $p_{t_0}(0) \leq 1$. We then have

$$\begin{aligned} p_{t_0}(x-y) &= p_{t_0}\left(\frac{2(x-y)}{2}\right) \\ &\geq p_{t_0}(2x)\, p_{t_0}(2y) \\ &= \frac{1}{2n} p_{\frac{t_0}{2^\alpha}}(x)\, p_{t_0}(2y)\,. \end{aligned}$$

The above follows from the decreasing nature of p_t and its scaling property. See [5][21] for the full proof. This immediately gives:

$$\begin{aligned} (P_{t_0}u)(x) &= \int_{\mathbb{R}^n} p_{t_0}(x-y)\, u_0(y) dy \\ &\geq c_1 p_{\frac{t_0}{\alpha}}(x) \int_{\mathbb{R}^n} p_{t_0}(2y)\, u_0(y) dy \end{aligned}$$

Using the semigroup property of the transition densities,

$$\begin{aligned} (P_{t+t_0}u)(x) &= \int_{\mathbb{R}^n} p_{t+t_0}(x-y)\, u_0(y) dy \\ &= \int_{\mathbb{R}^n} p_t(x-y)(P_{t_0}u)(y) dy \\ &\geq c_2 p_{\frac{t+t_0}{2}}(x) \end{aligned}$$

□

Now we present some renewal inequalities which will be used in the proving Theorem 11.1.3. The proofs can be found in [20].

Lemma 11.2.3 *Let* $0 < \rho \leq 1$*. Then there exists* $c_1 > 0$ *such that for all* $b \geq \left(\frac{\mathrm{e}}{\rho}\right)^\rho$

$$\sum_{i=0}^{\infty} \left(\frac{b}{i^\rho}\right)^i \geq \mathrm{e}^{c_1 b^{\frac{1}{\rho}}}.$$

Lemma 11.2.4 *Let $0 < \rho \leq 1$. Then there exists $c_1, c_2 > 0$ such that*

$$\sum_{i=0}^{\infty} \frac{b^i}{\Gamma(i\rho+1)} \leq c_1 \mathrm{e}^{c_2 b^{\frac{1}{\rho}}} \qquad \textit{for all } b > 0,$$

where Γ is the Gamma function.

Proposition 11.2.5 *Let $\rho > 0$ and $h(t) \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that*

$$h(t) \leq c_1 + \mu \int_0^t (t-s)^{\rho-1} f(s) ds \quad \forall t > 0,$$

for some $c_1 > 0$. Then, there exists $c_2, c_3 > 0$ such that:

$$h(t) \leq c_2 \mathrm{e}^{c_3 (\Gamma(\rho))^{\frac{1}{\rho}} \mu^{\frac{1}{\rho}} t} \quad \forall t > 0.$$

Proposition 11.2.6 *Let $\rho > 0$ and $h(t) \in L^1_{\mathrm{loc}}(\mathbb{R})$ and nonnegative such that*

$$h(t) \geq c_1 + \mu \int_0^t (t-s)^{\rho-1} f(s) ds \qquad \textit{for all } t > 0,$$

for some $c_1 > 0$. Then, there exists $c_2, c_3 > 0$ such that:

$$h(t) \geq c_2 \mathrm{e}^{c_3 (\Gamma(\rho))^{\frac{1}{\rho}} \mu^{\frac{1}{\rho}} t} \quad \forall t > \frac{\mathrm{e}}{\rho} (\Gamma(\rho)\mu)^{\frac{1}{\rho}}.$$

11.3 Proof Of Theorem 11.1.3

Proposition 11.3.1 *There exist $c_1, c_2 > 0$ such that $\forall t > 0$,*

$$\sup_{x \in \mathbb{R}^n} \mathbf{E} |u_t(x)|^2 \leq c_1 + c_2 (\lambda L_\sigma)^2 \int_0^t \frac{\sup_{x\in\mathbb{R}^n} \mathbf{E} |u_s(x)|^2}{(t-s)^{\frac{\beta}{\alpha}}} ds$$

Proof. Taking the second moment of the mild solution,

$$\mathbf{E} |u_t(x)|^2 = |(P_t u)(x)|^2 + \lambda^2 \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_{t-s}(x-y) p_{t-s}(x-z) f(y-z) \mathbf{E} \left(\sigma\left(u_s(y)\right) \sigma\left(u_s(z)\right)\right) dydzds$$

