Isoperimetric inequalities in the hyperbolic plane

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Isoperimetric inequalities in the hyperbolic plane towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Gaurav Kumar at Indian Institute of Science Education and Research under the supervision of Dr. Anisa Chorwadwala, Associate Professor, Department of Mathematics , during the academic year 2023-2024.

Anisa

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To my Family and Friends

Declaration

I hereby declare that the matter embodied in the report entitled Isoperimetric inequalities in the hyperbolic plane are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anisa Chorwadwala and the same has not been submitted elsewhere for any other degree.

गौरव
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Abstract

Isoperimetric problem is a mathematical problem where we enclose a given area with the shortest possible curve. The isoperimetric problem in the Euclidean plane states "Among all piecewise smooth simple closed planar curves enclosing a fixed area $A > 0$, the circle enclosing area A is the unique perimeter minimizer".

The history of the Isoperimetric Problem is fascinating. Many mathematicians, starting from the ancient Greek mathematician Zenodorous (c. 200 - c. 140 BC), contributed to the development of the proof. The proof of this classical theorem was settled completely by the year 1882.

Inspired by the legend of Queen Dido mentioned in the Latin epic poem Aeneid (c. 29 - c. 19 BC), where she wanted to enclose the largest piece of land using an ox hide thong, the isoperimetric problem was later considered in other settings and proven for non-Euclidean surfaces. The theorem was proved for curved surfaces such as the unit sphere (by Bernstein [1] in 1905 and by Fiala [4] in 1940-41) and the hyperbolic space (by Schmidt [10] in 1940) among several other Riemannian manifolds with nonzero curvature.

In this thesis, we present a proof of The Isoperimetric problem and the isoperimetric inequality for the hyperbolic plane $\mathbb{H}^2$. The isoperimetric inequality states that for any piecewise smooth simple closed curve $\mathcal{C}$ in $\mathbb{H}^2$ with arc-length ℓ and enclosing area $A > 0$ we have $\ell^2 \geq 4\pi A + A^2$ and equality holds if and only if $\mathcal{C}$ is a circle in $\mathbb{H}^2$ of radius $\sinh^{-1}\left(\dfrac{\sqrt{A(4\pi + A)}}{2\pi}\right)$.

Contents

Chapter 1

Introduction

Which shape should one draw on paper so that it encloses the maximum area among all possible shapes that can be drawn using a thread of a fixed length? It turns out that the circular shape is the unique area maximizer among all the possible planar shapes using this particular thread of a given length. Another similar question is the following: what should be the shape of a planar domain enclosing a given area $A > 0$ such that its perimeter is minimum among all the shapes enclosing the same area A. The unique perimeter minimizer in this particular family of domains considered is again the circular shape enclosing area A. These problems are referred to as the isoperimetric problems in the Euclidean plane. Knowing the solution to the problem in the plane, it is natural to ask the same question for domains in higher-dimensional Euclidean spaces.

The history of the isoperimetric problem is fascinating. It goes back to the story of Queen Dido mentioned in the Latin epic poem called Aeneid, written by the Roman poet Virgil between 29 BCE and 19 BCE, where she wanted to enclose the largest piece of land using an ox hide thong. The proof of the isoperimetric problem for the polygonal case was first attempted by Zenodorous (200 BC- 140 BC), known for authoring the treatise “On isoperimetric figures”. In this, using a sequence of images, Zenodorous indicated a proof of the uniqueness of the perimeter minimizer among all polygons with n sides enclosing a fixed area.

The original Isoperimetric problem of Queen Dido was for a piece of land on the planet earth, and hence in the setting of non-Euclidean domains. Many later mathematicians have considered the Isoperimetric problem on curved surfaces. In this thesis, we present a proof of the Isoperimetric Theorem for domains on a surface of constant negative curvature, namely the Hyperbolic plane. The isoperimetric Inequality, which is a succinct expression of the Isoperimetric Theorem in terms of a sharp inequality, is also derived.

Motivated by the non-Euclidean case of the Hyperbolic plane, we begin with an exposition of Riemannian geometry in general and of important theorems like the Hopf-Rinow theorem and the Hadamard theorem in particular. These theorems are useful as they guarantee the existence of a unique geodesic between two distinct points in the hyperbolic plane. We work with the hyperboloid model of the hyperbolic plane. We then carry out the study of the isoperimetric problem and the isoperimetric inequality in the hyperbolic plane. It was first proved by E. Schmidt in 1940 [10]. Since all the simply connected Riemannian manifolds with constant sectional curvature $\kappa = -1$ are isometric, the results that are proved in one particular model of a hyperbolic space also hold true in its other models.

The thesis is organized as follows:

In Chapter 2, we briefly introduce Riemannian manifolds and Riemannian metrics. We then discuss vector fields on a Riemannian manifold, their Lie bracket, and the corresponding Lie algebra. We then talk about the arc length of a curve and the volume of a domain on a Riemannian manifold.

In Chapter 3, we discuss two fundamental concepts of Riemannian geometry, namely, the geodesics on a Riemannian manifold and the curvature tensor of a Riemannian manifold.

For a Riemannian manifold (M, g), consider $p \in M$ and σ a two-dimensional subspace of T_pM, the tangent space at p. Then, its sectional curvature $\kappa(p, \sigma)$ determines how fast the geodesics that emanate from p and are tangent to σ deviate from each other. The study of Jacobi fields, which is the content of Chapter 4, helps us understand this. We also discuss the Hopf-Rinow theorem and the theorem of Hadamard in this chapter.

In chapter 5, we introduce Riemannian manifolds of constant sectional curvature because we will be studying the isoperimetric problem on hyperbolic plane $\mathbb{H}^2$, which is a complete simply connected Riemannian manifold of constant sectional curvature -1.

Chapters 2-5 of this thesis are based on the book Riemannian Geometry by Do Carmo [3]. A general reference for this chapter is [3].

In Chapter 6, we study the two-dimensional Hyperboloid model of the hyperbolic space via the hyperbolic geometry in the three-dimensional Minkowski space.

Chapter 7 accomplishes the main objective of this thesis, namely, the presentation of the proofs of the Isoperimetric theorem and the Isoperimetric inequality in $\mathbb{H}^2$.

Original Contribution: This thesis has no claim of new results. We give a literature review of the isoperimetric problem and the isoperimetric inequality in the hyperbolic plane $\mathbb{H}^2$. We briefly develop the necessary mathematical background such as the theory of Riemannian manifolds, its sectional curvature, geodesics on them, etc. We give a simple and detailed description of the distance function in the hyperboloid model of $\mathbb{H}^2$, its orthogonal transformations, and its isometry group. We then give a simple exposition of circles, parabolas, and hyperbolas on this model of $\mathbb{H}^2$. We consider the hyperbolic plane with constant sectional curvature $\kappa = -1$ and present a geometric proof of the isoperimetric inequality in $\mathbb{H}^2$ based on the expository article [2].

Chapter 2

Riemannian Manifolds

In this chapter, we briefly introduce Riemannian manifolds and Riemannian metrics on them. We then discuss vector fields on a Riemannian manifold, their Lie bracket, and the corresponding Lie algebra. We then talk about the arc length of a curve and the volume of a domain on a Riemannian manifold. A general reference for this chapter is [3].

2.1 Some basic definitions and results

Definition 2.1.1. *A differentiable manifold of dimension n is a set M and a family of one-to-one mappings $f_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open subsets U_α of $\mathbb{R}^n$ into M such that*

1. $\cup_\alpha f_\alpha(U_\alpha) = M$.
2. *The sets $f_\alpha^{-1}(W)$ and $f_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $f_\beta^{-1} \circ f_\alpha$ are differentiable, for any pair α, β, with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \emptyset$.*
3. *The family $\{(U_\alpha, f_\alpha)\}$ is maximal with respect to the conditions 1 and 2.*

The pair (U_α, f_α)(or the mapping f_α) with $p \in f_\alpha(U_\alpha)$ is called a *parametrization*(or system of coordinates) of M at p; $f_\alpha(U_\alpha)$ is called a *coordinate neighbourhood* at p. A family $\{(U_\alpha, f_\alpha)\}$ satisfying conditions 1 and 2 is called a *differentiable structure* on M.

Definition 2.1.2. *Let M and N be differentiable manifolds of dimension m and n, respectively. A mapping $\varphi : M \to N$ is differentiable at $p \in M$ if given a parametrization $g : V \subset \mathbb{R}^n \to N$ at $\varphi(p)$ there exits a parametrization $f : U \subset \mathbb{R}^m \to M$ at p such that $\varphi(f(U)) \subset g(V)$ and the mapping*

$$g^{-1} \circ \varphi \circ f : U \subset \mathbb{R}^m \to \mathbb{R}^n$$

is differentiable at $f^{-1}(p)$. If φ is differentiable at all points of an open set of M, then it is differentiable on this open set.

Definition 2.1.3. *Suppose that M is a differentiable manifold. A differentiable mapping $c : (-\varepsilon, \varepsilon) \to M$ is termed a (differentiable) curve in M. Let $c(0) = p \in M$, and denote by $\mathcal{D}$ the set of functions on M that are differentiable at p. The tangent vector to the curve c at $t = 0$ is a function $c'(0) : \mathcal{D} \to \mathbb{R}$ given by*

$$c'(0)f = \left.\frac{d(f \circ c)}{dt}\right|_{t=0}, \qquad f \in \mathcal{D} \tag{2.1}$$

A *tangent vector* at p is the tangent vector at $t = 0$ of some curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$. The set of all tangent vectors to M at p will be indicated by T_pM.

Proposition 2.1.1. *Let M and N be differentiable manifolds of dimensions m and n respectively. Suppose that $\varphi : M \to N$ is a differentiable mapping. For every $p \in M$ and for each $v \in T_PM$, choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M$, with $\alpha(0) = p$, $\alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p : T_pM \to T_{\varphi(p)}N$ given by $d\varphi_p(v) = \beta'(0)$ is a linear mapping that is independent of the choice of α.*

Definition 2.1.4. *The linear mapping $d\varphi_p$ defined by Proposition 2.1.1 is called the differential of φ at p.*

Definition 2.1.5. *Let M and N be differentiable manifolds. A mapping $\varphi : M \to N$ is a diffeomorphism if it is differentiable, bijective, and its inverse φ^{-1} is also differentiable. φ is said to be a local diffeomorphism at $p \in M$ it there exist neighbourhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffeomorphism.*

Theorem 2.1.2. *Suppose that $\varphi : M^m \to N^m$ is a differentiable mapping. φ is a local diffeomorphism at $p \in M$ if $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is an isomorphism.*

Definition 2.1.6. *Let M and N be differentiable manifolds of dimensions m and n, respectively. For a differentiable map $\varphi : M \to N$, if $d\varphi_p : T_pM \to T_{f(p)}N$ is one-to-one for all $p \in M$ then φ is called an immersion. If φ is an immersion and also a homeomorphism onto $\varphi(M) \subset N$, where $\varphi(M)$ has the subspace topology induced from N, we say that φ is an embedding. If $M \subset N$ and the inclusion $i : M \subset N$ is an embedding, we say that M is a submanifold of N.*

Example 1. (*The tangent bundle*). *Suppose that M is an n-dimensional differentiable manifold. Consider the set $TM = \{(p, v); p \in M,\ v \in T_pM\}$. It is easy to see that TM is a $2n$-dimensional differentiable manifold and we call it the tangent bundle of M.*

Definition 2.1.7. *Suppose that M is a differentiable manifold and it is said to be orientable if it has smooth structure $\{(U_\alpha, f_\alpha)\}$ such that whenever $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \emptyset$, the determinant of the differential of the change of coordinate $d(f_\beta^{-1} \circ f_\alpha)$ is positive.*

2.2 Lie Brackets

Definition 2.2.1. *A vector field A on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $A(p)$ in the tangent space of M at p. If the mapping $A : M \to TM$ is differentiable, then the vector field is differentiable.*

Lemma 2.2.1. *Suppose that A and B are two differentiable vector fields on a differentiable manifold M. Then there exists a unique differentiable vector field C, for all $f \in \mathcal{D}$ given by,*

$$Cf = (AB - BA)f$$

,

The vector field C given by the above lemma is called the *Lie bracket* $[A, B] = AB - BA$ of A and B.

Proposition 2.2.2. *If A, B, C are differentiable vector fields on M, $x, y \in \mathbb{R}$, and f, g are differentiable functions, then*

1. $[A, B] = -[B, A]$ (*anti commutativity*)

2. $[xA + yB, C] = x[A, C] + y[B, C]$ (*linearity*)

3. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$ (*Jacobi Identity*)

4. $[fA, gB] = fg[A, B] + fA(g)B - gB(f)A.$

We can think of The Lie bracket $[A, B]$ as a derivation of B along the "trajectories" of A.

Theorem 2.2.3. *Suppose that M is a differentiable manifold, let X be a differentiable vector field on M, and let $p \in M$. Then there exist a neighbourhood $U \subset M$ of p, an interval $(-\delta, \delta), \delta > 0$, and a differentiable mapping $f : (-\delta, \delta) \times U \to M$ such that the curve $t \to f(t, q)$, $t \in (-\delta, \delta)$, $q \in U$, is the unique curve which satisfies $\frac{\partial f}{\partial t} = X(f(t, q))$ and $f(0, q) = q$.*

2.3 Riemannian Metrics

Definition 2.3.1. *A Riemannian metric on a differentiable manifold M is correspondence which associates to each point p of M an inner product $\langle\ ,\ \rangle_p$(that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM, which varies differentiably, i.e., If $x : U \subset \mathbb{R}^n \to M$ is a system of coordinates around p, with $x(x_1, x_2, \ldots, x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0, \ldots, 1, \ldots, 0)$, then*

$$\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \right\rangle_q = g_{ij}(x_1, \ldots, x_n)$$

is a differentiable function on U.

A differentiable manifold with a given Riemannian metric will be called a Riemannian manifold.

Definition 2.3.2. *Suppose that M and N are two Riemannian manifolds. Consider the diffeomorphism $\phi : M \to N$. ϕ is called an isometry if ϕ satisfies following condition,*

$$\langle u, v \rangle_p = \langle d\phi_p(u), d\phi_p(v) \rangle_{\phi(p)}, \ \forall\ p \in M,\ u, v \in T_pM. \tag{2.2}$$

Example 2. *Lie groups: A Lie group is a group G with a differentiable structure such that the mapping $G \times G \to G$ given by $(x, y) \to xy^{-1}$, is differentiable for all $x, y \in G$. It follows that translation from the left L_x and translation from the right R_x given by $L_x : G \to G,\ L_x(y) = xy;\ R_x : G \to G,\ R_x(y) = yx$ are diffeomorphisms.*

To introduce a left invariant metric on G(Lie Group), take any arbitrary inner product $\langle\ ,\ \rangle_e$ on T_eG, the tangent space to G at e and define

$$\langle u, v\rangle_x = \langle (dL_{x^{-1}})_x(u), (dL_{x^{-1}})_x(v)\rangle_e\,, \quad x \in G,\ u, v \in T_xG. \tag{2.3}$$

where e is the identity element of G.
One can similarly introduce a right invariant metric by using right translations. A Riemannian metric on G is said to be *bi-invariant* if it is (simultaneously) both left and right invariant. A vector field V on G is said to be left-invariant (resp., right-invariant) if $dL_{x(V(y))} = V(xy)$ (resp. $dR_{x(V(y))} = V(yx)$). A left-invariant (resp. right-invariant) vector field is uniquely determined by $V(e) \in T_eG$. Conversely, $V \in T_eG$ defines a left-invariant (resp. right-invariant) vector field by the relation $V(x) = dL_x(V(e))$ (resp. $= dR_x(V(e))$). Thus, the set of left-invariant (resp. right-invariant) vector fields form an n-dimensional subspace of space of vector fields on G. Further, it turns out that Lie bracket of any two left-invariant (resp. right-invariant) vector fields is a left-invariant (resp. right-invariant) vector field. This sets up a vector space isomorphism between the tangent space T_eG to G at e and the space of left-invariant (resp. right-invariant) vector fields, and therefore, one can identify the two as vector spaces, and use a common notation $\mathcal{G}$ for this vector space. If G has a bi-invariant metric, the inner product that the metric determines on $\mathcal{G}$ satisfies the following relation: For any $A, B, C \in \mathcal{G}$,

$$\langle [A, C], B\rangle = -\langle A, [B, C]\rangle\,. \tag{2.4}$$

Now, we use the Riemannian metric to calculate the length of a curve.

Definition 2.3.3. *A (parametrized) curve is a differentiable mapping $c : I \subset \mathbb{R} \to M$ of an open set I of real numbers into a differentiable manifold M.*

Definition 2.3.4. *A vector field V along a curve $c : I \to M$ is a differentiable mapping that associates to every t in the interval I a tangent vector $V(t)$ in the tangent space of M at $c(t)$ i.e., $V(t) \in T_{c(t)}M$.*

Let M be a Riemannian manifold, and then the length of a segment is given by

$$\ell_a^b(c) = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{\frac{1}{2}} dt \tag{2.5}$$

Proposition 2.3.1. *Suppose that M is a differentiable manifold which is Hausdroff with countable basis then M has a Riemannian metric.*

We can use the Riemannian metric to calculate the volume on a given oriented manifold. Suppose that $R \subset M$ is a region(an open connected subset) with compact closure. Let $R \subset x(U)$, where $x(U)$ is coordinate neighborhood, with a positive parametrization $x : U \to M$, and the boundary of $x^{-1}(R) \subset U$ has measure 0 in $\mathbb{R}^n$. Now the volume of R, $\mathrm{Vol}(R)$, is given by the following integral in $\mathbb{R}^n$

$$\mathrm{Vol}(R) = \int_{x^{-1}(R)} \sqrt{\det(g_{ij})} dx_1 \dots dx_n \tag{2.6}$$

2.4 Affine Connections and Riemannian Connections

Suppose that X is a vector field along c tangent to $S \subset \mathbb{R}^3$(surface) then the orthogonal projection of $\frac{dX}{dt}(t)$ on $T_{c(t)}S$ is called the covariant derivative and we will denote it by $\frac{DX}{dt}(t)$. Let $\mathcal{X}(M)$ denotes the set of all $\mathcal{C}^\infty$ vector fields on M, and $\mathcal{D}$ denotes the ring of $\mathcal{C}^\infty$ real-valued functions on M.

Definition 2.4.1. *Let M be a differentiable manifold. An affine connection ∇ on M is a mapping*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

which is denoted by $(A, B) \xrightarrow{\nabla} \nabla_A B$ such that:

1. $\nabla_{fA+gB} C = f\nabla_A C + g\nabla_B C$.
2. $\nabla_A(B + C) = \nabla_A B + \nabla_A C$.
3. $\nabla_A(fB) = f\nabla_A B + A(f)Y$, *in which* $A, B, C \in \mathcal{X}(M)$ *and* $f, g \in \mathcal{D}(M)$.

