Subspace Detection in measurement induced quantum walks

A Thesis

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Certificate

This is to certify that this dissertation entitled Subspace Detection in measurement induced quantum walks towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Aashay Pandharpatte at Harish-Chandra Research Institute, Allahabad under the supervision of Dr. Aditi Sen De, Professor, , during the academic year 2019-2024.

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Committee:
Dr. Aditi Sen De

This thesis is dedicated to my parents...

Declaration

I hereby declare that the matter embodied in the report entitled Subspace Detection in measurement induced quantum walks are the results of the work carried out by me at the , Harish-Chandra Research Institute, Allahabad, in under the supervision of Dr. Aditi Sen De and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

Aashay Pandharpatte

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Contents

List of Figures

List of Tables

Abstract

This thesis extends the formalism of first detection in quantum walks from the detection of a single state to the detection of a subspace within the quantum system. Using the insight of bright/dark state formalism, which effectively calculates the state detection probability, further development is done to encompass subspace detection. We state the necessary conditions for stationary dark states in the subspace detection problem. We analytically calculate dark states and the subspace detection probability for position subspaces on a ring system and give expressions for the subspace detection probability for subspaces of various dimensions..

Chapter 1

Introduction to quantum walks

Classical random walks have been established as a great mathematical tool for algorithm design [16],[21][20]. Quantum walks [1], theorized as the quantum version of random walks, was studied for their interesting physical properties. Later, considerable interest was generated in understanding the properties of quantum walks, which are useful for developing quantum algorithms[3],[10],[2] that perform better than their classical counterparts. The field of quantum walks explores the interplay between classical random walks and quantum mechanics. Algorithms such as Grover's search [10] used the concept of amplitude amplification for efficient search. The two significant formalisms that define quantum walks are discrete-time quantum walks[4] and continuous-time quantum walks[7],[15]. Both of these models are used to describe evolution on discrete graphs. Various problems in computer science can be mapped as problems on graphs for the development of algorithms and is also considered a universal mode of quantum computation[6]. In particular, quantum walks provide a natural framework for developing spatial search [8] algorithms where the graph is used as a database.

1.1 Discrete-time Quantum walk

The discrete-time quantum walk[4] is one formulation of the quantum walk where the evolution occurs in discrete-time steps. Similar to the case of a classical random walk, we can formulate the theory of discrete quantum walk as a walker on a one-dimensional infinite line described by the basis $\{|n\rangle \, | n \in \mathbb{Z}\}$. This infinite-dimensional basis forms the position space of the walker, and the corresponding Hilbert space is denoted as $\mathcal{H}_p$. The walker starts its evolution from position $n = 0$. In addition to this, the motion of the walker is governed by a coin which acts as a spin-1/2 particle with basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, giving us the coin Hilbert space $\mathcal{H}_c$. Thus, the combined Hilbert space of the quantum walk is the tensor product of the position and the coin Hilbert space $\mathcal{H}_p \otimes \mathcal{H}_c$. The evolution operator is unitary and is composed of two independent operators, namely, the coin operator ($\hat{C}$) and the Shift operator ($\hat{S}$). The coin operator is a rotation operator acting on the coin, thus "tossing" the coin into superposition. During this coin operation, the position state of the walker remains unaltered. Hence, the coin-toss operation looks like $\mathbb{I}_p \otimes \hat{C}$ where $\hat{C}$ is unitary. An example of a coin operator is the Hadamard matrix $H = 1/\sqrt{2}\,[|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|]$. The Hadamard operator changes the $|\uparrow\rangle$ and $|\downarrow\rangle$ into equal superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ with different phases. The shift operator changes the walker's position depending on the coin state. For a one-dimensional walk, the shift operator can be stated as

$$\hat{S} = \sum_{n=-\infty}^{\infty} |n+1\rangle\langle n| \otimes |\uparrow\rangle\langle\uparrow| + |n-1\rangle\langle n| \otimes |\downarrow\rangle\langle\downarrow| . \tag{1.1}$$

At each time step, the unitary operator $\hat{U} = \hat{S}(\mathbb{I} \otimes \hat{C})$ evolves the walk. So far, the evolution is deterministic, and therefore, the wavefunction at time-steps t can be calculated as

$$|\psi(t)\rangle = \hat{U}^t |\psi(0)\rangle , \tag{1.2}$$

thus, there is no randomness involved in the evolution. However, at any given time step, the wavefunction of the walk is de-localized over several position states. Hence, upon measurement, the detection of the walk at a particular position has a probability associated with it. The probability distribution over the position space is given by

$$p(n,t) = \sum_{n=-\infty}^{\infty} |\langle\uparrow n|\psi(t)\rangle|^2 + |\langle\downarrow n|\psi(t)\rangle|^2. \tag{1.3}$$

Thus, the randomness in quantum walks arises from the measurements and not from its evolution. Choosing an initial state $|\psi(0)\rangle$, one can calculate the probability distribution in position space using Eq. (1.3) for a particular time step t. In figure 1.1, we calculate the position probability distribution of a quantum walk on a 1-D infinite line after 100 time-steps with initial state,

$$|\psi(0)\rangle = |0\rangle \otimes \frac{|\uparrow\rangle + i\,|\downarrow\rangle}{\sqrt{2}}. \tag{1.4}$$

The distribution of quantum walk is symmetric but differs greatly compared to a classical random walk distribution, as shown in fig. 1.1. Another distinguishing fact is the standard deviation for a classical walk distribution grows as $\sqrt{t}$ with time-steps t in the case of the quantum walk; it grows

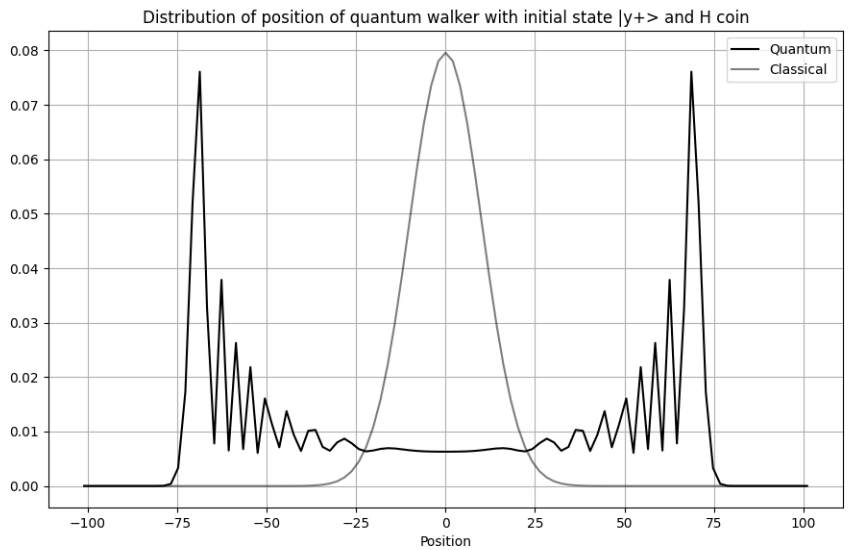


Figure 1.1: Probability distribution of quantum walks with the initial state as Eq. (1.4) and Hadamard coin compared to a random walk unbiased coin starting at the origin.

linearly with t [11] [18]. Note that, unlike classical random walks, the probability distribution depends on the initial coin state of the walk. In the above example with Hadarmard coin, the state Eq. (1.4) gives a symmetric distribution as seen in Fig. 1.1. For the coin states $|0\rangle\,|\uparrow\rangle$ and $|0\rangle\,|\downarrow\rangle$, the resulting probability distribution is asymmetric due to interference. The coin operator also affects the evolution of a quantum walk. Similar to a biased coin in the classical walk, which has a higher probability for one of its outcomes, a biased coin operator is such that the action of the operator on the basis states $\{|\uparrow\rangle\,, |\downarrow\rangle\}$ result into states with unequal amplitudes.

1.2 Continuous-time quantum walks

Another formalism for quantum walks on graphs is the continuous-time quantum walks[7],[15]. As the name suggests, the unitary operator governing the evolution depends on the time t of the evolution. Defining a unitary operator as a function of time is a regular practice in quantum mechanics. The generator for this evolution is the Hamiltonian H, and the corresponding unitary is defined as $U(t) = \exp(-iHt)$. This unitary operator evolves a given initial state $|\psi(0)\rangle$ upto time t as $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$. Such a generator should be unique to each graph and encapsulate the graph structure within its definition. The Hamiltonian is defined using the graph's adjacency matrix. On a graph $G(V, E)$ where V is the vertex set and E is the edge set, an adjacency matrix is defined as

$$A_{ij} = \begin{cases} 1 & \text{if, } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

The adjacency matrix describes the graph in a $|V| \times |V|$ matrix. E.g., the adjacency matrix for a cyclic graph with 6 vertices is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{1.6}$$

Using A as our Hamiltonian provides a natural way of generating unitary evolution between adjacent vertices. One defines the Hamiltonian as $H = \gamma A$ where γ controls the dispersion of the walk. An initial state $|\psi(0)\rangle$ is defined on the graph. After a time t, the state evolves as $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$ and the probability to detect at vertex v at time t is given by $p_v(t) = |\langle v|\psi(t)\rangle|^2$. Many spatial search problems are tackled using continuous-time quantum walk [8],[26].

1.3 Measurement-induced quantum walks

Measurements play a major role in quantum information processing tasks to extract information from the state. These measurement outcomes are probabilistic, and after measurement, the state collapses into the eigenstate corresponding to the measurement outcome. Performing repeated measurements in between the unitary evolution forces the state to collapse and has shown improved success probability in regular graphs [19]. Hence, the overall evolution of the quantum walk is non-unitary. In a typical quantum walk evolution, measurements help determine the arrival of the quantum walk to a desired target state. Typically, the quantum walk is initialized to a particular state (e.g., a localized vertex state). Repeated detection attempts are carried out throughout the evolution process to detect the walk in a target state (e.g., a different localized vertex state). Using these measurement outcomes, one can calculate statistics such as the detection probability of a state and the first detected arrival time for a particular state [24],[17],[5]. Both of these quantities are dependent on the measurement protocol. Note that measurement-induced quantum walks involve periodic strong measurements to detect the quantum particle, which differs from the time-of-arrival operator [9] approach and other quantum search algorithms [22].

1.3.1 First detection of state under stroboscopic measurements

We consider a stroboscopic measurement protocol in which the quantum walk is evolved for a fixed time interval τ between strong projective measurements. Considering the state to be measured is $|\psi_d\rangle$, a projector can be constructed which projects any wavefunction into $|\psi_d\rangle$ upon successful detection. This projector is

$$D = |\psi_d\rangle \langle\psi_d| \, . \tag{1.7}$$

Such a detection has a probability associated with it. Similarly, we can fail to detect the walk in the desired state, leading to the collapse of the state into a subspace orthogonal to $|\psi_d\rangle$. This collapse can be modeled by the projector $D_p = \mathbb{I} - D$, which projects the onto the subspace orthogonal to $|\psi_d\rangle$. Repeated measurements are performed on the quantum walk evolving as per stroboscopic measurement protocol until the state $|\psi_d\rangle$ is successfully detected. This process can be represented in a diagram as shown in fig. 1.2. The unnormalized state after $(n-1)$ unsuccessful measurements

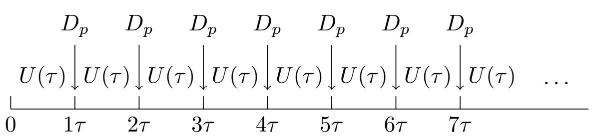


Figure 1.2: Stroboscopic protocol

can be written as

$$|\psi(n\tau)\rangle \equiv |\psi(n)\rangle = U(\tau)[(\mathbb{I} - D)U(\tau)]^{n-1} |\psi(0)\rangle \, . \tag{1.8}$$

The first detection probability of $|\psi_d\rangle$ at the n-th measurement, F_n [13],[14] is defined as,

$$\begin{aligned} F_n &= \langle\psi(n)| D |\psi(n)\rangle \, , && (1.9) \\ &= | \langle\psi_d| U(\tau) \, [(\mathbb{I} - D)U(\tau)]^{n-1} |\psi(0)\rangle |^2, && (1.10) \end{aligned}$$

with $|\psi(0)\rangle$ as the initial state and $U(\tau)$ being the unitary operator. The probability of detecting the walk eventually can be calculated by adding all the F_n-. This is called the detection probability, given by

$$P_{det} = \sum_{i=1}^{\infty} F_n. \tag{1.11}$$

The survival probability of the quantum walk at n-th measurement is the probability of the walk being un-detected for the first n measurements. This is calculated using the survival operator

$\mathbb{S} = (\mathbb{I} - D)U(\tau)$ as follows

$$S_n = ||\mathbb{S}^n |\psi(0)\rangle ||^2 = 1 - \sum_{i=1}^{n} F_n \tag{1.12}$$

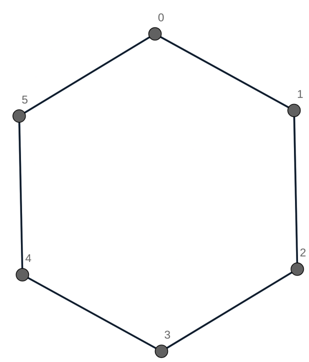
The total survival probability P_{sur} is the probability of a quantum walk to evade the measurement process,

$$P_{\text{sur}} = \lim_{n\to\infty} S_n = 1 - P_{det} \tag{1.13}$$

Another quantity of importance is the average number of measurements, which is defined as

$$\overline{n} = \frac{1}{P_{det}} \sum_{i=1}^{\infty} nF_n. \tag{1.14}$$

To further analyze the first detection probability for quantum walks, let us take a ring with six vertices (L=6) given in Fig. 1.3.1. The Hamiltonian for this system is given by



$$H = -\gamma \sum_{i=0}^{L} |i\rangle \langle i+1| + |i+1\rangle \langle i| \,. \tag{1.15}$$

One can calculate F_n using the formula given in Eq(1.10). Generally, we expect F_n to decrease with increasing n as seen in Fig. 1.3. An approximate value of P_{det} and $\overline{n}$ can be calculated by truncating the sum in Eq. (1.11) and (1.14) up to some large N as shown in Fig. 1.4. P_{det} for the state $|\psi_d\rangle = |1\rangle$ and $|\psi_{in}\rangle = |0\rangle$ is independent of the underlying time of evolution τ and equal to 1/2. As the system is finite, we expect the quantum walk to be detected at any vertex as long as we measure repeatedly. However, $P_{det} = 1/2$ implies we, in an ensemble of systems prepared in $|0\rangle$ and measured using the stroboscopic protocol as seen in Fig. 1.2, only half of the

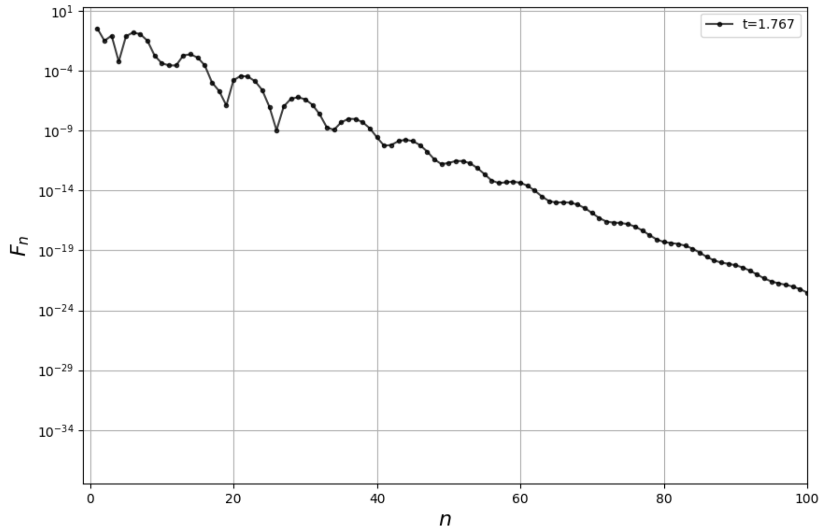


Figure 1.3: Plot of F_n vs. n for quantum walk on on the ring $L = 6$ with Hamiltonian as Eq. (1.15). Stroboscopic measurement protocol was followed with $\tau = 1.767$, and projective measurements were done to detect the state $|1\rangle$. The initial state of the walk was set to $|0\rangle$.