Since u_0 is bounded, so is $|(P_t u)(x)|^2$. As for the second term, using Hölder's inequality and

Assumption (11.1.1),

$$\mathbf{E}\left(\sigma\left(u_s(y)\right)\sigma\left(u_s(z)\right)\right) \leq L_\sigma^2\mathbf{E}\left(|u_s(y)u_s(z)|\right) \leq L_\sigma^2\left(\mathbf{E}\left|u_s(y)\right|^2\right)^{1/2}\left(\mathbf{E}\left|u_s(z)\right|^2\right)^{1/2} \leq L_\sigma^2 \sup_{x\in\mathbb{R}^n}\mathbf{E}\left|u_s(x)\right|^2$$

As for $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p_{t-s}(x-y)p_{t-s}(x-z)f(y-z)dydz$, changing the variables first from $y \to y+z$ and then from $z \to x-z$ followed by the fact that p_t is symmetric,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p_{t-s}(x-y)p_{t-s}(x-z)f(y-z)dydz = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p_{t-s}(y-z)p_{t-s}(z)f(y)dydz = \int_{\mathbb{R}^n} p_{2(t-s)}(y)f(y)dy$$

By the scaling property of the heat kernel,

$$\int_{\mathbb{R}^n} p_{2(t-s)}(y)f(y)dy = \frac{1}{2(t-s)^{\frac{n}{\alpha}}}\int_{\mathbb{R}^n}\frac{p_1\left(\frac{y}{2(t-s)^{\frac{1}{\alpha}}}\right)}{|y|^\beta}dy = \frac{c}{(t-s)^{\frac{\beta}{\alpha}}}\int_{\mathbb{R}^n}\frac{p_1(r)}{|r|^\beta}dr \leq \frac{c_2}{(t-s)^{\frac{\beta}{\alpha}}}$$

by Lemma 11.2.1. Combining the above bounds proves the proposition. □

From here on out, fix a $t_0 > 0$ and let

$$\inf_{x\in B\left(0,(t+t_0)^{\frac{1}{\alpha}}\right)} (P_{s+t_0}u)(x) \geq m(t) \tag{11.3}$$

for $0 \leq s \leq t$. Proposition 11.2.2 guarantees the existence of such an m. We then have the following key proposition:

Proposition 11.3.2 *There exists $c_1 > 0$ such that the following holds $\forall t \geq 0$:*

$$\mathbf{E}\left|u_{t+t_0}(x)\right|^2 \geq m(t)^2\sum_{k=0}^{\infty}(c_1\lambda l_\sigma)^{2k}\left(\frac{t}{k}\right)^{\frac{k(\alpha-\beta)}{\alpha}}.$$

$\forall x \in B\left(0,(t+t_0)^{\frac{1}{\alpha}}\right)$

Proof. Taking the second moment on both sides of (11.2) and using Assumption 11.1.1,

$$\begin{aligned}\mathbf{E}|u_{t+t_0}(x)|^2 &\geq |P_{t+t_0}u(x)|^2 \\ &\quad + \lambda^2 l_\sigma^2\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p_{t-s_1}(x-x_1)p_{t-s_1}(x-x_1')f(x_1-x_1')\mathbf{E}|u_{s_1+t_0}(x_1)u_{s_1+t_0}(x_1')|dx_1dx_1'ds_1\end{aligned}$$

Now,

$$\begin{aligned}\mathbf{E}|u_{s_1+t_0}(x_1)u_{s_1+t_0}(x_1')| \geq \mathbf{E}u_{s_1+t_0}(x_1)u_{s_1+t_0}(x_1')\\ = (P_{s_1+t_0}u)(x_1)(P_{s_1+t_0}u)(x_1') + \lambda^2 l_\sigma^2 \int_0^{s_1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p_{s_1-s_2}(x_1-x_2)\\ p_{s_1-s_2}(x_1'-x_2')f(x_2-x_2')\mathbf{E}|u_{s_2+t_0}(x_2)u_{s_2+t_0}(x_2')|dx_2dx_2'ds_2\end{aligned}$$