Proposition 2.4.1. *Assume that ∇ is an affine connection on a differentiable manifold M. Let $c : I \to M$ be a differentiable curve in M and X be a vector field along c then there exists* *a unique mapping that associates to X another vector field $\frac{DX}{dt}$ along c, called the covariant derivative of X along c such that*

1. $\frac{D}{dt}(X+Y) = \frac{DX}{dt} + \frac{DY}{dt}$
2. $\frac{D}{dt}(fX) = \frac{Df}{dt}X + f\frac{DX}{dt}$
   where Y is a vector field along c and f is a differentiable function on I.
3. *If X is induced by a vector field $Z \in \mathcal{X}(M)$, then $\frac{DX}{dt} = \nabla_{\frac{dc}{dt}} Z$.*

Definition 2.4.2. *Assume that M is a differentiable manifold. Let ∇ be an affine connection on M. Suppose that $c : I \to M$ is a curve in M and X is a vector field along c then we say X is parallel when $\frac{DX}{dt} = 0 \ \forall \ p \in I$.*

Proposition 2.4.2. *Suppose that M is a differentiable manifold. Let ∇ be an affine connection defined on M. Let $c : I \to M$ be a differentiable curve in M and let X_0 be a vector tangent to M at $c(t_0), t_0 \in I$ (i.e. $X_0 \in T_{c(t_0)}M$). Then three exists a unique parallel vector field X along c, such that $X(t_0) = X_0$, (($X(t)$ is called the parallel transport of $X(t_0)$ along c.)*

Definition 2.4.3. *Suppose that M is a differentiable manifold with an affine connection ∇ and a Riemannian metric $\langle \ , \ \rangle$. A connection said to be compatible with metric $\langle \ , \ \rangle$ when for any differentiable curve c and any pair of parallel vector fields R and R' along c, we have*

$$\langle R, R' \rangle = constant \tag{2.7}$$

Proposition 2.4.3. *Suppose that M is a Riemannian manifold with affine connection ∇. Let $c : I \to M$ be a differentiable curve in M. ∇ is compatible with a metric if and only if for any vector field, X and Y along c we have*

$$\frac{d}{dt}\langle X, Y \rangle = \left\langle \frac{DX}{dt}, Y \right\rangle + \left\langle X, \frac{DY}{dt} \right\rangle, \quad t \in I. \tag{2.8}$$

Corollary 2.4.4. *Suppose that M is a Riemannian manifold with affine connection ∇ then ∇ is compatible with a metric if and only if*

$$A\langle B, C \rangle = \langle \nabla_A B, C \rangle + \langle B, \nabla_A C \rangle, \qquad A, B, C \in \mathcal{X}(M). \tag{2.9}$$

Definition 2.4.4. *An affine connection ∇ on a smooth manifold M is said to be symmetric when $\nabla_A B - \nabla_B A = [A, B] \quad \forall \ \ A, B \in \mathcal{X}(M)$.*

Theorem 2.4.5. (*Levi-Civita*). *Given a Riemannian manifold M, there exits a unique affine connection ∇ on M satisfying the conditions:*

1. *∇ is symmetric.*

2. *∇ is compatible with the Riemannian metric.*

The connection given by the above theorem is called the Levi-Civita connection or the Riemannian connection on M.

Suppose that X and Y are two differentiable vector fields on a differentiable manifold M and (U, x) be a coordinate system about $p \in M$ then it is customary to call the functions Γ_{ij}^k defined on U by $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$, the coefficients of the connection ∇ on U or the Christoffel symbols of the connection. It follows from the proof of the above theorem(see [3]) that

$$\sum_{\ell} \Gamma_{ij}^{\ell} g_{\ell k} = \frac{1}{2} \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} \tag{2.10}$$

where $g_{ij} = \langle X_i, X_j \rangle$. From the above equation, we get

$$\Gamma_{ij}^{\ell} = \frac{1}{2} \sum_m \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} g^{km} \tag{2.11}$$

where (g^{km}) is the inverse of matrix (g_{ij}). Note that for the Euclidean space $\mathbb{R}^n$, we have $\Gamma_{ij}^{\ell} = 0$ and in terms of the Christoffel symbols, the covariant derivative has the following expression

$$\frac{DV}{dt} = \sum_k \left\{ \frac{dv^k}{dt} + \sum_{i,j} \Gamma_{ij}^k v^j \frac{dx_i}{dt} \right\} X_k \tag{2.12}$$

Chapter 3

Geodesics and Curvature

Suppose that M is a Riemannian manifold. Let ∇ be a Riemannian connection on M. In this chapter, we introduce two fundamental concepts of Riemannian geometry, namely, the geodesics in a Riemannian manifold and the curvature tensor. Geodesics are nothing but curves on the Riemannian manifold with zero acceleration. The primary reference for this chapter is [3].

3.1 Geodesics

Definition 3.1.1. *Suppose that M is a Riemannian manifold and $c : I \to M$ is a parametrized curve in M. We say c is geodesic at $t_0 \in I$ if $\frac{D}{dt}\left(\frac{dc}{dt}\right) = 0$ at t_0. If c is a geodesic for all points in I then c is a geodesic. If $[a, b] \subset I$ and $c : I \to M$ is a geodesic, the restriction of c to $[a, b]$ is then called a geodesic segment joining $c(a)$ to $c(b)$.*

The arc-length $\ell(c(t)) = \int_{t_0}^{t} \left|\frac{dc}{dt}\right| dt$ is proportional to the parameter of a geodesic. If we consider a geodesic c in a system of local coordinates (U, x) about $c(t_0)$. In U a curve $c(t) = (x_1(t), \ldots, x_n(t))$ will be a geodesic if only if $0 = \frac{D}{dt}\left(\frac{dc}{dt}\right) = \sum_k \left\{\frac{dx_k^2}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt}\right\} \frac{\partial}{\partial x^k}$. Hence, we have second-order system

$$\frac{dx_k^2}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \quad k = 1, 2, \ldots, n. \tag{3.1}$$

It is convenient to study (3.1) on tangent bundle TM. Define, the differentiable canonical projection $\pi : TM \to M$ given by $\pi(q, v) = q$ for $q \in M, v \in T_qM$. Any differentiable curve $t \to \big(c(t), \frac{dc}{dt}(t)\big)$ in TM. If c is a geodesic then, on TU, the curve $t \to \big(x_1(t), \dots, x_n(t), \frac{dx_1}{dt}(t), \dots, \frac{dx_n}{dt}(t)\big)$ satisfies the first order system

$$\begin{aligned} \frac{dx_k}{dt} &= y_k \\ \frac{dy_k}{dt} &= -\sum_{i,j} \Gamma^k_{ij} y_i y_j \quad k = 1, 2, \dots, n \end{aligned}$$

in terms of coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on TU where as usual (U, x) is a system of coordinates on M. The existence and uniqueness of ODE make sure that geodesics exist on any Riemannian manifold.

Theorem 3.1.1. *If $X \in \mathcal{X}$, is a vector field on the open set V in the manifold M and $p \in V$, then there exist an open set $V_0 \subset V, p \in V_0$, a number $\delta > 0$, and a $\mathcal{C}^\infty$ mapping $f(-\delta, \delta) \times V_0 \to V$ such that the curve $t \to f(t, q)$, $t \in (-\delta, \delta)$, is the unique trajectory of X which at the instant t=0 passes through the point q, for very $q \in V_0$.*

The mapping $f_t : V_0 \to V$ given by $f_t(q) = f(t, q)$ is called the *flow* of X on V.

Lemma 3.1.2. *Consider the tangent bundle TM then $\exists$ G which is a unique vector field on TM such that for a geodesic c in M the trajectories of G are of the form $t \to (c(t), c'(t))$.*

Definition 3.1.2. *The vector field G on TM as given by above lemma is called the geodesic field on TM, and the flow of G is said to be the geodesic flow on TM.*

We now apply Theorem 3.1.1 to G around a point $(p, 0) \in TM$. Then,there exist $\mathcal{U} \subset TU$, numbers $\delta > 0$, and $\varepsilon > 0$, and a differentiable map $f : (-\delta, \delta) \times \mathcal{U} \to TU$ such that $t \mapsto f(t, q, v)$ is the unique trajectory of G with $f(0, q, v) = (q, v)$ for each $(q, v) \in \mathcal{U}$. It is possible to choose the following form of $\mathcal{U}$,

$$\mathcal{U} = \{(q, v) \in TU \mid q \in V \text{ and } v \in T_qM \text{ with } |v| < \varepsilon\}$$

where $V \subset U$ is a neighbourhood of $p \in M$. Define $c = \pi \circ f$. We can describe this result in the following way.

Proposition 3.1.3. *Given a point $p \in M$, there exist an open set $V \subset M$, numbers $\delta > 0$, $\varepsilon > 0$ and a differentiable mapping $c : (-\delta, \delta) \times \mathcal{U} \to M$ ($\mathcal{U} = \{(q, v) \in TU \mid q \in V$ and*

$v \in T_qM$ with $|v| < \varepsilon\}$) such that the curve $t \mapsto c(t,q,v)$, $\;t \in (-\delta,\delta)$ is the unique geodesic of M which passes through q at $t = 0$ with a velocity v for all $q \in V$ and for all $v \in T_qM$ with $|v| < \varepsilon$.

This proposition states that a unique geodesic exists which starts from any point in a Riemannian manifold in any given direction. The following lemma claims that it is possible to decrease the velocity by increasing the size of the interval in which the geodesic is defined or vice-versa.

Lemma 3.1.4. (*Homogeneity of a geodesic*). *Let $c(t,q,v)$ be a geodesic defined on the interval $(-\delta,\delta)$. Let $a > 0$, $a \in \mathbb{R}$ then the geodesic $c(t,q,av)$ is defined on the interval $\left(-\frac{\delta}{a}, \frac{\delta}{a}\right)$ and $c(t,q,av) = c(at,q,v)$.*

Using the above lemma, one can make the velocity at which a geodesic is defined uniformly large. Now, we can define the exponential map using the above lemma.

Definition 3.1.3. *Suppose that $p \in M$ and $\mathcal{U} \subset TM$ are an open set as in Proposition 3.1.3. Consider the map $\exp : \mathcal{U} \to M$ defined as $\exp(q,v) = c(1,q,v) = c\left(|v|, q, \frac{v}{|v|}\right)$, $\;(q,v) \in \mathcal{U}$. We call it the exponential map on $\mathcal{U}$.*

Frequently, we consider the mapping $\exp_q : B_\varepsilon(0) \subset T_qM \to M$, given by $\exp_q(v) = \exp(q,v)$, where $B_\varepsilon(0)$ represents the open ball of radius ε centered at 0 in T_qM. Geometrically, $\exp_q(v)$ is the point obtained by going a distance $|v|$ along the geodesic, which starts from q with the velocity $\frac{v}{|v|}$.

Proposition 3.1.5. *Given $q \in M$ there exists an $\varepsilon > 0$ such that $\exp_q : B_\varepsilon(0) \subset T_qM \to M$ is a diffeomorphism of $B_\varepsilon(0)$ onto an open subset of M.*

If the exponential map $\exp_p$ defines a diffeomorphism of a neighborhood V of the origin in T_pM, denoted as $\exp_p V = U$, then U is termed a normal neighborhood of p. Furthermore, if the closed ball $\overline{(B_\varepsilon(0))}$ is contained within V, we define $\exp_p(B_\varepsilon(0)) = B_\varepsilon(p)$ as the normal ball (or geodesic ball) with center p and radius ε. The boundary of a normal ball forms a hypersurface in M that is orthogonal to the geodesic originating from p, denoted by $S_\varepsilon(p)$, and referred to as the normal sphere (or geodesic sphere) at p. Geodesics within $B_\varepsilon(p)$ that emanate from p are called radial geodesics.

3.1.1 Minimizing properties of geodesics

Definition 3.1.4. *A piecewise differentiable curve is a continuous function $c : [a, b] \to M$, where $[a, b]$ is a closed interval in $\mathbb{R}$, mapped into a manifold M. It satisfies the condition that there exists a partition $a = t_0 < t_1 < t_2 < \ldots < t_{k-1} < t_k = b$ of $[a, b]$, such that the restrictions $c|_{[t_i,t_{i+1}]}$, for $i = 0, 1, 2, \ldots, k-1$, are differentiable.*

Definition 3.1.5. *Let $c : [a, b] \subset \mathbb{R} \to M$ be any arbitrary piecewise differentiable curve joining $\gamma(a)$ to $\gamma(b)$. We say that a segment of the geodesic $\gamma : [a, b] \to M$ is called minimizing if $\ell(\gamma) \leq \ell(c)$.*

Lemma 3.1.6. (*Symmetry*). *If M is a differentiable manifold with a symmetric connection and $s : A \subset \mathbb{R}^2 \to M$ is a parametrized surface, then:*

$$\frac{D}{\partial v}\frac{\partial s}{\partial u} = \frac{D}{\partial u}\frac{\partial s}{\partial v}$$

where A is a connected set in $\mathbb{R}^2$, and the boundary of A is a piecewise differentiable curve with any vertex angle not equal to π.

Lemma 3.1.7. (*Gauss*). *Suppose that $p \in M$ and $v \in T_pM$ such that $\exp_p v$ is defined. Assume, $w \in T_pM \approx T_v(T_pM)$. Then*

$$\big\langle (dexp_p)_v(v), dexp_p)_v(w)\big\rangle = \langle v, w\rangle$$

We now have the following proposition that asserts that the geodesics locally minimize the arc length.

Proposition 3.1.8. *Let $p \in M$, U a normal neighbourhood of p, and $B \subset U$ a normal ball of center p. Let $\gamma : [0, 1] \to B$ be a geodesic segment with $\gamma(0) = p$. If $c : [a, b] \to M$ is any piecewise differentiable curve joining $\gamma(0)$ to $\gamma(1)$ then $\ell(\gamma) \leq \ell(c)$ and if equality holds then $\gamma([0, 1]) = c([0, 1])$.*

The above proposition is not global; for example, take a point p on the sphere then the geodesics on the sphere which start from this point are no longer minimizing after they pass through the antipodal point of p. On the other hand, if a piecewise differentiable curve c is minimizing, we will prove that c is a geodesic. For this, we need a refinement of the above Proposition, where we prove the existence of normal neighborhoods. The following theorem says that for any $p \in M$, there exists a neighborhood W of p, which is a normal neighborhood of each $q \in W$.

Theorem 3.1.9. *For any point p in the manifold M, there exists a neighborhood W of p and a positive number δ, such that for every point q in W, the exponential map $\exp_q$ is a diffeomorphism on the ball $B_\delta(0)$ contained in the tangent space T_qM, and $\exp_q(B_\delta(0))$ contains W. In other words, W is a normal neighborhood around each of its points.*

Corollary 3.1.10. *If a piecewise differentiable curve $\gamma : [a, b] \to M$, with parameter proportional to arc length, has length less or equal to the length of any other piecewise differentiable curve joining $\gamma(a)$ to $\gamma(b)$ then γ is a geodesic.*

Example 3. *Suppose that $\mathbb{H}^2$ is the upper half-plane, i.e., $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$ and its Riemannian metric is given by $g_{11} = g_{22} = \frac{1}{y^2}, g_{12} = g_{21} = 0$ We are going to show that the segment $\gamma : [a, b] \to \mathbb{H}^2, a > 0$ of the y axis, given by $\gamma(t) = (0, t)$ is the image of a geodesic. In fact for any arc $c : [a, b] \to \mathbb{H}^2$ given by $c(t) = (x(t), y(t))$ with $c(a) = (0, a)$, and $c(b) = (0, b)$, we have that*

$$\begin{aligned}\ell(c) = \int_a^b \left|\frac{dc}{dt}\right| dt &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\frac{dt}{y} \\ &\geq \int_a^b \left|\frac{dy}{dt}\right| \frac{dt}{y} \geq \int_a^b \frac{dy}{y} = \ell(\gamma).\end{aligned}$$

we can see that γ minimizes arc length for piecewise differentiable curves. From Corollary 3.1.10, we can say that the image of γ is a geodesic.

3.2 Curvature

In this section, we discuss the curvature of an arbitrary manifold from [9], and for more intuition about curvature, see Chapter 1 of [5].

Definition 3.2.1. *Let M be a Riemannian manifold and R be the curvature of M then R is a correspondence that associates to every pair $A, B \in \mathcal{X}(M)$, a mapping $R(A, B) : \mathcal{X}(M) \to \mathcal{X}(M)$ defined by $R(A, B)C = \nabla_A\nabla_BC - \nabla_B\nabla_AC + \nabla_{[A,B]}C$*

Note that if $M = \mathbb{R}^n$; then $R(A,B)C = 0 \ \forall \ A,B,C \in \mathcal{X}(M)$. From now on, we will write $\langle R(A,B)C, D\rangle := (A,B,C,D)$ where $A,B,C,D \in \mathcal{X}(M)$.

Proposition 3.2.1. *Let M be a Riemannina manifold and R be the curvature of M then for $A,B,C,D \in \mathcal{X}(M)$ we have:*

1. $(D,A,B,C) = -(A,D,B,C)$
2. $(D,A,B,C) = -(D,A,C,B)$
3. $(D,A,B,C) = (B,C,D,A)$
4. $(D,A,B,C) + (A,B,D,C) + (B,D,A,C) = 0 \qquad$ *(Bianchi Identity)*

We now define the sectional curvature.

Definition 3.2.2. *Suppose that M is a Riemannian manifold and σ is a two-dimensional subspace of T_pM. Suppose that $A,B \subset \sigma$ be two linearly independent vectors, then the sectional curvature is defined to be $\kappa(\sigma) := \kappa(A,B) = \frac{(A,B,A,B)}{|A|^2|B|^2-\langle A,B\rangle^2}$*

It is not hard to see that the definition of $\kappa(\sigma)$ is independent of the choice of the basis.

Lemma 3.2.2. *Consider a vector space V of dimension n, with $n \geq 2$, that possesses an inner product $\langle \ , \ \rangle$. Assume two tri-linear mappings, $R : V\times V\times V \to V$ and $R' : V\times V\times V \to V$, with their associated forms given by $(A,B,C,D) = \langle R(A,B)C,D\rangle$ and $(A,B,C,D)' = \langle R'(A,B)C,D\rangle$, both satisfying to proposition 3.2.1. For any two-dimensional subspace $\sigma \subset V$, spanned by linearly independent vectors A and B, the sectional curvatures $\kappa(\sigma)$ and $\kappa'(\sigma)$ are defined as $\kappa(\sigma) = \frac{(A,B,B,A)}{|A|^2|B|^2-\langle A,B\rangle^2}$ and $\kappa'(\sigma) = \frac{(A,B,B,A)'}{|A|^2|B|^2-\langle A,B\rangle^2}$ respectively. If the sectional curvatures $\kappa(\sigma)$ and $\kappa'(\sigma)$ are equal for all subspaces $\sigma \subset V$, then $R = R'$.*

Suppose that $h : A \subset \mathbb{R}^2 \to M$ be a parametrized surface and suppose (s,t) is the usual coordinates of $\mathbb{R}^2$. Assume that $X = X(s,t)$ is a vector field along h. For each (s,t) it is possible to define $R\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right) X$. We now have the following lemma.

Lemma 3.2.3. $\frac{D}{\partial t}\frac{D}{\partial s}X - \frac{D}{\partial s}\frac{D}{\partial t}X = R\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right) X.$

Chapter 4

Jacobi Fields, The Hopf-Rinow Theorem and The Theorem of Hadamard

We recall that M is a Riemannian manifold with Riemannian connection ∇. Let p be a point on M and σ be a two-dimensional subspace of T_pM. We will see that the sectional curvature $\kappa(p,\sigma)$ defined in Chapter 2 determines how fast the geodesics that originate from p and are tangent to σ deviate from each other. In order to formalize precisely the velocity of this deviation of the geodesics, it is necessary to introduce the concept of *Jacobi-fields*. This study is based on the content available in [3].

4.1 Jacobi Fields

4.1.1 The Jacobi Equation

Let M be a Riemannian manifold, $p \in M, v \in T_pM, w \in T_v(T_pM)$, then we can write

$$\left(\mathrm{dexp}_p\right)_v(w) = \frac{\partial f}{\partial s}(1,0),$$

where f is parametrized surface given by $f(t,s) = \exp_p tv(s)$, $\quad 0 \leq t \leq 1, \quad -\varepsilon \leq s \leq \varepsilon$ and $v(s)$ is a curve in T_pM with $v(0) = v, v'(0) = w$. Intuitively $\left|(d\exp_p)_v w\right|$ is rate of spreading of the geodesics $t \to \exp_p tv(s)$ which starts from p. It is convenient to study the

$$\left(d\exp_p\right)_{tv}(tw) = \frac{\partial f}{\partial s}(t,0)$$

along the geodesic $\gamma(t) = \exp_p(tv), 0 \leq t \leq 1$. Since γ is a geodesic, we have for all (t,s), $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$. Thus, from Lemma 3.2.3,

$$\begin{aligned} 0 = \frac{D}{\partial s}\left(\frac{D}{\partial t}\frac{\partial f}{\partial t}\right) &= \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} \\ &= \frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} \end{aligned}$$

Putting $\frac{\partial f}{\partial s}(t,0) = J(t)$, we obtain that J satisfies the equation

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0 \tag{4.1}$$

Definition 4.1.1. *The above equation is called the Jacobi equation. Let $\gamma : [0,a] \to M$ be a geodesic in M. If a vector field J along γ satisfies the Jacobi equation for all $t \in [0,a]$, then J is called a Jacobi field.*

Proposition 4.1.1. *Let $p \in M$ and $\gamma : [0,a] \to M$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$. Let $w \in T_v(T_pM)$ with $|w| = 1$ and let J be a Jacobi field along γ given by*

$$J(t) = \left(d\exp_p\right)_{tv}(tw), \quad 0 \leq t \leq a.$$

Then the Taylor expansion of $|J(t)|^2$ about $t = 0$ is given by

$$|J(t)|^2 = t^2 - \frac{1}{3}\langle R(v,w)v, w\rangle t^4 + R(t)$$

where $\lim_{t\to 0}\frac{R(t)}{t^4} = 0$.