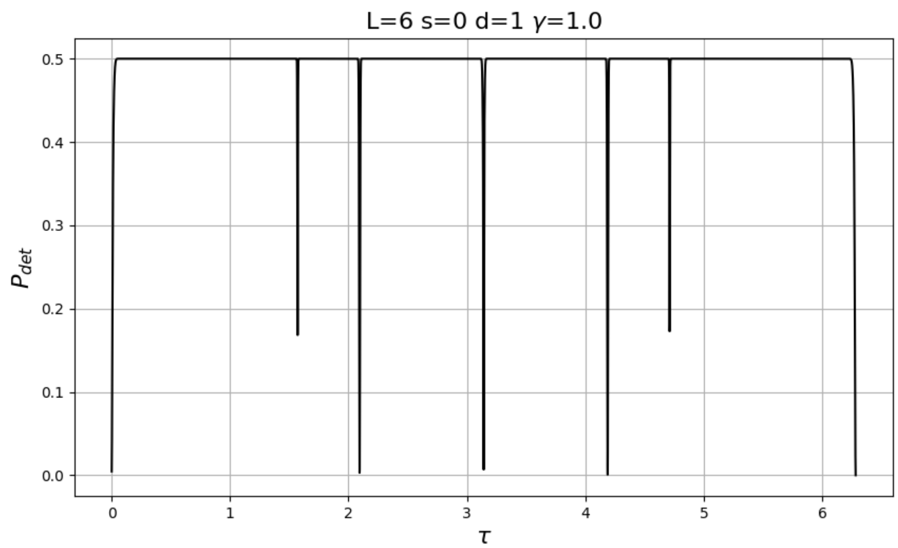
ensemble will successfully detect the particle in state $|1\rangle$. Another important feature is that P_{det} drops discontinuously to a very low value and, in some cases, is zero at certain exceptional τ. At some of these exceptional points, e.g., $\tau = 0, 2\pi$, $P_{det} = 0$, there is a zero probability of the walk being present at the detected state. But, at time $\tau = \pi$, the initial state evolves by a global phase as $U(\pi) = \exp(-iH\pi) = -\mathbb{I}$, thus the probability of detection is again zero.
At these exceptional points, we also observe divergences in $\overline{n}$ due to $P_{det} \to 0$ as $\tau \to \tau_c$. These discontinuities in P_{det} and divergences in $\overline{n}$ occur at resonant times, where the unitary operator used for evolution is degenerate. Considering $\hbar = 1$, if two eigenvalues of $e^{-iE_i\tau}$ and $e^{-iE_j\tau}$ are degenerate where E_i and E_j are eigenvalues of the Hamiltonian and $E_i \neq E_j$ then,

$$|E_i - E_j|\gamma\tau_c = 2n\pi. \tag{1.16}$$

Using Eq. (1.16), we can find these exceptional times for any system with Hamiltonian H

1.3.2 Dividing the Energy spectrum into Dark and Bright subspace

The stroboscopic measurement protocol is characterized by the unitary operator and the projective measurement D defined by the detection state $|\psi_d\rangle$. Starting from certain initial states, it is possible



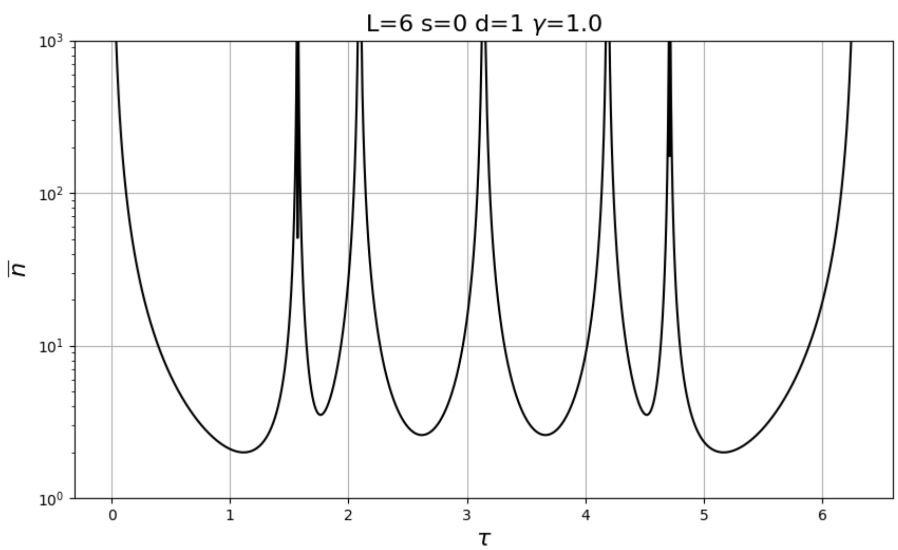


Figure 1.4: Approximate P_{det} and $\overline{n}$ calculated by truncating the sum up to $N = 10000$. The quantum walk is defined on a cyclic graph with $L = 6$, initial state $|s\rangle = |0\rangle$ and the detected state $|d\rangle = |1\rangle$.

to obtain $F_n = 0 \; \forall \; n \implies P_{det} = 0$, which implies, depending on the initial state, $|\psi_d\rangle$ is never detected. Such states are called dark states. On the contrary, there may be certain initial states such that $P_{det} = 1$, i.e., $|\psi_d\rangle$ can always be detected during the measurement protocol. These states are called the bright states. A stationary dark state is one that remains invariant under the action of $U(\tau)$ and detector D. Such a dark state must be an eigenvector of the unitary. The eigenvectors of the corresponding unitary operator have eigenvalues of the form $e^{-i\tau E_k/\hbar}$ where E_k is the eigenvalue of Hamiltonian H. Due to symmetries in the graph, the eigenvalue E_k may degenerate with degeneracy g_k. Labelling the degenerate eigenstates as $|E_{k,m}\rangle$, one can write the Hamiltonian as

$$H = \sum_k \sum_{m=1}^{g_k} E_k \, |E_{k,m}\rangle \langle E_{k,m}| \, . \tag{1.17}$$

The eigenvectors of $U(\tau)$ are the same as eigenvectors of H except with an eigenvalue of $e^{\frac{-iE_k\tau}{\hbar}}$. However, the degeneracy of $e^{\frac{-iE_k\tau}{\hbar}}$ also depends on τ. As far as full spectrum of U (including degenerate subspaces) matches with that for H, $U(\tau)$ can be written as

$$U(\tau) = \sum_k \sum_{m=1}^{g_k} e^{-i\tau E_k/\hbar} \, |E_{k,m}\rangle \langle E_{k,m}| \, , \tag{1.18}$$

which is diagonal. Note that at exceptional τs, defined by Eq. (1.16), the degeneracy of $U(\tau)$ increases compared to H. Consequently, Eq. (1.18) is no more diagonal. Apart from these exceptional τs, we can classify eigenstates of $U(\tau)$ into two categories depending on the detector D.

Consider the initial state to be one of the eigenstates of $U(\tau)$, $|\psi_{in}\rangle = |E_k\rangle$. For $n = 1$, the first detection probability using Eq. (1.10) is $F_1 = |\langle\psi_d|E_k\rangle|^2$, which depends entirely on the overlap $\langle\psi_d|E_k\rangle$. If this overlap is zero, no detection occurs on the measurement $n = 1$. In fact, upon measurement, the wavefunction $|E_k\rangle$ remains unchanged as it exists entirely in the orthogonal space of $|\psi_d\rangle$. Hence, for such an initial state, $F_n = 0$ and then eventually $P_{det} = 0$ and $\overline{n} \to \infty$. Therefore, the initial state is a dark state and an eigenstate of the unitary operator. Hence, a dark energy eigenstate $|E_k\rangle$ satisfies $D\,|E_k\rangle = 0$. Due to this,

$$\begin{aligned} \mathbb{S}\,|E_k\rangle = (\mathbb{I} - D)U(\tau)\,|E_k\rangle &= e^{\frac{-iE_k\tau}{\hbar}} \left(|E_k\rangle - \langle\psi_d|E_k\rangle\,|d_i\rangle\right) \\ &= e^{\frac{-iE_k\tau}{\hbar}}\,|E_k\rangle \, . \end{aligned} \tag{1.19}$$

Hence, the state $|E_k\rangle$ is invariant under the action of $\mathbb{S}$. We define different types of dark levels as follows:

Completely dark levels : For an energy E_k with degeneracy g_k, if $D\,|E_{k,m}\rangle = 0$ for all degenerate eigenstates of E_k, the energy level E_k is completely dark.

Partially dark levels : For an energy E_k such that $D\,|E_{k,m}\rangle \neq 0$ for degenerate states $|E_{k,m}\rangle$, we construct the projector corresponding to the energy level E_k as

$$\mathbb{E}_k = \sum_{m=1}^{g_k} |E_{k,m}\rangle \langle E_{k,m}| \,. \tag{1.20}$$

Using this projector, we can construct a state $|\eta\rangle$ as

$$|\eta\rangle = \frac{\mathbb{E}_k\,|\psi_d\rangle}{\sqrt{\langle\psi_d|\,\mathbb{E}_k\,|\psi_d\rangle}}, \tag{1.21}$$

which is an eigenstate of $U(\tau)$. Ref[23] has shown that this is a bright state. Any orthogonal state $|\zeta\rangle$ to $|\eta\rangle$ will have zero overlap with $|\psi_d\rangle$ as

$$\langle\zeta|\psi_d\rangle = \langle\zeta|\,\mathbb{E}_k\,|\psi_d\rangle \propto \langle\zeta|\eta\rangle = 0. \tag{1.22}$$

Hence, $|\zeta\rangle$ is a dark state. As the degenerate energy subspace is g_k-dimensional, there exist $g_k - 1$ dark states denoted by $|\zeta_{k,m}\rangle$ for $D = |\psi_d\rangle\langle\psi_d|$ corresponding to energy E_k. Thus, the degenerate energy E_k with $\mathbb{E}_k\,|\psi_d\rangle \neq 0$ is partially dark as it contains $g_k - 1$ dark states. In this degenerate energy subspace, any dark state $|\zeta_{k,j}\rangle$ is constructed as

$$|\zeta_{k,j}\rangle = N_j \begin{vmatrix} |E_{k,1}\rangle & |E_{k,2}\rangle & \dots & |E_{k,m+1}\rangle \\ \langle\psi_d|E_{k,1}\rangle & \langle\psi_d|E_{k,2}\rangle & \dots & \langle\psi_d|E_{k,m+1}\rangle \\ \langle\zeta_{k,1}|E_{k,1}\rangle & \langle\zeta_{k,1}|E_{k,2}\rangle & \dots & \langle\zeta_{k,1}|E_{k,m+1}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\zeta_{k,j-1}|E_{k,1}\rangle & \langle\zeta_{k,j-1}|E_{k,2}\rangle & \dots & \langle\zeta_{k,j-1}|E_{k,m+1}\rangle \end{vmatrix}. \tag{1.23}$$

From Eq. (1.23), it can be easily shown that $|\zeta_{k,m}\rangle$ is orthogonal to $|\zeta_{k,m-1}\rangle$. Along with the one bright state found in Eq. (1.21) span an orthonormal set of the basis for the degenerate energy subspace.