Iterating this, we get,

$$\begin{aligned}\mathbf{E}|u_{t+t_0}(x)|^2 &\geq |P_{t+t_0}u(x)|^2\\ &+ \sum_{k=1}^{\infty}(\lambda l_\sigma)^{2k}\int_0^{s_0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cdots\int_0^{s_{k-1}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big((P_{s_k+t_0}u)(x_k)\\ &(P_{s_k+t_0}u)(x_k')\prod_{i=1}^{k}p_{s_{i-1}-s_i}(x_{i-1}-x_i)p_{s_{i-1}-s_i}(x_{i-1}'-x_i')f(x_i-x_i')\Big)dz_{k+1-i}dz_{k+1-i}'ds_{k+1-i}\end{aligned}$$

where $s_0 = t$. Restricting the integration to $B\left(0,(t+t_0)^{\frac{1}{\alpha}}\right)$ and using (11.3),

$$\begin{aligned}\mathbf{E}|u_{t+t_0}(x)|^2 \geq & m(t)^2 + m(t)^2\sum_{k=1}^{\infty}(\lambda l_\sigma)^{2k}\int_0^{s_0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cdots\\ &\cdots\int_0^{s_{k-1}}\int_{B_{t_0}}\int_{B_{t_0}}\prod_{i=1}^{k}p_{s_{i-1}-s_i}(x_{i-1}-x_i)p_{s_{i-1}-s_i}(x_{i-1}'-x_i')f(x_i-x_i')dz_{k+1-i}dz_{k+1-i}'ds_{k+1-i}\end{aligned}$$

where $B_{t_0} = B\left(0,(t+t_0)^{\frac{1}{\alpha}}\right)$. Regarding the integral on the right handside of the above equation, it is greater than or equal to

$$\begin{aligned}&\int_{s_0-\frac{t}{k}}^{s_0}\int_{\mathbb{R}^n\times\mathbb{R}^n}\cdots\int_{s_{k-1}-\frac{t}{k}}^{s_{k-1}}\int_{B_{t_0}\times B_{t_0}}\prod_{i=1}^{k}p_{s_{i-1}-s_i}(x_{i-1}-x_i)p_{s_{i-1}-s_i}(x_{i-1}'-x_i')f(x_i-x_i')dz_{k+1-i}dz_{k+1-i}'ds_{k+1-i}\\ &= \int_0^{\frac{t}{k}}\int_{\mathbb{R}^n\times\mathbb{R}^n}\cdots\int_0^{\frac{t}{k}}\int_{B_{t_0}\times B_{t_0}}\prod_{i=1}^{k}p_{s_i}(x_{i-1}-x_i)p_{s_i}(x_{i-1}'-x_i')f(x_i-x_i')dz_{k+1-i}dz_{k+1-i}'ds_{k+1-i}\end{aligned} \tag{11.4}$$

where in the second line, we have used change of variables from $s_{i-1}-s_i \to s_i$.

We want to use Lemma 11.2.1. To that end, choose $\{x_i\}$ and $\{x_i'\}$ according to the following rule:

$$x_i \in B\left(x, \frac{s_1^{\frac{1}{\alpha}}}{2}\right) \cap B\left(x_{i-1}, s_i^{\frac{1}{\alpha}}\right) = B_i$$

$$x_i' \in B\left(x, \frac{s_1^{\frac{1}{\alpha}}}{2}\right) \cap B\left(x_{i-1}', s_i^{\frac{1}{\alpha}}\right) = B_i'$$

The above choice of $\{x_i\}$ and $\{x_i'\}$ satisfy $|x_i - x_i'| \leq s_1^{\frac{1}{\alpha}}, |x_i - x_{i-1}| \leq s_i^{\frac{1}{\alpha}}$ and $|x_i' - x_{i-1}'| \leq s_i^{\frac{1}{\alpha}}$ which imply that $f(x_i - x_i') \geq s_1^{-\frac{\beta}{\alpha}}, p_{s_i}(z_{i-1} - z_i) \geq c_1 s_i^{\frac{-n}{\alpha}}$ and $p_{s_i}(z_{i-1}' - z_i') \geq c_1 s_i^{\frac{-n}{\alpha}}$ respectively. Note that $|B_i|, |B_i'| \geq c_2 s_i^{\frac{n}{\alpha}}$. Using this, we see that (11.4) is greater than or equal to