Corollary 4.1.2. *Suppose that $p \in M$ and $\gamma : [0,a] \to M$ is parametrized by arc-length,(i.e. $|v| = 1$) and $\langle w, v\rangle = 0$, the expression $\langle R(v,w)v, w\rangle$ is the sectional curvature at p with* *respect to the plane σ generated by v and w. Therefore, in this situation,*

$$|J(t)| = t - \frac{1}{6}\kappa(p,\sigma)t^3 + \tilde{R}(t)$$

where $\lim_{t\to 0} \frac{\tilde{R}(t)}{t^3} = 0$.

Above corollary tells us that geodesic $t \to \exp_p(tv(s))$ deviate from the geodesic $t \to \exp_p(tv(0))$ with a velocity that differs from t by a term of the third order in t given by $-\frac{1}{6}\kappa(p,\sigma)t^3$. This also asserts that if $\kappa(p,\sigma) > 0$, then locally the geodesics spread apart less than the rays in T_pM, and if $\kappa(p,\sigma) < 0$, then locally the geodesics spread apart more than the rays in T_pM.

4.1.2 Conjugate Points

Definition 4.1.2. *Given points p, q within a manifold M, suppose a geodesic γ links p and q. Then q is termed conjugate to p if there exists a non-identically zero Jacobi field J along γ such that $J(p) = J(q) = 0$. The multiplicity of the conjugate point q is defined as the highest number of such linearly independent Jacobi fields.*

Proposition 4.1.3. *Let $\gamma : [0,a] \to M$ be a geodesic. Let V_1 in $T_{\gamma(0)}M$ and $V_2 \in T_{\gamma(a)}M$. If $\gamma(a)$ is not conjugate to $\gamma(0)$ then there exists a unique Jacobi field J along γ, with $J(0) = V_1$ and $J(a) = V_2$.*

4.2 Complete Manifolds

So far, we have studied the local properties of Riemannian manifolds. However we are now interested in the global properties and the interplay that exists between the local properties and the global properties of a Riemannian manifold.

Definition 4.2.1. *Let M be a Riemannian manifold. For $p \in M$, if the exponential map, $\exp_p$, is defined for all $v \in T_pM$, i.e., if any geodesic $\gamma(t)$ starting from p is defined for all values of the parameter $t \in \mathbb{R}$, then we say M is (geodesically)complete.*

4.2.1 Hopf-Rinow Theorem

Proposition 4.2.1. *Suppose that p, q are two points on the Riemannian manifold M. Let $d : M \times M \to \mathbb{R}$ denote a distance map such that $d(p, q)$ is equal to the infimum of the length of all piecewise differentiable curves joining p and q, then d is a metric on M.*

Theorem 4.2.2. (*Hopf-Rinow*) *Let M be a Riemannian manifold and $p \in M$, then the following are equivalent.*

1. $\exp_p$ *is defined on all of T_pM.*
2. *If K is closed and bounded set in M then K is compact.*
3. *M is complete as a metric space.*
4. *M is geodesically complete.*

   In addition, any of the statements above implies that "For any $p, q \in M$ there exists a geodesic γ joining p and q such that $d(p, q) = L(\gamma)$."

This theorem has some useful corollaries. For example, each compact Riemannian manifold is inherently complete, and any closed submanifold within a complete manifold also exhibits completeness. The completeness of manifolds is significant for examining their global characteristics since it ensures the possibility of connecting any two points on the manifolds through a geodesic that minimizes distance.

4.2.2 Hadamard Theorem

Lemma 4.2.3. *Suppose that M is a complete manifold with non-positive sectional curvature for all the points $p \in M$ and for all $\sigma \subset T_pM$. Consider any geodesic starting form $p \; \forall \; p \in M$, then p does not have conjugate points along the geodesic, i.e., there is a local diffeomorphism given by* $\exp_p \; \forall \; p \in M$.

Lemma 4.2.4. *Let M be a complete Riemannian manifold and Let $\varphi : M \to N$ be a local diffeomorphism onto N, which has the following property: $|d\varphi_p(v)| \geq |v|$ for all p and for all $v \in T_pM$ then φ is a covering map.*

Definition 4.2.2. *A Riemannian manifold M is said to be simply connected if it is path-connected, and its first fundamental group is the trivial group for some point $p \in M$, and hence for every point of M (See chapter 9 of [8]).*

Theorem 4.2.5. (*Hadamard*). *Let M be a complete, simply connected Riemannian manifold of dimension n with sectional curvature $\kappa(p, \sigma) \leq 0 \ \forall \ p \in M$ and for all $\sigma \subset T_pM$ then M is diffeomorphic to $\mathbb{R}^n$; more precisely $\exp_p : T_pM \to M$ is a diffeomorphism.*

Chapter 5

Spaces of Constant Curvature

We study Riemannian manifolds with constant sectional curvature in this chapter. It is not difficult to see that when we multiply a Riemannian metric by a constant $c > 0$, then its sectional curvature gets divided by c. Therefore, without loss of generality, we can assume that the sectional curvature of any constant sectional curvature Riemannian manifold is either +1, 0, or -1. For proofs of the theorems in this chapter, please refer to [3].

5.1 Theorem by Cartan

Suppose that M and N are two Riemannian manifolds of dimension n with curvature $R, \tilde{R}$ respectively. Suppose that $p \in M, \tilde{p} \in N$ and $i : T_pM \to T_{\tilde{p}}N$ is a linear isometry. Suppose that $V \subset M$ of p such that we can define $\exp_{\tilde{p}}$ on $i \circ \exp_p^{-1}(V)$. Let $f : V \to N$ define by $f(q) = \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q)$. There exist a unit speed geodesic $\gamma : [0, t] \to M$ joining $\gamma(0) = p$ to $\gamma(t) = q$, as V is a normal neighbourhood. P_t is the parallel transport along γ from $\gamma(0) = p$ to $\gamma(t), t \in [0, a]$. Also consider the geodesic $\tilde{\gamma} : [0, a] \to \bar{M}$ with $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}'(0) = i\,(\gamma'(0))$. Similarly parallel transport along $\tilde{\gamma}(0)$ to $\tilde{\gamma}(t)$ as $\tilde{P}_t$. We now define $\phi_t : T_qM \to T_{f(q))}N$, as $\phi_t(v) = \tilde{P}_t^{-1} \circ P_t(v)\ v \in T_qM$.

Theorem 5.1.1. (*Cartan*). *Let M, N be Riemannian manifolds as above. If for all $q \in U$ and for all $x, y, u, v \in T_qM$ we have $\langle R(x, y)u, v\rangle = \langle \tilde{R}(\phi_t(x), \phi_t(y))\phi_t(u), \phi_t(v)\rangle$, then $f : V \to f(V) \subset N$ is a local isometry if and only if $df_p = i$*

Corollary 5.1.2. *Suppose that M and N are two Riemannian manifolds of the same dimension n and the same constant curvature. Let $p \in M$ and $\tilde{p} \in N$. Choose $\{e_i\}$ and $\{\tilde{e}_i\}$ be orthonormal basis of T_pM and $T_{\tilde{p}}N$ respectively. Then there exist a neighborhood of $V \subset M$ of p, $\tilde{V}$ of $\tilde{p}$ respectively, and an isometry $f : V \subset M \to \tilde{V}$ such that $df_p(e_i) = \tilde{e}_i$.*

Example 4. *Now we discuss an example of a space that is simply connected, and it is a space of constant sectional curvature -1. Consider the half-space of $\mathbb{R}^n$ given by*

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

with the Riemannian metric $g_{ij}(x_1, \ldots, x_n) = \frac{\delta_{ij}}{x_n^2}$. Let $f = \log(x_n)$ and denote $\frac{\partial f}{\partial x_i} = f_i$, it is not hard to see that the Christoffel symbols are given by

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}\sum_m \left\{\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij}\right\} g^{mk} \\ &= -\delta_{jk} f_i - \delta_{ki} f_j + \delta_{ij} f_k\end{aligned}$$

therefore if all three indices are distinct we have $\Gamma_{ij}^k = 0$ while if two indices are same, we have $\Gamma_{ij}^i = -f_j, \Gamma_{ii}^j = -f_j, \Gamma_{ij}^j = -f_i, \Gamma_{ii}^i = -f_i$. After some calculation, we can say that the sectional curvature of $\mathbb{H}^n$ is $\kappa = (-\frac{1}{x_n^2})x_n^2 = -1$.

5.2 Space Forms

Lemma 5.2.1. *Let M and N be two connected Riemannian manifolds. Let f, g be two local isometries between M and N. Suppose that there exists $p \in M$ such that $f(p) = g(p)$, $df_p = dg_p$ then $f = g$.*

Theorem 5.2.2. *Let M be an m dimensional complete Riemannian manifold with constant sectional curvature. Then the universal covering $\widetilde{M}$ of M with the pull-back metric of sectional curvature κ is isometric(up to a scaling of the metric by a constant) to:*

1. $\mathbb{H}^n$, *if* $\kappa = -1$,
2. $\mathbb{R}^n$, *if* $\kappa = 0$,
3. S^n, *if* $\kappa = 1$.

The complete manifolds with constant sectional curvature are called *space forms*.

Definition 5.2.1. *Suppose U is an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}^n$. If f preserves the angles(non-oriented) of the intersecting curves, i.e., the angle formed by v and w at $p \in U$ is the same as the angle formed by $df_p(v)$ with $df_p(w)$, then we say f is a conformal map.*

Let f be as in the above definition. Then, the necessary and sufficient condition for f to be conformal is that for all $p \in U$ and for all pairs of vectors v and w at p, the following conditions is True,

$$\langle df_p(v), df_p(w)\rangle = \lambda^2(p)\langle v, w\rangle, \qquad \lambda^2 \neq 0. \tag{5.1}$$

The function $\lambda : U \to \mathbb{R}$ is called the coefficient of conformality of f.

Remark 5.2.1. *Let $f(p) = \lambda I(p), p \in \mathbb{R}^n$ be linear transformation, where I is the identity matrix and $\lambda = cons. > 0$. Then f is conformal transformation and f is called a dilatation.*

Theorem 5.2.3. (*Liouville*). *Let $f : U \to \mathbb{R}^n$, $n \geq 3$, be a conformal transformation of an open set $U \subset \mathbb{R}^n$. Then f is the restriction to U of a composition of isometries, dilatations or inversions, at most one of each.*

Theorem 5.2.4. *The isometries of $\mathbb{H}^n$ are the restriction to $\mathbb{H}^n \subset \mathbb{R}^n$ of the conformal transformation of $\mathbb{R}^n$ that take $\mathbb{H}^n$ onto itself.*

In the next chapter, we study another model, the hyperboloid model of simply connected Riemannian manifold with constant section curvature -1.

Chapter 6

The Hyperboloid Model

In this chapter, we study the two-dimensional Hyperboloid model of the hyperbolic plane. The study of the hyperbolic geometry presented in this chapter is based on the content available in [9] and [11]. This hyperboloid model is introduced as a submanifold of the Minkowski 3-space $\mathcal{M}^3$. Therefore, we begin this chapter with the definition of the Minkowski 3-space.

6.1 The Minkowski 3-space

Definition 6.1.1. *Define a real-valued function q on $\mathbb{R}^3$ by following equation,*

$$q\left(\sum_{i=0}^{2} x_i U_i\right) = -x_0^2 + x_1^2 + x_2^2 \tag{6.1}$$

for some basis $\mathscr{U} = (U_0, U_1, U_2)$ of the space. $\mathbb{R}^3$ together with q, we get a new space, and we call this space Minkowski 3-space. We denote Minkowski 3-space by $\mathcal{M}^3$.

We can rewrite the above equation (6.1) as,

$$q\left(\sum_{i=0}^{2} x_i U_i\right) = \sum_{i=0}^{2} e_i {x_i}^2 \tag{6.2}$$

where $e_0 = -1, e_1 = e_2 = 1$; then $q(U_i) = e_i$. We define $\mathcal{M}^2$ with the basis (U_0, U_1) as a subspace of $\mathcal{M}^3$. If we replace (6.2) by following equation

$$q_{\mathbb{E}}\left(\sum_{i=0}^{2} x_i U_i\right) = \sum_{i=0}^{2} {x_i}^2 = x_0^2 + x_1^2 + x_2^2, \tag{6.3}$$

then we get usual Euclidean 3-space $\mathbb{E}^3$, and $q_{\mathbb{E}}$ gives the square of the length.

For $X = \sum_{i=0}^{2} x_i U_i$ and $Y = \sum_{i=0}^{2} y_i U_i$, two vectors in $\mathcal{M}^3$, define the bilinear form or pairing correspondence to q by following equation

$$\frac{1}{2}\left[q(X+Y) - q(X) - q(Y)\right] =: p(X, Y) \tag{6.4}$$

Proposition 6.1.1. *For any $X \in \mathcal{M}^3$, we have, $q(X) = p(X, X)$.*

Proof. Consider $p(X, X)$, we see that,

$$\begin{aligned} p(X, X) &= \frac{1}{2}\left[q(X+X) - q(X) - q(X)\right] \\ &= \frac{1}{2}\left[4q(X) - 2q(X)\right] \\ &= q(X) = LHS. \end{aligned}$$

□

Proposition 6.1.2. *For $X = \sum_{i=0}^{2} x_i U_i$ and $Y = \sum_{i=0}^{2} y_i U_i$ in $\mathcal{M}^3$, we have, $p(X, Y) = \sum e_i x_i y_i = -x_0 y_0 + x_1 y_1 + x_2 y_2$.*

Proof. Consider $p(X, Y)$, we see that,

$$\begin{aligned} p(X, Y) &= \frac{1}{2}\left[q\left(\sum_{i=0}^{2} x_i U_i + \sum_{i=0}^{2} y_i U_i\right) - q\left(\sum_{i=0}^{2} x_i U_i\right) - q\left(\sum_{i=0}^{2} y_i U_i\right)\right] \\ &= \frac{1}{2}\left[-(x_0+y_0)^2 + (x_1+y_1)^2 + (x_2+y_2)^2 - (-x_0^2 + x_1^2 + x_2^2) - (-y_0^2 + y_1^2 + y_2^2)\right] \\ &= \frac{1}{2}\left[-2x_0 y_0 + 2x_1 y_1 + 2x_2 y_2\right] = RHS \end{aligned}$$

□

It is easy to see that, if $i = j$ then $p(U_i, U_j) = e_i$ and if $i \neq j$ then $p(U_i, U_j) = 0$.

Remark 6.1.1. *Let $\mathscr{U}$ be an orthonormal basis of $\mathbb{E}^3$, observe that $p_{\mathbb{E}}(X, Y) = X \cdot Y = \sum_{i=0}^{2} x_i y_i$, hence bilinear form p is analogous to the usual dot product in $\mathbb{E}^3$ given by $p_{\mathbb{E}}$.*

6.2 The Hyperboloid Model

There are two types of hyperboloid models, one sheeted hyperboloid model, and two sheeted hyperboloid model. In this thesis, we take two sheeted hyperboloid model as our hyperboloid model. In $\mathcal{M}^3$, we define $\mathbb{H}^2$ be the set of all vectors X for which $q(X) = -1$. When we choose $x_0 > 0$, we pick only the upper sheet of this two-sheeted hyperboloid model. We can express the above two conditions as

$$x_0 = \sqrt{1 + x_1^2 + x_2^2} \tag{6.5}$$

Similarly for any arbitrary n, we can define $\mathbb{H}^n$ in $(n+1)$-dimensional Minkowski space $\mathcal{M}^{n+1}$. For $n = 1$, we take $\mathbb{H}^1$ as intersection of $\mathbb{H}^2$ with $\mathcal{M}^2$, i.e. $\mathbb{H}^1 = \mathbb{H}^2 \cap \mathcal{M}^2$. In flat space-time $\mathcal{M}^4$, it appears to each observer that $\mathbb{H}^2$ is a circle whose radius is increasing at a slightly greater speed than the speed of light!

Definition 6.2.1. *$\mathbb{H}$-points are defined as the elements of $\mathbb{H}^2$, we can also think of them as points in $\mathbb{H}^2$ and either points or vectors when considered in $\mathcal{M}^3$.*

Definition 6.2.2. *We defined the non-empty intersection of $\mathbb{H}^2$ with the 2-dimensional subspace of $\mathcal{M}^3$, which passes through the origin as line or $\mathbb{H}$-line of the model. Example of an $\mathbb{H}$-line is $\mathbb{H}^1$.*

In the hyperbolic plane $\mathbb{H}^2$, two distinct points A and B uniquely determine a hyperbolic line $\overleftrightarrow{AB}$, which is the intersection of $\mathbb{H}^2$ with the plane defined by OAB in the three-dimensional Minkowski space $\mathcal{M}^3$, with O as the origin. The hyperbolic segment $\overline{AB}$ and the hyperbolic ray $\overrightarrow{AB}$ are defined in a standard manner. The *sides* of a hyperbolic line in $\mathbb{H}^2$ are the two hyperbolic half-planes that are the complements of the line. Intersection properties of two distinct hyperbolic lines, which may share one or no common points, establish the hyperbolic parallel axiom. The axioms of completeness (or continuity) and the Archimedean axiom are also recognized (see [7] for details). The aim is to define the hyperbolic distance

$d(A, B)$, or the length of the hyperbolic segment $\overline{AB}$, for any two distinct points A and B on $\mathbb{H}^1$.

6.3 Distance

To define distance between two distinct A and B, we partition the $\overline{AB}$, by points $X_0 = A, X_1, \ldots, X_m = B$, and the distance is given by

$$d(A, B) = \lim_{m\to\infty} \sum_{k=1}^{m} \sqrt{q(X_k - X_{k-1})} \tag{6.6}$$

but, first we have to show that $q(X_k - X_{k-1}) > 0$, so that $q(X_k - X_{k-1})$ which is analogues to square length make sense. By Mean value theorem there exists a point of $\overline{X_k - X_{j-1}}$ such that at this point, the tangent vectors in $\mathcal{M}^3$ to this $\mathbb{H}$-segment are parallel to $X_k - X_{k-1}$, so it is enough to prove that all the vectors $V \neq 0$ tangent to $\mathbb{H}^2$ have $q(V) > 0$. In the following proposition, we prove that this is even true for $\mathbb{H}^n \subset \mathcal{M}^{n+1}$.