1.3.3 Computation of P_{det} from the perspective of dark and bright states

Using the definitions of dark and bright states given in Eq. (1.19) and Eq. (1.21), we can construct the corresponding dark and bright subspaces. The projectors of these subspaces are

$$\mathbb{P}_{\mathcal{H}_\eta} = \sum_k |\eta_k\rangle \langle\eta_k| , \tag{1.24}$$

$$\mathbb{P}_{\mathcal{H}_\zeta} = \sum_k |\zeta_k\rangle \langle\zeta_k| . \tag{1.25}$$

These bright/dark states split the Hilbert space $\mathcal{H}$ into two orthogonal subspaces $\mathcal{H}_\zeta$ and $\mathcal{H}_\eta$. An initial state $|\psi_{in}\rangle$ can be split as

$$|\psi_{in}\rangle = \mathbb{P}_{\mathcal{H}_\zeta} |\psi_{in}\rangle + \mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle . \tag{1.26}$$

Any dark state $|\zeta\rangle$ is invariant under the action of $\mathbb{S}$; therefore, the subspace $\mathcal{H}_\zeta$ is invariant under the action of $\mathbb{S}$. Therefore, one can infer that $\mathbb{P}_{\mathcal{H}_\zeta} |\psi_{in}\rangle$ is invariant under the action of $\mathbb{S}$. The overlap $||\mathbb{S}^n |\psi_{in}\rangle ||^2$ is the probability that $|\psi_{in}\rangle$ survives first n measurements. As $n \to \infty$ this is the survival probability of $|\psi_{in}\rangle$. As $\mathbb{P}_{\mathcal{H}_\zeta} |\psi_{in}\rangle$ stays invariant under $\mathbb{S}$, the survival probability will be given by

$$\begin{aligned} P_{\text{sur}}(\psi_{in}) &= \lim_{n\to\infty} ||\mathbb{S}^n |\psi_{in}\rangle ||^2 \\ &= ||\mathbb{P}_{\mathcal{H}_\zeta} |\psi_{in}\rangle ||^2 \\ &= 1 - P_{det}(\psi_{in}). \end{aligned} \tag{1.27}$$

Hence the only subspace contributing to first detection statistics is $\mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle$. Therefore the probability of detection for a state $|\psi_{in}\rangle$ is

$$P_{det}(\psi_{in}) = \langle\psi_{in}| \mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle . \tag{1.28}$$

Thus any initial state completely in the bright subspace $|\psi_{in}\rangle = \mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle$ and $\mathbb{P}_{\mathcal{H}_\zeta} |\psi_{in}\rangle = 0$, then

$$P_{\text{sur}}(\psi_{in}) = \langle\psi_{in}| \mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle = \langle\psi_{in}|\psi_{in}\rangle = 1. \tag{1.29}$$

On the other hand, a state completely in dark subspace $\mathbb{P}_{\mathcal{H}_\zeta} |\psi_{in}\rangle = |\psi_{in}\rangle$ then

$$P_{\text{sur}}(\psi_{in}) = \langle\psi_{in}| \mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle = 0. \tag{1.30}$$

Thus, we can give an explicit formula for P_{det} from Eq. (1.28) as also seen in [23]

$$P_{det}(\psi_{in}) = \sum_k{}' \frac{|\langle\psi_d| \mathbb{E}_k |\psi_{in}\rangle|^2}{\langle\psi_d| \mathbb{E}_k |\psi_d\rangle}, \tag{1.31}$$

where the sum is over only energy levels such that $\mathbb{E}_k |\psi_d\rangle \neq 0$.
In this thesis, we extend the formalism of the first detection of quantum walk from detecting a single state to detecting multiple states, i.e., the detection of subspace. Chapter 2 lays the formalism of the first detection of a subspace and gives formal definitions of dark and bright states for subspace detection. In Chapter 3, we calculate the subspace detection probability for the ring system and ring system with the addition of next-nearest connections and discuss the results. Finally, we conclude the thesis in Chapter 4.

Chapter 2

Methods

In all our previous analyses of quantum walks, the measurements were carried out on a single state $|\psi_d\rangle$. This split the Hilbert space into two orthogonal subspaces, bright and dark. The state in the dark subspace would remain invariant under the action of the survival operator. On the contrary, a bright state changes into a superposition of bright states. These (dark/bright) subspaces are determined by the detector D and are invariant under the survival operator $\mathbb{S}$ [23]. In this chapter, we will further explore the first detection statistics in the case of detection on a subspace.

2.1 First Detection of a subspace

In this section, we formulate the problem of detecting quantum walk in a subspace. Any subspace of the total Hilbert space can be defined by defining a projector of the Hilbert space. The subspace to be detected in the evolution can be similarly defined by a projector on the subspace D. The dimension of a subspace is equal to the number of vectors in the basis set of the subspace. Hence, for a subspace of dimension $\tilde{r}$, the projector D can be written as

$$D = \sum_{i=1}^{\tilde{r}} |d_i\rangle \langle d_i| \,, \tag{2.1}$$

where $\{|d_i\rangle\}_{i=1}^{\tilde{r}}$ are states such that $\langle d_i|d_j\rangle = \delta_{i,j}$ i.e. $\{|d_i\rangle\}_{i=1}^{\tilde{r}}$ is a basis for the subspace to be detected. In principle, measurement of the subspace given by Eq. (2.1) can be thought of as measuring $\tilde{r}$ outcomes of some observable corresponding eigenstates $\{|d_i\rangle\}_{i=1}^{\tilde{r}}$. In this thesis, we consider the set $\{|d_i\rangle\}_{i=1}^{\tilde{r}}$ as a set of localized positions on the graph, and the projector in Eq. (2.1) projects a state onto a subspace spanned by these position states. A "click" from the measurement device implies successfully detecting the walk in the subspace as we perform measurements on the walk. This implies the state of the quantum walk is either at a particular position in the subspace or is a linear combination of positions in this subspace. The post-measured state is localized on a few positions only. A "no-click" event describes the scenario of the device falling to detect the quantum walk. Therefore, in the case of a "no-click" event, the post-measured state is orthogonal to the detection subspace. This "no-click" event can be modeled as the $(\mathbb{I} - D)$ action on the state. As we detect multiple positions, we expect the first detection probability at n-th measurement to increase compared to single detection, and consequently, an increase is expected in the detection probability. Using the stroboscopic measurement protocol, the quantum walk state after $(n-1)$ unsuccessful measurement attempts is given by

$$|\psi(n-1)\rangle = U(\tau)[(\mathbb{I} - D)U(\tau)]^{n-1} |\psi_{in}\rangle \,. \tag{2.2}$$

At the n-th measurement, we successfully detect the quantum walk. Therefore, F_n is calculated as

$$F_n = ||DU(\tau)[(\mathbb{I} - D)U(\tau)]^{n-1} |\psi_{in}\rangle ||^2 = \langle\psi(n-1)| D |\psi(n-1)\rangle \,, \tag{2.3}$$

where D is the subspace projector as defined in Eq. (2.1). Using this definition of F_n, one can calculate the subspace detection probability and average measurement as per Eq. (1.11) and Eq. (1.14). One can approximate these first detection statistics by calculating their truncated expressions for large N measurements. By doing so, one can only obtain approximated values, and it is numerically intensive to calculate detection probability for a range of evolution time τ. A faster method to calculate first detection statistics for a single state was developed in Ref. [12] and [25], which uses matrix multiplication of $L^2 \times L^2$ matrices. The proposition below extends this method from single-state to subspace detection.

Proposition 1. *In a discrete bounded lattice of L sites, the probability of the first successful detection after an infinite number of measurements (with periodicity τ) in a subspace of dimension $\tilde{r}$ can be written as*

$$P_{det} = \sum_{i=1}^{\tilde{r}} \mathrm{Tr}\left[\mathcal{L}^{-1}\bar{U}\bar{S}\bar{W}\bar{T}_i^*\right]_{L^2\times L^2} \,. \tag{2.4}$$

where the $L \times L$ dimensional matrices are,

$$U_{ij} = \delta_{ij} e^{-iE_i\tau}, \tag{2.5}$$

$$S_{kp} = \delta_{kp} \langle E_p|\phi(0)\rangle, \tag{2.6}$$

$$W_{ij} = 1, \tag{2.7}$$

$$(T_j)_{kp} = \delta_{kp} \langle E_p|d_j\rangle, \tag{2.8}$$

$$C = \mathbb{I} - \sum_{j=1}^{\tilde{r}} T_j^* W T_j, \tag{2.9}$$

with any operator $\bar{O} \equiv O^* \otimes O$ and $\mathcal{L} = \mathbb{I} - \bar{U}\bar{C}$.

Proof. For any operator X, $\langle\psi| X |\phi\rangle$ in energy representation is

$$\begin{aligned} \langle\psi| X |\phi\rangle &= \sum_{i,j} \psi_i^* X_{ij} \phi_j, \\ &= \text{Tr}\left[XSWT^*\right], \end{aligned} \tag{2.10}$$

where $\psi_i = \langle E_i|\psi\rangle$, $\phi_i = \langle E_i|\phi\rangle$. The matrices inside trace are

$$\begin{aligned} W_{ij} &= 1, \\ S &= \text{diag}(\phi_1, \phi_2, ..., \phi_n), \\ T &= \text{diag}(\psi_1, \psi_2, ..., \psi_n). \end{aligned} \tag{2.11}$$

The unitary operator is diagonal in this representation i.e.,

$$U = \text{diag}(e^{-iE_1\tau}, e^{-iE_1\tau}, e^{-iE_2\tau}, ..., e^{-iE_n\tau}). \tag{2.12}$$

This first detection probability using Eqs. (2.3,2.2, 2.1) is for $n = 1$ is

$$\begin{aligned} F_1 &= \langle\psi(1)| D |\psi(1)\rangle \\ &= \sum_{i=1}^{r} \langle\psi(0)| U^\dagger |d_i\rangle \langle d_i| U |\psi(0)\rangle \\ &= \sum_{i=1}^{r} \text{Tr}\left[U^* T_i W S^*\right] \text{Tr}\left[USWT_i^*\right] \\ &= \sum_{i=1}^{r} \text{Tr}\left[USWT_i^*\right]^* \text{Tr}\left[USWT_i^*\right], \end{aligned} \tag{2.13}$$

where, $S_{ij} = \delta_{ij} \langle E_j|\psi(0)\rangle$, $(T_i)_{pq} = \delta_{pq} \langle E_p|d_i\rangle$.
Going from the third to the fourth line, we use the cyclic property of trace along with the fact that U, S, T_i commute as they are diagonal matrices.
Similarly, for $n = 2$,

$$\begin{aligned}
F_2 &= \langle\psi(2)| D |\psi(2)\rangle \\
&= \sum_{i=1}^{r} \langle\psi(0)| U^\dagger(\mathbb{I} - D)U^\dagger |d_i\rangle \langle d_i| U(\mathbb{I} - D)U |\psi(0)\rangle \\
&= \sum_{i=1}^{r} \mathrm{Tr}\left[U\left(\mathbb{I} - \sum_{j=1}^{r} T_j^* W T_j\right) USWT_i^*\right]^* \mathrm{Tr}\left[U\left(\mathbb{I} - \sum_{j=1}^{r} T_j^* W T_j\right) USWT_i^*\right] \\
&= \sum_{i=1}^{r} \mathrm{Tr}\left[UCUSWT_i^*\right]^* \mathrm{Tr}\left[UCUSWT_i^*\right],
\end{aligned} \tag{2.14}$$

where $C = \mathbb{I} - \sum_{j=1}^{r} T_j^* W T_j$. Moving forward for $n \geq 2$, F_n is given by,

$$\begin{aligned}
F_n &= \langle\psi(n-1)| D |\psi(n-1)\rangle \\
&= \sum_{i=1}^{r} \mathrm{Tr}\left[(UC)^{n-1}USWT_i^*\right]^* \mathrm{Tr}\left[(UC)^{n-1}USWT_i^*\right].
\end{aligned} \tag{2.15}$$

Using important properties of trace

$$\begin{aligned}
\mathrm{Tr}[A \otimes B] &= \mathrm{Tr}[A]\,\mathrm{Tr}[B], \\
\mathrm{Tr}[A^* \otimes A] &= \mathrm{Tr}[A^*]\,\mathrm{Tr}[A] \\
&= \mathrm{Tr}[A]^*\,\mathrm{Tr}[A].
\end{aligned} \tag{2.16}$$

and defining $\bar{\mathbb{O}} = \mathbb{O}^* \otimes \mathbb{O}$, we can rewrite Eq. (2.15) as

$$F_n = \sum_{i=1}^{\tilde{r}} \mathrm{Tr}\left[(\bar{U}\bar{C})^{n-1}\bar{U}\bar{S}\bar{W}\bar{T}_i^*\right]. \tag{2.17}$$

To derive an expression for P_{det} and $\overline{n}$, we first write $\mathcal{R} = \bar{U}\bar{C}$, and obtaining

$$\begin{aligned} P_{det} &= \sum_{n=1}^{\infty} F_n \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\tilde{r}} \mathrm{Tr}\left[(\mathcal{R})^{n-1}\bar{U}\bar{S}\bar{W}\bar{T}_i^*\right] \\ &= \sum_{i=1}^{\tilde{r}} \mathrm{Tr}\left[\mathcal{L}^{-1}\bar{U}\bar{S}\bar{W}\bar{T}_i^*\right], \end{aligned} \tag{2.18}$$

where we have summed the infinite geometric series and defined $\mathcal{L} = \mathbb{I} - \mathcal{R}$. The average number $\overline{n}$ of measurements conditioned on successful detection is

$$\begin{aligned} \overline{n} &= \frac{1}{P_{det}} \sum_{i=1}^{\infty} nF_n \\ &= \frac{1}{P_{det}} \sum_{n=1}^{\infty}\sum_{i=1}^{r} n\, \mathrm{Tr}\left[(\mathcal{R})^{n-1}\bar{U}\bar{S}\bar{W}\bar{T}_i^*\right] \\ &= \frac{1}{P_{det}} \sum_{i=1}^{r} \mathrm{Tr}\left[\mathcal{L}^{-2}\bar{U}\bar{S}\bar{W}\bar{T}_i^*\right]. \end{aligned} \tag{2.19}$$

□

Notice that we take an inverse of $\mathcal{L}$ in our calculations. If $\mathcal{L}$ is singular, we cannot invert $\mathcal{L}$. This is due to the existence of dark states [25]. In such cases, we take a pseudo-inverse of $\mathcal{L}$ and calculate P_{det}.

2.2 Finding Dark/Bright states in subspace detection

As previously seen in section (1.3.2), the dark and bright states arise in quantum walks due to the symmetries of the unitary operator and the detector. The dark/bright energy eigenstates help to determine the detection probability of a state. Similar to the detection of single states, we can develop the concept of dark and bright subspace in the case of subspace detection.

2.2.1 Dark States

The definition for dark state $|\zeta\rangle$ in given in Eq. (1.19) can be extended to the case of subspace

$$\begin{aligned} D\,|\zeta\rangle &= 0, \text{ and} \\ (\mathbb{I}-D)U(\tau)\,|\zeta\rangle &= \exp(-iE_k\tau)\,|\zeta\rangle\,. \end{aligned} \tag{2.20}$$

In Eq. (2.20), the operator D is the projector on the r-dimensional subspace as given in 2.1. Eq. (2.20) implies $\langle d_i|\zeta\rangle = 0 \;\forall\; i = 1, 2, ..., \tilde{r}$.

A non-degenerate energy E_k with eigenvector $|E_k\rangle$ is, therefore, dark if and only if $\langle d_i|E_k\rangle = 0\;\forall$ $i = 1, 2, ..., \tilde{r}$. For a degenerate energy E_k, let $\mathbb{E}_k$ be the projector corresponding to energy level E_k for the degenerate case. Hence if, E_k is degenerate with g_k degeneracy, then its projector can be written as

$$\mathbb{E}_k = \sum_{m=1}^{g_k} |E_{k,m}\rangle\,\langle E_{k,m}|\,. \tag{2.21}$$

A dark state corresponding to this energy E_k is the linear combination of eigenvectors within the energy subspace

$$|\zeta_k\rangle = \sum_{m=1}^{g_k} \alpha_m\,|E_{k,m}\rangle\,. \tag{2.22}$$

Using Eq. (2.20), we get

$$\begin{aligned} D\,|\zeta_k\rangle = 0 \Longrightarrow \\ \sum_{i=1,m=1}^{\tilde{r},g_k} \alpha_m\,|d_i\rangle\,\langle d_i|E_{k,m}\rangle = 0, \\ \mathcal{A}_k\,|\tilde{\alpha}\rangle = 0. \end{aligned} \tag{2.23}$$

The existence of $|\zeta_k\rangle$ is dependent on the overlaps $\langle d_i|E_{k,m}\rangle$. Let us analyse the situation by considering different cases.