$$\int_0^{\frac{t}{k}} \int_{B_1 \times B_1'} \cdots \int_0^{\frac{t}{k}} \int_{B_k \times B_k'} \prod_{i=1}^{k} p_{s_i}(x_{i-1} - x_i) p_{s_i}(x_{i-1}' - x_i') f(x_i - x_i') dz_{k+1-i} dz_{k+1-i}' ds_{k+1-i}$$
$$c^k \int_0^{\frac{t}{k}} \cdots \int_0^{\frac{t}{k}} \frac{1}{s_1^{\frac{k\beta}{\alpha}}}$$
$$= c^k \left(\frac{t}{k}\right)^{\frac{k(\alpha-\beta)}{\alpha}}$$

Combining the above results, we obtain

$$\mathbf{E}|u_{t+t_0}(x)|^2 \geq m(t)^2 + m(t)^2 \sum_{k=1}^{\infty} (c\lambda l_\sigma)^{2k} \left(\frac{t}{k}\right)^{\frac{k(\alpha-\beta)}{\alpha}} \tag{11.5}$$

□

Now we come to the proof of Theorem 11.1.3. For the upper bound, we use Propositions 11.3.2 and 11.2.5 with $\rho = \frac{\alpha-\beta}{\alpha}$ and $\mu = (\lambda L_\sigma)^2$. Now for the lower bound, setting $t \to t - t_0$ in Proposition 11.3.2 gives

$$\mathbf{E}\,|u_t(x)|^2 \geq m(t-t_0)^2 \sum_{k=0}^{\infty} (c_1 \lambda l_\sigma)^{2k} \left(\frac{t-t_0}{k}\right)^{\frac{k(\alpha-\beta)}{\alpha}}$$

Choosing a T large enough so that the assumptions of Lemma 11.2.3 are satisfied and using the same gives the required lower bound.

Chapter 12

Multifractality Of The Stochastic Heat Equation

12.1 Introduction

In Chapter 5, we studied the *intermittency* properties of the general nonlinear stochastic heat equation. In [28] the authors have studies a closely related concept called *multifractality* which is related to the fractal behaviour of the tall peaks of a process. This will be the content of this chapter.

Let $X = \{X_t\}_{t\in\mathbb{R}^n}$ be real-valued continuous stochastic process. We wish to study the fractal nature of the peaks of X. In this regard, let us first introduce an increasing real-valued function *gauge function* g such that $\lim_{x\to\infty} g(x) = \infty$ and

$$\limsup_{|t|\to\infty} \frac{|X_t|}{g(|t|)} = 1 \quad \text{a.s.}$$

where $|t|$ is the sup-norm on $\mathbb{R}^n$. Let us fix such a g and consider the random set:

$$\mathcal{P}_{X,g}(\gamma) = \left\{ t \in \mathbb{R}^n : |t| > 1, \frac{|X_t|}{g(|t|)} > \gamma \right\} \tag{12.1}$$

where $\gamma > 0$ is the length scale at which the tall peaks of the process X are viewed. Just like we have the usual Hausdorff dimension for ordinary fractals, there is an analogous measure for multifractals, called the macroscopic Hausdorff dimension originally due to Barlow and Taylor [1, 2].

Definition 12.1.1 *The peaks of a process X are multifractal in gauge g if there exists a strictly decreasing sequence of real numbers $\gamma_1 > \gamma_2 \ldots > 0$ such that*

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{P}_{X,g}(\gamma_{i+1})) > \mathrm{Dim}_{\mathrm{H}}(\mathcal{P}_{X,g}(\gamma_i)) \quad \forall i \geq 1$$

We will focus on two equations here: The linear stochastic heat equation:

$$\frac{\partial}{\partial t} Z_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z_t(x) + \dot{W}_t(x) \text{ on } (0,\infty) \times \mathbb{R}, \text{ subject to } Z_0(x) \equiv 0. \tag{12.2}$$

and the Anderson model:

$$\frac{\partial}{\partial t} u_t(x) = \frac{\partial^2}{\partial x^2} u_t(x) + u_t(x)\dot{W}_t(x) \quad \text{ for } t > 0, x \in \mathbb{R} \text{ subject to } u_0(x) \equiv 1 \tag{12.3}$$

Since $u_0(x) > 0$ for all $x \in \mathbb{R}$, Mueller's comparison principle tells us that $u_t(x) > 0$ for all t and x. Therefore, $h_t(x) = \log u_t(x)$ is well defined and the tall peaks of h are in one-to-one correspondence with those of u. For ease of calculations, we shall be working with h instead.