Proposition 6.3.1. *The vectors $V \neq 0$ in $\mathcal{M}^{n+1}$ tangent to $\mathbb{H}^n$ have $q(V) > 0$.*

Proof. Suppose that $X = (x_0, \hat{x})$ is in $\mathbb{H}^n$ and $V = (v_0, \hat{v}) \neq 0$ is in the tangent space of $\mathbb{H}^n$ at X, where $\hat{v}, \hat{x} \in \mathbb{R}^n$. If $v_0 = 0$, then $\langle V, V\rangle = V \cdot V$. Hence $\langle V, V\rangle = q(V) > 0$, so we may assume that $v_0 \neq 0$. Then

$$0 = \langle V, X\rangle = -v_0x_0 + \hat{v} \cdot \hat{x},$$

and

$$-1 = \langle x, x\rangle = -x_0^2 + \hat{x} \cdot \hat{x}.$$

Where $\langle\ ,\ \rangle$ denotes the hyperbolic inner product and $(\ \cdot\ )$ denotes the usual Euclidean inner product. From now on, we will use $\langle\ ,\ \rangle$ notation to denote the hyperbolic inner product. The first equality follows since V and X are orthogonal(Please see Pg. 65-67 of [6]). Hence, by the Cauchy–Schwarz inequality,

$$(\hat{v} \cdot \hat{v})(\hat{x} \cdot \hat{x}) \geq (\hat{v} \cdot \hat{x})^2 = (v_0x_0)^2 = v_0^2(\hat{x} \cdot \hat{x} + 1).$$

Therefore, $\langle V, V\rangle\, (\hat{x} \cdot \hat{x}) \geq x_0^2$, which implies that $\langle V, V\rangle = q(V) > 0$. $\square$

To evaluate the limit in (6.6), we parametrize $\overline{AB}$. Suppose that T is a differentiable injective mapping of an interval $a \leq u \leq b$ onto $\mathbb{H}$-line segment, $\overline{AB}$, such that $T(a) = A, T(b) = B$. It is easy to see that the derivative of T exists and $X_k = T(u_k)$. For a partition $a = u_0 < \ldots < u_m = b$, we have

$$\begin{aligned}
d(A,B) &= \lim_{m\to\infty} \sum_{k=1}^{m} \sqrt{q(X_k - X_{k-1})} \\
&= \lim_{m\to\infty} \sum_{j=1}^{m} \sqrt{\frac{q(T(u_k) - T(u_{k-1}))}{(u_k - u_{k-1})^2}}\ (u_k - u_{k-1}) \\
&= \lim_{m\to\infty} \sum_{j=1}^{m} \sqrt{q\left(\frac{T(u_k) - T(u_{k-1})}{(u_k - u_{k-1})}\right)}\ (u_k - u_{k-1}) \\
&= \int_a^b \sqrt{q(T'(u))}\ du.
\end{aligned}$$

Thus we get distance between two distinct $\mathbb{H}$-points A and B by parametrizing segment $\overline{AB}$ and we parametrize $\overline{AB}$ by $T(u)$, then

$$d(A,B) = \int_a^b \sqrt{q(T'(u))}\ du. \tag{6.7}$$

By (6.7) we can calculate the $\mathbb{H}$-distances along $\mathbb{H}^1$. We will first calculate the distance along $\mathbb{H}^1$ before giving a general formula for distance in $\mathbb{H}^2$. To develop a general formula of distance in $\mathbb{H}^2$, we will discuss some orthogonal transformations which will be helpful for this. Suppose that $\overline{AB}$ is any $\mathbb{H}$-line segment of $\mathbb{H}^1$. Let $a_1 < b_1$ and the points on $\mathbb{H}^1$ are $A = a_0U_0 + a_1U_1,\ B = b_0U_0 + b_1U_1$. Now we can use (6.5) to parametrize $\overline{AB}$ by setting parameter $u = x_1$ with $T(u) = \sqrt{1+u^2}U_0 + uU_1$, $a_1 \leq u \leq b_1$. Then we have

$$\begin{aligned}
d(A,B) &= \int_{a_1}^{b_1} \sqrt{q(T'(u))}\ du \\
&= \int_{a_1}^{b_1} \sqrt{-\frac{u^2}{1+u^2} + 1}\ du \\
&= \ln(u + \sqrt{u^2+1})\Big|_{a_1}^{b_1}.
\end{aligned}$$

Let $\sinh^{-1}(u) = \ln(u + \sqrt{u^2+1})$, we get

$$d(A, B) = \sinh^{-1}(b_1) - \sinh^{-1}(a_1).$$

Take a special case when $A = U_0$, then we see that for $b_1 > 0$, $d(A = U_0, B) = \sinh^{-1}(b_1)$. Hence, we see that the $\mathbb{H}$-length of $\mathbb{H}^1$ is infinite. Say, $r = \sinh^{-1}(u)$ then $u = \frac{e^r - e^{-r}}{2} = \sinh(r)$, the *hyperbolic sine* of r. On $\mathbb{H}^1$, $x_0 = \sqrt{1+u^2} = \frac{e^r + e^{-r}}{2} = \cosh(r)$, the *hyperbolic cosine* of r. Let $r \in \mathbb{R}$, then we can parametrize $\mathbb{H}^1$ by

$$P(r) = (\cosh(r))U_0 + (\sinh(r))U_1, \tag{6.8}$$

which is helpful to develop a general distance formula.

6.4 Orthogonal Transformations

Definition 6.4.1. *Let T be a linear transformation of 3-dimensional Minkowski space to 3-dimensional Minkowski such that q is preserved, i.e.*

$$q(T(X)) = q(X) \quad \forall\ X \in \mathcal{M}^3, \tag{6.9}$$

then T is called an orthogonal linear transformation.

We can also represent the above equation (6.9) in terms of matrices. Let $X = \sum_{i=0}^{2} X_i U_i$ be a vector in $\mathcal{M}^3$ with $\mathscr{U}$ to be the usual basis of $\mathcal{M}^3$. We can write X as a column of coordinates, i.e.,

$$[X] = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

We know that T has a matrix $[T] = [t_{ij}]$, hence for $X \in \mathcal{M}^3$, we can write

$$[T(X)] = [T][X]. \tag{6.10}$$

Theorem 6.4.1. *Let $T : \mathcal{M}^3 \to \mathcal{M}^3$ be a linear transformation and $A, B \in \mathcal{M}^3$. Then T is* *an orthogonal linear transformation if and only if*

$$p(T(A), T(B)) = p(A, B) \tag{6.11}$$

Proof. Suppose that $T : \mathcal{M}^3 \to \mathcal{M}^3$ is an orthogonal linear transformation. From the definition of p, for any $A, B \in \mathcal{M}^3$ we can write,

$$\begin{aligned}
p(T(A), T(B)) &= \frac{1}{2}\left[q(T(A) + T(B)) - q(T(A)) - q(T(B))\right] \\
&= \frac{1}{2}\left[q(T(A + B)) - q(T(A)) - q(T(B))\right] \\
&= \frac{1}{2}\left[q(A + B) - q(A) - q(B)\right] \\
&= p(A, B).
\end{aligned}$$

To prove converse, let $p(T(A), T(B)) = p(A, B)$ for every $A, B \in \mathcal{M}^3$, then

$$q(T(A)) = p(T(A), T(A)) = p(A, A) = q(A)$$

□

It is easy to see that (6.11) is equivalent to

$$p\left(T\left(U_j\right), T\left(U_k\right)\right) = \sum_{i=0}^{2} e_i t_{ij} t_{ik} = \begin{cases} e_j & \text{when } j = k \\ 0 & \text{when } j \neq k \end{cases} \tag{6.12}$$

We can rewrite equation (6.12) as

$$T^t J_0 T = J_0, \tag{6.13}$$

where T^t denotes the transpose of the matrix T and

$$J_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, if any given linear transformation T satisfies (6.13), then T is an orthogonal linear transformation and vice-versa.

Theorem 6.4.2. *The set of all linear orthogonal transformations of $\mathcal{M}^3$ is a group, it is called the orthogonal group, and it is denoted by* $\mathbf{O}(\mathcal{M}^{\mathbf{3}})$.

Proof. It is easy to see that the identity, $T(A) = A \ \forall \ A \in \mathcal{M}^3$, is a linear orthogonal transformation. Let $T \in \mathbf{O}(\mathcal{M}^{\mathbf{3}})$; then the inverse transformation also exists because each orthogonal transformation preserves distance(See Theorem 6.5.1) hence it is necessarily injective and onto. We know that $T^t J_0 T = J_0$ and $J_0 J_0 = I$(identity) hence, $T^{-1} = J_0 T^t J_0$. Now, if T^{-1} satisfies equation (6.13) then we can say that $T^{-1} \in \mathbf{O}(\mathcal{M}^{\mathbf{3}})$. Thus, we have

$$\begin{aligned}(T^{-1})^t J_0 (T)^{-1} &= (J_0 T^t J_0)^t J_0 (J_0 T^t J_0)\\ &= J_0 T (J_0 T^t J_0) = J_0 T T^{-1} = J_0\end{aligned}$$

Hence, $T^{-1} \in \mathbf{O}(\mathcal{M}^{\mathbf{3}})$. At last, we want to show that the composition of two linear orthogonal transformations is also an element of $\mathbf{O}(\mathcal{M}^{\mathbf{3}})$. So, suppose that $T, S \in \mathbf{O}(\mathcal{M}^{\mathbf{3}})$ and $(T \circ S)(X) = (TS)X$, then

$$(TS)^t J_0 (TS) = S^t T^t J_0 T S = S^t J_0 S = J_0.$$

Thus, $\mathbf{O}(\mathcal{M}^{\mathbf{3}})$ satisfies all the conditions of a group, hence $\mathbf{O}(\mathcal{M}^{\mathbf{3}})$ is a group. $\square$

Definition 6.4.2. $\mathbf{O}^{+}(\mathcal{M}^{\mathbf{3}}) = \{T \in \mathbf{O}(\mathcal{M}^{\mathbf{3}}) : \det(T) = 1\}$ *is a subgroup of index 2, and we call it special orthogonal group.*

Note that either $T(Y) \in \mathbb{H}^2 \ \forall \ Y \ \in \mathbb{H}^2$, or $-T(Y) \in \mathbb{H}^2 \ \forall \ Y \ \in \mathbb{H}^2$.

Definition 6.4.3. *The set* $\mathbf{G}(\mathcal{M}^{\mathbf{3}}) = \{T \in \mathbf{O}(\mathcal{M}^{\mathbf{3}}) \mid T : \mathbb{H}^2 \to \mathbb{H}^2\}$ *is a group. This has a subgroup define as,*

$$\mathbf{G}^{+}(\mathcal{M}^{\mathbf{3}}) = \mathbf{G}(\mathcal{M}^{\mathbf{3}}) \cap \mathbf{O}^{+}(\mathcal{M}^{\mathbf{3}}).$$

Note that $T \in \mathbf{G}(\mathcal{M}^{\mathbf{3}})$ maps each $\mathbb{H}$-line onto another $\mathbb{H}$-line because T maps each 2-dimensional subspace of $\mathcal{M}^3$ onto another keeping origin fixed.

Theorem 6.4.3. *Let* $T \in \mathbf{G}(\mathcal{M}^{\mathbf{3}})$ *then T preserves distance or T is an $\mathbb{H}$-isometry.*

Proof. Suppose that $T \in \mathbf{G}(\mathcal{M}^{\mathbf{3}})$ and A and B are points of $\mathbb{H}^2$ then, by (6.6) and (6.9), we have

$$d(T(A),T(B)) = \lim_{m\to\infty} \sum_{k=1}^{m} \sqrt{q\left(T\left(X_k\right) - T\left(X_{k-1}\right)\right)}$$
$$= \lim_{m\to\infty} \sum_{k=1}^{m} \sqrt{q\left(T\left(X_k - X_{k-1}\right)\right)} = \lim_{m\to\infty} \sum_{k=1}^{m} \sqrt{q\left(X_k - X_{k-1}\right)} = d(A,B)$$

□

6.5 Generalization of Distance in $\mathbb{H}^2$

We first consider the case of distance on $\mathbb{H}^1$ in $\mathcal{M}^2$. Similar to $\mathbf{O}(\mathcal{M}^3)$, $\mathbf{O}^+(\mathcal{M}^3)$ there exists $\mathbf{O}(\mathcal{M}^2)$, $\mathbf{O}^+(\mathcal{M}^2)$, etc.

Theorem 6.5.1. *Let T be a linear transformation of $\mathcal{M}^2$ then T is orthogonal transformation if and only if*

$$T = \begin{pmatrix} a\cosh(r) & b\sinh(r) \\ a\sinh(r) & b\cosh(r) \end{pmatrix}$$

where $a = \pm 1, b = \pm 1$, and r are uniquely determined by T.

Proof. Let

$$T = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

then it follows from $J_0 T^t J_0 = T^{-1}$ that

$$J_0 T^t J_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & g \\ f & h \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & -g \\ -f & h \end{pmatrix} = \pm \begin{pmatrix} h & -f \\ -g & e \end{pmatrix}$$

Thus, we have two cases:
1. When $\det(T) = 1$, then

$$\begin{pmatrix} e & -g \\ -f & h \end{pmatrix} = \begin{pmatrix} h & -f \\ -g & e \end{pmatrix}$$

which implies that $e = h$ and $f = g$, so $eh - fg = e^2 - f^2 = 1$.

2. When $\det(T) = -1$ then

$$\begin{pmatrix} e & -g \\ -f & h \end{pmatrix} = \begin{pmatrix} -h & f \\ g & -e \end{pmatrix}$$

which implies that $e = -h$ and $f = -g$ so $eh - fg = -e^2 + f^2 = -1$.
For both cases, we have to $a = \pm\cosh(r)$ and $b = \pm\sinh(r)$ for some r. Hence, for the first case we have, $T = \begin{pmatrix} \pm\cosh(r) & \pm\sinh(r) \\ \pm\sinh(r) & \pm\cosh(r) \end{pmatrix}$. For the second case, we have $T = \begin{pmatrix} \pm\cosh(r) & \pm\sinh(r) \\ \mp\sinh(r) & \mp\cosh(r) \end{pmatrix}$ or $T = \begin{pmatrix} \pm\cosh(r) & \mp\sinh(r) \\ \pm\sinh(r) & \mp\cosh(r) \end{pmatrix}$. □

Corollary 6.5.2. *For a and b as in theorem 6.5.1, we have*

1. $T \in \mathbf{G}(\mathcal{M}^{\mathbf{2}}) \iff a = 1$,
2. $T \in \mathbf{O}^{+}(\mathcal{M}^{\mathbf{2}}) \iff a = b$, *and*
3. $T \in \mathbf{G}^{+}(\mathcal{M}^{\mathbf{2}}) \iff a = b = 1$.

Definition 6.5.1. *Let $\mathbf{G_1} = \{T \in \mathbf{O}(\mathcal{M}^{\mathbf{3}}) \mid T(\mathbb{H}^1) = \mathbb{H}^1\}$, i.e., it fixes $\mathbb{H}^1$. It is a subgroup of $\mathbf{G}(\mathcal{M}^{\mathbf{3}})$.*

Theorem 6.5.3. *Let $T \in \mathbf{G_1}$ then T fixes the subspace spanned by U_2, i.e., it fixes $\mathcal{M}^2$.*

Proof. Suppose that $T \in \mathbf{G_1}$, and T is given by

$$T = \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}.$$

Now, to show that T fixes $\mathcal{M}^2$, we need to show that $t_{1,3} = t_{2,3} = t_{3,1} = t_{3,2} = 0$. Suppose that $X \in \mathbb{H}^1$, we have

$$TX = \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 t_{1,1} + x_1 t_{1,2} \\ x_0 t_{2,1} + x_1 t_{2,2} \\ x_0 t_{3,1} + x_1 t_{3,2} \end{pmatrix} = Y$$

is also on $\mathbb{H}^1$. Hence $x_0 t_{3,1} + x_1 t_{3,2} = 0\ \forall X \in \mathbb{H}^1$. Assume $X = (1,0,0)$, hence $t_{3,1} = 0$. Then $x_1 t_{3,2} = 0\ \forall X \in \mathbb{H}^1$, which implies that $t_{3,2} = 0$.
Since $T^{-1} = J_0 T^t J_0$, we have

$$T^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{2,1} & 0 \\ t_{1,2} & t_{2,2} & 0 \\ t_{1,3} & t_{2,3} & t_{3,3} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t_{1,1} & -t_{2,1} & 0 \\ -t_{1,2} & t_{2,2} & 0 \\ -t_{1,3} & t_{2,3} & t_{3,3} \end{pmatrix}.$$

Since $T^{-1}(\mathbb{H}^1) = \mathbb{H}^1$, hence $t_{1,3} = t_{2,3} = 0$. Thus for any $T \in \mathbf{G_1}$ and $X = (x_0, x_1, 0)$ we have

$$TX = \begin{pmatrix} t_{1,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} & 0 \\ 0 & 0 & t_{3,3} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 t_{1,1} + x_1 t_{1,2} \\ x_0 t_{2,1} + x_1 t_{2,2} \\ 0 \end{pmatrix}.$$

Thus, if $T \in \mathbf{G_1}$ then T fixes $\mathcal{M}^2$, or it fixes the subspace spanned by U_2. □

We know that $T^t J_0 T = J_0$ then,

$$\begin{aligned} T^t J_0 T &= \begin{pmatrix} t_{1,1} & t_{2,1} & 0 \\ t_{1,2} & t_{2,2} & 0 \\ 0 & 0 & t_{3,3} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} & 0 \\ 0 & 0 & t_{3,3} \end{pmatrix} \\ &= \begin{pmatrix} -t_{1,1}^2 - t_{2,1}^2 & -t_{1,1}t_{1,2} - t_{2,1}t_{2,2} & 0 \\ t_{1,1}t_{1,2} + t_{2,1}t_{2,2} & t_{1,1}^2 + t_{2,1}^2 & 0 \\ 0 & 0 & t_{3,3}^2 \end{pmatrix} = J_0, \end{aligned}$$

hence, $t_{3,3} = \pm 1$. With this fact and Theorem 6.5.1 we can say that if $T \in \mathbf{G_1}$ then

$$T = L_r J_1^i J_2^j \tag{6.14}$$

where $i, j = 0, 1$ are exponents, and for $r \in \mathbb{R}$

$$L_r = \begin{pmatrix} \cosh(r) & \sinh(r) & 0 \\ \sinh(r) & \cosh(r) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.15}$$

and

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

From here, we have

$$\begin{aligned} L_r L_s &= \begin{pmatrix} \cosh(r) & \sinh(r) & 0 \\ \sinh(r) & \cosh(r) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh(s) & \sinh(s) & 0 \\ \sinh(s) & \cosh(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(r+s) & \sinh(r+s) & 0 \\ \sinh(r+s) & \cosh(r+s) & 0 \\ 0 & 0 & 1 \end{pmatrix} = L_{r+s}. \end{aligned}$$

Thus, we have $L_r L_s = L_{r+s}$ and $L_0 = I$.