Case 1: $\langle d_i|E_{k,m}\rangle = 0\;\forall\; i = 1, 2, ..., \tilde{r}$ and $m = 1, 2, ..., g_k$.

In this case, the entire energy subspace is dark, and the dark states are the eigenstates $|\zeta_{k,m}\rangle = |E_{k,m}\rangle$. Hence, the entire energy level does not contribute in P_{det}.

Case 2: In this case we assume $\langle d_i|E_{k,m}\rangle = 0$ for some i or m but not all of them. From Eq.

(2.23), finding a dark state is the same as finding non-trivial solutions to $\mathcal{A}_k \left|\tilde{\alpha}\right\rangle = 0$.

$$\begin{bmatrix} \langle d_1|E_{k,1}\rangle & \langle d_1|E_{k,2}\rangle & \dots & \langle d_1|E_{k,g_k}\rangle \\ \langle d_2|E_{k,1}\rangle & \langle d_2|E_{k,2}\rangle & \dots & \langle d_2|E_{k,g_k}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle d_{\tilde{r}}|E_{k,1}\rangle & \langle d_{\tilde{r}}|E_{k,2}\rangle & \dots & \langle d_{\tilde{r}}|E_{k,g_k}\rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{g_k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.24}$$

where

$$\mathcal{A}_k = \begin{bmatrix} \langle d_1|E_{k,1}\rangle & \langle d_1|E_{k,2}\rangle & \dots & \langle d_1|E_{k,g_k}\rangle \\ \langle d_2|E_{k,1}\rangle & \langle d_2|E_{k,2}\rangle & \dots & \langle d_2|E_{k,g_k}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle d_{\tilde{r}}|E_{k,1}\rangle & \langle d_{\tilde{r}}|E_{k,2}\rangle & \dots & \langle d_{\tilde{r}}|E_{k,g_k}\rangle \end{bmatrix} \tag{2.25}$$

where $\left|\tilde{\alpha}\right\rangle = (\alpha_1, \alpha_2, \dots, \alpha_{g_k})^T$ is in the basis $\{\left|E_{k,m}\right\rangle\}_{m=1}^{g_k}$. Notice if a column of $\mathcal{A}_k$ is zero, then the corresponding eigenstate is dark by definition. In such cases, to find the non-trivial dark states, we should solve Eq. (2.25) by deleting the column with zero entries. These non-trivial solutions to Eq. (2.25) are $\left|\tilde{\alpha}\right\rangle \in \mathcal{N}_{\mathcal{A}_k}$, the nullspace of $\mathcal{A}_k$. Therefore, the existence of dark states in an energy subspace depends entirely on the existence of a non-trivial nullspace of $\mathcal{A}_k$. If there are no dark states, $\mathcal{N}_{\mathcal{A}_k}$ is the vector $\left|\tilde{\alpha}\right\rangle = (0, 0, 0, ..., 0)^T$. The dimension of the nullspace $\dim(\mathcal{N}_{\mathcal{A}_k})$ is equal to the number of orthonormal dark states present in the energy subspace. These dark states are energy eigenstates of the Hamiltonian with eigenvalue E_k and the unitary operator with the corresponding eigenvalue $e^{-iE_k\tau/\hbar}$. To find the non-trivial solutions of $\mathcal{A}_k \left|\tilde{\alpha}\right\rangle = 0$, we can convert $\mathcal{A}_k$ into its reduced echelon form,

$$REF(\mathcal{A}_k) = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,l} & \dots & a_{1,g_k} \\ 0 & a_{2,2} & \dots & a_{2,l} & \dots & a_{2,g_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \dots & a_{l_k,l_k} & \dots & a_{l_k,g_k} \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}_{\tilde{r}\times g_k}. \tag{2.26}$$

where $a_{i,i} \neq 0$s are the pivots and $l_k = \mathrm{rank}(\mathcal{A}_k)$.

Proposition 2. *Number of dark states in the subspace $\{\left|E_{k,m_k}\right\rangle\}_{m_k=1}^{g_k}$ is equal to the dimension of the null space of $\mathcal{A}_k$, denoted as $\dim(\mathcal{N}_{\mathcal{A}_k}) = g_k - \mathrm{rank}(\mathcal{A}_k) = g_k - l_k$. The basis for $\mathcal{N}_{\mathcal{A}_k}$, i.e,* *the orthonormal dark states are given by*

$$|\zeta_{k,1}\rangle = N_1 \begin{vmatrix} |E_{k,1}\rangle & |E_{k,2}\rangle & \dots & |E_{k,l_k}\rangle & |E_{k,l_k+1}\rangle \\ a_{1,1} & a_{1,2} & \dots & a_{1,l_k} & a_{1,l_k+1} \\ 0 & a_{2,2} & \dots & a_{2,l_k} & a_{2,l_k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{l_k,l_k} & a_{l_k,l_k+1} \end{vmatrix},$$

$$|\zeta_{k,j}\rangle = N_j \begin{vmatrix} |E_{k,1}\rangle & |E_{k,2}\rangle & \dots & |E_{k,l_k}\rangle & |E_{k,l_k+1}\rangle & \cdots & |E_{k,l_k+j}\rangle \\ a_{1,1} & a_{1,2} & \dots & a_{1,l_k} & a_{1,l_k+1} & \cdots & a_{1,l_k+j} \\ 0 & a_{2,1} & \cdots & a_{2,l_k} & a_{2,l_k+1} & \cdots & a_{2,l_k+j} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{l_k,l_k} & a_{l_k,l_k+1} & \cdots & a_{l_k,l_k+j} \\ \langle\zeta_k^1|E_{k,1}\rangle & \langle\zeta_k^1|E_{k,2}\rangle & \cdots & \langle\zeta_k^1|E_{k,l_k}\rangle & \langle\zeta_k^1|E_{k,l_k+1}\rangle & \cdots & \langle\zeta_k^1|E_{k,l_k+j}\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \left\langle\zeta_k^{j-1}\middle|E_{k,1}\right\rangle & \left\langle\zeta_k^{j-1}\middle|E_{k,2}\right\rangle & \dots & \left\langle\zeta_k^{j-1}\middle|E_{k,l_k}\right\rangle & \left\langle\zeta_k^{j-1}\middle|E_{k,l_k+1}\right\rangle & \dots & \left\langle\zeta_k^{j-1}\middle|E_{k,l_k+j}\right\rangle \end{vmatrix}, \tag{2.27}$$

where, $\mathrm{rank}(\mathcal{A}_k) = l_k \leq g_k$

Proof. It is evident from Eq. (2.23) that the existence of dark states is equivalent to finding non-trivial solutions (trivial solution is $\alpha_i = 0$ for $i = 1, 2, \dots, g_k$) of the matrix $\mathcal{A}_k$ which in turn, is linked to the determination of its nullspace, denoted as $\mathcal{N}_{\mathcal{A}_k}$. Therefore, number of dark states in the corresponding energy subspace is equal to the dimension of the nullspace, i.e., $\dim(\mathcal{N}_{\mathcal{A}_k})$. Moreover, from the rank-nullity theorem, we know

$$\begin{aligned} \mathrm{rank}(\mathcal{A}_k) + \dim(\mathcal{N}_{\mathcal{A}k}) &= \text{Number of columns of } \mathcal{A}_k = g_k. \\ \dim(\mathcal{N}_{\mathcal{A}k}) &= g_k - \mathrm{rank}(\mathcal{A}_k). \end{aligned} \tag{2.28}$$

Therefore, the number of dark states is determined by the rank of matrix $\mathcal{A}_k$, which is equal to the number of independent rows or columns of the matrix $\mathcal{A}_k$.

Observe that the rows of $\mathcal{A}_k$ are the projections of $|d_i\rangle$ in the energy subspace E_k. These are unnormalised and can be written as $\mathbb{E}_k |d_i\rangle$, where $\mathbb{E}_k$ is the corresponding energy subspace projector. The set $\{\mathbb{E}_k |d_i\rangle\}_{i=1}^{\tilde{r}}$ may have some linearly dependent vectors, especially when $\tilde{r} \geq g_k$. The $\mathrm{rank}(\mathcal{A}_k)$ is equal to the dimension of rowspace the of $\mathcal{A}_k$, which is equal to the number of linearly independent vectors in the rowspace, i.e. the number of linearly independent vectors in the set

$\{\mathbb{E}_k\,|d_i\rangle\}_{i=1}^{\tilde{r}}$. Considering the REF($\mathcal{A}_k$) as given in Eq. (2.26) where, $\mathrm{rank}(\mathcal{A}_k) = l_k$. The first l_k rows of REF($\mathcal{A}_k$) are linearly independent by definition and hence form the basis of rowspace of $\mathcal{A}_k$. Therefore, any dark state can be defined as a state orthogonal to the basis of rowspace of $\mathcal{A}_k$. A total of $g_k - l_k$. Such orthonormal dark states can be found. These dark states can be constructed iteratively as

$$|\zeta_{k,1}\rangle = N_1 \begin{vmatrix} |E_{k,1}\rangle & |E_{k,2}\rangle & \dots & |E_{k,l_k}\rangle & |E_{k,l_k+1}\rangle \\ a_{1,1} & a_{1,2} & \dots & a_{1,l_k} & a_{1,l_k+1} \\ 0 & a_{2,2} & \dots & a_{2,l_k} & a_{2,l_k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{l_k,l_k} & a_{l_k,l_k+1} \end{vmatrix}, \tag{2.29}$$

where N_1 is the normalization constant. Similarly,

$$|\zeta_{k,j}\rangle = N_j \begin{vmatrix} |E_{k,1}\rangle & |E_{k,2}\rangle & \dots & |E_{k,l_k}\rangle & |E_{k,l_k+1}\rangle & \cdots & |E_{k,l_k+j}\rangle \\ a_{1,1} & a_{1,2} & \dots & a_{1,l_k} & a_{1,l_k+1} & \cdots & a_{1,l_k+j} \\ 0 & a_{2,1} & \cdots & a_{2,l_k} & a_{2,l_k+1} & \cdots & a_{2,l_k+j} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{l_k,l_k} & a_{l_k,l_k+1} & \cdots & a_{l_k,l_k+j} \\ \langle \zeta_k^1 | E_{k,1}\rangle & \langle \zeta_k^1 | E_{k,2}\rangle & \cdots & \langle \zeta_k^1 | E_{k,l_k}\rangle & \langle \zeta_k^1 | E_{k,l_k+1}\rangle & \cdots & \langle \zeta_k^1 | E_{k,l_k+j}\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \zeta_k^{j-1} | E_{k,1}\rangle & \langle \zeta_k^{j-1} | E_{k,2}\rangle & \dots & \langle \zeta_k^{j-1} | E_{k,l_k}\rangle & \langle \zeta_k^{j-1} | E_{k,l_k+1}\rangle & \dots & \langle \zeta_k^{j-1} | E_{k,l_k+j}\rangle \end{vmatrix}. \tag{2.30}$$

Due to such a construction, the inner product of two dark states $\langle \zeta_{k,j} | \zeta_{k,j'}\rangle$ is a determinant with two equal rows which turns out to be zero. Therefore, $\langle \zeta_{k,j} | \zeta_{k,j'}\rangle = 0$ and hence the dark states are orthogonal to each other. Using this method, we can construct all the $g_k - l_k$ dark states in the energy subspace. □

This proposition gives a method to calculate dark states in an energy subspace arising in the case of subspace detection. The method relies on the information of eigenvectors, and their degeneracy requires the calculation of row echelon form of $\mathcal{A}_k$. Converting $\mathcal{A}_k$ into its echelon form, one can identify the non-zero rows of REF($\mathcal{A}_k$) as the basis for rowspace of $\mathcal{A}_k$. This basis set is not orthogonal but can be orthogonalized using the Gram-Schmidt technique. Let this orthogonal basis be $\{|\eta_{k,m}\rangle\}_{m=1}^{l_k}$. Each $|\eta_{k,m}\rangle$ is a linear combination of the projections $\mathbb{E}_k\,|d_i\rangle$. This can be easily proved as the rows of REF($\mathcal{A}_k$) are linear combinations of $\mathbb{E}_k\,|d_i\rangle$ the rows of $\mathcal{A}_k$. Therefore, after the Gram-Schmidt process, $|\eta_{k,m}\rangle$ can still be represented as a linear combination of $\mathbb{E}_k\,|d_i\rangle$. Thus a degenerate energy subspace E_k with basis $\{|E_{k,m_k}\rangle\}_{m_k=1}^{g_k}$ is split into two orthogonal subspaces spanned by $\left\{\{|\zeta_{k,j}\rangle\}_{j=1}^{g_k-l_k}, \{|\eta_{k,j}\rangle\}_{j=1}^{l_k}\right\}$, where $l_k = \mathrm{rank}(\mathcal{A}_k)$.