Now, we present the main theorems of this chapter.

Theorem 12.1.1 *Let $g(x) = (\log x)^{\frac{1}{2}}$ and consider $Z_t(x)$ as in (12.2). Define $\forall t, \gamma > 0$,*

$$\mathcal{P}_{Z(t),g}(\gamma) = \left\{ x \geq \mathrm{e}^{\mathrm{e}} : \frac{Z_t(x)}{(\log x)^{\frac{1}{2}}} \geq \gamma t^{\frac{1}{4}} \right\} \tag{12.4}$$

Then,

$$\mathrm{Dim}_{\mathrm{H}}\left(\mathcal{P}_{Z(t),g}(\gamma)\right) = 1 - \frac{\sqrt{\pi}}{2}\gamma^2 \quad \text{a.s.}$$

where $\mathrm{Dim}_{\mathrm{H}}(E) < 0$ means that E is bounded. Hence the tall peaks of $Z_t(x)$ are multifractal a.s.

Theorem 12.1.2 *Let* $g(x) = (\log x)^{\frac{2}{3}}$ *and consider* $h_t(x)$ *as in* (12.3). *Define* $\forall t, \gamma > 0$

$$\mathcal{P}_{h(t),g}(\gamma) = \left\{ x \geq \mathrm{e}^{\mathrm{e}} : \frac{h_t(x)}{(\log x)^{\frac{2}{3}}} \geq \gamma t^{\frac{1}{3}} \right\} \tag{12.5}$$

Then,

$$\mathrm{Dim_H}\left(\mathcal{P}_{h(t),g}(\gamma)\right) = 1 - \frac{4\sqrt{2}}{3}\gamma^{\frac{3}{2}} \qquad a.s. \tag{12.6}$$

where $\mathrm{Dim_H}(E) < 0$ *means that* E *is bounded. Hence the tall peaks of* $h_t(x)$ *are multifractal a.s.*

The idea of the proof is to derive the upper and lower bounds on the dimension using results from Section 3. Before that, we give a brief introduction to macroscopic Hausdorff dimension.

12.2 Barlow-Taylor Dimension

We recall the macroscopic dimension for $E \subset \mathbb{R}^n$ as expounded in [1, 2]. Define for $n \geq 0$,

$$V_k = \left[-\mathrm{e}^k, \mathrm{e}^k\right)^n, \quad \mathcal{S}_0 = \mathcal{V}_0 \quad \text{and} \quad \mathcal{S}_{k+1} = \mathcal{V}_{k+1} \backslash \mathcal{V}_k \tag{12.7}$$

Definition 12.2.1 *Let* $\mathbb{B}$ *be the family of sets of the form:*

$$Q(x,r) = [x_1, x_1 + r) \times [x_2, x_2 + r) \times \cdots [x_n, x_n + r) \tag{12.8}$$

where $x = (x_1, x_2, \cdots x_n) \in \mathbb{R}^n$ *and* $r > 0$. *Here the side length of* Q *is set as* $\mathrm{side}(Q) = r$.

Now, define for every $E \subset \mathbb{R}^n$, $\rho > 0$ and integer $n \geq 0$,

$$\nu_\rho^k(E) = \inf \sum_{i=1}^m \left(\frac{\mathrm{side}\,(Q_i)}{\mathrm{e}^n}\right)^\rho \tag{12.9}$$

where the infimum is taken over all $\{Q_i\}_{i=1}^m \subset \mathbb{B}$ such that $\mathrm{side}(Q_i) \geq 1$ and $E \cap S_k \subset \bigcup_{i=1}^m Q_i$. The reader should compare this with the definition of the usual Hausdorff measure of E.

Definition 12.2.2 *The* Barlow-Taylor macroscopic Hausdorff dimension *of a set E is*

$$\mathrm{Dim}_{\mathrm{H}}(E) = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \nu_{\rho}^{k}(E) < \infty \right\} \tag{12.10}$$

Clearly, any bounded set has $\mathrm{Dim}_{\mathrm{H}}(E) = 0$.