Definition 6.5.2. *Let* $\mathbf{G_0} = \{T \in \mathbf{O}(\mathcal{M}^\mathbf{3}) \mid T(U_0) = U_0\}$*. It is subgroup of* $\mathbf{G}(\mathcal{M}^\mathbf{3})$*.*

Suppose that $T \in \mathbf{G_0}$ then its inverse also fixes U_0, hence T has following form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & t_{2,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix}$$

Observe that $q(x_1U_1 + x_2U_2) = x_1^2 + x_2^2$. We know that if T is an orthogonal transformation of an Euclidean plane, then it is a rotation or reflection. Hence,

$$T = \begin{pmatrix} \cos(\theta) & -c\sin(\theta) \\ \sin(\theta) & c\cos(\theta) \end{pmatrix}$$

for $c = \pm 1$. $T \in \mathbf{O}(\mathbb{E}^\mathbf{2}) \iff c = 1$. Let $T \in \mathbf{G_0}$, we can write

$$T = R_\alpha J_2^j \tag{6.16}$$

where $j = 0, 1$ for some α, and R_α is given as

$$R_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

Theorem 6.5.4. *Let* $\mathbf{G_0}$, $\mathbf{G_1}$ *as before, then* $\mathbf{G_0} \cap \mathbf{G_1} = \{I, R_\pi, J_1, J_2\}$ *where* I, R_π, J_1, J_2 *are as before.*

Proof. Let $T \in \mathbf{G_0} \cap \mathbf{G_1}$, then T must satisfy both equations (6.14) and (6.16), hence

$$T = L_r J_1^i J_2^j = R_\alpha J_2^k$$

for some r and α, $i, j, k = 0, 1$. This implies $\cosh(r) = 1$, hence $r = 0$. Hence, T has the following form,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}$$

where $e, f = \pm 1$. For the different choices of e, f we have set the $\{I, R_\pi, J_1, J_2\}$ of possible choices of T such that $T \in \mathbf{G_0} \cap \mathbf{G_0}$. $\square$

Now, we apply R_θ to the parametrization of $\mathbb{H}^1$, (6.8), we get

$$P(r, \theta) = R_\theta(P(r)) = U_0 \cosh(r) + U_1 \cos(\theta) \sinh(r) + U_2(\sin(\theta) \sinh(r)). \tag{6.17}$$

Thus, we can parametrize $\mathbb{H}^2$ by equation (6.17). Here, r and θ are *hyperbolic polar* coordinates for $\mathbb{H}^2$. The meaning of r follows from (6.8), but the meaning of θ we will discuss in the next section. Using properties of the hyperbolic functions and some matrix multiplication, we have

$$L_t(P(r, 0)) = P(t + r, 0) \tag{6.18}$$

and

$$R_\alpha(P(r, \beta)) = P(r, \alpha + \beta). \tag{6.19}$$

We define L_t as $\mathbb{H}$-translation along $\mathbb{H}^1$ by t and R_α is $\mathbb{H}$-rotation about U_0 by α. Now, we can get a general formula for distance in $\mathbb{H}^2$. Suppose that X and Y are two distinct points in $\mathbb{H}^2$. Let $X = P(t, \alpha)$, then we apply $L_{-t} R_{-\alpha}$ to X we can reduce to the case $A = U_0$; similarly we apply $T \in \mathbf{G}$ to Y such that we can map Y to some point Y' of $\mathbb{H}^1$. Let $Y' = P(r, 0)$. Since $T \in G$ preserve bilinear form p (6.11), hence we have

$$p(X, Y) = p(U_0, Y') = -\cosh(r)$$

Since, r is the distance from U_0 to Y', and by theorem 6.4.3, we have

$$r = d(X, Y) = \cosh^{-1}(-p(X, Y)). \tag{6.20}$$

Thus we get a basic formula for $\mathbb{H}$-distance in $\mathbb{H}^2$ by (6.20). We know that, in spherical geometry, the distance $d_S(X, Y)$ between two points of S^2 is the radian measure of the Euclidean angle $\angle XOY$, which is given by $\cos^{-1} p_E(X, Y)$, hence see that equation (6.20) is similar to the case of distance in spherical geometry with different bilinear form.

6.6 Angles

Definition 6.6.1. *Let $\overrightarrow{XY}$ and $\overrightarrow{XZ}$ be rays in $\mathbb{H}^2$ with $\overleftrightarrow{XY} \neq \overleftrightarrow{XZ}$. Suppose that $V, W \in \mathcal{M}^3$ such that they are tangent to $\overline{XY}$ and $\overline{XZ}$ respectively at X with $q(V) = q(W) = 1$, then the measure of the angle $\angle YXZ$ ($\mathbb{H}$-angle) is given by*

$$m\angle YXZ = \cos^{-1}(p(V, W)). \tag{6.21}$$

It is easy to check that $|p(V, W)| \leq 1$. Let $T \in \mathbf{G}$. Now, transform everything such that $T(X) = U_0$, we see that these vectors exist and are unique. Euclidean plane $x_0 = 1$ is the tangent plane to $\mathbb{H}^2$, hence (6.21) make sense. By the usual law of cosine, we can deduce that the $\mathbb{H}$-angle measure of $\angle YU_0Z$ is equal to the Euclidean angle. Since $T \in \mathbf{G}$ preserves the bilinear form, hence by equation (6.21), we can say that T also preserves angle.

Till now, we have used some element of $\mathbf{G}$ to move points of $\mathbb{H}^2$ to U_0. The following theorem is useful to say that there always exists a unique some T to do this.

Theorem 6.6.1. *Suppose that l_1 and l_2 are two lines in $\mathbb{H}^2$, $\overrightarrow{A_1B_1}$ and $\overrightarrow{A_2B_2}$ are rays of l_1 and l_2 respectively, and, S_1 and S_2 are sides of l_1 and l_2 respectively. Then, there exits unique $T \in \mathbf{G}$ such that $T(A_1) = A_2, T(l_1) = l_2$, $T(\overrightarrow{A_1B_1}) = \overrightarrow{A_2B_2}$, and $T(S_1) = S_2$.*

Proof. Within the group $\mathbf{G}$, a transformation T can align any ray within the hyperbolic line $\mathbb{H}^1$ such that it initiates from the point U_0 with $x_1 \geq 0$. Transformations T_1 and T_2 in $\mathbf{G}$ can map the rays $\overrightarrow{A_1B_1}$ and $\overrightarrow{A_2B_2}$, respectively, to a shared ray in $\mathbb{H}^1$, with $T_1(A_1) = U_0$ and $T_2(A_2) = U_0$. Thus, lines l_1 and l_2 are transformed onto $\mathbb{H}^1$ by T_1 and T_2. If sides S_1 and S_2

are transferred to the same side of $\mathbb{H}^1$ via T_1 and T_2, then the composite $T = T_2^{-1} \circ T_1$ acts as the required transformation in $\mathbf{G}$, ensuring $T(A_1) = A_2$, $T(l_1) = l_2$, $T(\overrightarrow{A_1B_1}) = \overrightarrow{A_2B_2}$, and $T(S_1) = S_2$. Alternatively, if S_1 and S_2 are positioned on opposite sides of $\mathbb{H}^1$ after application by T_1 and T_2, then J_2 maps $T_1(S_1)$ to $T_2(S_2)$. Hence, $T = T_2^{-1} \circ J_2 \circ T_1$ becomes the desired element in $\mathbf{G}$, aligning A_1 with A_2, l_1 with l_2, $\overrightarrow{A_1B_1}$ with $\overrightarrow{A_2B_2}$ and S_1 with S_2.

□

6.7 Isometries of $\mathbb{H}^2$

Theorem 6.7.1. $\mathbf{G} = \{T \in \mathbf{O}(\mathcal{M}^{\mathbf{2}}) | \mathbf{T} : \mathbb{H}^{\mathbf{2}} \to \mathbb{H}^{\mathbf{2}}\}$ *is the group of isometries of* $\mathbb{H}^2$.

Proof. Let $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ be an isometry of $\mathbb{H}^2$. Since $T \in \mathbf{G}$ preserves distance hence T is an isometry. Let $\triangle PQR$ be a triangle on $\mathbb{H}^2$. Suppose that $\phi(P) = P'$, $\phi(Q) = Q'$ and $\phi(R) = R'$. Since ϕ is an isometry, $\triangle PQR \cong \triangle P'Q'R'$ by the SSS congruence theorem. According to Theorem 6.6.1, there exists a transformation T in the group $\mathbf{G}$ such that $T(P) = P'$ and $T(\overrightarrow{PQ}) = \overrightarrow{P'Q'}$. As T is an isometry, it guarantees that the segment $\overline{P'Q'}$ is congruent to $\overline{PQ}$, and thus, congruent to $\overline{P'T(Q)}$, which implies $T(Q) = Q'$. By the same theorem, it is feasible to assert that T maps point R on the analogous side of the segment $\overline{P'Q'}$ as point R'. Consequently, this leads to $T(R) = R'$, thus T as the isometry ϕ. Hence, it can be inferred that every isometry in the hyperbolic plane $\mathbb{H}^2$ essentially represents an element of $\mathbf{G}$.

□

Now we are familiar with distance and angles on $\mathbb{H}^2$; we can now do trigonometry in $\mathbb{H}^2$.

6.8 Trigonometry in $\mathbb{H}^2$

Let $\triangle PQR$ be a triangle in $\mathbb{H}^2$, and $d_{\mathbb{H}}(P, Q) = r$, $d_{\mathbb{H}}(R, Q) = p$, $d_{\mathbb{H}}(P, R) = q$, and $m\angle PQR = \alpha_2$, $m\angle QRP = \alpha_3$, and $m\angle RPQ = \alpha_1$. Transform by a appropriate element of $\mathbf{G}$ such that $R = U_0$, P is on the ray of $\mathbb{H}^1$ for which $x_1 \geq 0$, and that Q is on the side of $\mathbb{H}^1$

for which $x_2 > 0$; then $P = P(q, 0)$ and $Q = P(p, \alpha_3)$. Now L_{-q} will map P to $P' = U_0$, R to $R' = P(-q, 0) = P(q, \pi)$. As for $Q = P(p, \alpha_3)$ to the point $Q' = P(r, \pi - \alpha_1)$, this comes from the fact that our parametrization gives a point X on $\mathbb{H}^2$ and the angle formed by $\overrightarrow{U_0X}$ and $\mathbb{H}^1$ for $x_1 \geq 0$. Since the distance from Q' to $P' = U_0$ is r, and the angle formed by $\overrightarrow{U_0Q'}$ and $\mathbb{H}^1$ with $x_1 \leq 0$ is α_1, it follows that $Q' = P(r, \pi - \alpha_1)$. Together with (6.15) and (6.17), the equation $Q' = L_{-q}[Q]$ yields,

$$\begin{bmatrix} \cosh(r) \\ -\sinh(r)\cos(\alpha_1) \\ \sinh(r)\sin(\alpha_1) \end{bmatrix} = \begin{bmatrix} \cosh(q) & -\sinh(q) & 0 \\ -\sinh(q) & \cosh(q) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh(p) \\ \sinh(p)\cos(\alpha_3) \\ \sinh(p)\sin(\alpha_3) \end{bmatrix}$$

Solve the right side of the equation, we get

$$\begin{bmatrix} \cosh(r) \\ -\sinh(r)\cos(\alpha_1) \\ \sinh(r)\sin(\alpha_1) \end{bmatrix} = \begin{bmatrix} \cosh(q)\cosh(p) - \sinh(q)\sinh(p)\cos(\alpha_3) \\ -\sinh(q)\cosh(p) + \cosh(q)\sinh(p)\cos(\alpha_3) \\ \sinh(p)\sin(\alpha_3) \end{bmatrix}$$

1. The hyperbolic law of cosines,

$$\cos(\alpha_3) = \frac{\cosh(p)\cosh(q) - \cosh(r)}{\sinh(p)\sinh(q)}$$

2. The hyperbolic law of sines,

$$\frac{\sin(\alpha_1)}{\sinh(p)} = \frac{\sin(\alpha_3)}{\sinh(r)} = \frac{\sin(\alpha_2)}{\sinh(q)}.$$

From the hyperbolic law of cosine, we get the following corollary.

Corollary 6.8.1. *Let $\triangle PQR$ be a right triangle, with $m\angle PQR = 90°$, then letting $d_{\mathbb{H}}(P, Q) = r$, $d_{\mathbb{H}}(R, Q) = p$, $d_{\mathbb{H}}(P, R) = q$, then*

$$\cosh(q) = \cosh(p)\cosh(r)$$

6.9 Equation of Lines and Poles

The equation of an $\mathbb{H}$-line l, where l is the intersection of $\mathbb{H}^2$ with a subspace of $\mathcal{M}^3$, is given by

$$p(V,X) = -v_0x_0 + v_1x_1 + v_2x_2 = 0 \tag{6.22}$$

for some non-zero $V = \sum v_iU_i \in \mathcal{M}^3$.

Proposition 6.9.1. *For V as defined in (6.22) then $q(V) > 0$.*

Proof. Let T be a linear orthogonal transformation such that $T(X) = U_0$ for some fixed $X \in \mathbb{H}^2$ and suppose that $T(V) = V' = \begin{pmatrix} v_0' \\ v_1' \\ v_2' \end{pmatrix}$. We know that the bilinear form p is preserved by T, then

$$p(V', U_0) = p(T(V), T(X)) = p(V, X) = 0 \implies v_0' = 0$$

Now we have

$$q(T(V)) = -v_0'^2 + v_1'^2 + v_2'^2 = v_1'^2 + v_2'^2 > 0$$

for some $V \neq 0$. Hence $q(T(V)) = p(T(V), T(V)) = p(V,V) = q(V) > 0$. $\square$

Now, say $q(V) = -v_0^2 + v_1^2 + v_2^2 = k > 0$, then dividing V by $\sqrt{q(V)}$ we get

$$\begin{aligned} q\left(\frac{V}{\sqrt{q(V)}}\right) &= \left(\frac{-v_0}{\sqrt{k}}\right)^2 + \left(\frac{v_1}{\sqrt{k}}\right)^2 + \left(\frac{v_2}{\sqrt{k}}\right)^2 \\ &= \frac{-v_0^2 + v_1^2 + v_2^2}{k} = \frac{k}{k} = 1 \end{aligned}$$

Hence, we can assume the following

$$D^2 = \{V \in \mathcal{M}^3 | q(V) = 1\}, \tag{6.23}$$

be a subset of $\mathcal{M}^3$. D^2 is a hyperboloid of one sheet. Each line corresponds to exactly two points $\pm V$ of D^2; because if X satisfies $p(V,X) = 0$ then X also satisfies $p(-V,X) = 0$. We can call these the *poles* of the line.

6.10 Cycles

Consider the curves C on $\mathbb{H}^2$ that are the intersections of $\mathbb{H}^2$ with planes of $\mathcal{M}^3$ which are not passing through Origin, we call these curves the cycles (or *generalized circles*) of $\mathbb{H}^2$. Any three non-collinear points of $\mathbb{H}^2$ are contained in only one cycle. The equation of these planes has the following form

$$p(V, X) = -v_0x_0 + v_1x_1 + v_2x_2 = -k \tag{6.24}$$

for fixed $V \in \mathcal{M}^3$ and real k, neither V nor k being 0. There are three cases.

6.10.1 Circles

Case 1: $q(V) < 0$. If we divide equation (6.24) by $\pm\sqrt{-q(V)}$ we get $V \in \mathbb{H}^2$. By theorem 6.6.1 $\exists\ T \in \mathbf{G}$ such that $T(V) = U_0$. Since T is an isometry, then the curve $C_k = T(C)$ is congruent to C, has equation

$$p(U_0, X) = -k \ \text{ or } x_0 = k > 0. \tag{6.25}$$

We can write equation (6.25) in the polar co-ordinates as

$$r = \operatorname{arccosh}(k). \tag{6.26}$$

In the hyperbolic plane $\mathbb{H}^2$, define a circle C with center at V and radius s specified by $s = \operatorname{arccosh}(k)$. Within the framework of the three-dimensional Minkowski space $\mathcal{M}^3$, C manifests as either an ellipse or a circle. According to equation (6.21), the ensemble of all circles in $\mathbb{H}^2$ with center V and the set of congruent hyperbolic lines passing through V serve as orthogonal trajectories to each other. Since the plane $x_0 = k$ is characterized by Euclidean geometry, the hyperbolic arc length of circle C_k corresponds to its circumference in the Euclidean space $\mathbb{E}^3$. Thus, the radius of the circle in $\mathbb{E}^3$ is $\sinh(s)$, with a circumference of $2\pi\sinh(s)$.

Theorem 6.10.1. *The curve C defined in (6.25) is a circle with radius $\sinh(s)$ and its circumference is $2\pi\sinh(s)$ in $\mathbb{E}^3$, where $s =$ arccosh(k).*

Proof. By (6.25), we know that $x_0 = k$, substituting this value in $q(X) = -1$ we get,

$$x_1^2 + x_2^2 = k^2 - 1 = \sinh^2(s).$$

Hence in the plane $x_0 = k$, C is a circle with radius $\sinh(s)$ and circumference $2\pi \sinh(s)$ in $\mathbb{E}^3$. $\square$

6.10.2 Hyperbola

Case 2: $q(V) > 0$. We can assume $V \in D^2$. Let $T \in \mathbf{G}$ such that $l[T(V)] = T(l) = \mathbb{H}^1$. We multiply (6.24) by -1 if necessary, so $T(V) = U_2$. Then it is easy to see that by (6.24), $T(C) = C_k$ has following equation,

$$x_2 = k \tag{6.27}$$

for $k \neq 0$, whence C_k and C are branches of hyperbolas in $\mathcal{M}^3$.

Theorem 6.10.2. *The curve $T(C) = C_k$ defined by (6.27) is a hyperbola in $\mathcal{M}^3$.*

Proof. Since the curve given by (6.27) lies on $\mathbb{H}^2$. Hence, substituting $x_2 = k$ in $q(X) = -1$ we get

$$-x_0^2 + x_1^2 + k^2 = -1$$

we can rewrite it as

$$\frac{x_0^2}{1+k^2} - \frac{x_1^2}{1+k^2} = 1 \tag{6.28}$$

Thus, we get an equation of a hyperbola. $\square$

Now, consider the $\mathbb{H}$-translation L_t along $\mathbb{H}^1$. Let $P_s = U_0 \cosh(s) + U_2 \sinh(s)$ be a point on $\mathbb{H}^1$-line coordinatized by $\mathbb{H}$-length,

$$\begin{aligned} L_t(P_s) &= \begin{pmatrix} \cosh(t) & -\sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh(s) \\ 0 \\ \sinh(s) \end{pmatrix} \\ &= (\cosh(s)\cosh(t))U_0 + (\cosh(s)\sinh(t))U_1 + (\sinh(s))U_2 \end{aligned}$$

The transformation L_t preserves the curve C_k by mapping it onto itself, where the constant k is specified as $k = \pm\sinh(s)$, with s denoting the distance orthogonal to the hyperbolic line $\mathbb{H}^1$. Curves characterized by $x_2 = k$ (or $x_2 = \pm k$) are known as equidistant curves with respect to the axis $\mathbb{H}^1$. Through the application of mappings L_t, the line $x_1 = 0$ yields a series of lines described as "divergent" or "hyperparallel" hyperbolic lines, orthogonal to $\mathbb{H}^1$. After considering translations along $\mathbb{H}^1$, we want to know how much translation is done along the hyperbolic arc length of C_k. Hence, for fixed $s > 0$ the $\mathbb{H}$-arc length of C_k from P_s to $L_t(P_s))$ is

$$\begin{aligned}\int_0^t \sqrt{q(L_u'(P_s))}du &= \int_0^t \sqrt{\cosh^2(s)(\cosh^2(u) - \sinh^2(u)}du \\ &= \int_0^t \cosh(s)du = t\cosh(s),\end{aligned}$$

6.10.3 Parabola

Case 3: $q(V) = 0$. It is easy to see that V is on the cone $v_0^2 = v_1^2 + v_2^2$. We can assume that $v_0 = 1$, so that C has equation

$$-x_0 + \cos(\theta)x_1 + \sin(\theta)x_2 = -k. \tag{6.29}$$

Then applying $R_{-\theta}$ to (C) we get,

$$-x_0 + x_1 = -k. \tag{6.30}$$

Theorem 6.10.3. *The curves C defined by (6.30) are parabolas in $\mathcal{M}^3$.*

Proof. We have $x_0 = x_1 + k$ by (6.30), substituting this value in $-x_0^2 + x_1^2 + x_2^2 = -1$ we get,

$$x_2^2 - 2x_1k = k^2 - 1 \implies x_1 = \frac{x_2^2}{2k} - \frac{k^2-1}{2k}$$

□

A similar calculation to the calculation we have done in section 6.5 shows that the elements of $\mathbf{G}$ that map any one of the curves C_k on itself are the transformations $N_tJ_2^j$

where

$$N_u = \begin{pmatrix} 1+\frac{u^2}{2} & -\frac{u^2}{2} & u \\ \frac{u^2}{2} & 1-\frac{u^2}{2} & u \\ u & -u & 1 \end{pmatrix} \tag{6.31}$$

here $N_u N_t = N_{u+t}$. It is easy to see that,

$$N_u(L_r(U_0)) = \left(\cosh r + \frac{u^2}{2}e^{-r}\right) U_0 + \left(\sinh r + \frac{u^2}{2}e^{-r}\right) + ue^{-r}. \tag{6.32}$$

We can parametrize the horocycle C_k, for which $k = \cosh r - \sinh r = e^{-r}$, by equation 6.32. It is not hard to see that the $\mathbb{H}$-arc length on C_k between $L_r(U_0)$ and $N_u(L_r(U_0)$ is $|ue^{-r}|$. The $\mathbb{H}$-line $N_u(\mathbb{H}^1)$ has equation $ux_0 + ux_1 + x_2 = 0$ and we can parameterize it using $N_u(L_r(U_0))$ with u fixed; these lines form a family of *"parallel"*(in the asymptotic sense) lines which are orthogonal trajectories of the curves C_k. The distance between C_1 and C_k along any of these lines, e.g., $\mathbb{H}^1$ is $|r|$. It is easy to see that $N_u L_r = L_r N_{ue^{-s}}$.