2.2.2 Bright States

As defined earlier in the introduction, bright states with respect to the detection state are initial states such that the detection probability $P_{det} = 1$. In single-state detection, we observed the division of total Hilbert space into mutually orthogonal dark and bright subspaces through which we can calculate the detection probability as $P_{det}(|\psi\rangle) = \langle\psi| \mathbb{P}_{\mathcal{H}_\eta} |\psi\rangle$ where, $\mathbb{P}_{\mathcal{H}_\eta}$ was the bright subspace projector. In the case of subspace detection, we have found dark states in the previous section and also define a mutually orthonormal set $\{|\eta_{k,j}\rangle\}_{j=1}^{l_k}$ which is orthogonal to the dark states. By construction, $|\eta_{k,m}\rangle$ is a linear combination of the projections $\mathbb{E}_k |d_i\rangle$. We will prove $|\eta_{k,m}\rangle$ to be bright. Let $\mathbb{P}_{\mathcal{H}_\zeta}$ be the dark subspace projector in case of subspace detection. One can construct $\mathbb{P}_{\mathcal{H}_\zeta}$ as,

$$\mathbb{P}_{\mathcal{H}_\zeta} = \sum_k \sum_{m=1}^{g_k-l_k} |\zeta_{k,m}\rangle \langle\zeta_{k,m}| \,. \tag{2.31}$$

For a degenerate energy E_k the state $|\eta_{k,m}\rangle$ lies in the space orthogonal to the dark subspace as

$$\begin{aligned}
\mathbb{P}_{\mathcal{H}_\zeta} |\eta_{k,m}\rangle &= \sum_{k'} \sum_{m'=1}^{g_{k'}-l_{k'}} |\zeta_{k',m'}\rangle \langle\zeta_{k',m'}|\eta_{k,m}\rangle \\
&= \sum_{m'=1}^{g_k-l_k} |\zeta_{k,m'}\rangle \langle\zeta_{k,m'}|\eta_{k,m}\rangle \\
&= \sum_{m'=1}^{g_{k'}-l_{k'}} |\zeta_{k',m'}\rangle \langle\zeta_{k',m'}| \sum_i c_{i,k}\mathbb{E}_k |d_i\rangle \\
&= \sum_{m'=1}^{g_{k'}-l_{k'}} \sum_i |\zeta_{k',m'}\rangle\, c_{i,k} \langle\zeta_{k',m'}| \mathbb{E}_k |d_i\rangle \\
&= \sum_{m'=1}^{g_{k'}-l_{k'}} \sum_i c_{i,k} |\zeta_{k',m'}\rangle \langle\zeta_{k',m'}|d_i\rangle \\
&= 0 \text{ ....By definition of dark states.}
\end{aligned} \tag{2.32}$$

Therefore, the subspace survival probability of $|\eta_{k,m}\rangle$,

$$P_{\text{sur}}(|\eta_{k,m}\rangle) = \langle\eta_{k,m}| \mathbb{P}_{\mathcal{H}_\zeta} |\eta_{k,m}\rangle = 0. \tag{2.33}$$

This implies the subspace detection probability of any state $|\eta_{k,m}\rangle$ is $P_{det}(|\eta_{k,m}\rangle) = 1$ and hence $|\eta_{k,m}\rangle$ is part of the bright subspace. As $|\eta_{k,m}\rangle$ are mutually orthogonal, these can be used to construct the bright state projector,

$$\mathbb{P}_{\mathcal{H}_\eta} = \sum_k \sum_{m=1}^{l_k} |\eta_{k,m}\rangle \langle \eta_{k,m}| \, . \tag{2.34}$$

Thus, the orthonormal set $\{|\eta_{k,j}\rangle\}_{j=1}^{l_k}$ are bright states for the degenerate energy subspace E_k.

2.2.3 Calculation of subspace detection probability

Using the bright states and dark states found in the previous section, we can construct the bright subspace projector and the dark subspace projector as follows,

$$\mathbb{P}_{\mathcal{H}_\zeta} = \sum_k \sum_{m=1}^{g_k - l_k} |\zeta_{k,m}\rangle \langle \zeta_{k,m}| \, , \tag{2.35}$$

$$\mathbb{P}_{\mathcal{H}_\eta} = \sum_k \sum_{m=1}^{l_k} |\eta_{k,m}\rangle \langle \eta_{k,m}| \, . \tag{2.36}$$

These projectors are useful in calculating the subspace detection probability for a given initial state $|\psi_{in}\rangle$ as,

$$P_{det}(|\psi_{in}\rangle) = \langle \psi_{in}| \, \mathbb{P}_{\mathcal{H}_\eta} \, |\psi_{in}\rangle \, . \tag{2.37}$$

Eq. (2.37) implies the detection probability for a given initial state depends on its projection in the bright subspace. If $|\psi_{in}\rangle = \mathcal{H}_\eta \, |\psi_{in}\rangle$, $P_{det}(|\psi_{in}\rangle) = 1$. The quantum walk starting from such an initial state will always be detected deterministically in the subspace. One such class of initial states is,

$$|\psi_{in}\rangle = \sum_{i=1}^{\tilde{r}} c_i \, |d_i\rangle \, . \tag{2.38}$$

This initial state belongs to the subspace to be detected and hence the subspace detection probability corresponds to the probability of a quantum walk to return to the subspace. Another class of initial states with subspace $P_{det}(|\psi_{in}\rangle) = 1$ are states such as,

$$|\psi_{in}\rangle = \sum_{i=1}^{\tilde{r}} \sum_k c_{i,k} \mathbb{E}_k \, |d_i\rangle \, . \tag{2.39}$$

Note these states may not belong to the detection subspace spanned by $\{|d_i\rangle\}_{i=1}^{\tilde{r}}$. Nevertheless, these initial states lie completely in the bright subspace and can guarantee subspace detection with certainty. Apart from the initial state, the subspace detection probability also depends on the size of the dark subspace spanned by the $|\zeta_{k,m}\rangle$. For certain non-trivial subspace $\{|d_i\rangle\}_{i=1}^{\tilde{r}}$ we may find no dark states. Therefore, the subspace detection probability is equal to 1 as all the energy eigenstates are bright. An important feature is that any initial state of the quantum walk is detected in this subspace with detection probability $P_{det} = 1$. This subspace is characterized by certain conditions, as stated in the theorem below.

Theorem 1. *Let H be a degenerate bounded Hamiltonian of a quantum walk with a discrete spectrum. The Hamiltonian has degeneracy g_k for each energy level E_k and, let g be the largest degeneracy of this Hamiltonian. Measurements are performed on the subspace corresponding to projector $D = \sum_{i=1}^{\tilde{r}} |d_i\rangle \langle d_i|$ where $\langle d_i|d_j\rangle = \delta_{i,j}$ and $\langle d_i|E_{k,m}\rangle \neq 0$. The quantum walk starting from $|\psi_{in}\rangle = |s\rangle$ is detected within the subspace deterministically $P_{det}(|s\rangle) = 1$ if,*

- $\tilde{r} \geq g$
- *rank($\mathcal{A}_k$) = g_k for all degenerate energy levels E_k.*

Proof. We prove the if side of the theorem first. Consider, a subspace of dimension $\tilde{r}$ with basis $\{|d_i\rangle\}_{i=1}^{\tilde{r}}$, $\langle d_i|d_j\rangle = \delta_{i,j}$. $\tilde{r} \geq g$ and for each degenerate energy level E_k $\text{rank}(\mathcal{A}_k) = g_k$. Along with this, we had also assumed $\langle E_{k,m}|d_i\rangle \neq 0$. Therefore, all the energy states are bright. The corresponding bright states are the energy eigenstates of the Hamiltonian

$$|\eta_{k,m}\rangle = |E_{k,m}\rangle \tag{2.40}$$

Therefore the bright state projector is,

$$\begin{aligned}\mathbb{P}_{\mathcal{H}_\eta} &= \sum_k \sum_{m=1}^{\text{rank}(\mathcal{A}_k)} |\eta_{k,m}\rangle \langle \eta_{k,m}| \\ &= \sum_k \sum_{m=1}^{g_k} |E_{k,m}\rangle \langle E_{k,m}| \\ &= \mathbb{I}\end{aligned} \tag{2.41}$$

Therefore, any initial state $|s\rangle$ has is detected in the subspace by probability

$$\begin{aligned}P_{det}(|s\rangle) &= \langle s| \mathbb{P}_{\mathcal{H}_\eta} |s\rangle \\ &= \langle s|s\rangle = 1\end{aligned} \tag{2.42}$$

We will refer to $P_{det}(|\psi_{in}\rangle) = \langle\psi_{in}| \mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle$ as our definition of detection probability. Notice,

$$P_{det}(|\psi_{in}\rangle) = 1 \implies \mathbb{P}_{\mathcal{H}_\eta} = \mathbb{I}, \text{ the identity operator.}$$

Therefore, the bright states span the complete Hilbert space without dark energy states. Therefore, each energy subspace E_k degenerate or not, must be bright. Going back to the dark states condition Eq. (2.23), for a non-degenerate energy level to be bright $\langle d_i|E_k\rangle \neq 0$. In case of degenerate energy levels, dark states exist if and only if dim($\mathcal{N}_{\mathcal{A}_k}$) > 0, where $\mathcal{N}_{\mathcal{A}_k}$ is the nullspace of $\mathcal{A}_k$ Eq. (2.25). This implies rank($\mathcal{A}_k$) $= g_k$ for all degenerate energy levels E_k. The matrix $\mathcal{A}_k$ is an $\tilde{r} \times g_k$ matrix defined as

$$\mathcal{A}_k = \begin{bmatrix} \langle d_1|E_{k,1}\rangle & \langle d_1|E_{k,2}\rangle & \dots & \langle d_1|E_{k,g_k}\rangle \\ \langle d_2|E_{k,1}\rangle & \langle d_2|E_{k,2}\rangle & \dots & \langle d_2|E_{k,g_k}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle d_{\tilde{r}}|E_{k,1}\rangle & \langle d_{\tilde{r}}|E_{k,2}\rangle & \dots & \langle d_{\tilde{r}}|E_{k,g_k}\rangle \end{bmatrix} \tag{2.43}$$

The $\mathrm{rank}(\mathcal{A}_k) \leq \min(\tilde{r}, g_k)$. Therefore, $\mathrm{rank}(\mathcal{A}_k) = g_k$ is possible if $\tilde{r} \geq g$ where g is the largest degeneracy.

□

Chapter 3

Results and Discussion

So far, we have formulated the problem of subspace detection in quantum walks. To calculate the first detection statistics such as F_n, P_{det} and $\overline{n}$, we reformulated the problem and derived an exact expression for the same in Eq. (2.17) and Eq. (2.18), (2.19). We use this formulation to calculate P_{det} for subspaces in on a ring system. Throughout our analysis the detection subspace $\{|d_i\rangle_{i=1}^{r}\}$ is defined by choosing $|d_i\rangle$ to be localised position on ring.

3.1 System: Ring with size $L = even$

The Hamiltonian of a ring system is the adjacency matrix of a cyclic graph with L sites Fig. (1.3.1). The Hamiltonian is

$$H = -\gamma \sum_{i=0}^{L-1} |i+1\rangle \langle i| + |i\rangle \langle i+1| \tag{3.1}$$

For L =even, the eigenvectors and eigenvalues are given by

$$\begin{aligned} \epsilon_p &= -2\gamma \cos\left(\frac{2\pi p}{L}\right) \\ |\epsilon_p\rangle &= \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} \exp\left(\frac{i2\pi pj}{L}\right) |j\rangle \end{aligned} \tag{3.2}$$

where $p = 0, 1, \ldots, L-1$. The states $|\epsilon_0\rangle$ and $\left|\epsilon_{L/2}\right\rangle$ are non-degenerate. $|\epsilon_k\rangle$ has the same energy as $|\epsilon_{L-k}\rangle$. For convenience, we will relabel the spectrum with respect to distinct energy levels as follows

$$
\begin{aligned}
\{E_k,\ |E_k\rangle\, | k = 0, L/2\} \\
\left\{E_k,\ \{|E_{k,m}\rangle\}_{m=1}^{2}\, | k = 1, 2, \ldots, \frac{L}{2} - 1\right\}
\end{aligned} \tag{3.3}
$$

To study the effects of the size of the detection subspace on the detection probability, we will vary the dimension of our detection subspace $\tilde{r}$. Therefore, the projector of the detection subspace D is of rank $\tilde{r}$. Keeping the rank constant and changing the states $|d_i\rangle$ used to construct the subspace, we can analyze and compare detection probabilities in different detection subspaces but of the same dimension. We have used the reformulation technique Eq. (2.18) to calculate detection probability in our results.

3.1.1 $\tilde{\mathbf{r}} = \mathbf{1}$

A 1-dimensional subspace is just a state. The result seen here is exactly the result of state detection as discussed in [23]. We repeat the analysis of P_{det} and calculate bright and dark states here for $|\psi_{in}\rangle = |s\rangle$, $|\psi_d\rangle = |d\rangle$ where d, s are some localized positions on the lattice. The measurement operator is the projector $D = |d\rangle\langle d|$. As shown in [23]

$$
P_{det} = \sum_k \frac{|\langle d|\, \mathbb{E}_k\, |s\rangle|^2}{\langle d|\, \mathbb{E}_k\, |d\rangle} \tag{3.4}
$$

In this case, the non-degenerate energy states are completely bright as $\langle E_0|d\rangle \neq 0$ and $\langle E_{L/2}|d\rangle \neq 0$. The bright states are the energy eigenstates themselves $|\eta_0\rangle = |E_0\rangle$, $|\eta_{L/2}\rangle = |E_{L/2}\rangle$ The doubly degenerate energy levels $\epsilon_k = \epsilon_{L-k}$ are partially bright, giving one dark and one bright state each.

$$
|\eta_k\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i2\pi kd}{L}}\, |\epsilon_k\rangle + e^{\frac{i2\pi kd}{L}}\, |\epsilon_{L-k}\rangle\right) \tag{3.5}
$$

$$
|\zeta_k\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i2\pi kd}{L}}\, |\epsilon_k\rangle - e^{\frac{i2\pi kd}{L}}\, |\epsilon_{L-k}\rangle\right) \tag{3.6}
$$

Using Eq. (3.4) we can calculate P_{det} as

$$
P_{det} = \frac{2}{L} + \frac{2}{L}\sum_{k=1}^{\frac{L}{2}-1} \cos^2\left[\frac{2\pi k(d-s)}{L}\right] = \begin{cases} 1 & s = d, d + \frac{L}{2} \\ 1/2 & o/w \end{cases} \tag{3.7}
$$

3.1.2 $\tilde{\mathbf{r}} = 2$

Here, we detect subspaces of dimension 2. The subspaces are constructed using two localized positions $|d_1\rangle$ and $|d_2\rangle$. Notice that the system Eq. (3.1) has a translation symmetry; we can change the detectors $|d_1\rangle$, $|d_2\rangle$ and the initial state $|\psi_{in}\rangle = |s\rangle$ by a constant number as $|d_1 + c\rangle$, $|d_2 + c\rangle$ and $|s + c\rangle$ and the detection probability is independent of this translation. This is helpful as we can fix one of the detectors, say $|d_1\rangle = |0\rangle$ and vary the other one as $|d_2\rangle = |d\rangle$ with $d \neq 0$. Now we can calculate P_{det} for initial states $|\psi_{in}\rangle = |s\rangle$ with $s = 0, 1, \ldots L$ and cover all possibilities of subspaces of dimension 2. The projector over these subspaces is

$$D = |0\rangle\langle 0| + |d\rangle\langle d| \text{ with } d \neq 0 \tag{3.8}$$

Let's find the dark/bright states in the system. The non-degenerate levels are again completely bright as $\langle\epsilon_k|0\rangle \neq 0$ and $\langle\epsilon_k|d\rangle \neq 0$ for $k = 0, L/2$. Therefore, the bright states are

$$|\eta_0\rangle = |E_0\rangle \tag{3.9}$$

$$|\eta_{L/2}\rangle = |E_{L/2}\rangle \tag{3.10}$$

In case of doubly degenerate energy levels $\epsilon_k = \epsilon_{L-k}$ the level is not completely dark as $\mathbb{E}_k|0\rangle \neq 0$ and $\mathbb{E}_k|d\rangle \neq 0$. Therefore, using Eq. (2.23) to find dark states the matrix $\mathcal{A}_k$ is

$$\mathcal{A}_k = \begin{bmatrix} \langle 0|\epsilon_k\rangle & \langle 0|\epsilon_{L-k}\rangle \\ \langle d|\epsilon_k\rangle & \langle d|\epsilon_{L-k}\rangle \end{bmatrix} \tag{3.11}$$

$$= \frac{1}{L}\begin{bmatrix} 1 & 1 \\ \exp\left(\frac{i2\pi kd}{L}\right) & \exp\left(\frac{-i2\pi kd}{L}\right) \end{bmatrix} \tag{3.12}$$

$$REF(\mathcal{A}_k) = \begin{bmatrix} 1 & 1 \\ 0 & 2i\sin\left(\frac{2\pi kd}{L}\right) \end{bmatrix} \tag{3.13}$$