12.3 General Bounds

In this section, we state some methods to estimate find upper and lower bounds for the macroscopic Hausdorff dimension of the sets in Theorems 12.1.1,12.1.2 to prove the corresponding theorems.

Let $X = \{X_t\}_{t \in \mathbb{R}^n}$. For $b \in (0, \infty)$, define

$$c(b) = -\limsup_{y \to \infty} \frac{\sup_{t \in \mathbb{R}^n} \log \mathbf{P}\{X(t) > y\}}{y^b} \tag{12.11}$$

$$C(b) = -\liminf_{y \to \infty} \frac{\inf_{t \in \mathbb{R}^n} \log \mathbf{P}\{X(t) > y\}}{y^b} \tag{12.12}$$

Clearly, $0 \leq c(b) \leq C(b) \leq \infty$.

First, we have a theorem to estimate the upper bound of the macroscopic Hausdorff dimension of a random set.

Theorem 12.3.1 *Let $b > 0$ be such that $c(b) > 0$ and $\forall \gamma \in (0, n)$,*

$$\sup_{w \in \mathbb{R}^n} \mathbf{P} \left\{ \sup_{t \in Q(w,1)} X(t) > \left(\frac{\gamma}{c(b)} \log s \right)^{\frac{1}{b}} \right\} \leq s^{-\gamma + o(1)} \qquad \text{as } s \to \infty. \tag{12.13}$$

Then,

$$\mathrm{Dim}_{\mathrm{H}} \left\{ t \in \mathbb{R}^n : |t| > \mathrm{e}^{\mathrm{e}}, X(t) \geq \left(\frac{\gamma}{c(b)} \log |t| \right)^{\frac{1}{b}} \right\} \leq n - \gamma \tag{12.14}$$

The corresponding theorem for the lower bound is:

Theorem 12.3.2 *Let $b > 0$ be such that $C(b) < \infty$. Also assume there is a $\delta \in (0, 1)$ and an increasing measurable function $S : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \max_{\{t_i\}_{i=1}^m \in \Pi_n(\delta)} \max_{1\le j\le m} \inf_{\{Y_i\}_{i=1}^m \in \mathcal{I}} \log \mathrm{P}\left\{\left|S\left(X\left(t_j\right)\right) - S\left(Y_j\right)\right| > 1\right\} = -\infty$$

where $m = |\Pi_n(\delta)|$ and $\mathcal{I}$ is the set of all independent finite sequences of independent random variables. Then for all $\gamma \in (0, n)$

$$\mathrm{Dim_H}\left\{ t \in T : |t| > \mathrm{e}^{\mathrm{e}}, \frac{X(t)}{(\log t)^{\frac{1}{b}}} \ge \left(\frac{\gamma}{C(b)}\right)^{\frac{1}{b}} \right\} \ge n - \gamma \quad a.s. \tag{12.15}$$

Here $\Pi_n(\delta)$ is a collection of m points such that $\{t_i\}_{i=1}^m \subset \mathcal{S}_n$ and $|t_{i+1} - t_i| > \mathrm{e}^{\delta n}$.

12.4 The Linear Stochastic Heat Equation

In this section, we focus on the linear stochastic heat equation (12.2) and give a proof of Theorem 12.1.1. Recall that the solution to (12.2) is given by

$$Z_t(x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) W(dsdy)$$

where $p_t(x)$ is the usual heat kernel. It is known that $b = 2$ and $C(b) = c(b) = \left(\frac{\pi}{4t}\right)^{\frac{1}{2}}$. In order to use Theorem 12.3.2 for $S(x) = x$ for the lower bound we need to find an appropriate coupling. In this regard, define the following random fields for every $B > 0$,

$$Z_t^{(B)}(x) = \int_0^t \int_{\left[x-\sqrt{Bt}, x+\sqrt{Bt}\right]} p_{t-s}(y-x) W(dsdy) \tag{12.16}$$

If $x_1, x_2, \ldots, x_m \in \mathbb{R}$ are such that $|x_i - x_j| > \sqrt{2(Bt)}$ for1 $\le i \ne j \le m$, then the random variables $Z_t^{(B)}(x_1), \ldots, Z_t^{(B)}(x_m)$ are independent as the integrands are nonrandom with disjoint supports.