Chapter 7

The Isoperimetric Inequality in the Hyperbolic Plane

In this last chapter, we study the goal of this thesis, namely 'The isoperimetric inequality in the hyperbolic plane'. This study is completely based on the expository article [2].

We have discussed the Hopf-Rinow theorem and the theorem of Hadamard in chapter 4. As a consequence of these theorems, we can join any two distinct points of $\mathbb{H}^2$ by a unique geodesic. This allows us to draw a polygon with n sides in the hyperbolic plane $\mathbb{H}^2$. Here, we consider the two-sheeted hyperboloid model of the hyperbolic plane. In accordance with [2], we work with the upper sheet of this two-sheeted hyperboloid model. In other words, for us, $\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1, x_3 > 0\}$ equipped with the Riemannian metric induced from the quadratic form $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$ where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are points in $\mathbb{H}^2$.

We now state the main results we aim to prove:

Theorem 7.0.1. *The perimeter minimizer in the family of all polygons with n sides in $\mathbb{H}^2$ enclosing area $0 < A < (n-2)\pi$ exists and is the regular n-gon. Here, the natural number $n \geq 3$ is fixed.*

Theorem 7.0.2. *The perimeter minimizer in the family of all piecewise smooth simple closed curves in $\mathbb{H}^2$ having a fixed area $A > 0$ exists and is the circle of radius* $\sinh^{-1}\left(\frac{\sqrt{4\pi A + A^2}}{2\pi}\right)$.

Corollary 7.0.3. *If a domain in $\mathbb{H}^2$ has n components each having area $A_k > 0$ such that*

$A = \sum_{k=1}^{n} A_k$, then the perimeter minimizer is the unique circle in the family of all piecewise smooth simple closed curves in $\mathbb{H}^2$ having fixed area A, and the radius of this minimizing circle is $\sinh^{-1}\left(\frac{\sqrt{4\pi A + A^2}}{2\pi}\right)$.

Theorem 7.0.4. (*The Isoperimetric Inequality for* $\mathbb{H}^2$) *Let $\mathcal{C}$ be a 'piecewise smooth simple closed curve' in $\mathbb{H}^2$. Let ℓ be arc-length of $\mathcal{C}$ and $A > 0$ be the area enclosed by $\mathcal{C}$. Then we have following inequality,*

$$\ell^2 \geq 4\pi A + A^2$$

and $\ell^2 = 4\pi A + A^2$ if and only if $\mathcal{C}$ is a circle in $\mathbb{H}^2$ of radius $\sinh^{-1}\left(\frac{\sqrt{4\pi A + A^2}}{2\pi}\right)$.

We start with the following definition.

Definition 7.0.1. *The reflection r_ℓ through an $\mathbb{H}$-line ℓ in $\mathbb{H}^2$ is defined as*

$$r_\ell(x) = x - 2\langle x, n\rangle n.$$

where n is a unit normal vector to the $\mathbb{H}$-line ℓ.

Proposition 7.0.5. *Let m be a fix positive number and $A = \{X_1, \ldots, X_m\}$, $B = \{Y_1, \ldots, Y_m\}$ be two sets of m points in $\mathbb{H}^2$ such that the pairwise distance $d(X_k, X_l)$ is equal to the pairwise distance $d(Y_k, Y_l)\ \forall\ k, l \in \{1, \ldots, m\}$. Then $\exists$ an isometry ϕ of $\mathbb{H}^2$ such that $\phi(X_k) = Y_k\ \forall\ k \in \{1, \ldots, m\}$. We can write ϕ as the composition of at most m reflections through $\mathbb{H}$-lines.*

7.1 Geodesic Segments, Triangle and Polygons in $\mathbb{H}^2$

Definition 7.1.1. *Let ℓ be an $\mathbb{H}$-line in $\mathbb{H}^2$ then we define geodesic segments in $\mathbb{H}^2$ as connected subsets of ℓ.*

For $x \neq y \in \mathbb{H}^2$, define $w \in T_x\mathbb{H}^2$ as

$$w := y + \langle y, x\rangle x.$$

The following set is a geodesic segment in $\mathbb{H}^2$ which joins x to y,

$$\left\{ (\cosh t)x + \sinh t \frac{w}{\sqrt{\langle w, w\rangle}} \mid 0 \leq t \leq d(x,y) \right\},$$

and we denote it by $[[x, y]]$.

Consider $p \in \mathbb{H}^2$ and a unit vector $w \in T_p\mathbb{H}^2 \setminus \{\underline{0}\}$ where $\underline{0}$ is the origin of $T_p\mathbb{H}^2$. Assume $c_{p,w}$ is the geodesic such that $c_{p,w}(0) = p$ and $c'_{p,w}(0) = w$. Then $c_{p,w}$ can be given by

$$c_{p,w}(t) = (\cosh t)p + \sinh(t)\frac{w}{\sqrt{\langle w, w\rangle}} \quad (t \in \mathbb{R}).$$

Definition 7.1.2. *Let $\wp$ be an open region with a simple closed curve called boundary $\partial\wp$ which consists of geodesic segments, then $\wp$ is said to be a polygon in $\mathbb{H}^2$.*

Definition 7.1.3. *If the angle at the point p of $\partial\wp$ is other than π, then p is called a vertex of $\wp$.*

Definition 7.1.4. *The sides of $\wp$ are the geodesic segments of $\partial\wp$.*

Definition 7.1.5. *Suppose that c_{q,w_1} are c_{q,w_2} two sides of $\wp$. Clearly, these two sides have a common vertex q. Give positive orientation to $\partial\wp$ then the angle at vertex q is given by*

$$\measuredangle \textit{ at } q := \begin{cases} \measuredangle\{w_1, w_2\} & \textit{if } \det(w_1, w_2) < 0, \\ 2\pi - \measuredangle\{w_1, w_2\} & \textit{if } \det(w_1, w_2) > 0 \end{cases}$$

Definition 7.1.6. *For a polygon $\wp$, if the geodesic segment $[[x, y]]$ is contained in $\wp$ for any $x, y \in \wp$, then $\wp$ is said to be a convex polygon.*

Definition 7.1.7. *For a polygon $\wp$, if $B(z, \epsilon) \cap \wp$ is convex $\forall\ \epsilon > 0$ for any $z \in \wp$ then $\wp$ is said to be locally convex.*

Remark 7.1.1. *A convex polygon is a connected locally convex polygon, and the converse is also true.*

Definition 7.1.8. *An n-gon in $\mathbb{H}^2$ is a polygon of n sides in $\mathbb{H}^2$, $n \geq 3$.*

Definition 7.1.9. *A triangle in $\mathbb{H}^2$ is a convex 3-gon in $\mathbb{H}^2$. A triangle in $\mathbb{H}^2$ whose vertices are $a, b, c \in \mathbb{H}^2$ is denoted by $[[a, b, c]]$.*

Theorem 7.1.1. *Let T be a triangle in $\mathbb{H}^2$ with angles α, β, γ, then the area of T is equal to $\pi - (\alpha + \beta + \gamma)$.*

Proof. It easily follows from the Gauss-Bonnet Formula that

$$-\int_T dV = (a + b + c - \pi),$$

where dV is the area element of $\mathbb{H}^2$. □

Theorem 7.1.2. *Let A be the area of a triangle in $\mathbb{H}^2$ whose sides are a, b, c then*

$$\tanh(A/4) = \sqrt{\tanh(s/2)\tanh[(s-a)/2]\tanh[(s-b)/2]\tanh[(s-c)/2]}$$

where $s := (a + b + c)/2$.

Proof. The proof follows from theorem 7.1.1 and by using some elementary trigonometric identities. The complete derivation is available in [2]. □

Proposition 7.1.3. *Suppose that A is the area of a triangle in $\mathbb{H}^2$ whose two sides are a, b and the included angle is γ then,*

$$\cot(A/2) = \frac{\coth(a/2)\coth(b/2) - \cos\gamma}{\sin\gamma}$$

Proof. Suppose that the remaining angles of the triangle, which are opposite to sides a, b, are α, β, respectively. We have

$$\sin\tfrac{A}{2} \quad = \sin\left(\tfrac{-(\alpha+\beta+\gamma-\pi)}{2}\right) = -\sin\left(\tfrac{\alpha+\beta+\gamma-\pi}{2}\right) \quad = \tfrac{\sin\gamma}{\cosh\left(\frac{c}{2}\right)}\sinh\left(\tfrac{a}{2}\right)\sinh\left(\tfrac{b}{2}\right)$$

Similarly,

$$\cos\frac{A}{2} = \frac{\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right) - \sinh\left(\frac{a}{2}\right)\sinh\left(\frac{b}{2}\right)\cos\gamma}{\cosh\left(\frac{c}{2}\right)}$$

Thus

$$\cot\left(\frac{A}{2}\right) = \frac{\coth\left(\frac{a}{2}\right)\coth\left(\frac{b}{2}\right) - \cos\gamma}{\sin\gamma}.$$

□

Definition 7.1.10. *Suppose that $T := [[A, B, C]], T' := [[A', B', C']]$ are two triangles in $\mathbb{H}^2$. Let ϕ be an isometry of $\mathbb{H}^2$ such that $\phi(A) = A', \phi(B) = B'$ and $\phi(C) = C'$ then we say that the triangle T is congruent to T'.*

Proposition 7.1.4. *and Let T_1, T_2 be triangles in $\mathbb{H}^2$. Suppose that sides of T_1 are a_1, b_1, c_1 and angles of T_1 opposite to these sides are $\alpha_1, \beta_1, \gamma_1$ respectively. Similarly, assume a_2, b_2, c_2 be the sides of T_2 and angles opposite to these sides are $\alpha_2, \beta_2, \gamma_2$ respectively. Then, the following statements are equivalent:*

1. *T_1 is congruent to T_2.*
2. *$a_1 = a_2, b_1 = b_2, c_1 = c_2$.*
3. *$\alpha_1 = \alpha_2, b_1 = b_2, c_1 = c_2$.*
4. *$\alpha_1 = \alpha_2, \beta_1 = \beta_2, c_1 = c_2$.*
5. *$\alpha_1 = \alpha_2, \beta_1 = \beta_2, \gamma_1 = \gamma_2$.*

Proposition 7.1.5. *Consider a family $\mathcal{U}$ of all triangles in $\mathbb{H}^2$ such that two sides of every triangle in this family have length a, b. Then, the triangle, which is circumscribed in a circle with the center as the midpoint of the 'remaining side', is the area maximizer in $\mathcal{U}$.*

Proof. Suppose $r := a + b$.
Existence: Suppose that $p \in \mathbb{H}^2$. All the triangles in $\mathcal{U}$ lie inside $B(p, r)$ because the triangles that are outside and belong in the family are congruent to these triangles. There exists, T_0 in $\mathcal{U}$, an 'area maximizer' as $\overline{B(p, r)}$ is compact in $(\mathbb{H}^2, d)$.

Suppose that $\gamma = \gamma(T)$ is the angle of a triangle $T \in \mathcal{U}$ that is included between the sides of length a, b. Suppose, $A_\gamma = \text{area}(T)$. By proposition 7.1.3,

$$\cot(A_\gamma/2) = \frac{\coth\left(\frac{a}{2}\right)\coth\left(\frac{b}{2}\right) - \cos(\gamma)}{\sin(\gamma)}$$

Consider the unit circle S^1 in $\mathbb{E}^2$, its center is $O := (0, 0)$. Suppose that Q is a point in $\mathbb{E}^2$ such that Euclidean distance of Q from the origin is equal to $\coth(a/2)\coth(b/2)$. Q lies 'outside' S^1 because $\coth\left(\frac{a}{2}\right)\coth\left(\frac{b}{2}\right) > 1$. Extend the line segment $[[Q, O]]$ and denote the intersection point with S^1 as R. Assume that P is the point on S^1 such that $\measuredangle POR = \pi - \gamma$. Suppose that PN is the perpendicular on QR. Then,

$$\|Q - N\|_{\mathbb{E}^2} = \begin{cases} \|Q - O\|_{\mathbb{E}^2} - \|N - O\|_{\mathbb{E}^2} & \text{if } \gamma \in \left(0, \frac{\pi}{2}\right] \\ \|Q - O\|_{\mathbb{E}^2} + \|N - O\|_{\mathbb{E}^2} & \text{if } \gamma \in \left[\frac{\pi}{2}, \pi\right). \end{cases}$$

Therefore,

$$\begin{aligned} \|Q - N\|_{\mathbb{E}^2} &= \coth\left(\frac{a}{2}\right)\coth\left(\frac{b}{2}\right) + \cos\left(\frac{\pi}{2} - \gamma\right) \\ &= \coth\left(\frac{a}{2}\right)\coth\left(\frac{b}{2}\right) - \cos(\gamma) \end{aligned}$$

We have,

$$\measuredangle PQR = A_\gamma/2. \tag{7.1}$$

Suppose, $M \in S^1$ with QM being tangent to S^1 at M. Hence, when $P = M$ then $\measuredangle PQR$ is maximum. Let $\gamma_0 := \gamma(T_0)$ and denote the area of T_0 as A_0. By equation 7.1, $P(\gamma_0) = M(\gamma_0)$. Now, we have

$$\pi - \gamma_0 = \frac{\pi}{2} + \frac{A_0}{2}.$$

Suppose that α_0, β_0 are two remaining angles of T_0. Since $A_0 = (\pi - (\alpha_0 + \beta_0 + \gamma_0))$, hence $\gamma_0 = \alpha_0 + \beta_0$. Let A, B, and C represent the vertices of T_0. There exists a distinct point D on the geodesic segment $[[A, B]]$ such that $\angle ACD = \alpha_0$ (given $\gamma_0 > \alpha_0$). Consequently, $\angle BCD = \beta_0$. This results in the triangles $\triangle ADC$ and $\triangle BDC$ being isosceles. Therefore, the geodesic segments $[[A, D]]$, $[[D, B]]$, and $[[D, C]]$ are all of equal length. Thus, the vertices of T_0 are circumscribed by a circle, with the circle's center located at the midpoint of the segment $[[A, B]]$.

$\square$

Proposition 7.1.6. *In the hyperbolic plane $\mathbb{H}^2$, an isosceles triangle exists with a base of length $a > 0$ and base angles θ, where $\theta \in (0, \pi)$, if and only if θ satisfies the condition $\theta \in (0, \arccos(\tanh(\frac{a}{2})))$.*

Proof. By theorem 7.1.1 it is easy to see that an isosceles triangle in $\mathbb{H}^2$ with base angles θ exists only if $\theta \in (0, \pi/2)$. Now suppose that $p = (0, 0, 1)$ is a point in $\mathbb{H}^2$. The geodesic segment joining p to $q := (\sinh(a), 0, \cosh(a))$ is given by

$$c(u) = c_{p,e_1}(u) = (\sinh(u), 0, \cosh(u)), u \in [0, a]$$

Let $w_1 = (\cos(\theta), \sin(\theta), 0) \in T_p\mathbb{H}^2$ be a vector and define, $n = c'\left(\frac{a}{2}\right)$, $\tilde{\ell} = \{x \in \mathbb{R}^3 \mid \langle x, n\rangle = 0\}$ and $\ell = \tilde{\ell} \cap \mathbb{H}^2$. Let r_ℓ be the reflection in $\mathbb{H}^2$ through $\mathbb{H}$-line ℓ. Let

$$w_2 = d\left(r_\ell\right)_p (w_1) = r_\ell (w_1) = (-\cosh(a)\cos(\theta), \sin(\theta), -\sinh(a)\cos(\theta))$$

It is easy to see that w_2 makes an angle θ with $-c'(a)$ in $T_q\left(\mathbb{H}^2\right)$. Consider the geodesics $c_1 = c_{p,w_1}$ and $c_2 = c_{q,w_2}$ of $\mathbb{H}^2$. Then,

$$c_1(u) = (\sinh(u)\cos(\theta), \sinh(u)\sin(\theta), \cosh(u))$$

and
$c_2(u) = (\cosh(u)\sinh(a) - \sinh(u)\cosh(a)\cos(\theta), \sinh(u)\sin(\theta), \cosh(u)\cosh(a) - \sinh(u)\sinh(a)\cos(\theta))$.
So, $c_1(u) = c_2(u)$ for some $u \in \mathbb{R}\backslash\{0\}$ if and only if

$$\cos(\theta) = \coth(u)\tanh\left(\frac{a}{2}\right). \tag{7.2}$$

we get $u > 0$. Now, for all $u > 0$,

$$\tanh(a/2)\coth(u) \in (\tanh(a/2), \coth(u)) \subset (0, \infty)$$

since $\tanh(a/2) \in (0, 1)$ and $\coth(u) \in (1, \infty) \forall\, u > 0$. By equation (7.2) we can conclude that $\cos(\theta) \in (0, 1) \cap (\tanh(a/2), \coth(u)) = (\tanh(a/2), 1)$. Thus, the proposition is proved. □

Proposition 7.1.7. *In the hyperbolic plane $\mathbb{H}^2$, an isosceles triangle exists with a base of length $a > 0$ and equal sides $s - \frac{a}{2}$.*

Proof. Let $h(a, s) = \frac{\tanh(a/2)}{\tanh\left(s - \frac{a}{2}\right)}$. Then, $h(a, s) > \tanh(a/2) \in (0, 1)$ since $\coth(u) > 1 \,\forall\, u > 0$. Now suppose $\theta(a, s) = \arccos(h(a, s))$. Then $\theta(a, s) \in \left(0, \arccos(\tanh(\frac{a}{2}))\right)$. Therefore, by proposition 7.1.6, in the hyperbolic plane $\mathbb{H}^2$, an isosceles triangle exists with a base of length $a > 0$ and equal sides $s - \frac{a}{2}$. □

Proposition 7.1.8. *In a family of all triangles in $\mathbb{H}^2$ with base a and perimeter $2s$, then the 'area maximizer' in that family is the isosceles triangle.*

Proof. Clearly, $s > a$. By Proposition 7.1.7, $\exists$ an isosceles triangle T_0 with base a and equal sides $s - \frac{a}{2}$. Suppose that T is a triangle in $\mathbb{H}^2$ with sides a, b, c such that $a + b + c = 2s$. The areas of triangles T, T_0 are denoted by A, A_0 respectively. Then,

$$\begin{aligned}
\tan\left(\tfrac{A}{4}\right) &= \sqrt{\tanh\left(\tfrac{s}{2}\right)\tanh\left(\tfrac{s-a}{2}\right)\tanh\left(\tfrac{s-b}{2}\right)\tanh\left(\tfrac{s-c}{2}\right)} \quad \text{and,} \\
\tan\left(\tfrac{A_0}{4}\right) &= \sqrt{\tanh\left(\tfrac{s}{2}\right)\tanh\left(\tfrac{s-a}{2}\right)\tanh\left(\tfrac{a}{4}\right)\tanh\left(\tfrac{a}{4}\right)}.
\end{aligned}$$

We show that $A \le A_0$: Note that $\frac{A}{4}, \frac{A_0}{4} \in (0, \frac{\pi}{4})$, and tan is increasing on $(0, \frac{\pi}{4})$. Hence $A \le A_0$ if and only if $\tan\left(\frac{A}{4}\right) \le \tan\left(\frac{A_0}{4}\right)$. Then, it suffices to show that

$$\tanh\left(\frac{s-b}{2}\right)\tanh\left(\frac{s-c}{2}\right) \le \tanh^2\left(\frac{a}{4}\right). \qquad \text{(A)}$$