For $kd \neq mL/2$ where m is an integer, the rank$(\mathcal{A}_k) = 2$. Therefore, the nullspace is trivial, and no dark states exist in the energy subspace. If $kd \neq mL/2$ for all k then there are no dark states in the entire energy spectrum. The bright subspace projector $\mathbb{P}_{\mathcal{H}_\eta}$ is the $\mathbb{P}_{\mathcal{H}_\eta} = \sum_k \mathbb{E}_k = \mathbb{I}$. This is not possible if there exist some d for which $kd = mL/2$ for all k. One such example is $d = \frac{L}{2}$, where matrix $\mathcal{A}_k$ is

$$\mathcal{A}_k = \frac{(-1)^k}{L}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We can easily calculate rank($\mathcal{A}_k$) $= 1$ and the dim($\mathcal{N}_{\mathcal{A}k}$) = 1. Therefore, the energy subspace is contains one dark state and one bright state. The dark state is

$$|\zeta_k\rangle = \frac{1}{\sqrt{2}} \left(|\epsilon_k\rangle - |\epsilon_{L-k}\rangle\right). \tag{3.14}$$

The bright state of the energy subspace, which is orthogonal to the dark state $|\zeta_k\rangle$ is

$$|\eta_k\rangle = \frac{1}{\sqrt{2}} \left(|\epsilon_k\rangle + |\epsilon_{L-k}\rangle\right). \tag{3.15}$$

Using this information of bright sates we can provide an expression for detection probability as $P_{det} = \langle\psi_{in}| \mathbb{P}_{\mathcal{H}_\eta} |\psi_{in}\rangle$.
For $kd \neq mL/2$, we found no dark energy eigenstates and

$$\mathbb{P}_{\mathcal{H}_\eta} = \sum_{k,m} |E_{k,m}\rangle \langle E_{k,m}| = \mathbb{I}. \tag{3.16}$$

As $\mathbb{P}_{\mathcal{H}_\eta} = \mathbb{I}$, $P_{det}(|s\rangle) = 1$. But if $d = L/2$ then each doubly degenerate energy has one dark state and on bright state as shown in Eq. (3.15) and Eq. (3.14). The bright subspace projector is

$$\mathbb{P}_{\mathcal{H}_\eta} = |E_0\rangle\langle E_0| + \left|E_{L/2}\right\rangle\left\langle E_{L/2}\right| + \sum_{k=1}^{L/2-1} |\eta_k\rangle\langle\eta_k|. \tag{3.17}$$

Hence, $P_{det}(|s\rangle)$ is

$$P_{det}(|s\rangle) = \frac{2}{L} + \frac{2}{L}\sum_{k=1}^{L/2-1} \cos^2\left[\frac{2\pi ks}{L}\right] = \begin{cases} 1 & s = 0, L/2 \\ 1/2 & o/w. \end{cases} \tag{3.18}$$

Notice that the result for subspace detection probability in the case of $d = L/2$ matches exactly with the result of subspace detection on a single state Eq. (3.4) with $d = 0$ or $d = L/2$. The dark and bright states given in Eq. (3.6) and Eq. (3.5) for $d = 0$ or $d = L/2$ match precisely with the bright/dark states Eq. (3.14) and Eq. (3.15) up to an overall phase. This similarity in results will be discussed later. Using Eq. (2.18) we calculate detection probability for a combination of $|d\rangle$ and initial state $|s\rangle$ on the ring Eq. (3.1) with $L = 10$.

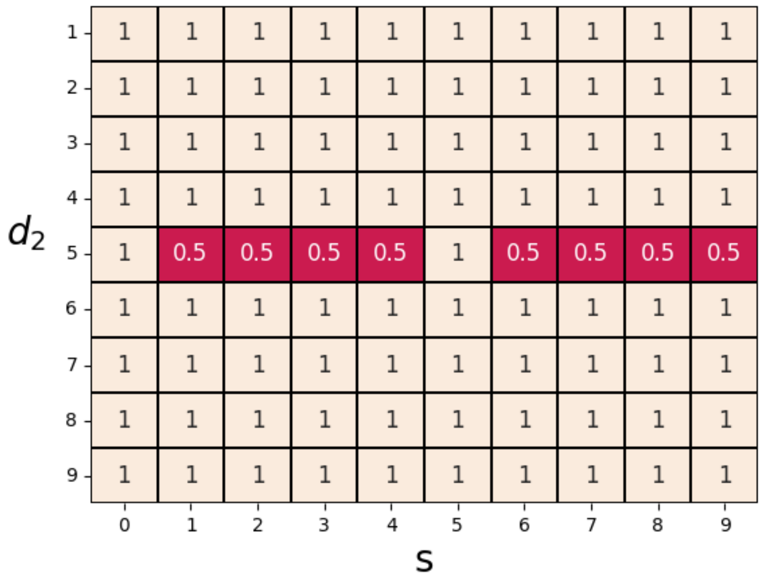


Figure 3.1: (Color online) 2-dimensional subspace detection probability for a ring system (L=10) as defined in Eq. (3.1) and detector as Eq. (3.8)

3.1.3 $\tilde{\mathrm{r}} = 3$

We now move on to 3-dimensional subspaces spanned by position states $|d_1\rangle, |d_2\rangle, |d_3\rangle$. Using translation symmetry as done previously, we pick $d_1 = 0$, $d_2 = 1, 2, \ldots L/2 - 1$, $d_3 = 1, 2, \ldots L$ with $d_2 \neq d_3$. Notice we only vary d_2 from 1 to $L/2$ as the detectors are symmetric with respect to reflection about $(0 - L/2)$ axis. The subspace projector is

$$D = |0\rangle\langle 0| + |d_2\rangle\langle d_2| + |d_3\rangle\langle d_3|. \tag{3.19}$$

We can calculate the detection probability by analyzing the dark/bright energy eigenstates of each energy level. The non-degenerate energy levels ϵ_0 and $\epsilon_{L/2}$ are completely bright as $\langle\epsilon_k|0\rangle \neq 0$, $\langle\epsilon_k|d_2\rangle \neq 0$, $\langle\epsilon_k|d_3\rangle \neq 0$ for $k = 0, L/2$.

In case of the doubly degenerate energy levels $\epsilon_k = \epsilon_{L-k}$ using the dark state condition 2.23

the matrix $\mathcal{A}_k$ is

$$\begin{aligned}
\mathcal{A}_k &= \begin{bmatrix} \langle 0|\epsilon_k\rangle & \langle 0|\epsilon_{L-k}\rangle \\ \langle d_2|\epsilon_k\rangle & \langle d_2|\epsilon_{L-k}\rangle \\ \langle d_3|\epsilon_k\rangle & \langle d_3|\epsilon_{L-k}\rangle \end{bmatrix} && (3.20) \\
&= \frac{1}{L^{3/2}} \begin{bmatrix} 1 & 1 \\ \exp\left(\frac{i2\pi k d_2}{L}\right) & \exp\left(\frac{-i2\pi k d_2}{L}\right) \\ \exp\left(\frac{i2\pi k d_3}{L}\right) & \exp\left(\frac{-i2\pi k d_3}{L}\right) \end{bmatrix} && (3.21) \\
\mathrm{ref}(\mathcal{A}_k) &= \begin{bmatrix} 1 & 1 \\ 0 & 2i\sin\left(\frac{2\pi k d_2}{L}\right) \\ 0 & 2i\sin\left(\frac{2\pi k d_3}{L}\right) \end{bmatrix} && (3.22)
\end{aligned}$$

We can see that $\mathrm{rank}(\mathcal{A}_k) = 2$ as long as $d_2 \neq d_3$ and $kd_2 \neq m_2 L/2$ or $kd_3 \neq m_3 L/2$ for some integer m_2, m_3. Therefore, the energy subspace has no dark states; consequently if $kd_2 \neq m_2 L/2$ or $kd_3 \neq m_3 L/2$ is satisfied for all k then the entire spectrum has no dark energy states. Therefore, we can conclude that such a subspace will certainly be detected.

In the Fig. (3.2) we calculate 3-dimensional subspace detection probability using Eq. (2.18) for a quantum walk on a ring $(L = 10)$. Considering the initial state as $|\psi_{in}\rangle = |s\rangle$ a localized position, and the projector used is Eq. (3.19).

From Fig. (3.2), we see detection probability is independent of the initial state and the detected position subspace. Thus, here, we can conclude that a quantum walk on a ring ($L = 10$) is certainly detected in a 3-dimensional position subspace independent of the initial position state. Note we state this result in subspaces with projectors given by 3.19.

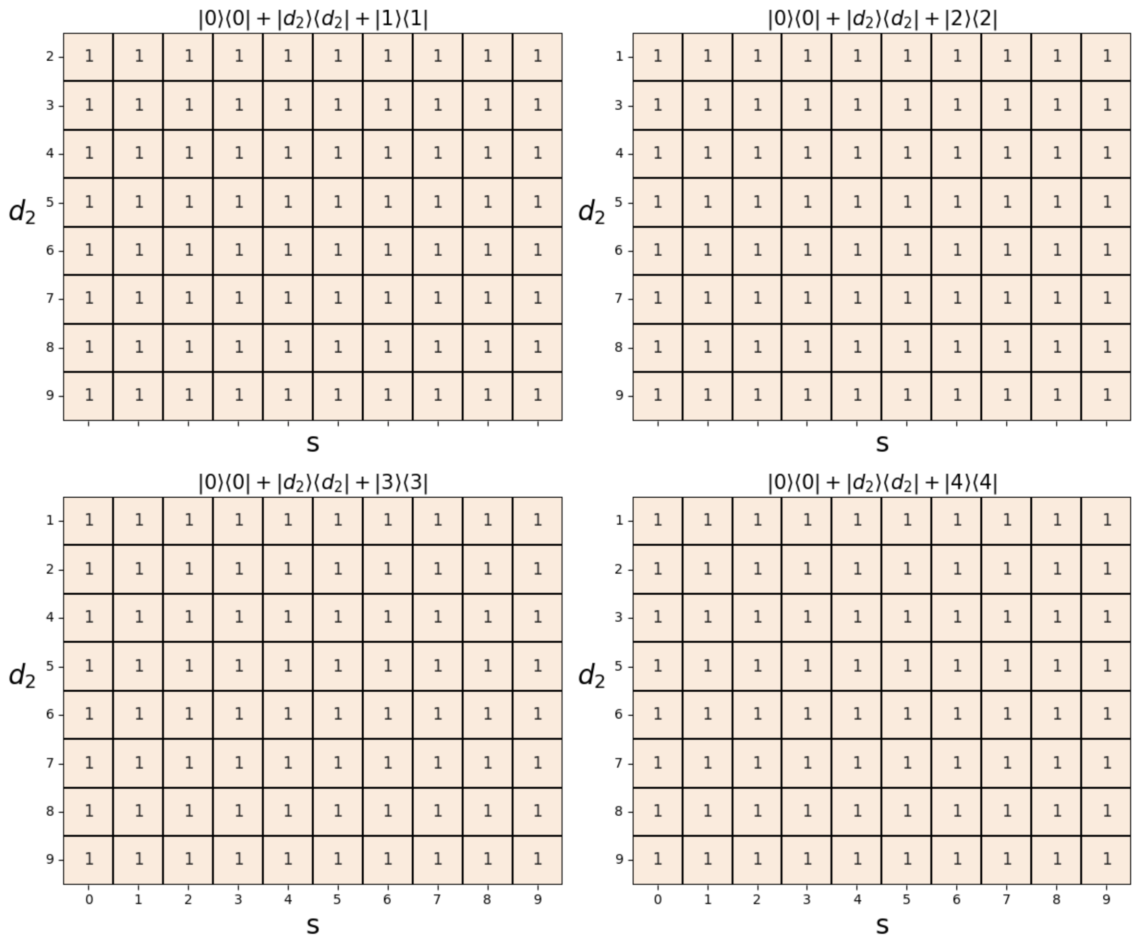


Figure 3.2: (Color online) 3-dimensional subspace detection probability on a ring system (L=10) as defined in Eq. (3.1) and detector as Eq. (3.19)

3.2 System: Ring with additional next nearest neighbor interactions $L = 10n$

To change the degeneracy in the ring systme, we introduce additional edges. These edges connect each vertex in the cycle to its next-nearest neighbor. The Hamiltonian is given by

$$H = -\gamma \sum_{i=0}^{L-1} |i+1\rangle \langle i| + |i\rangle \langle i+1| + |i+2\rangle \langle i| + |i\rangle \langle i+2| \tag{3.23}$$

The eigenvectors and eigenvalues are as follows

$$\begin{aligned} \epsilon_p &= -2\gamma \Big[\cos\Big(\frac{2\pi p}{L}\Big) + \cos\Big(\frac{4\pi p}{L}\Big) \Big] \\ |\epsilon_p\rangle &= \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} \exp\Big(\frac{i2\pi p j}{L}\Big) |j\rangle \end{aligned} \tag{3.24}$$

Due to the addition of next-nearest neighbor interactions, there is a change in the degeneracy of the spectrum.

For $L = 10n$ **and** $n \geq 1$, we have $L/2$ distinct energy levels with degeneracy as follows. It includes non-degenerate levels, doubly degenerate levels, and a 4-degenerate level. They are as follows

Non-degenerate: ϵ_0 and $\epsilon_{L/2}$ are non-degenerate.

4-degenerate: $\epsilon_{L/5} = \epsilon_{2L/5} = \epsilon_{3L/5} = \epsilon_{4L/5}$

2-degenerate: $\epsilon_k = \epsilon_{L-k}$ $\forall k \neq 0, L/2, L/5, 2L/5, 3L/5, 4L/5$ We can re-label the eigenstates as follows for $L = 10$.