Lemma 12.4.1

$$\sup_{x\in\mathbb{R}} \mathbf{P}\left\{\left|Z_t(x) - Z_t^{(B)}(x)\right| > \lambda\right\} \le 2\mathrm{e}^{\left(-\frac{\lambda^2}{2}\sqrt{\frac{\pi}{8t}}\mathrm{e}^{B/2}\right)} \tag{12.17}$$

Proof. Clearly,

$$\mathbf{E}\left(\left|Z_t(x) - Z_t^{(B)}(x)\right|^2\right) = \int_0^t \int_{|z|>\sqrt{(Bt)}} [p_s(z)]^2 \, dsdz$$
$$\leq \int_0^t \frac{1}{\sqrt{2\pi s}} \mathbf{P}|X| > \sqrt{\frac{Bt}{s}} ds$$

where X is a standard normal variable. We then use the following fact

$$\mathbf{P}\left\{|X| > \left(\sqrt{\frac{Bt}{s}}\right)\right\} \leq \mathbf{P}\{|X| > \sqrt{B}\} \tag{12.18}$$

along with an elementary Gaussian tail bound to conclude the lemma. □

Now fix $\delta \in (0,1)$. Choose n large enough such that $2\sqrt{nt} < \mathrm{e}^{\delta n}$ and m points in $\mathcal{S}_n$ such that $\mathrm{e}^n \leq x_1 < \cdots < x_m < \mathrm{e}^{n+1}$ and $x_{i+1} - x_i \geq \exp\{\delta n\}$. By the first equation and Chebyshev's inequality we have

$$\max_{1\leq i\leq m} \mathbf{P}\{|Z_t(x_i) - Y_i| > 1\} \leq 2\mathrm{e}^{\left(-\frac{1}{2}\sqrt{\frac{\pi}{8t}}\mathrm{e}^{\frac{n}{2}}\right)}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{n} \max_{1\leq i\leq m} \log \mathbf{P}\left\{|Z_t(x_i) - Y_i| > 1\right\} = -\infty$$

By Theorem 12.3.2,

$$\mathrm{Dim_H}\left\{x \geq \mathrm{e}^{\mathrm{e}} : \frac{Z_t(x)}{(\log x)^{\frac{1}{2}}} \geq t^{\frac{1}{4}}\left(\frac{2\gamma}{\sqrt{\pi}}\right)^{\frac{1}{2}}\right\} \leq 1 - \gamma$$

Substituting $\gamma \to \frac{\sqrt{\pi}\gamma^2}{2}$, we obtain the required bound.

12.5 The Anderson Model

We know from the results of Chapter 9 that

$$\limsup_{x\to\infty} \frac{h_t(x)}{(\log x)^{\frac{2}{3}}} < \infty \quad \text{a.s.}$$

Hence a good candidate for the gauge function g is $g(x) = (\log x)^{\frac{2}{3}}$. For the upper bound, we begin with a tail probability estimate akin to (9.4) in Chapter 9.

Lemma 12.5.1 *For all $t > 0$,*

$$\liminf_{z\to\infty} \frac{\inf_{x\in\mathbb{R}} \log \mathbf{P}\{h_t(x) \geq z\}}{z^{\frac{3}{2}}} = \limsup_{z\to\infty} \frac{\sup_{x\in\mathbb{R}} \log \mathbf{P}\{h_t(x) \geq z\}}{z^{\frac{3}{2}}} = -\frac{4\sqrt{2}}{3\sqrt{t}}$$

The above lemma suggest we can use $b = \frac{3}{2}$ and $C(b) = c(b) = -\frac{4\sqrt{2}}{3\sqrt{t}}$ in Theorems 12.3.1 and 12.3.2.

Lemma 12.5.2 *For all $\gamma \in (0, 1)$,*

$$\sup_{y\in\mathbb{R}} \mathbf{P}\left\{ \sup_{x\in[y,y+1]} \log u_t(x) > \left(t^{\frac{1}{3}} \left(\frac{3\gamma \log s}{4\sqrt{2}} \right)^{\frac{2}{3}} \right) \right\} \leq s^{-\gamma+o(1)}$$