Now, after simplification using trigonometric identities, we see that.

$$\text{LHS of (A)} = \frac{\cosh\left(\frac{a}{2}\right) - \cosh\left(\frac{c-b}{2}\right)}{\cosh\left(\frac{a}{2}\right) + \cosh\left(\frac{c-b}{2}\right)}.$$

If $b = c = s - \frac{a}{2}$ then

$$\text{RHS of } (A) = \tanh^2\left(\frac{a}{4}\right) = \frac{\cosh\left(\frac{a}{2}\right) - 1}{\cosh\left(\frac{a}{2}\right) + 1}.$$

Since $\cosh(\theta) \in [1, \infty)$ we get

$$\text{LHS of (A)} = \frac{\cosh\left(\frac{a}{2}\right) - \cosh\left(\frac{c-b}{2}\right)}{\cosh\left(\frac{a}{2}\right) + \cosh\left(\frac{c-b}{2}\right)} \le \frac{\cosh\left(\frac{a}{2}\right) - 1}{\cosh\left(\frac{a}{2}\right) + 1} = \text{RHS of (A)}.$$

$\square$

Theorem 7.1.9. *For a polygon $\wp$ in $\mathbb{H}^2$, the following statements are equivalent:*

1. *The angle at each vertex of $\wp$ lies in $(0, \pi)$.*
2. *$\wp$ is convex.*
3. *$\wp$ is intersection of finitely many closed half-spaces.*

Proof. For the proof, please see [2]. $\square$

Proposition 7.1.10. *Suppose that $\wp$ is an m-sided convex polygon. Let $\alpha_1, \ldots, \alpha_m$ be the angles of $\wp$ at its vertices. Then area of $\wp$ is equal to $\{(n-2)\pi - (\sum_{k=1}^{m} \alpha_i)\}$.*

Proof. This is followed by Theorem 7.1.1. □

Lemma 7.1.11. (*Lemma of Cauchy*)*: Suppose that $\wp$ and $\bar{\wp}$ are two convex n-gons in $\mathbb{H}^2$. Denote the vertices of $\wp$ and $\bar{\wp}$, in a cyclic order as, $\{P_k\}_{k=1,\ldots,n}$ and $\{\bar{P}_k\}_{k=1,\ldots,n}$ respectively. Let $a_k := d(P_k, P_{k+1})$ and $\overline{a_k} = d(\bar{P}_k, \bar{P}_{k+1})$ $(1 \leq k \leq n-1)$ denote the lengths of $(n-1)$ sides of $\wp$ and $\bar{\wp}$ respectively. Let α_k and $\overline{\alpha_k}$ denote the angle of $\wp$ and $\bar{\wp}$ at vertex P_k and $\bar{P}_k$ of $\wp$ and $\bar{\wp}$ respectively for $2 \leq k \leq n-1$. Let a_n and $\overline{a_n}$ be the length of 'remaining' side of $\wp$ and $\bar{\wp}$ respectively. If $a_k = \overline{a_k}$ $\forall$ $k = 1, \ldots, n-1$ and $\alpha_k \leq \overline{\alpha_k}$ $\forall$ $k = 2, \ldots, n-1$ then $a_n \leq \overline{a_n}$ holds. Moreover, $\exists$ $k \in \{2, \ldots, n-1\}$ such that $\alpha_k < \overline{\alpha_k}$ then $a_n < \overline{a_n}$.*

Proof. For the proof, please see [2]. □

Lemma 7.1.12. *Consider the family $\mathcal{U}$ of all convex m-gons in $\mathbb{H}^2$ with side lengths $a_1, \ldots, a_{m-1}$, the area maximizer in $\mathcal{U}$ is the convex m-gon whose vertices lie on a circle and center of the circle is the midpoint of the 'remaining side'.*

Proof. If $m = 3$, then this result is already proved in Proposition 7.1.5. Hence, let $m \geq 4$. Denote $r = a_1 + \cdots + a_{m-1}$.

Existence: Let $p \in \mathbb{H}^2$, then upto congruence all polygons in $\mathcal{U}$ lie inside $B(p, r)$. Since $\overline{B(p,r)}$ is compact in $(\mathbb{H}^2, d)$ there exists an 'area maximizer' $\wp$ in $\mathcal{U}$.

We denote the vertices of $\wp$, which occur in a cyclic order, as $A_1, \ldots, A_m$ $(A_{m+1} = A_1)$ with $d(A_k, A_{k+1}) = a_k$ $\forall$ $k = 1, \ldots, m-1$. Let $r' = d(A_1, A_m)/2$ and O be the mid-point of geodesic segment $[[A_1, A_m]]$. We show that $d(O, A_k) = r'$ $\forall$ $k = 1, \ldots, m$

Suppose there exists a point A_k among $A_2, \ldots, A_{m-1}$ such that $d(O, A_k) \neq r'$. Denote $a = d(A_1, A_k)$ and $b = d(A_m, A_k)$. As d is a metric on $\mathbb{H}^2$, we have $a + b \leq a_1 + a_2 + \cdots + a_{m-1}$. Moreover, since A_k is assumed not to lie on the circle with center O and radius r', according to Proposition 7.1.5, there exists a triangle $[[A_1', A_k, A_m']]$ such that $d(A_1', A_k) = a$, $d(A_m', A_k) = b$, and

$$\text{area}([[A_1', A_k, A_m']]) > \text{area}([[A_1, A_k, A_m]]).$$

Further we can suppose that the angles at vertices A'_1, A_k, A'_m are close to the angles at vertices A_1, A_k, A_m for triangles $[[A'_1, A_k, A'_m]]$ and $[[A_1, A_k, A_m]]$ respectively.

Consider the triangle T' constituted by vertices $[[A'_1, A_k, A'_m]]$. Let H_1 be the closed half-space in $\mathbb{H}^2$ that encompasses A'_m, and its boundary is $[[A'_1, A_k]]$, with H_2 representing the complementary half-space. Within H_2, a polygon $\wp'_1$ with vertices $A'_1, A'_2, \ldots, A'_k$ (where $A'_k = A_k$) is configured in a cyclic manner, mirroring the convex polygon $[A_1, A_2, \ldots, A_k]$. Analogously, H_3 is identified as the closed half-space containing A'_1 with its boundary along $[[A_k, A'_n]]$, and H_4 as the opposite half-space. Positioned in H_4, the polygon $\wp'_2$ includes vertices from $A'_k(= A_k)$ to A'_m, sequenced cyclically to reflect the convex polygon $[A'_k, A_{k+1}, \ldots, A_m]$. Neither $\wp'_1$ nor $\wp'_2$ intersects the interior of T'. Thus, we have produced $\wp'$ with $A'_1, A'_2, \ldots, A'_m$ arranged in that order with $A'_{m+1} = A'_1$, such that $\wp' = \wp'_1 \cup T' \cup \wp'_2$. This construction yields a polygon $\wp'$ whose area surpasses that of $\wp$. Additionally, the angles at $A'_1, A_k(= A'_k), A'_m$ in $\wp'$ closely match those in the triangle $[[A_1, A_k, A_m]]$, ensuring all angles are less than π. By Theorem 7.1.9, we can say that $\wp'$ is a convex polygon, placing $\wp'$ within a set $\mathcal{U}$ and marking its area as greater than that of $\wp$, contradicting the assumption of $\wp$ as the area maximizer within $\mathcal{U}$. Consequently, it is deduced that the distance from O to each A_k is uniformly r' for every k ranging from 1 to m.

□

7.2 Regular polygon in $\mathbb{H}^2$

Definition 7.2.1. *For a polygon $\wp$ in $\mathbb{H}^2$, if all the sides of $\wp$ are of equal length, then $\wp$ is called an equilateral polygon.*

Definition 7.2.2. *For a polygon $\wp$ in $\mathbb{H}^2$, if all the angles of $\wp$ are equal then $\wp$ is called an equiangular polygon.*

Definition 7.2.3. *For a polygon $\wp$ in $\mathbb{H}^2$, if $\wp$ is convex, equilateral, and equiangular, then $\wp$ is called a regular polygon. A regular polygon of m sides is called a regular m-gon.*

Construction of regular polygons in $\mathbb{H}^2$: Let $p = (0, 0, 1)$ be a point in $\mathbb{H}^2$. Fix $r > 0$ and $m \geq 3$. Consider the disc $B(p, r) \subset \mathbb{H}^2$. Denote the boundary of $B(p, r)$ by $\mathcal{C}(p, r)$.

We see that, in $\mathbb{R}^3$, $\mathcal{C}\,(p,r)$ is a Euclidean circle in the plane $\{(x_1, x_2, \cosh(r)) : x_1, x_2 \in \mathbb{R}\}$. Denote the center of $\mathcal{C}\,(p,r)$ by $\underline{c} = \cosh(r)p$ and its radius is $\sinh(r)$.

Let $P_1, \ldots, P_m$ be points situated on a circle $\mathcal{C}(p,r)$ within the hyperbolic plane $\mathbb{H}^2$, arranged clockwise such that the angle $\measuredangle\{P_k - \underline{c}, P_{k+1} - \underline{c}\} = \frac{2\pi}{m}$ for each $k = 1, \ldots, m$ (with $P_{m+1} = P_1$). These points define the vertices of a convex polygon, denoted by $\wp_{m,r}$, in $\mathbb{H}^2$. By construction, a rotation $\rho_{\frac{2\pi}{m}}$ about an axis through $\underline{c}$ and perpendicular to the plane of $\mathcal{C}(p,r)$—the axis orientation being defined by vector p—acts as a symmetry for $\wp_{m,r}$. Thus $\wp_{m,r}$ is a regular m-gon in $\mathbb{H}^2$. Furthermore, for any specific $m \geq 3$ and $r > 0$, all convex polygons formed as described are congruent. Let $\wp_{m,r}$ be a regular polygon in the hyperbolic plane $\mathbb{H}^2$ with m sides, each of length a. Consider the segment $[[P_1, P_2]]$ and denote its midpoint by M. The triangles $[[\underline{c}, M, P_1]]$ and $[[\underline{c}, M, P_2]]$ are congruent. Furthermore, the angle at vertex M in both triangles is $\frac{\pi}{2}$, indicating they are right-angled triangles. Apply The Law of hyperbolic sine to the triangle $[[\underline{c}, M, P_1]]$, we get

$$\frac{\sinh(r)}{\sin\left(\frac{\pi}{2}\right)} = \frac{\sinh\left(\frac{a}{2}\right)}{\sin\left(\frac{\pi}{m}\right)}$$

Hence,

$$a = a(m,r) = 2A\sinh^{-1}\left(\sinh(r)\sin\left(\frac{\pi}{m}\right)\right).$$

Now we compute the angle $\theta = \theta(m,r)$ at the vertices of $\wp_{m,r}$. When we apply the Law of hyperbolic sine to the triangle $[[P_1, P_2, \underline{c}]]$ we get

$$\frac{\sinh(r)}{\sin\left(\frac{\theta}{2}\right)} = \frac{\sinh(a)}{\sin\left(\frac{2\pi}{m}\right)}$$

and

$$\frac{\sinh(r)}{\sin\left(\frac{\theta}{2}\right)} = \frac{\sinh(a)}{\sin\left(\frac{2\pi}{m}\right)} = \frac{2\sinh\left(\frac{a}{2}\right)\cosh\left(\frac{a}{2}\right)}{2\sin\left(\frac{\pi}{m}\right)\cos\left(\frac{\pi}{m}\right)} = \frac{\sinh(r)\sin\left(\frac{\pi}{m}\right)\cosh\left(\frac{a}{2}\right)}{\sin\left(\frac{\pi}{m}\right)\cos\left(\frac{\pi}{m}\right)}.$$

Thus

$$\sinh\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{m}\right)}{\cosh\left(\frac{a}{2}\right)}$$

Hence,

$$\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{\cosh^2\left(\frac{a}{2}\right) - \cos^2\left(\frac{\pi}{m}\right)}}{\cosh\left(\frac{a}{2}\right)} = \frac{\sqrt{1 + \sinh^2\left(\frac{a}{2}\right) - \cos^2\left(\frac{\pi}{m}\right)}}{\cosh\left(\frac{a}{2}\right)}$$
$$= \frac{\sqrt{\sinh^2\left(\frac{\pi}{m}\right) + \sinh^2(r)\sin^2\left(\frac{\pi}{m}\right)}}{\cosh\left(\frac{a}{2}\right)}$$
$$= \frac{\sqrt{\sin^2\left(\frac{\pi}{m}\right)\left(1 + \sinh^2(r)\right)}}{\cosh\left(\frac{a}{2}\right)} = \frac{\sqrt{\sin^2\left(\frac{\pi}{m}\right)\cosh^2(r)}}{\cosh\left(\frac{a}{2}\right)}.$$

Thus

$$\cos\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\pi}{m}\right)\cosh(r)}{\cosh\left(\frac{a}{2}\right)}$$

We get,

$$\tan\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{m}\right)}{\cosh(r)\sin\left(\frac{\pi}{m}\right)}$$

Hence,

$$\theta = \theta(m, r) = 2\arctan\left(\frac{\cot\left(\frac{\pi}{m}\right)}{\cosh(r)}\right)$$

Suppose that A denotes the area of the regular m-gon $\wp_{m,r}$. By Proposition 7.1.10, $A = (m-2)\pi - m\theta$. Therefore,

$$A = A(m, r) = \left\{(m-2)\pi - 2m\arctan\left(\frac{\cot\left(\frac{\pi}{m}\right)}{cos(r)}\right)\right\}.$$

Theorem 7.2.1. *For a unique $r > 0$, any regular m-gon in $\mathbb{H}^2$ is congruent to $\wp_{m,r}$.*

Proof. Suppose that $\wp'$ is a regular m-gon in $\mathbb{H}^2$. Since $\wp'$ is a regular m-gon, hence $m \geq 3$. Assume a' is the length of a side of $\wp'$. We know $a' \in (0, \infty)$. Suppose, $J := (0, \infty)$ and $J' := (0, \infty)$. Consider the function $h : J \to J'$ given by $h(r) = 2\sinh^{-1}\left(\sinh\left(\frac{\pi}{m}\right)\sinh(r)\right)$. For a fixed $m \geq 3$, $h : J \to J'$ is a bijective map. Hence for $a' \in J'$, there exists a unique $r \in J$ such that $a' = h(r)$. Thus, for a unique $r \in J$, $a' = a(m, r)$.

To demonstrate that polygon $\wp'$ is congruent to $\wp_{m,r}$, consider vertices $P_1', \ldots, P_m'$ of $\wp'$ and $P_1, \ldots, P_m$ of $\wp_{m,r}$, arranged in a cyclic order. Denote the angles at the vertices of $\wp'$ and $\wp_{m,r}$ as ϕ' and ϕ, respectively. If $\phi' < \phi$ (or $\phi' > \phi$), according to the Lemma of Cauchy, this would imply $d(P_1', P_m') < d(P_1, P_m)$ (or $d(P_1', P_m') > d(P_1, P_m)$), contradicting the fact that $a' = a(m, r)$. Hence, we must have $\phi' = \phi$. By applying the Lemma of Cauchy again to the sequences of vertices $[P_1, P_2, \ldots, P_k]$ and $[P'_1, P'_2, \ldots, P'_k]$, we find that $d(P'_1, P'_k) = d(P_1, P_k)$ for all $k = 2, \ldots, m$. Similarly, for any pair of indices $i, k \in \{1, \ldots, m\}$, $d(P'_i, P'_k) = d(P_i, P_k)$. Consequently, according to proposition 7.0.5, there exists an isometry φ of $\mathbb{H}^2$ such that $\varphi(\wp') = \wp_{m,r}$, confirming the congruence of the two polygons. $\square$

Proposition 7.2.2. *For any regular polygon $\wp$ in the hyperbolic plane $\mathbb{H}^2$, with m sides, each of length a, an internal angle θ, and an area A, there exists a singular value $r > 0$ such that $\wp$ can be exactly circumscribed by a circle with radius r. Further, equations*

1. $r = r(m, a) = \sinh^{-1}\left(\frac{\sinh(a/2)}{\sinh(\pi/m)}\right)$,

2. $r = r(m, \theta) = \cosh^{-1}\left(\frac{\cot(\pi/m)}{\tan(\theta/2)}\right)$,

3. $r = r(m, A) = \cosh^{-1}\left(\cot\left(\frac{\pi}{m}\right)\tan\left(\frac{2\pi + A}{2m}\right)\right)$

hold. Any regular m-gon in $\mathbb{H}^2$ can be determined (uniquely up to congruence) by any one of three :

$$a \in (0, \infty) \quad , \quad \theta \in (0, \pi), \quad A \in (0, (m-2)\pi).$$

Remark 7.2.1. *The regular m-gons in $\mathbb{H}^2$ with angle θ are congruent.*

Corollary 7.2.3. *Consider a sequence of regular polygons $\left(\tilde{\wp_k}^2\right)$ in the hyperbolic plane $\mathbb{H}^2$, where each $\tilde{\wp_k}$ possesses k vertices for each $k \in \mathbb{N}$. Let $A_k = (\text{area}\,(\tilde{\wp_k}))$ denote the area of each polygon $\tilde{\wp_k}$, with this sequence of areas converging to a limit A' as $k \to \infty$. For each polygon $\tilde{\wp_k}$, let r_k indicate the radius of the circle in which $\tilde{\wp_k}$ is inscribed, for all $k \in \mathbb{N}$. Denote the periemeter of $\tilde{\wp_k}$ by $\ell(\tilde{\wp_k})$. Then*

1. $\lim_{k\to\infty} r_k = \sinh^{-1}\left(\sqrt{A'\,(4\pi + A')}/(2\pi)\right)$

2. $\lim_{k\to\infty} (\ell(\tilde{\wp_k})) = \sqrt{A'\,(4\pi + A')}$.

Proof. (1) We have

$$\frac{\cot\left(\frac{\pi}{k}\right)}{\cosh\left(r_k\right)} = \tan\left(\frac{A_k - (k-2)\pi}{-2k}\right) = \tan\left(\frac{\pi}{2} - \frac{2\pi + A_k}{2k}\right) = \cot\left(\frac{2\pi + A_k}{2k}\right).$$

Hence, $\cosh(r_k) = \cot\left(\frac{\pi}{k}\right)\tan\left(\frac{2\pi+A_k}{2k}\right)$. Therefore, $\sinh^2(r_k) = \frac{\sin\left(\frac{A_k}{2k}\right)\sin\left(\frac{4\pi+A_k}{2k}\right)}{\sin^2\left(\frac{\pi}{k}\right)\cos^2\left(\frac{2\pi+A_k}{2k}\right)}$. So,

$$\lim_{k\to\infty}\sinh^2(r_k) = \frac{(4\pi + A')\,A'}{4\pi^2}.$$

Hence,

$$\lim_{k\to\infty} r_k = \sinh^{-1}\left(\frac{\sqrt{(4\pi + A')\,A'}}{2\pi}\right).$$

(2) Let $r_0 = \sinh^{-1}\left(\frac{\sqrt{(4\pi A' + A'^2)}}{2\pi}\right)$. Since, each $\tilde{\wp_k}$ is a regular k-gon inscribed in a circle of radius r_k in $\mathbb{H}^2$. By (1), (r_k) converges to r_0 . Hence,

$$\lim_{k\to\infty}(\ell(\tilde{\wp_k})) = 2\pi\sinh(r_0) = \sqrt{A'\,(4\pi + A')}.$$

□

7.3 Isoperimetric problem for polygons in $\mathbb{H}^2$

Proof. of Theorem 7.0.1: Fix $n \geq 3$ in the natural numbers and $A \in (0, (n-2)\pi)$. We will denote the perimeter of $\wp$ by $\ell(\wp)$. Suppose that $\mathcal{G}$ is the family of all n-gons $\mathbb{H}^2$ in which every polygon has an area greater than or equal to A. $\mathcal{G}$ is a nonempty family because by Proposition 7.2.2 there exists a regular n-gon $\wp_{n,r}$ of area equal to A. Define $L = \text{glb}\{\ell(\wp) : \wp \in \mathcal{G}\}$. Suppose that $(\wp_i)_{i\in\mathbb{N}}$ is a sequence in $\mathcal{G}$ such that $(\ell(\wp_i))$ decreases to L as $i \to \infty$. Let $p = (0, 0, 1)$ be a vertex of $\wp_i \forall\ i \in \mathbb{N}$.