Degeneracy g_k	Eigenvalues E_k	Eigenvectors $\lvert E_{k,m}\rangle$
1	$E_0 = \epsilon_0$	$\lvert E_0\rangle = \lvert\epsilon_0\rangle$
1	$E_5 = \epsilon_5$	$\lvert E_5\rangle = \lvert\epsilon_5\rangle$
2	$E_1 = \epsilon_1 = \epsilon_9$	$\lvert E_{1,1}\rangle = \lvert\epsilon_1\rangle , \lvert E_{1,2}\rangle = \lvert\epsilon_9\rangle$
2	$E_3 = \epsilon_3 = \epsilon_7$	$\lvert E_{3,1}\rangle = \lvert\epsilon_3\rangle , \lvert E_{3,2}\rangle = \lvert\epsilon_7\rangle$
4	$E_2 = \epsilon_2 = \epsilon_8 = \epsilon_6 = \epsilon_4$	$\lvert E_{2,1}\rangle = \lvert\epsilon_2\rangle , \lvert E_{2,2}\rangle = \lvert\epsilon_4\rangle$, $\lvert E_{2,3}\rangle = \lvert\epsilon_6\rangle , \lvert E_{2,4}\rangle = \lvert\epsilon_8\rangle$

Table 3.1: Spectrum of Hamiltonian Eq. (3.23) with $L = 10$

3.2.1 $\tilde{\mathbf{r}} = \mathbf{1}$

We will briefly state the result in case of single state detection for the Hamiltonian Eq. (3.23) with $L = 10$ is.

$$\begin{aligned}
P_{det}(|s\rangle) &= \sum_k \frac{|\langle d|\,\mathbb{E}_k\,|\psi_{in}\rangle|^2}{\langle d|\,\mathbb{E}_k\,|d\rangle} \\
&= \frac{1}{5} + \frac{1}{5}\cos^2\left[\frac{\pi(d-s)}{5}\right] \\
&\quad + \frac{1}{5}\cos^2\left[\frac{3\pi(d-s)}{5}\right] \\
&\quad + \frac{1}{10}\left[\cos^2\left[\frac{2\pi(d-s)}{5}\right] + \cos^2\left[\frac{4\pi(d-s)}{5}\right]\right] && (3.25) \\
&= \begin{cases} 1 & \text{for } s = d, d + L/2 \\ 3/8 & o/w \end{cases} && (3.26)
\end{aligned}$$

One can also find the dark/bright states for this system. The non-degenerate energy levels ϵ_0 and ϵ_5 are completely bright as $\langle \epsilon_k | d\rangle \neq 0$ for $k = 0, 5$. Next, 4-degenerate level $\epsilon_2 = \epsilon_4 = \epsilon_6 = \epsilon_8$ is not dark as $\mathbb{E}_2\,|d\rangle \neq 0$. The bright state is

$$|\eta_2\rangle = \frac{1}{2}\left[e^{\frac{-i\pi 2d}{5}}\,|E_{2,1}\rangle + e^{\frac{-i\pi 4d}{5}}\,|E_{2,2}\rangle + e^{\frac{-i\pi 6d}{5}}\,|E_{2,3}\rangle + e^{\frac{-i\pi 8d}{5}}\,|E_{2,4}\rangle\right] \quad (3.27)$$

This energy subspace will have three dark states orthogonal to the bright state.

$$\begin{aligned}
\left|\zeta_{L/5,1}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|E_{L/5,2}\right\rangle - e^{\frac{i2\pi d}{5}}\left|E_{L/5,1}\right\rangle\right] \\
\left|\zeta_{L/5,2}\right\rangle &= \frac{1}{\sqrt{6}}\left[2\left|E_{L/5,3}\right\rangle - e^{\frac{i2\pi d}{5}}\left|E_{L/5,2}\right\rangle - e^{\frac{i4\pi d}{5}}\left|E_{L/1}\right\rangle\right] && (3.28) \\
\left|\zeta_{L/5,3}\right\rangle &= \frac{1}{2\sqrt{3}}\left[3\left|E_{L/5,4}\right\rangle - e^{\frac{i2\pi d}{5}}\left|E_{L/5,3}\right\rangle - e^{\frac{i4\pi d}{5}}\left|E_{L/5,2}\right\rangle - e^{\frac{i6\pi d}{5}}\left|E_{L/5,1}\right\rangle\right]
\end{aligned}$$

In the case of the doubly degenerate levels, there will be one bright and one dark state.

$$|\eta_k\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{-i\pi kd}{5}}|E_{k,1}\rangle + e^{\frac{i\pi kd}{5}}|E_{k,2}\rangle\right) \tag{3.29}$$

$$|\zeta_k\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{-i\pi kd}{5}}|E_{k,1}\rangle - e^{\frac{i\pi kd}{5}}|E_{k,2}\rangle\right) \tag{3.30}$$

3.2.2 $\tilde{\mathbf{r}} = 2$

In the case of 2-dimensional subspaces, the projector is constructed as done previously

$$D = |0\rangle\langle 0| + |d\rangle\langle d| \text{ and } d \neq 0 \tag{3.31}$$

We will analyze this by finding dark/bright states for each energy level.
In the case of non-degenerate energy levels E_0 and $E_{L/2}$ no dark states as $\langle 0|E_k\rangle \neq 0$ and $\langle d|E_k\rangle \neq 0$ for $k = 0, \frac{L}{2}$.
In the case of a 4-degenerate energy level using Eq. (2.23) and Eq. (2.25) matrix $\mathcal{A}_{L/5}$ is

$$\begin{aligned}
\mathcal{A}_{L/5} &= \begin{bmatrix} \langle 0|E_{L/5,1}\rangle & \langle 0|E_{L/5,2}\rangle & \langle 0|E_{L/5,3}\rangle & \langle 0|E_{L/5,4}\rangle \\ \langle d|E_{L/5,1}\rangle & \langle d|E_{L/5,2}\rangle & \langle d|E_{L/5,3}\rangle & \langle d|E_{L/5,4}\rangle \end{bmatrix} \\
&= \frac{\exp\left(\frac{i2\pi d}{5}\right)}{L^2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \exp\left(\frac{i2\pi d}{5}\right) & \exp\left(\frac{i3\pi d}{5}\right) & \exp\left(\frac{i4\pi d}{5}\right) \end{bmatrix}
\end{aligned} \tag{3.32}$$

$$\text{ref}(\mathcal{A}_k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \exp\left(\frac{i2\pi d}{5}\right) - 1 & \exp\left(\frac{i3\pi d}{5}\right) - 1 & \exp\left(\frac{i4\pi d}{5}\right) - 1 \end{bmatrix} \tag{3.33}$$

For $d \neq 10m$ where m is some integer, rank($\mathcal{A}_{L/5}) = 2$. Hence, we can construct two bright and two dark states of the energy subspace.

$$\begin{aligned}
|\zeta_{L/5,1}\rangle &= \mathcal{N}_1\Big[e^{(i\pi d)}|E_{L/5,1}\rangle - 2\cos\left(\frac{\pi d}{5}\right)e^{\left(\frac{i4\pi d}{5}\right)}|E_{L/5,2}\rangle \\
&\quad + e^{\left(\frac{i3\pi d}{5}\right)}|E_{L/5,3}\rangle\Big] \\
\mathcal{N}_1 &= \frac{1}{\sqrt{2[1+\cos^2\left(\frac{\pi d}{5}\right)]}}
\end{aligned} \tag{3.34}$$

$$
\begin{aligned}
\left|\zeta_{L/5,2}\right\rangle &= \mathcal{N}_2\Bigg[2i\cos^2\left(\frac{\pi d}{5}\right)\sin\left(\frac{\pi d}{5}\right)\left|E_{L/5,1}\right\rangle \\
&\quad -\frac{i}{2}\left(e^{\left(\frac{i8\pi d}{5}\right)}\sin\left(\frac{3\pi d}{5}\right)+e^{\left(\frac{i12\pi d}{5}\right)}\sin\left(\frac{\pi d}{5}\right)\right)\left|E_{L/5,2}\right\rangle \\
&\quad +\frac{e^{\left(\frac{i6\pi}{5}\right)}}{2}\left[2\left(e^{\left(\frac{i6\pi}{5}\right)}-1\right)\cos\left(\frac{\pi d}{5}\right)+i\sin\left(\frac{3\pi d}{5}\right)\right]\left|E_{L/5,3}\right\rangle \\
&\quad -ie^{\left(\frac{i8\pi d}{5}\right)}\sin\left(\frac{3\pi d}{5}\right)\left|E_{L/5,4}\right\rangle\Bigg] \qquad (3.35)\\
\mathcal{N}_2 &= \sqrt{\frac{2}{\left[20+27\cos\left(\frac{2\pi d}{5}\right)+6\cos\left(\frac{4\pi d}{5}\right)+7\cos\left(\frac{6\pi d}{5}\right)\right]\sin^2\left(\frac{\pi d}{5}\right)}}
\end{aligned}
$$

The bright states are orthogonal to dark states and hence, can be constructed using the basis $\mathbb{E}_{L/5}\left|0\right\rangle$ and $\mathbb{E}_{L/5}\left|d\right\rangle$. The Gram-Schmidt procedure is used to orthogonalize this basis giving us the bright states for the energy subspace.

$$
\begin{aligned}
\left|\eta_{L/5,1}\right\rangle &= \frac{\mathbb{E}_{L/5}\left|0\right\rangle}{\sqrt{\left\langle 0\right|\mathbb{E}_{L/5}\left|0\right\rangle}} \\
\left|\eta_{L/5,2}\right\rangle &= N\Big(\mathbb{E}_{L/5}\left|d\right\rangle-\left\langle\eta_{L/5,1}\right|\mathbb{E}_{L/5}\left|d\right\rangle\left|\eta_{L/5,1}\right\rangle\Big) \\
N &= \frac{1}{\sqrt{\left\langle d\right|\mathbb{E}_{L/5}\left|d\right\rangle-\left\langle d\right|\mathbb{E}_{L/5}\left|\eta_{L/5,1}\right\rangle\left\langle\eta_{L/5,1}\right|\mathbb{E}_{L/5}\left|d\right\rangle}} \\
\left|\eta_{L/5,1}\right\rangle &= \frac{1}{2}\left(\left|E_{L/5,1}\right\rangle+\left|E_{L/5,2}\right\rangle+\left|E_{L/5,3}\right\rangle+\left|E_{L/5,4}\right\rangle\right) \qquad (3.36)\\
\left|\eta_{L/5,2}\right\rangle &= N\Big[\left(e^{\frac{i4\pi d}{5}}+e^{\frac{i6\pi d}{5}}+e^{\frac{i8\pi d}{5}}\right)\left|E_{L/5,1}\right\rangle \qquad (3.37)\\
&\quad +\left(e^{\frac{i2\pi d}{5}}+e^{\frac{i6\pi d}{5}}+e^{\frac{i8\pi d}{5}}\right)\left|E_{L/5,2}\right\rangle \\
&\quad +\left(e^{\frac{i2\pi d}{5}}+e^{\frac{i4\pi d}{5}}+e^{\frac{i8\pi d}{5}}\right)\left|E_{L/5,3}\right\rangle \\
&\quad +\left(e^{\frac{i2\pi d}{5}}+e^{\frac{i4\pi d}{5}}+e^{\frac{i6\pi d}{5}}\right)\left|E_{L/5,4}\right\rangle\Big] \\
N &= \frac{1}{2\sqrt{1-1\left[\cos\left(\frac{2\pi d}{5}\right)+\cos\left(\frac{4\pi d}{5}\right)\right]^2}}.
\end{aligned}
$$

Next, for $d = 10m$ Rank($\mathcal{A}$) $= 1$ hence there exist three dark and one bright state.

$$\begin{aligned}
\left|\zeta_{L/5,1}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|E_{L/5,2}\right\rangle - \left|E_{L/5,1}\right\rangle\right] \\
\left|\zeta_{L/5,2}\right\rangle &= \frac{1}{\sqrt{6}}\left[2\left|E_{L/5,3}\right\rangle - \left|E_{L/5,2}\right\rangle - \left|E_{L/5,1}\right\rangle\right] \\
\left|\zeta_{L/5,3}\right\rangle &= \frac{1}{2\sqrt{3}}\left[3\left|E_{L/5,4}\right\rangle - \left|E_{L/5,3}\right\rangle - \left|E_{L/5,2}\right\rangle - \left|E_{L/5,1}\right\rangle\right]
\end{aligned} \tag{3.38}$$

$$\left|\eta_{L/5}\right\rangle = \frac{1}{2}\left[\left|E_{L/5,1}\right\rangle + \left|E_{L/5,2}\right\rangle + \left|E_{L/5,3}\right\rangle + \left|E_{L/5,4}\right\rangle\right] \tag{3.39}$$

Now we move on to doubly degenerate energy levels, the dark state condition Eq. (2.23) gives matrix $\mathcal{A}_k$ as

$$\begin{aligned}
\mathcal{A}_k &= \begin{bmatrix} \langle 0|E_{k,1}\rangle & \langle 0|E_{k,2}\rangle \\ \langle d|E_{k,1}\rangle & \langle d|E_{k,2}\rangle \end{bmatrix} \\
&= \frac{1}{L}\begin{bmatrix} 1 & 1 \\ \exp\left(\frac{i2\pi kd}{L}\right) & \exp\left(\frac{-i2\pi kd}{L}\right) \end{bmatrix}
\end{aligned} \tag{3.40}$$

$$\mathrm{ref}(\mathcal{A}_k) = \begin{bmatrix} 1 & 1 \\ 0 & 2i\sin\left(\frac{2\pi kd}{L}\right) \end{bmatrix} \tag{3.41}$$

$\mathrm{rank}(\mathcal{A}_k) = 1$ if $kd = mL/2$ for some integer m. One such example is $d = L/2$ where the energy subspace contains one dark state and one bright state. The corresponding dark and bright states are as follows

$$|\zeta_k\rangle = \frac{|E_{k,1}\rangle - |E_{k,2}\rangle}{\sqrt{2}}, \tag{3.42}$$

$$|\eta_k\rangle = \frac{|E_{k,1}\rangle + |E_{k,2}\rangle}{\sqrt{2}}. \tag{3.43}$$

In all other cases $kd \neq mL/2$ we find Rank($\mathcal{A}_k$)$=$ 2. $\mathcal{A}_k$ has trivial nullspace, and the doubly degenerate levels are completely bright. The dark and bright found in the 4-degenerate levels Eq. (3.39) and Eq. (3.38) are an exact match to the dark and bright states calculated for the problem of single state detection Eq.(3.27) and Eq.(3.28) with $d = 0$. Similarly, the dark and bright states of doubly degenerate levels for $d = L/2$ Eq. (3.43) and Eq. (3.42) match exactly with dark and bright state calculated in single state detection in Eq. (3.29) and Eq. (3.30).
Using these bright states, we can give an expression for P_{det} for a system $L = 10$ using Eq. (3.36).