The proof of the above lemma uses some continuity estimates ([10]) similar to those used in proof of 9.1.2. Thus the conditions of Theorem 12.3.1 are satisfied and we can use it to see that

$$\mathrm{Dim}_{\mathrm{H}} \left\{ x \in \mathbb{R} : x > \mathrm{e}^{\mathrm{e}}, h_t(x) \geq \left(\frac{3\sqrt{t}\gamma}{4\sqrt{2}} \log x \right)^{\frac{2}{3}} \right\} \leq 1 - \gamma$$

Hence,

$$\mathrm{Dim}_{\mathrm{H}} \left\{ x \in \mathbb{R} : x > \mathrm{e}^{\mathrm{e}}, \frac{h_t(x)}{(\log x)^{\frac{2}{3}}} \geq t^{\frac{1}{3}} \left(\frac{3\gamma}{4\sqrt{2}} \right)^{\frac{2}{3}} \right\} = \mathrm{Dim}_{\mathrm{H}} \left(\mathcal{P}_{h(t),g} \left(\left(\frac{3\gamma}{4\sqrt{2}} \right)^{\frac{2}{3}} \right) \right) \leq 1 - \gamma$$

Substituting $\gamma \to \frac{4\sqrt{2}\gamma^{\frac{3}{2}}}{3}$, we obtain the required upper bound.

Now we proceed to derive the lower bound. For this, we would need some results from Chapter 9. Recall the definition of the random fields $\{U^{(\beta,n)}\}_{\beta>0,n\geq 0}$. Our first step towards proving the lower bound is the following lemma:

Lemma 12.5.3 *There exists $K > 1$ such that $\forall t > 0$ and $\lambda > 1$, and all integers $B \geq 1$,*

$$\sup_{x\in\mathbb{R}} \mathbf{P}\left\{ \left| u_t(x) - U_t^{(B,B)}(x) \right| > \lambda \right\} \leq K \mathrm{e}^{\left(-\frac{(B+\log\lambda)^{3/2}}{K\sqrt{t}} \right)}$$

The proof of the above lemma follows from a minor variation of the moment bound (9.3.3) and Chebyshev's inequality.

To use Theorem 12.3.2, set $S(x) = \mathrm{e}^x$ and $B = n^2$. Choose and fix $\delta \in (0,1)$ and consider m points $\{x_i\}_{i=1}^m$ such that $\mathrm{e}^n \le x_1 \le x_2 \cdots x_m < \mathrm{e}^{n+1}$ and $|x_{i+1} - x_i| \ge \mathrm{e}^{\delta n}$. Define $Y_j = U_t^{(n^2,n^2)}(x_j)$. By Lemma 9.3.3, $Y_1, Y_2 \cdots Y_m$ are independent as long as $\mathrm{e}^{\delta n} > 2n^3 t$. And by Lemma 12.5.3,

$$\max_{1\le j\le m} \mathbf{P}\left\{\left|S\left(h_t\left(x_j\right)\right) - S\left(Y_j\right)\right| > 1\right\} \le K\mathrm{e}^{\left(-\frac{n^3}{K\sqrt{t}}\right)}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{n} \max_{1\le j\le m} \mathbf{P}\left\{\left|S\left(h_t\left(x_j\right)\right) - S\left(Y_j\right)\right| > 1\right\} = -\infty$$

By Theorem 12.3.2, we have

$$\mathrm{Dim}_{\mathrm{H}}\left\{x \in \mathbb{R} : x > \mathrm{e}^{\mathrm{e}}, h_t(x) \ge \left(\frac{3\sqrt{t}\gamma}{4\sqrt{2}}\log x\right)^{\frac{2}{3}}\right\} \ge 1-\gamma$$

Hence,

$$\mathrm{Dim}_{\mathrm{H}}\left\{x \in \mathbb{R} : x > \mathrm{e}^{\mathrm{e}}, \frac{h_t(x)}{(\log x)^{\frac{2}{3}}} \ge t^{\frac{1}{3}}\left(\frac{3\gamma}{4\sqrt{2}}\right)^{\frac{2}{3}}\right\} = \mathrm{Dim}_{\mathrm{H}}\left(\mathcal{P}_{h(t),g}\left(\left(\frac{3\gamma}{4\sqrt{2}}\right)^{\frac{2}{3}}\right)\right) \ge 1-\gamma$$

Substituting $\gamma \to \frac{4\sqrt{2}\gamma^{\frac{3}{2}}}{3}$, we obtain the required lower bound.

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