For each $i \in \mathbb{N}$, let $A_i^{(1)}, A_i^{(2)}, \ldots, A_i^{(n)}$ denote the vertices of the polygon $\wp_i$, with $A_i^{(1)} = p$ and the vertices following the cyclic order prescribed by the orientation of $\partial\wp_i$. It is possible to assume that each polygon $\wp_i$ resides within the ball $B(p, L+1)$ for all i by considering the compactness of the closed ball $\overline{B(p, L+1)}$ in the hyperbolic plane $\mathbb{H}^2$. Then for each $j = 1, \ldots, n$, the vertex sequence $\left(A_i^{(j)}\right)_{i\in\mathbb{N}}$ has at least one converging subsequence. Consequently, we may assume that the sequence $\left(A_i^{(j)}\right)_{i\in\mathbb{N}}$ converges to a specific point B_j in $\mathbb{H}^2$ for every $j = 1, \ldots, n$.

Clearly, $B_1 = p$. Let $B_{n+1} := B_1$. Then $\cup_{k=1}^{n}[[B_k, B_{k+1}]]$ is a 'simple closed curve in $\mathbb{H}^2$'.

Suppose that $\wp_0$ is the polygon in $\mathbb{H}^2$ and its boundary is $\cup_{k=1}^{n}[[B_k, B_{k+1}]]$. Then $\ell(\wp_0) = L$ and area $(\wp_0) \geq A$. We now to prove that $\wp_0$ is a regular n-gon.

$\underline{\wp_0 \text{ is convex}}$: If this not the case then, by Theorem 7.1.9, $\exists\, j \in \{1, \ldots, n\}$ such that $\wp_0$, at the vertex B_j, is not locally convex. Let $B_0 = B_n$ and $B_{n+1} = B_1$. A point B'_j can be chosen on the segment $[[B_{j-1}, B_j]]$ such that the triangle $[[B'_j, B_j, B_{j+1}]]$ does not intersect the interior of the polygon $\wp_0$. Consider constructing a polygon $\wp$ in the hyperbolic plane $\mathbb{H}^2$ with vertices $B_1, B_2, \ldots, B_{j-1}, B'_j, B_{j+1}, \ldots, B_n$, positioned in a cyclic sequence. Thus $\wp$ has a perimeter shorter than $\wp_0$, i.e., $\ell(\wp) < \ell(\wp_0)$, and an area greater than or equal to that of $\wp_0$. This is not possible. Hence, $\wp$ is a convex n-gon.

$\underline{\text{area}(\wp_0) = A}$: For positive δ, let area $(\wp_0) = A + \delta$. We choose a point $B'_2 \in [[B_2, B_3]]$ such that $B'_2 \notin \{B_2, B_3\}$ and $\text{area}([[B_1, B_2, B'_2]]) < \delta$. The triangle $[[B_1, B_2, B'_2]]$ is contained in $\wp_0$ because $\wp_0$ is convex. Let $\wp''$ be the polygon with vertices $B_1, B'_2, B_3, \ldots, B_n$ occurring in a cyclic order. Then, $\text{area}(\wp'') > A$ and $\ell(\wp'') < \ell(\wp_0)$. This is a contradiction, hence, area $(\wp_0) = A$.

$\underline{\wp_0 \text{ is equilateral}}$: This is not hard to see that it follows from the Proposition 7.1.8. Thus, $\wp_0$ is an equilateral n-gon.

Suppose that 'a' is the side length of the convex equilateral n-gon $\wp_0$.

$\underline{\wp_0 \text{ is equiangular}}$: We show this by considering n even, n odd cases separately.

$\underline{\text{The } n \text{ even case}}$: Suppose that $n = 2i$. Join B_1 to B_{1+i} by the geodesic segment $[[B_1, B_{1+i}]]$ contained in $\wp_0$. Let M be the mid-point of $[[B_1, B_{1+i}]]$. It is suffice to show that $d(M, B_k) = r = d(M, B_1)\, \forall\, k = 2, \ldots, n$. The segment $[[B_1, B_{1+i}]]$ divides polygon $\wp_0$ into two convex sub-polygons, $\wp_1$ and $\wp_2$, each with $i + 1$ vertices. Let ρ represent the reflection in $\mathbb{H}^2$ across the hyperbolic line containing $[[B_1, B_{1+i}]]$. If $\text{area}(\wp_1) > \text{area}(\wp_2)$, then define $\wp_2$ as $\wp_1 \cup \rho(\wp_1)$. Thus, $\wp_2$ has a perimeter L and an area greater than A, which is contradictory. Consequently, the areas of $\wp_1$ and $\wp_2$ must be equal.

Consider the family of all convex polygons in $\mathbb{H}^2$ with $i + 1$ vertices, where i sides are of length a, but one side is distinct. If $\wp'_1 = [[B'_1, \ldots, B_{i+1}]]'$, with $[[B'_1, B_{i+1}]]'$ is the remaining side, maximizes the area in this family, then $\text{area}(\wp'_1) \geq \text{area}(\wp_1)$. If $\text{area}(\wp'_1) > \text{area}(\wp_1)$, then reflecting $\wp'_1$ across $[[B'_1, B_{i+1}]]'$ would yield a polygon with perimeter $L = 2ia$ and area greater than A, which is a contradiction. Therefore, $\text{area}(\wp'_1) = \text{area}(\wp_1)$. By lemma

7.1.12, it follows that $d(M, B_k) = r$ for all $k = 1, \ldots, i+1$. Similarly, it can be shown that $d(O, B_k) = r$ for all $k = i+1, \ldots, 2i$.

The n odd case: It is not hard to follow that by adjoining triangles congruent to T on each side of $\wp_0$ and $\wp_R^n$, where $\wp_R^n$ is a regular n-gon. Now, by even case, we arrive at a contradiction. $\square$

7.4 The isoperimetric problem in $\mathbb{H}^2$

Let $\ell(c)$ denote the arc-length of c, where c is a simple and closed piecewise smooth curve in $\mathbb{H}^2$. Such a curve c always will enclose a unique relatively compact domain in $\mathbb{H}^2$. By $A(c)$, we denote the area enclosed by c.

Proof. of Theorem 7.0.2: Let $p = (0,0,1)$ be a point in $\mathbb{H}^2$. For $r > 0$, the boundary of $B(p, r)$ is a circle $\mathcal{C}_r$ of area $4\pi \sinh^2\left(\frac{r}{2}\right)$ and perimeter $2\pi \sinh(r)$.

(Existence): There exists a perimeter minimizer among all piecewise smooth simple closed curves in $\mathbb{H}^2$ with fix area A and it is a circle of radius of $\sinh^{-1}\left(\frac{\sqrt{4\pi A + A^2}}{2\pi}\right)$.

Suppose that $\mathcal{F}$ is the family of 'all piecewise smooth simple closed curves in $\mathbb{H}^2$ enclosing area at least A'. Let $\mathcal{C} \in \mathcal{F}$ be arbitrary. If $X_1, X_2, \ldots, X_n$ are points on a curve $\mathcal{C} \in \mathcal{F}$ which appear in a cyclic order, then they determine a polygon with vertices $X_1, X_2, \ldots, X_n$. Define $L := \mathrm{glb}\{\ell(\mathcal{C}) \mid \mathcal{C} \in \mathcal{F}\}$. Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ such that $\ell(\mathcal{C}_n) \searrow L$ as $n \to \infty$. We may assume that $p \in \mathcal{C}_n$ and that $l(\mathcal{C}_n) \leq L+1, \forall\, n \in \mathbb{N}$. Hence $\mathcal{C}_n \subset B(p, L+1), \forall\, n \in \mathbb{N}$. Then $A(\mathcal{C}_n) \leq A(B(p, L+1)) = 4\pi \sinh^2\left(\frac{L+1}{2}\right) \forall\, n \in \mathbb{N}$. Therefore, we may assume, after taking a subsequence of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ if necessary, that

$$\ell(\mathcal{C}_n) \searrow L \text{ and } \left(A(\mathcal{C}_n)\right)_{n\in\mathbb{N}} \to A' \geq A.$$

As each $\mathcal{C}_n$ is a piecewise smooth simple closed curve, we can approximate $\mathcal{C}_n$ by the boundary of a polygon $\wp_{i(n)}$ in $\mathbb{H}^2$ with $i(n)$ sides: i.e., vertices of $\wp_{i(n)}$ lie on $\mathcal{C}_n, 0 < \ell(\mathcal{C}_n) - \ell\left(\partial\wp_{i(n)}\right) < \frac{1}{n}$ and $\left|\mathrm{area}\left(\wp_{i(n)}\right) - A(\mathcal{C}_n)\right| < \frac{1}{n} \quad \forall\, n \in \mathbb{N}$. (Here $\partial\wp$ denotes the boundary of $\wp$). Then it follows that

$$\lim_{n\to\infty} \ell\left(\partial\wp(n)\right) = L \quad \text{ and } \lim_{n\to\infty} \text{area}\left(\wp_{i(n)}\right) = A'.$$

Let $A_n = \text{area}\left(\wp_{i(n)}\right) \forall\, n \in \mathbb{N}$. By Proposition 7.2.2, for each $n \in \mathbb{N}$ there exists regular $i(n)$-gon $\tilde{\wp}_{i(n)}$ in $\mathbb{H}^2$ of area A_n. By Theorem 7.0.1 $\ell\left(\partial\tilde{\wp}_{i(n)}\right) \leq \ell\left(\wp_{i(n)}\right) \forall\, n \in \mathbb{N}$. Let $r_{i(n)}$ be the radius of the circle in $\mathbb{H}^2$ in which $\tilde{\wp}_{i(n)}$ is inscribed. By Corollary 7.2.3,

$$\lim_{n\to\infty} r_{i(n)} = \sinh^{-1}\left(\frac{\sqrt{4\pi A + A^2}}{2\pi}\right) = r.$$

Thus,

$$A' = 4\pi \sinh^2\left(\frac{r}{2}\right).$$

Again by Corollary 7.2.3, $\lim_{n\to\infty} \ell\left(\tilde{\wp}_{i(n)}\right) = \sqrt{A'\left(4\pi + A'\right)} = 2\pi\sinh\left(r\right)$. Note that $\ell\left(\partial\tilde{\wp}_{i(n)}\right) \leq \ell\left(\partial\wp_{i(n)}\right) \leq \ell\left(\mathcal{C}_n\right) \forall\, n \in \mathbb{N}$. Therefore,

$$L = \lim_{n\to\infty} \ell\left(\mathcal{C}_n\right) \geq \lim_{n\to\infty} \ell\left(\partial\tilde{\wp}_{i(n)}\right) = 2\pi\sinh\left(r\right).$$

Let $\mathcal{C}_r$ denote the circle in $\mathbb{H}^2$ of radius r. Then $A\left(\mathcal{C}_r\right) = 4\pi\sinh^2\left(\frac{r}{2}\right) = A'$. So, $\mathcal{C}_r \in \mathcal{F}$, and by the definition of L,

$$L \leq \ell\left(\mathcal{C}_r\right) = 2\pi\sinh\left(r\right).$$

We have $L = 2\pi\sinh\left(r\right) = \ell\left(\mathcal{C}_r\right)$. Hence $\mathcal{C}_r$ is a perimeter minimizer in $\mathcal{F}$. Now we show that $A\left(\mathcal{C}_r\right) = A$. If $A\left(\mathcal{C}_r\right) \neq A$ then, $A\left(\mathcal{C}_r\right) = 4\pi\sinh^2\left(\frac{r}{2}\right) = A' > A$. Then we can replace a small portion of circle $\mathcal{C}_r$ by a geodesic arc and produce a curve $\tilde{\mathcal{C}}$ in $\mathbb{H}^2$ with $\ell(\tilde{\mathcal{C}}) < \ell\left(\mathcal{C}_r\right) = L$ and $A < A(\tilde{\mathcal{C}}) < A\left(\mathcal{C}_r\right)$. Then $\tilde{\mathcal{C}} \in \mathcal{F}$ with $\ell(\tilde{\mathcal{C}}) < L$. This is a contradiction. Hence $A\left(\mathcal{C}_r\right) = A$.
(Uniqueness): 'In the family of all closed simple piecewise smooth curves in $\mathbb{H}^2$ bounding area A, any perimeter minimizer is a circle in $\mathbb{H}^2$ of radius $\sinh^{-1}\left(\frac{\sqrt{4\pi A + A^2}}{2\pi}\right)$'. Let $\mathcal{F}$ be the family of all such curves in $\mathbb{H}^2$ in which every element has an area greater or equal to A. Let $L = glb\{\ell(\mathcal{C}) \mid \mathcal{C} \in \mathcal{F}\}$. We have already shown the existence of a perimeter minimizer in $\mathcal{F}$ in step 1. Suppose that $\mathcal{C}_0 \in \mathcal{F}$ is a perimeter minimizer. Hence, $\ell\left(\mathcal{C}_0\right) = L$. Suppose that $\mathcal{C}_0$ encloses the domain $\mathbb{H}_0$ in $\mathbb{H}^2$. By the arguments similar to in the proof of Theorem 7.0.1, it follows that $\mathbb{H}_0$ is convex and area of $\mathbb{H}_0$ is equal to A.

Fix a point P on $\mathcal{C}_0$. Suppose that D is the point on $\mathcal{C}_0$ which divides $\mathcal{C}_0$ into two arcs $\mathcal{C}_0^+$, $\mathcal{C}_0^-$ of equal length. Denote the geodesic segment joining P and D in $\mathbb{H}_0$ by $[[P, D]]$. This segment divides $\mathbb{H}_0$ into two regions $\mathbb{H}_0^+$ and $\mathbb{H}_0^-$. If area $\left(\mathbb{H}_0^+\right) <$ area $\left(\mathbb{H}_0^-\right)$, then we can define a new domain using reflection ρ through the $\mathbb{H}$-line containing $[[P, Q]]$ as $\mathbb{H}_0^- \cup \rho\left(\mathbb{H}_0^-\right)$. Now, it is easy to see that we get a contradiction.

Let O be the midpoint of the geodesic segment $[[P, D]]$, and let us define r as the half of the distance between P and D, i.e., $r = \frac{d(P,D)}{2}$.

We aim to prove that for any point M on the curve $\mathcal{C}_0$, the distance from O to M is consistently r. Consider a scenario where there exists a point M on $\mathcal{C}_0$ such that $d(O, M) \neq r$. If M is located within the region $\mathbb{H}_0^+$, bounded by $\mathcal{C}_0^+ \cup [[P, D]]$, then due to the convex nature of $\mathbb{H}_0$, the triangle $[[P, M, D]]$ must be contained within $\mathbb{H}_0^+$. This implies that the sum of distances $d(P, M) + d(M, D)$ is at most $L/2$, the semi-perimeter of $\mathcal{C}_0$, hence M does not belong to the hyperbolic circle with center O and radius r.

By the arguments similar to the proof of Lemma 7.1.12, we can produce a domain $\widetilde{\mathbb{H}_0^+}$ in $\mathbb{H}^2$ such that area$(\widetilde{\mathbb{H}_0^+})>$ area $\left(\mathbb{H}_0^+\right)$ and its boundary consists of a curve $\widetilde{\mathcal{C}_0^+}$ and a geodesic segment $[[P', D']]$, where $\widetilde{\mathcal{C}_0^+}$ is congruent to $\mathcal{C}_0^+$ and P', D' are the endpoints of $\widetilde{\mathcal{C}_0^+}$. Thus, we can construct a domain $\widetilde{\mathbb{H}_0}$ such that area$(\widetilde{\mathbb{H}_0})> A$ and $\ell(\widetilde{\mathbb{H}_0}) = L$ by reflecting $\widetilde{\mathbb{H}_0^+}$ through the $\mathbb{H}$-line containing $[[P', D']]$. This is a contradiction, hence $d(O, M) = r$ and $\mathcal{C}_0 = \partial B\left(O, r\right)$. $\square$

Proof. of Corollary 7.0.3: Suppose that $\mathcal{C}$ is a 'piecewise smooth simple closed curve' having n components each enclosing area $A_k > 0$. Let $r_k = \sinh^{-1}\left(\frac{\sqrt{A(4\pi+A)}}{2\pi}\right)$ $(1 \leq k \leq n)$. For $1 \leq k \leq n$, denote the disjoint union of the circles $\tilde{\mathcal{C}}_k$ of radius r_k by $\tilde{\mathcal{C}}$. When we apply Theorem 7.0.2 to each component of $\mathcal{C}$ we get, $\ell(\tilde{\mathcal{C}}) \leq \ell(\mathcal{C})$. Now, it is easy to see that a circle of radius $\sinh^{-1}\left(\frac{\sqrt{A(4\pi+A)}}{2\pi}\right)$ is the 'perimeter minimizer'. $\square$

Proof. of Theorem 7.0.4 : Suppose that $\mathcal{C}$ is 'any piecewise smooth simple closed curve' in $\mathbb{H}^2$ having enclosed area $A = A(\mathcal{C}) > 0$, and denote the are-length of $\mathcal{C}$ by $\ell = \ell(\mathcal{C})$. Let $\mathcal{F}$ and L be as in the proof of Theorem 7.0.2 for the respective case. Let $r = \sinh^{-1}\left(\frac{\sqrt{A(4\pi+A)}}{2\pi}\right)$. By Theorem 7.0.2, the unique 'perimeter minimizer' in $\mathcal{F}$ is the circle $\mathcal{C}_r$ of radius r. Hence, $L^2 = \left(2\pi \sinh\left(r\right)\right)^2 = 4\pi A + A^2$, thus $\ell^2 = [\ell(\mathcal{C})]^2 \geq L^2 = 4\pi A + A^2$ is True for all $\mathcal{C}$ in $\mathcal{F}$.

Let $\mathcal{C}$ be a curve in $\mathcal{F}$,such that $\ell(\mathcal{C}) = \sqrt{4\pi A + A^2} = L$ and $\mathcal{C}$ is a perimeter minimizer in $\mathcal{F}$. By Theorem 7.0.2, it's a circle of radius r in $\mathbb{H}^2$. $\square$

Summary

Picking up from the basics of Riemannian geometry, we studied the essential notions of the Riemannian manifolds, Riemannian metrics, geodesics on a Riemannina manifold, the curvature tensor and the sectional curvature of a Riemannian manifold. Along the way, we also studied the Jacobi fields, the Hopf-Rinow theorem, and the theorem of Hadamard. We then confined our attention to the spaces of constant sectional curvature. After gaining the background necessary for the topic of our thesis, we narrowed down to a detailed understanding of the hyperbolic space $\mathbb{H}^n$ using the hyperboloid model of hyperbolic geometry.

Our main goal of the thesis is to present a proof of the isoperimetric theorem in $\mathbb{H}^2$ using a geometric approach. This proof is completely based on the expository article [2]. The isoperimetric theorem states that the *circle* in $\mathbb{H}^2$ is the unique perimeter minimizer among all piecewise smooth simple closed curves in $\mathbb{H}^2$ enclosing a given area. We prove this theorem for domains in a complete simply connected Riemannian manifold with constant sectional curvature -1. In view of Theorem 5.2.2, we can say that the isoperimetric theorem holds true even for domains in any hyperbolic plane of constant negative sectional curvature.

In the process of proving the isoperimetric problem for piecewise smooth, simple closed curves enclosing a given area, we first prove the isoperimetric problem for polygons in $\mathbb{H}^2$, which states that: For a fixed $n \geq 3$ in $\mathbb{N}$ and $A \in (0, (n-2)\pi)$, among all polygons with n sides in $\mathbb{H}^2$ having area A, the perimeter minimizer is the regular proper polygon with n sides. The result of general domains in $\mathbb{H}^2$ follows from the standard approximation argument which we have described in this thesis.

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