For $d \neq 5 = L/2$.

$$\begin{aligned}
P_{det}(|s\rangle) &= \langle s|\,\mathbb{P}_{\mathcal{H}_\eta}\,|s\rangle \\
&= \frac{6}{10} + \frac{1}{10}\Big[\cos\Big(\frac{2s\pi}{5}\Big) + \cos\Big(\frac{4s\pi}{5}\Big)\Big]^2 \\
&+ \frac{N}{10}\Big[\frac{3}{4}\cos\Big(\frac{2(s-d)\pi}{5}\Big) + \frac{3}{4}\cos\Big(\frac{4(s-d)\pi}{5}\Big) \\
&- \frac{1}{4}\cos\Big(\frac{2(s+d)\pi}{5}\Big) - \frac{1}{4}\cos\Big(\frac{4(s+d)\pi}{5}\Big) \\
&- \frac{1}{4}\cos\Big(\frac{2(s+2d)\pi}{5}\Big) - \frac{1}{4}\cos\Big(\frac{2(s-2d)\pi}{5}\Big) \\
&- \frac{1}{4}\cos\Big(\frac{2(2s+d)\pi}{5}\Big) - \frac{1}{4}\cos\Big(\frac{2(2s-d)\pi}{5}\Big)\Big]^2 \qquad (3.44) \\
&= \begin{cases} 1 & s = 0, d, 5, d+5 \\ 2/3 & o/w. \end{cases} \qquad (3.45)
\end{aligned}$$

And for $d = 5 = L/2$,

$$\begin{aligned}
P_{det} &= \frac{2}{10} + \frac{2}{10}\cos^2\left(\frac{2\pi s}{10}\right) + \frac{2}{10}\cos^2\left(\frac{6\pi s}{10}\right) \\
&\quad + \frac{1}{10}\left[\cos^2\left(\frac{4\pi s}{10}\right) + \cos^2\left(\frac{8\pi s}{10}\right)\right] \\
&= \begin{cases} 1 & s = 0, 5 \\ 3/8 & o/w. \end{cases} \qquad (3.46)
\end{aligned}$$

For $L = 10$, a numerical calculation is also performed Fig. (3.3) for subspace detection probability using Eq. (2.18) which matches with the analytical result. Eq.(3.44) shows an increase in the 2-dimensional subspace probability in the case $d \neq 5$ compared to single state detection in the same system Eq.(3.25). This is expected as an increased number of bright energy states are found in each degenerate energy subspace, thus increasing the detection probability.

In the case of the subspace $d_1 = 0$ and $d_2 = 5$ no increase is seen in the subspace detection probability Eq. (3.46) compared to earlier single-state detection Eq.(3.25).

The subspace $d_1 = 0$ and $d_2 = 5$ is peculiar as it detection probability is no better than its individual states $|0\rangle$ and $|5\rangle$ in its basis.

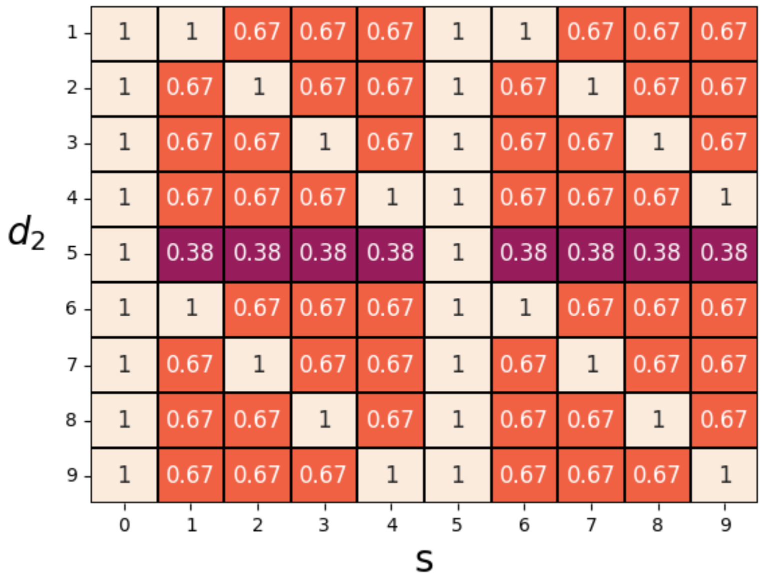


Figure 3.3: (Color Online) 2-dimensional subspace detection probability for a ring with additional next-nearest neighbor interactions (L=10) as defined by Eq. (3.23) and detector Eq. (3.31). Note: The values are rounded up to 2 decimal places

3.2.3 $\tilde{\mathbf{r}} = 3$

Next, we will calculate the detection probability of 3-dimensional subspace constructed by the position states $|d_1\rangle, |d_2\rangle, |d_3\rangle$. Due to symmetry, we will fix $d_1 = 0$, $d_2 = 1, 2, \ldots L/2 - 1$ and $d_3 = 1, 2, \ldots, L$. This covers all 3-dimensional subspaces spanned by position states. The projector for the subspace is given by

$$D = |0\rangle\langle 0| + |d_2\rangle\langle d_2| + |d_3\rangle\langle d_3| \tag{3.47}$$

In the system with $L = 10$ and Hamiltonian given by Eq. (3.23), we can calculate 3-dimensional detection probability using the Eq. (2.18). The results are displayed in Fig .(3.4)

$$P_{det}(|s\rangle) = \begin{cases} 1 & s = 0, d_1, d_2, 5, 5 + d_1, 5 + d_2 \\ 2/3 & d_2 = 5, d_1 + 5 \\ 3/4 & o/w. \end{cases} \tag{3.48}$$

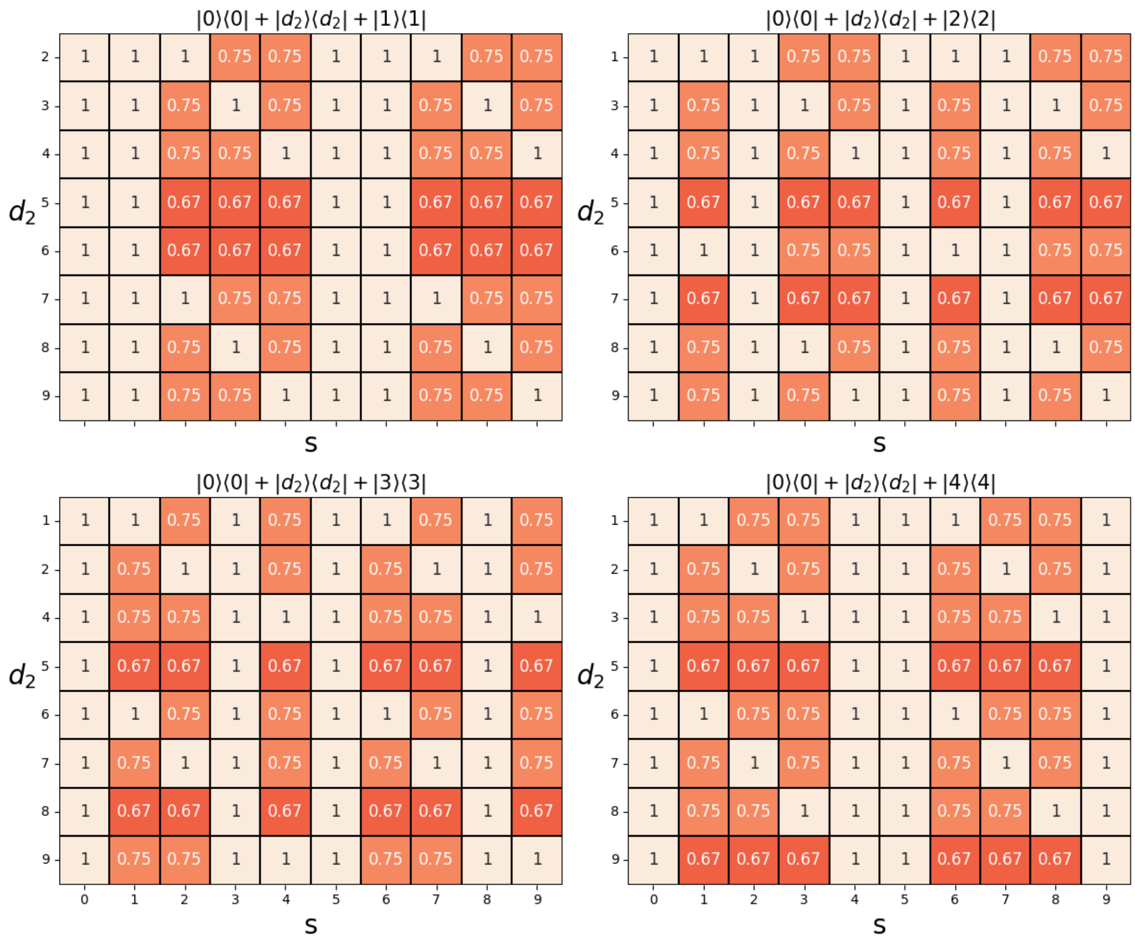


Figure 3.4: (Color Online) 3-dimensional subspace detection probability for for a ring with additional next-nearest neighbor interactions (L=10) as defined by Eq. (3.23) and detector Eq. (3.47). Note: The values are rounded up to 2 decimal places

3.3 Discussion on P_{det}

In the previous section, we calculated the subspace detection probability using the matrix method given in Eq. (2.18) for specific sizes and found the dark/bright states for degenerate subspaces of various dimensions. These dark/bright states determine the detection probability of the subspace for a given initial state $|\psi_{in}\rangle$. For initial states $|\psi_{in}\rangle = |d_i\rangle, |d_i + L/2\rangle$ the subspace detection probability is $P_{det}(|\psi_{in}\rangle) = 1$ on the ring. This result was also seen in the case of single state detection as shown in Ref[23] for a ring, and we have found this result to be true in the case of subspace detection. The initial state $|d_i + L/2\rangle$ is unique as starting from this state, we can detect the quantum walk in the subspace with probability $P_{det} = 1$. It is the only initial state other than $|d_i\rangle$ to have $P_{det} = 1$ independent of the subspace dimension of the system size. This is due to the fact that any projection of $|d_i + L/2\rangle$ in a degenerate energy subspace can be spanned by the bright state basis of the subspace, and this is only possible for ring systems. Therefore, $|d_i + L/2\rangle$ has no finite projection in the dark subspace. For general graphs possessing symmetry, one can find similar states outside the subspace with a detection probability equal to one.

Naturally, in the case of subspace detection, the detection probability is higher than the detection of a particular state. This is intuitive due to the fact that in subspace detection, the bright energy states corresponding to each energy subspace generally increase for subspace detection. However, in a ring, we find subspaces where this general rule is not followed. These exceptional subspaces are the subspaces that consist of a pair of diametrically opposite position states. Due to the symmetry of ring systems, the detection probability of a subspace consisting of a pair of diametrically opposite positions is no better than the individual state detection probability. The bright energy states of these two subspaces are equal to an overall phase factor. This hints at a redundancy of certain higher-dimensional subspaces to subspaces of lower dimension with respect to detection probability. For example, in subsection [3.1.2], the calculation of detection probability for the 2-dimensional subspaces with basis $|0\rangle$ and $|5\rangle$ show no increase in P_{det} for any localized initial state $|s\rangle$ compared to detection probability found for single state detection in subsection [3.1.2]. The rank($\mathcal{A}_k$) is the dimension of bright subspace in the energy eigensubspace of E_k which is spanned by $\mathbb{E}_k |d_i\rangle$. Hence, going from a r-dimensional subspace to $r+1$, the detection probability increases if the $\mathbb{E}_k |d_i\rangle$ spans a larger bright subspace than the previous one.

Subsection [3.1.3] is an example where the subspace detection probability is one and independent of the initial state. This is an ideal scenario where no dark states are observed; all energy states are, therefore, bright, and hence, a quantum walk starting from any initial state is detected with certainty. Such subspaces are essential and can be deemed targets in quantum search problems involving multiple solutions.

Chapter 4

Conclusion

In this thesis, we have formulated the concept of the first detection of a subspace on quantum walks and defined statistics such as first detection probability and subspace detection probability in case of detection on subspace. Through stroboscopic measurement protocol, the desired subspace is detected, and it is found that the subspace detection probability is independent of the time of unitary evolution between the measurements. The subspace projector D splits the Hilbert space into two mutually orthogonal subspaces: Dark and Bright. These subspaces are invariant under the survival operator $\mathbb{S}$. To define the dark subspace, we first find the dark energy states as the nullspace of the matrix $\mathcal{A}_k$ for each degenerate energy level and also provide definitions for non-degenerate dark states. Following this, we identify the dark energy states as the states belonging to nullspace of $\mathcal{A}_k$ for each degenerate energy level and subsequently provide a method to find all the dark energy states in the degenerate energy subspace using an iterative cross-product formula. The $\mathrm{rank}(\mathcal{A}_k)$ determines the bright energy states in the degenerate energy subspace and are mutually orthogonal to dark energy states. These bright states lie in the row space of $\mathcal{A}_k$, and an orthogonal set of bright states can be found using the Gram-Schmidt procedure. The projection of the initial state onto the total bright subspace determines the subspace detection probability P_{det} of the quantum walk. Subsequently, we have found dark and bright energy states in a ring hamiltonian for position subspaces of various dimensions. If the initial state lies completely inside the bright subspace, the subspace detection probability is $P_{det} = 1$. This is seen trivially for initial states belonging to the detection subspace. Another class of initial states with $P_{det} = 1$ are the states in the span of the bright energy states but are not from the detection subspace. In a ring system, such initial states are position states diametrically opposite to the detected states.

Increasing the dimension of the detected subspace generally increases the detection probability. This increment can be obtained by adding an orthogonal state to the existing subspace. However, this procedure may not increase the detection probability generally. If the added state belongs to the bright subspace, then the subspace detection probability does not change. This insight is useful in quantum searches involving multiple target states. The encoding of the target states should be done such that the subsequent additions of target states increase the total bright subspace dimension. At last, we find certain subspaces for which the detection probability is independent of the initial state, and the quantum walk is detected deterministically in the subspace $P_{det} = 1$. This implies that the bright subspace spans the total Hilbert space. Such subspaces are of unique importance from a quantum search perspective, as these are perfect candidates for encoding the target states in a typical quantum multiple search problem. Such a subspace depends on the system size and degeneracy, and we provide a theorem listing all the conditions such a subspace needs to satisfy. One of these conditions requires the dimension of the subspace $\tilde{r}$ to be greater than or equal to the largest degeneracy in the system. The degeneracy in a system is usually attributed to symmetries present in a system. Hence, this condition relates the existence of such subspaces to the symmetries present in the system. Further research is necessary to find such subspaces. The degeneracies in the spectrum are related to the symmetries of the graph and future work must focus on the same.

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